Chapter 8
Analytic Geometry

Section 8.1: Parabolas

- Equations of Parabolas

Equations of Parabolas

We recall from Section 2.1 that the graph of a function of the form $y = ax^2 + bx + c$ with $a \neq 0$ is a parabola. However, not all parabolas can be described by such an equation since not all parabolas are graphs of functions.

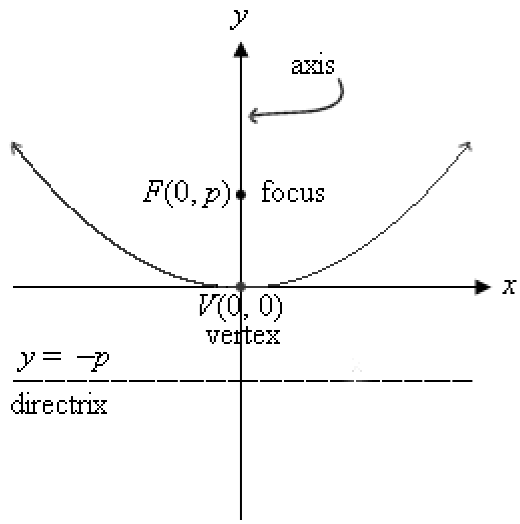
In this section, we give a more general definition of a parabola and use that definition to derive additional equations of parabolas.

Definition of a Parabola:

A parabola is the set of all points in the plane that are equidistant from a fixed line (called the directrix) and a fixed point not on the line (called the focus).

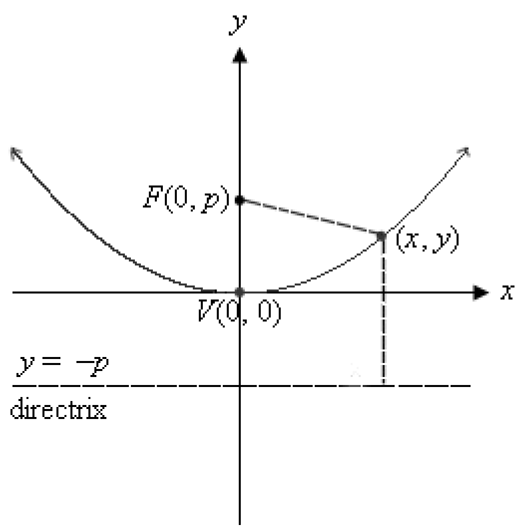
Equations of Parabolas with Vertex at the Origin:

To see how to derive the equation of a parabola, we consider a special case where the focus is the point $F(0, p)$ and the directrix is the line $y = -p$, where p is any positive number.



The line that passes through the focus and is perpendicular to the directrix is called the <u>axis</u> of the parabola. The <u>vertex</u> of the parabola is the point on the parabola that is halfway between the focus and the directrix. This is the point that is closest to both the focus and the directrix.

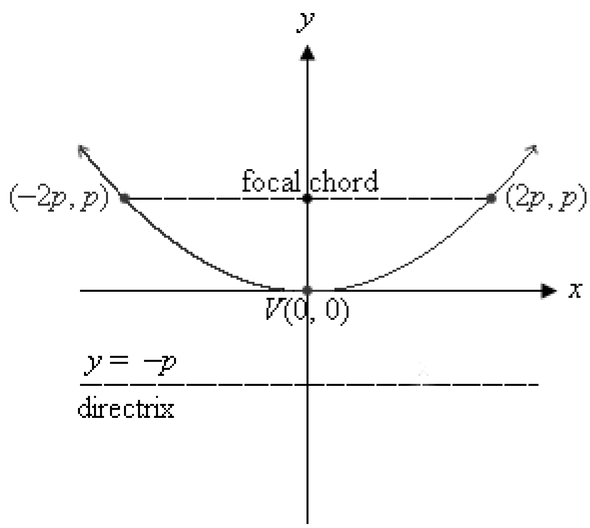
Now, let (x, y) be any point on the parabola. By definition, the distance from the point (x, y) to the focus $(0, p)$ is equal to the distance from the point (x, y) to the directrix $y = p$. Thus, $\sqrt{(x-0)^2 + (y-p)^2} = y + p$. (See the figure below.)



Square both sides of the equation $\sqrt{(x-0)^2+(y-p)^2}=y+p$ to obtain the equation.

$$\begin{aligned}(x-0)^2+(y-p)^2&=(y+p)^2\\x^2+y^2-2py+p^2&=y^2+2py+p^2\\x^2-2py&=2py\\x^2&=4py\end{aligned}$$

The line segment that passes through the focus and perpendicular to the axis with endpoints on the parabola is called the focal chord. Its length (called the focal width) is $4p$.



Example:

Graph the parabola $x^2=\frac{7}{4}y$. Specify the focus, the directrix, the focal width, and the vertex.

Solution:

The equation is of the form $x^2=4py$, where $4p=\frac{7}{4}$. Thus, $p=\frac{7}{16}$.

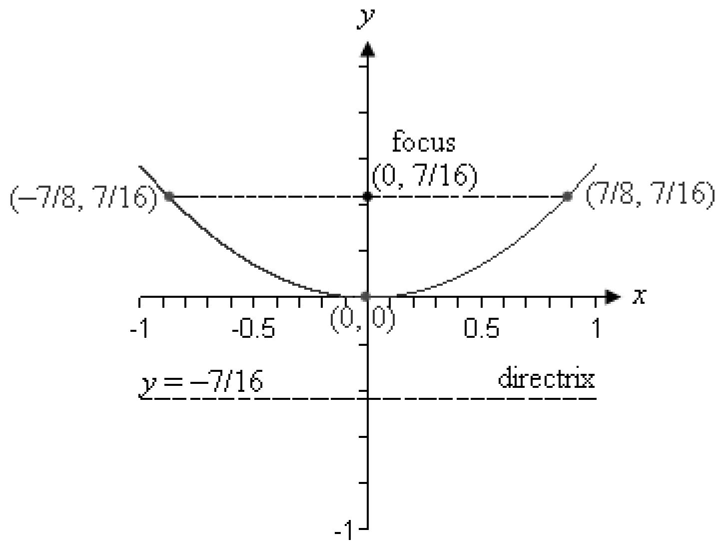
Focus: $(0,p)=\left(0,\frac{7}{16}\right)$

Directrix: $y=-p\Rightarrow y=-\frac{7}{16}$

Focal width: $4p=\frac{7}{4}$

Vertex: $(0,0)$

The graph is shown below.

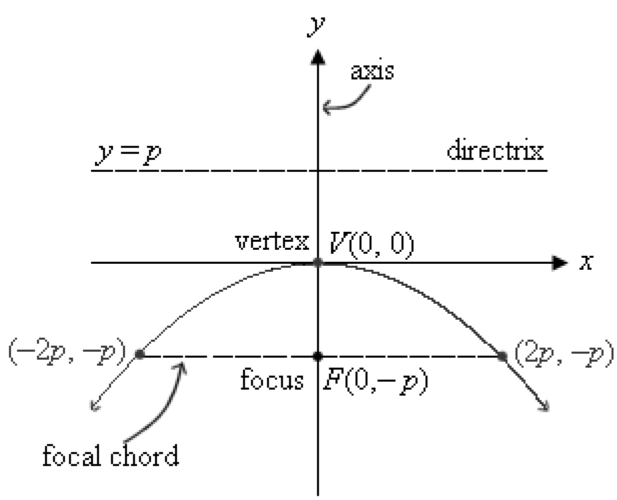


Additional Equations of Parabolas with Vertex at the Origin:

Basic equations can be derived in a similar manner for parabolas that have vertex at the origin and open downward, to the right, and to the left. The results are summarized below. We assume that $p > 0$.

Opening Downward:

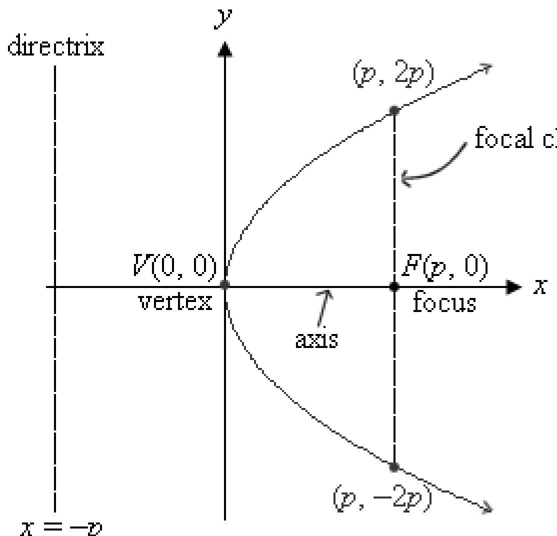
Parabola opening downward with vertex at the origin: $x^2 = -4py$



Graph of the parabola $x^2 = -4py$.

Opening to the Right:

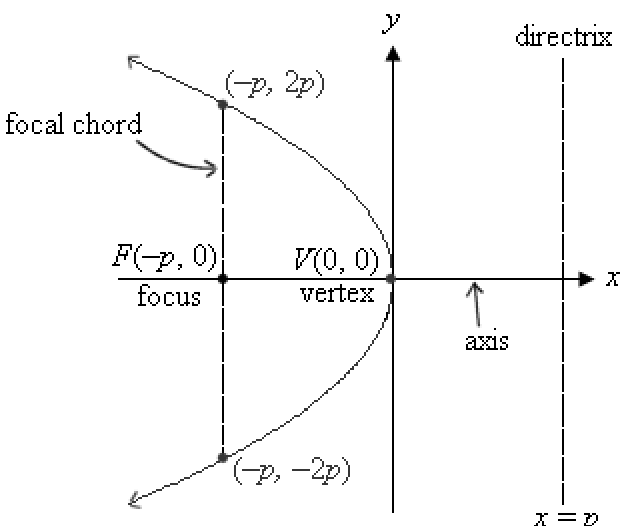
Parabola opening to the right with vertex at the origin: $y^2 = 4px$



Graph of the parabola $y^2 = 4px$.

Opening to the Left:

Parabola opening to the left with vertex at the origin: $y^2 = -4px$



Graph of the parabola $y^2 = -4px$.

Example:

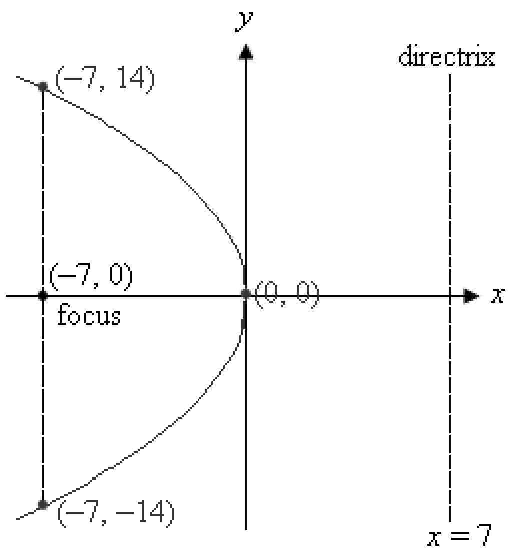
Graph the parabola $y^2 + 28x = 0$. Specify the focus, the directrix, the focal width, the vertex, and the endpoints of the focal chord.

Solution:

The given equation can be written as $y^2 = -28x$ by subtracting $28x$ from both sides of the equation. Thus, the equation is of the form $y^2 = -4px$, where $-4p = -28$. Hence, $p = 7$. The parabola opens to the left.

Focus: $(-p, 0) = (-7, 0)$
Directrix: $x = p \Rightarrow x = 7$
Focal width: $4p = 28$
Vertex: $(0, 0)$
Endpoints of the Focal Chord: $(-p, 2p) = (-7, 14)$ and $(-p, -2p) = (-7, -14)$

The graph is shown below.



The Standard Form for the Equation of a Parabola:

The types of parabolas considered above all have a vertex at the origin and a directrix that is parallel to one of the coordinate axes. We now consider parabolas that have a vertex at the point (h, k) and a directrix parallel to one of the coordinate axes.

The standard form of a parabola is represented by one of the following equations:

$$(x-h)^2 = 4p(y-k) \qquad (x-h)^2 = -4p(y-k)$$

$$(y-k)^2 = 4p(x-h) \qquad (y-k)^2 = -4p(x-h)$$

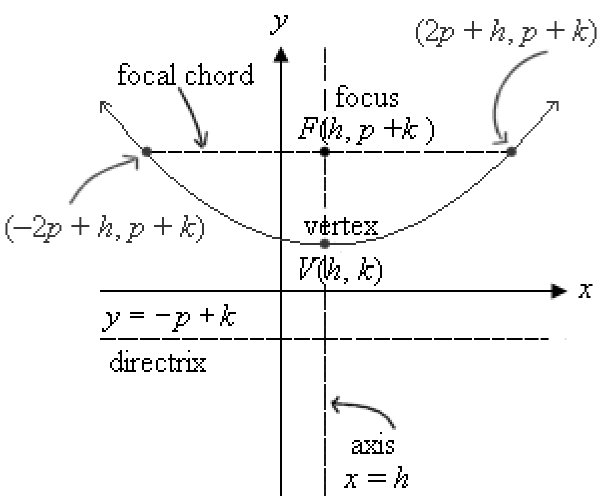
Opening Upward:

Parabola opening upward with vertex at (h, k): $(x-h)^2 = 4p(y-k)$

To graph the parabola $(x-h)^2 = 4p(y-k)$, shift the graph of the parabola $x^2 = 4py$ horizontally $|h|$ units and vertically $|k|$ units. If $h > 0$, the horizontal shift is to the right. If $h < 0$, the horizontal shift is to the left. If $k > 0$, the vertical shift is upward. If $k < 0$, the vertical shift is downward.

The following list reflects the changes in translating the parabola $x^2 = 4py$ to the parabola $(x-h)^2 = 4p(y-k)$:

Focus: The point $(0, p)$ changes to the point $(h, p+k)$.
Directrix: The line $y = -p$ changes to the line $y = -p + k$.
Focal width: The focal width remains $4p$.
Vertex: The point $(0, 0)$ changes to the point (h, k).
Axis: The line $x = 0$ changes to the line $x = h$.
Endpoints of focal chord: The points $(-2p, p)$ and $(2p, p)$ change to the points $(-2p+h, p+k)$ and $(2p+h, p+k)$.



Graph of the parabola $(x-h)^2 = 4p(y-k)$.

Example:

Graph the parabola $(x-2)^2 = 8(y+1)$. Specify the focus, the directrix, the focal width, the vertex, and the axis.

Solution:

The equation is of the form $(x-h)^2 = 4p(y-k)$, where $h = 2$, $k = -1$, and $4p = 8$. Thus, $p = 2$.

Focus: $(h, p+k) = (2, 1)$

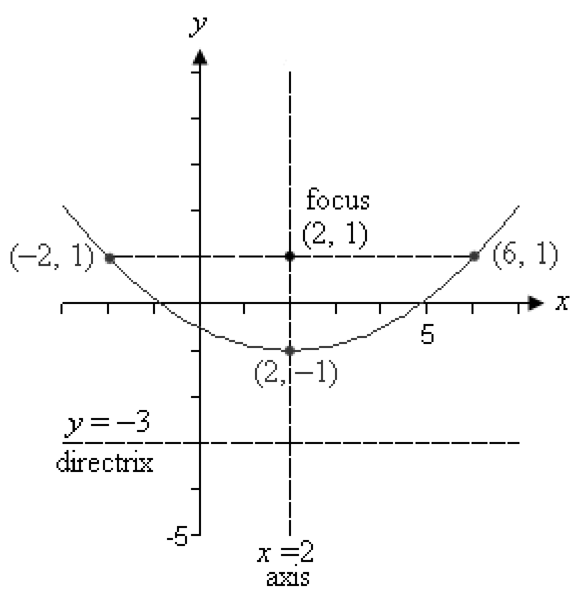
Directrix: $y = -p + k \Rightarrow y = -3$

Focal width: $4p = 8$

Vertex: $(h, k) = (2, -1)$

Axis: $x = h \Rightarrow x = 2$

The graph is shown below.



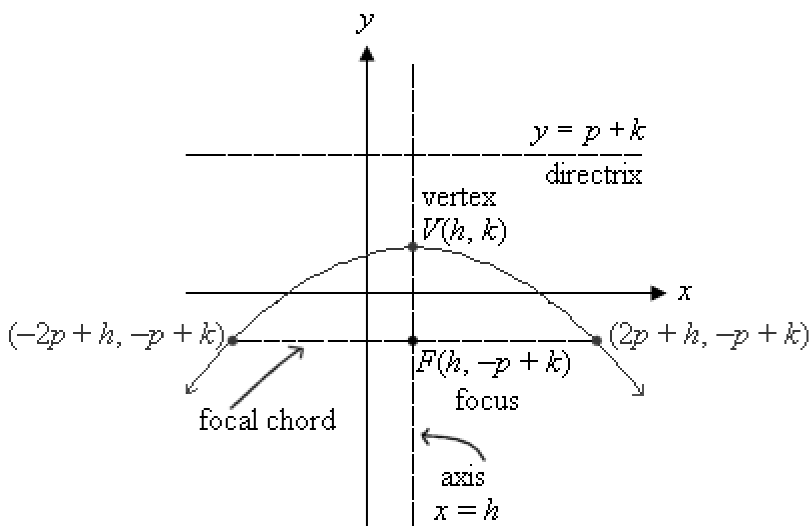
Opening Downward:

Parabola opening downward with vertex at (h, k): $(x-h)^2 = -4p(y-k)$

To graph the parabola $(x-h)^2 = -4p(y-k)$, shift the graph of the parabola $x^2 = -4py$ horizontally $|h|$ units and vertically $|k|$ units. If $h > 0$, the horizontal shift is to the right. If $h < 0$, the horizontal shift is to the left. If $k > 0$, the vertical shift is upward. If $k < 0$, the vertical shift is downward.

The following list reflects the changes in translating the parabola $x^2 = -4py$ to the parabola $(x-h)^2 = -4p(y-k)$:

Focus: The point $(0, -p)$ changes to the point $(h, -p+k)$.
Directrix: The line $y = p$ changes to the line $y = p + k$.
Focal width: The focal width remains $4p$.
Vertex: The point $(0, 0)$ changes to the point (h, k).
Axis: The line $x = 0$ changes to the line $x = h$.
Endpoints of focal chord: The points $(-2p, -p)$ and $(2p, -p)$ change to the points $(-2p+h, -p+k)$ and $(2p+h, -p+k)$.



Graph of the parabola $(x-h)^2 = -4p(y-k)$.

Opening to the Right:

Parabola opening to the right with vertex at (h, k): $(y-k)^2 = 4p(x-h)$

To graph the parabola $(y-k)^2 = 4p(x-h)$, shift the graph of the parabola $y^2 = 4px$ horizontally $|h|$ units and vertically $|k|$ units. If $h > 0$, the horizontal shift is to the right. If $h < 0$, the horizontal shift is to the left. If $k > 0$, the vertical shift is upward. If $k < 0$, the vertical shift is downward.

The following list reflects the changes in translating the parabola $y^2 = 4px$ to the parabola $(y-k)^2 = 4p(x-h)$:

Focus: The point $(p, 0)$ changes to the point $(p+h, k)$.

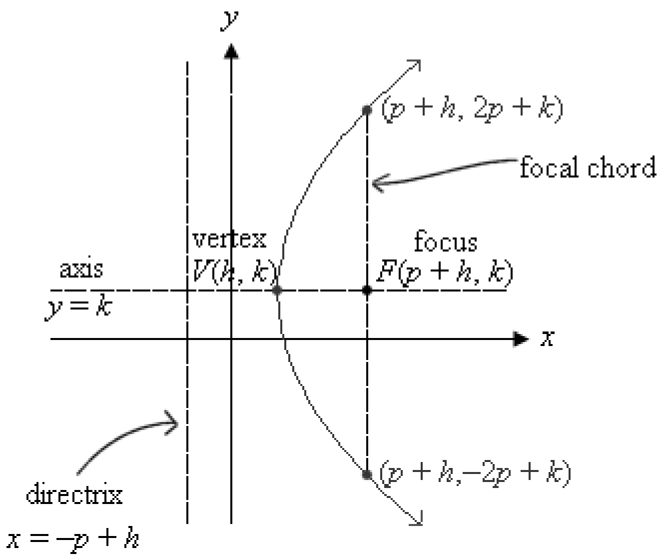
Directrix: The line $x = -p$ changes to the line $x = -p + h$.

Focal width: The focal width remains $4p$.

Vertex: The point $(0, 0)$ changes to the point (h, k).

Axis: The line $y = 0$ changes to the line $y = k$.

Endpoints of focal chord: The points $(p, 2p)$ and $(p, -2p)$ change to the points $(p+h, 2p+k)$ and $(p+h, -2p+k)$.



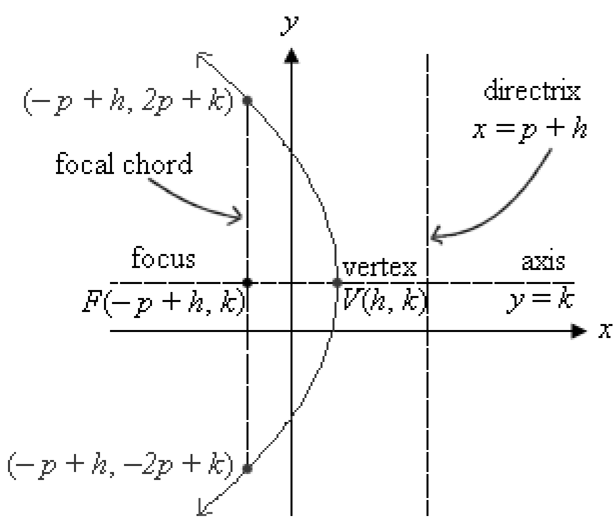
Graph of the parabola $(y-k)^2 = 4p(x-h)$.

Opening to the Left:

To graph the parabola $(y-k)^2 = -4p(x-h)$, shift the graph of the parabola $y^2 = -4px$ horizontally $|h|$ units and vertically $|k|$ units. If $h > 0$, the horizontal shift is to the right. If $h < 0$, the horizontal shift is to the left. If $k > 0$, the vertical shift is upward. If $k < 0$, the vertical shift is downward.

The following list reflects the changes in translating the parabola $y^2 = -4px$ to the parabola $(y-k)^2 = -4p(x-h)$:

Focus: The point $(-p, 0)$ changes to the point $(-p+h, k)$.
Directrix: The line $x = p$ changes to the line $x = p + h$.
Focal width: The focal width remains $4p$.
Vertex: The point $(0, 0)$ changes to the point (h, k).
Axis: The line $y = 0$ changes to the line $y = k$.
Endpoints of focal chord: The points $(-p, 2p)$ and $(-p, -2p)$ change to the points $(-p+h, 2p+k)$ and $(-p+h, -2p+k)$.



Graph of the parabola $(y-k)^2 = -4p(x-h)$.

Example:

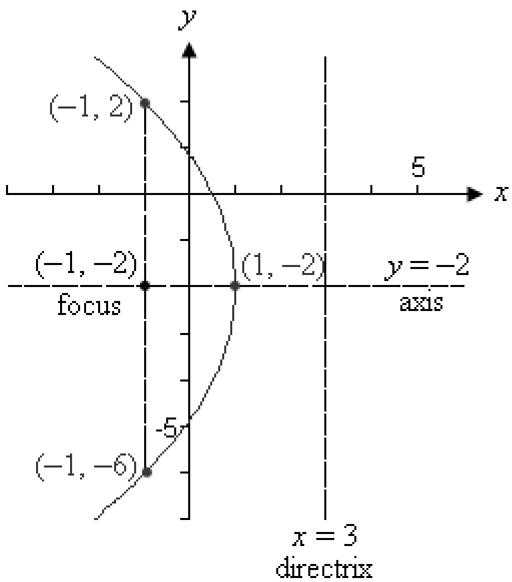
Graph the parabola $(y+2)^2 = -8(x-1)$. Specify the focus, the directrix, the focal width, the vertex, and the axis.

Solution:

The equation is of the form $(y-k)^2 = -4p(x-h)$, where $h = 1$, $k = -2$, and $-4p = -8$. Thus, $p = 2$. The parabola opens to the left.

Focus: $(-p+h, k) = (-1, -2)$
Directrix: $x = p + h \Rightarrow x = 3$
Focal width: $4p = 8$
Vertex: $(h, k) = (1, -2)$
Axis: $y = k \Rightarrow y = -2$

The graph is shown below.



Example:

Write an equation for the parabola in standard form if the vertex is at the point $(-3, 2)$, the parabola passes through the point $(-2, -1)$, and the axis is parallel to the y-axis.

Solution:

We see from the given information that the equation in standard form of a parabola in this position is $(x-h)^2 = -4p(y-k)$.

We are given $(h, k) = (-3, 2)$ so that $h = -3$ and $k = 2$. Substituting these values into the standard form, we have $(x+3)^2 = -4p(y-2)$.

To find p, we use the fact that the parabola passes through the point $(-2, -1)$. Substitute $x = -2$ and $y = -1$ into the equation $(x+3)^2 = -4p(y-2)$ and solve for p.

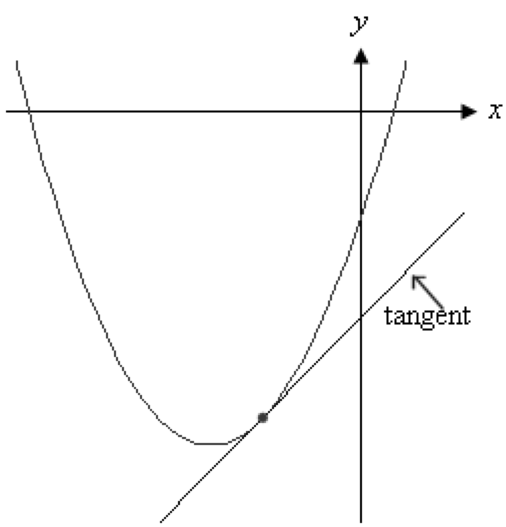
$$\begin{aligned}
(x+3)^2 &= -4p(y-2) \\
(-2+3)^2 &= -4p(-1-2) \\
1 &= -4p(-3) \\
1 &= 12p \\
\frac{1}{12} &= p
\end{aligned}$$

Substitute $p = \frac{1}{12}$ into the equation $(x+3)^2 = -4p(y-2)$ to obtain the desired result.

$$(x+3)^2 = -4\left(\frac{1}{12}\right)(y-2)$$
$$(x+3)^2 = -\frac{1}{3}(y-2)$$

Tangents to Parabolas:

A line through a point that lies on a parabola is said to be tangent to the parabola at that point provided that the line intersects the parabola only at that point and the line is not parallel to the axis of the parabola.



It can be shown that a tangent to a point (x_0, y_0) that lies on the parabola with equation $y = ax^2 + bx + c$ has slope $m = 2ax_0 + b$.

Example:

Write an equation of the line tangent to the parabola with equation $y = 3x^2 + 6x + 1$ at the point $(-2, 1)$.

Solution:

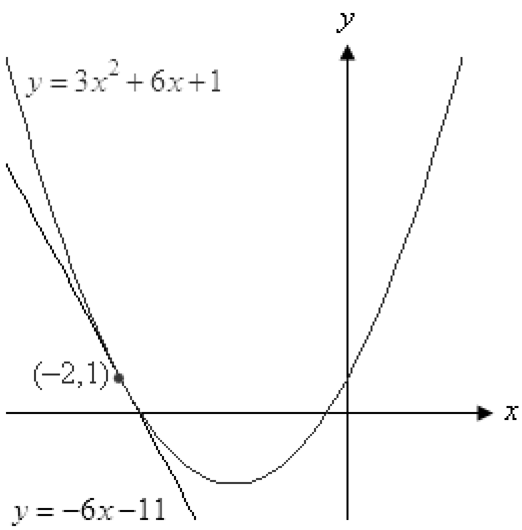
Substitute $a = 3,\ b = 6,$ and $x_0 = -2,$ into the formula $m = 2ax_0 + b$ to find the slope of the tangent line.

$$m = 2ax_0 + b = 2(3)(-2) + 6 = -6$$

Use the point-slope form to write an equation of the line that passes through the point $(-2, 1)$ with slope $m = -6$.

$$y - 1 = -6(x - (-2))$$
$$y - 1 = -6(x + 2)$$
$$y - 1 = -6x - 12$$
$$y = -6x - 11$$

The graphs of the parabola and tangent line are shown in the figure below.



Additional Example 1:

Graph the parabola $x^2 = -20y$. Specify the focus, the directrix, the focal width, and the vertex.

Solution:

The equation is of the form $x^2 = -4py$, where $-4p = -20$. Hence, $p = 5$. The parabola opens downward.

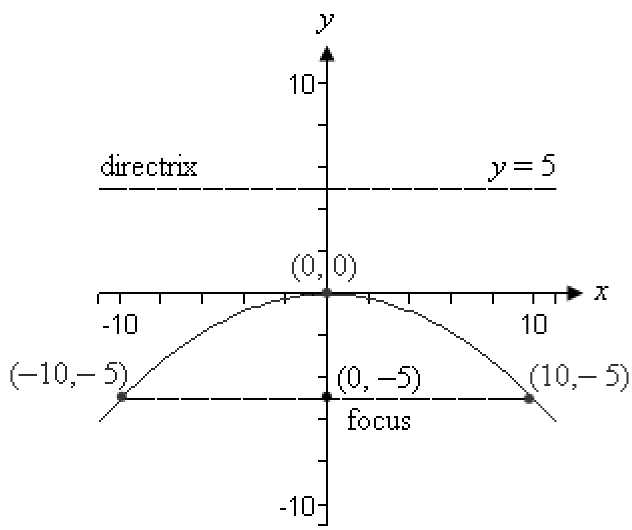
Focus: $(0,-p)=(0,-5)$

Directrix: $y=p \Rightarrow y=5$

Focal width: $4p=20$

Vertex: $(0,0)$

The graph is shown below.

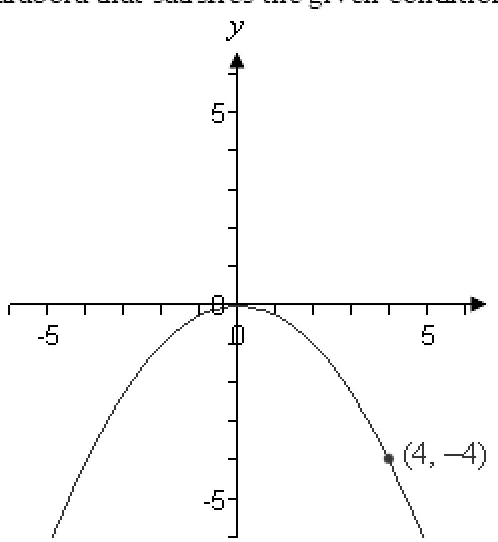


Additional Example 2:

Write an equation of the parabola satisfying the conditions that the vertex is at the origin, the focus lies on the y-axis, and the parabola passes through the point $(4,-4)$.

Solution:

Sketch a parabola that satisfies the given conditions.



A parabola in the position shown above satisfies an equation of the form $x^2 = -4py$.

The coordinates of the point $(4, -4)$ must satisfy the equation $x^2 = -4py$. Substitute $x = 4$ and $y = -4$ into the equation to determine p.

$$\begin{aligned} x^2 &= -4py \\ (4)^2 &= -4p(-4) \\ 16 &= 16p \\ 1 &= p \end{aligned}$$

With $p = 1$, the equation $x^2 = -4py$ becomes $x^2 = -4y$.

Additional Example 3:

Graph the parabola $y^2 + 2y - x + 1 = 0$. Specify the focus, the directrix, the focal width, the vertex, and the axis.

Solution:

Associate the terms involving the variable that is squared on one side of the equation. Then complete the square for the variable that is squared.

$$\begin{aligned} y^2 + 2y - x + 1 &= 0 \\ y^2 + 2y &= x - 1 \\ y^2 + 2y + 1 &= x - 1 + 1 \quad \text{Take one-half of the coefficient on } y\text{, square it, and add to both sides.} \\ (y+1)^2 &= x \end{aligned}$$

The equation is of the form $(y-k)^2 = 4p(x-h)$, where $h = 0$, $k = -1$, and $4p = 1$.

Thus, $p = \frac{1}{4}$. The parabola opens to the right.

Focus: $(p+h, k) = \left(\frac{1}{4}, -1\right)$

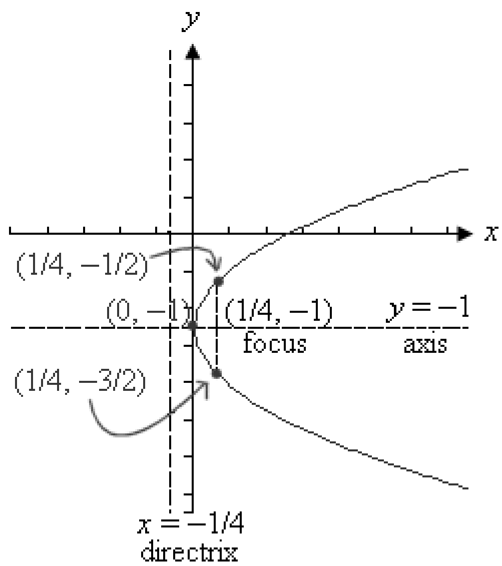
Directrix: $x = -p + h \Rightarrow x = -\frac{1}{4}$

Focal width: $4p = 1$

Vertex: $(h, k) = (0, -1)$

Axis: $y = k \Rightarrow y = -1$

The graph is shown below.



Additional Example 4:

Graph the parabola $x^2 - 2x - 8y + 17 = 0$. Specify the focus, the directrix, the focal width, the vertex, and the axis.

Solution:

Associate the terms involving the variable that is squared on one side of the equation. Then complete the square for the variable that is squared.

$$x^2 - 2x - 8y + 17 = 0$$
$$x^2 - 2x = 8y - 17$$
$$x^2 - 2x + 1 = 8y - 17 + 1$$ Take one-half of the coefficient on x, square it, and add to both sides.
$$(x-1)^2 = 8y - 16$$
$$(x-1)^2 = 8(y-2)$$ Factor on the RHS.

The equation is of the form $(x-h)^2 = 4p(y-k)$, where $h = 1$, $k = 2$, and $4p = 8$. Thus, $p = 2$. The parabola opens upward.

Focus: $(h, p+k) = (1, 4)$

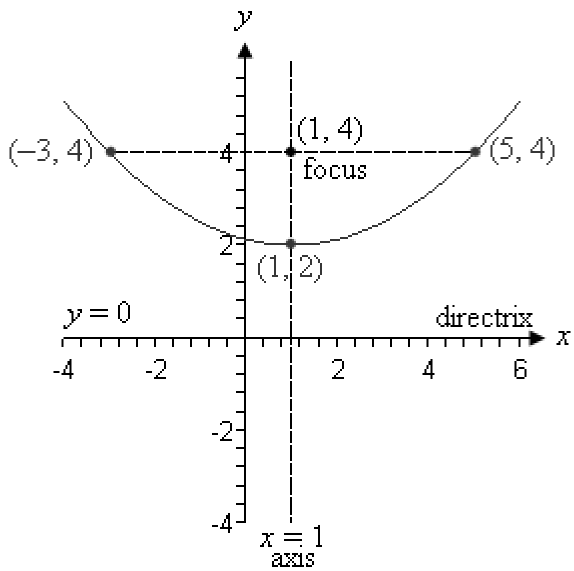
Directrix: $y = -p + k \Rightarrow y = 0$

Focal width: $4p = 8$

Vertex: $(h, k) = (1, 2)$

Axis: $x = h \Rightarrow x = 1$

The graph is shown below.



Additional Example 5:

Write an equation of the line tangent to the parabola with equation $y = -4x^2 + 3x - 2$ at the point $(1, -3)$.

Solution:

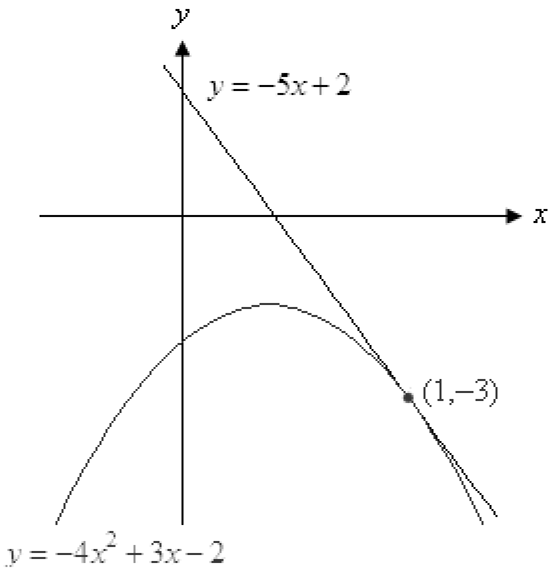
Substitute $a = -4,\ b = 3,$ and $x_0 = 1$ into the formula $m = 2ax_0 + b$ to find the slope of the tangent line.

$$m = 2ax_0 + b = 2(-4)(1) + 3 = -5$$

Use the point-slope form to write an equation of the line that passes through the point $(1, -3)$ with slope $m = -5$.

$$\begin{aligned} y - (-3) &= -5(x - 1) \\ y + 3 &= -5(x - 1) \\ y + 3 &= -5x + 5 \\ y &= -5x + 2 \end{aligned}$$

The graphs of the parabola and tangent line are shown in the figure below.



Additional Example 6:

Find the points of intersection of the parabola and the line whose equations are given.

$y = 2x^2 + 4x - 3;\quad y = 6x + 1$

Solution:

To find the x-coordinates of the points of intersection, we must solve the quadratic equation $2x^2 + 4x - 3 = 6x + 1$.

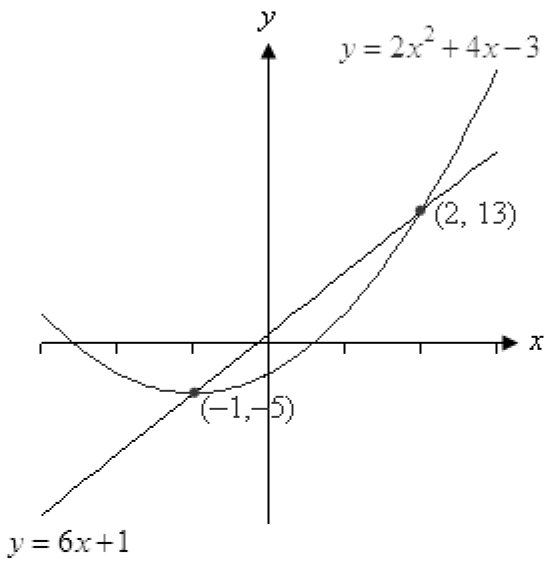
$2x^2 + 4x - 3 = 6x + 1$	
$2x^2 - 2x - 4 = 0$	Subtract $(6x+1)$ from both sides.
$2(x^2 - x - 2) = 0$	Factor out a common monomial factor on the LHS.
$2(x+1)(x-2) = 0$	Factor the trinomial on the LHS.
$x + 1 = 0$ or $x - 2 = 0$	Use the zero-product property.
$x = -1$ $\quad$ $x = 2$	Solve each equation.

To find the y-coordinates of the points of intersection, substitute $x = -1$ and $x = 2$ into either the equation of the line or the equation of the parabola. We will work with the simpler equation.

$$x = -1:\quad y = 6x + 1 = 6(-1) + 1 = -5$$
$$x = 2:\quad y = 6x + 1 = 6(2) + 1 = 13$$

The points of intersection are $(-1, -5)$ and $(2, 13)$.

The graphs of the parabola and line are shown in the figure below.



Exercise Set 8.1: Parabolas

Write each of the following equations in the standard form for the equation of a parabola, where the standard form is represented by one of the following equations:

$$(x-h)^2 = 4p(y-k) \qquad (x-h)^2 = -4p(y-k)$$
$$(y-k)^2 = 4p(x-h) \qquad (y-k)^2 = -4p(x-h)$$

1. $y^2 - 14y - 2x + 43 = 0$

2. $x^2 + 10x - 12y = -61$

3. $-9y = x^2 - 8x - 10$

4. $-7x = y^2 - 10y + 24$

5. $x = 3y^2 - 24y + 50$

6. $y = 2x^2 + 12x + 15$

7. $3x^2 - 3x - 5 - y = 0$

8. $5y^2 + 5y = x - 6$

For each of the following parabolas,

(a) **Write the given equation in the standard form for the equation of a parabola. (Some equations may already be given in standard form.)**

It may be helpful to begin sketching the graph for part (h) as a visual aid to answer the questions below.

(b) **State the equation of the axis.**

(c) **State the coordinates of the vertex.**

(d) **State the equation of the directrix.**

(e) **State the coordinates of the focus.**

(f) **State the focal width.**

(g) **State the coordinates of the endpoints of the focal chord.**

(h) **Sketch a graph of the parabola which includes the features from (c)-(e) and (g). Label the vertex V and the focus F.**

9. $x^2 - 4y = 0$

10. $y^2 - 12x = 0$

11. $-10x = y^2$

12. $-6y = x^2$

13. $(x-2)^2 = 8(y+5)$

14. $(y-4)^2 = 16x$

15. $y^2 = 4(x-3)$

16. $(x+3)^2 = 4(y-1)$

17. $(x+5)^2 = -2(y-4)$

18. $(y+1)^2 = -10(x+3)$

19. $(y-6)^2 = -1(x-2)$

20. $(x-1)^2 = -8(y-6)$

21. $x^2 + 12x - 6y + 24 = 0$

22. $x^2 - 2y = 8x - 10$

23. $y^2 - 8x = 4y + 36$

24. $y^2 + 6y - 4x + 5 = 0$

25. $x^2 + 25 = -16y - 10x$

26. $y^2 + 10y + x + 28 = 0$

27. $y^2 - 4y + 2x - 4 = 0$

28. $12y + x^2 - 4x = -16$

29. $3y^2 + 30y - 8x + 67 = 0$

30. $5x^2 + 30x - 16y = 19$

31. $2x^2 - 8x + 7y = 34$

32. $4y^2 - 8y + 9x + 40 = 0$

Use the given features of each of the following parabolas to write an equation for the parabola in standard form.

33. Vertex: $(-2, 5)$
Focus: $(4, 5)$

34. Vertex: $(1, -3)$
Focus: $(1, 0)$

35. Vertex: $(2, 0)$
Focus: $(2, -4)$

36. Vertex: $(-4, -2)$
Focus: $(-6, -2)$

37. Focus: $(-2, -3)$
Directrix: $y = -9$

38. Focus: $(4, 1)$
Directrix: $y = 5$

39. Focus: $(4, -1)$
Directrix: $x = 7\frac{1}{2}$

40. Focus: $(-3, 5)$
Directrix: $x = -4$

41. Focus: $(-4, -2)$
Opens downward
$p = 7$

42. Focus: $(1, 5)$
Opens to the right
$p = 3$

43. Vertex: $(5, 6)$
Opens upward
Length of focal chord: 6

44. Vertex: $\left(0, \frac{1}{2}\right)$
Opens to the left
Length of focal chord: 2

45. Vertex: $(-3, 2)$
Horizontal axis
Passes through $(6, 5)$

46. Vertex: $(2, 1)$
Vertical axis
Passes through $(8, 5)$

47. Endpoints of focal chord: $(0, 5)$ and $(0, -5)$
Opens to the left

48. Endpoints of focal chord: $(-2, 3)$ and $(6, 3)$
Opens downward

Answer the following.

49. Write an equation of the line tangent to the parabola with equation $f(x) = x^2 + 5x + 4$ at:
(a) $x = 3$
(b) $x = -2$

50. Write an equation of the line tangent to the parabola with equation $f(x) = -3x^2 + 6x + 1$ at:
(a) $x = 0$
(b) $x = -1$

51. Write an equation of the line tangent to the parabola with equation $f(x) = 2x^2 + 5x - 1$ at:
(a) $x = -1$
(b) $x = \frac{1}{2}$

52. Write an equation of the line tangent to the parabola with equation $f(x) = -x^2 + 4x - 2$ at:
(a) $x = -3$
(b) $x = \frac{3}{2}$

53. Write an equation of the line tangent to the parabola with equation $f(x) = 4x^2 - 5x - 3$ at the point $(1, -4)$.

54. Write an equation of the line tangent to the parabola with equation $f(x) = 2x^2 + 5x + 4$ at the point $(-3, 7)$.

55. Write an equation of the line tangent to the parabola with equation $f(x) = -x^2 - 6x + 1$ at the point $(-5, 6)$.

56. Write an equation of the line tangent to the parabola with equation $f(x) = -3x^2 + 4x + 9$ at the point $(2, 5)$.

Exercise Set 8.1: Parabolas

Give the point(s) of intersection of the parabola and the line whose equations are given.

57. $f(x) = x^2 - 4x + 11$
$g(x) = 5x - 3$

58. $f(x) = x^2 - 8x + 1$
$g(x) = -2x - 10$

59. $f(x) = x^2 + 10x + 10$
$g(x) = 8x + 9$

60. $f(x) = x^2 - 5x - 10$
$g(x) = -7x + 14$

61. $f(x) = -x^2 + 6x - 5$
$g(x) = 6x - 14$

62. $f(x) = -x^2 + 3x - 2$
$g(x) = -5x + 13$

63. $f(x) = -3x^2 + 6x + 1$
$g(x) = -3x + 7$

64. $f(x) = -2x^2 + 8x - 5$
$g(x) = 6x - 5$

Section 8.2: Ellipses

- Equations of Ellipses

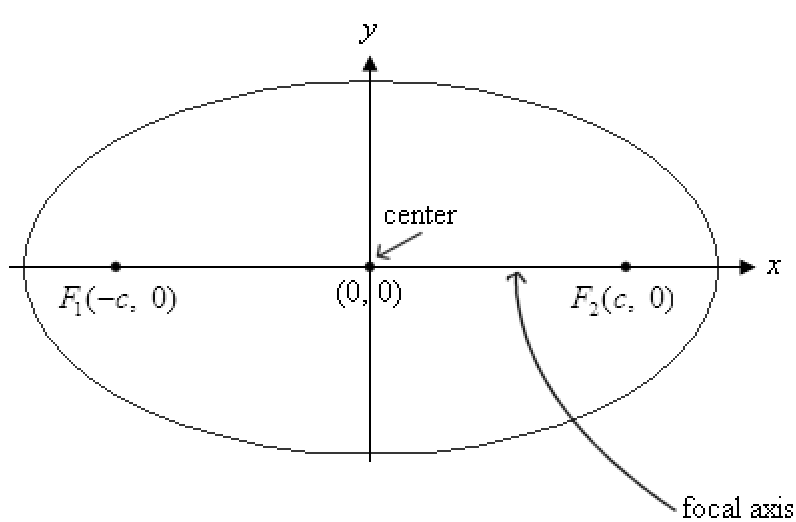
Equations of Ellipses

Definition of an Ellipse:

An ellipse is the set of all points in the plane so that for every point on the ellipse, the sum of its distances from two fixed points is a constant. The fixed points are called the foci of the ellipse.

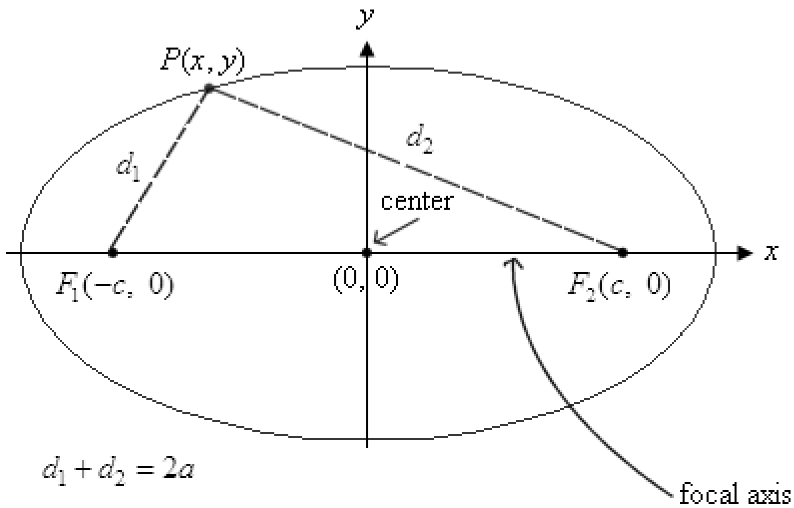
Equations of Ellipses with Center at the Origin:

To see how to derive the equation of an ellipse, we consider a special case where the foci are the points $F_1(-c,\ 0)$ and $F_2(c,\ 0)$ and the constant distance is $2a$. (It is assumed that a and c are positive numbers.)



The line that passes through the foci is called the focal axis. The center of the ellipse is the point that is midway between the foci.

Now, let (x, y) be any point on the ellipse. By definition, the sum of the distances from the point (x, y) to the foci is equal to $2a$. Thus, $d_1 + d_2 = 2a$. (See the figure below.)



Note: PF_1 and PF_2 are called focal radii. Based on the equation $d_1 + d_2 = 2a$ and the figure above, it can be seen that the sum of the focal radii of an ellipse is always equal to $2a$.

We have the following:

$$d_1 + d_2 = 2a$$

$$d_1 = 2a - d_2$$

$$\sqrt{(x+c)^2 + (y-0)^2} = 2a - \sqrt{(x-c)^2 + (y-0)^2}$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \qquad \text{Square both sides.}$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx$$

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x \qquad \text{Divide both sides by } 4a.$$

$$(x-c)^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2 \qquad \text{Square both sides.}$$

$$x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$$

$$\left(1 - \frac{c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

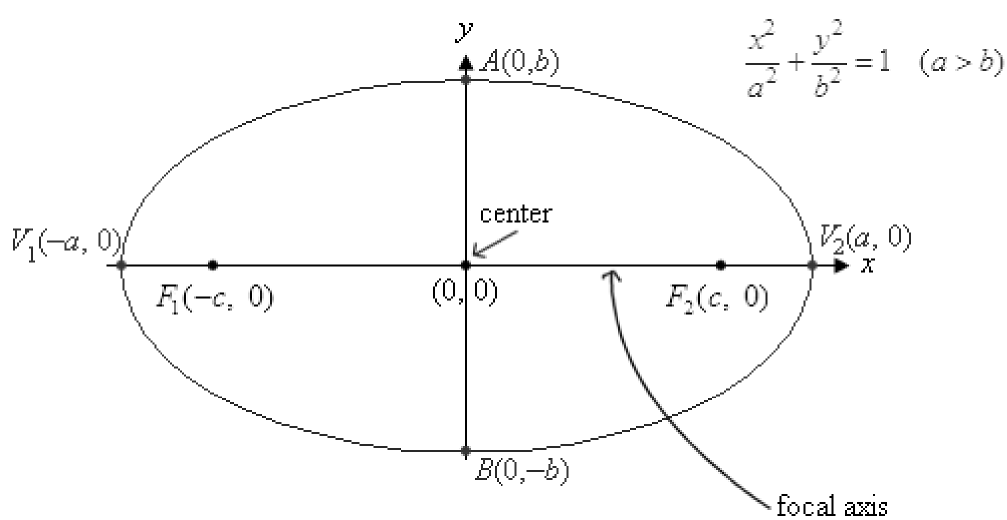
$$\left(\frac{a^2 - c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad \text{Divide both sides by } a^2 - c^2.$$

Now, define the positive number b by $b^2 = a^2 - c^2$. Substituting b^2 for $a^2 - c^2$ in the equation derived above, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Letting $y = 0$, we obtain the x-intercepts: $\frac{x^2}{a^2} = 1 \Rightarrow x = \pm a$. The points $(-a, 0)$ and $(a, 0)$ are the points of intersection of the focal axis and the ellipse. These points are called the vertices of the ellipse. The line segment joining the vertices is called the major axis.

Letting $x = 0$, we obtain the y-intercepts: $\frac{y^2}{b^2} = 1 \Rightarrow y = \pm b$. The minor axis of the ellipse is the line segment through the center of the ellipse and perpendicular to the major axis with endpoints on the ellipse. These endpoints are the points $(0, -b)$ and $(0, b)$.



Center: $(0, 0)$

Foci: $F_1(-c, 0)$ and $F_2(c, 0)$, where $c^2 = a^2 - b^2$

Vertices: $V_1(-a, 0)$ and $V_2(a, 0)$

Major Axis: $\overline{V_1V_2}$ Length of Major Axis: $2a$

Minor Axis: $\overline{AB}$ Length of Minor Axis: $2b$

The eccentricity of an ellipse is a number between 0 and 1 that describes its shape. (We use the letter e to denote the eccentricity.) As e gets closer to 1, the ellipse becomes increasingly flat. The eccentricity is given by the formula $e = \dfrac{c}{a}$.

Example:

Graph the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

The equation is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a = 5$ and $b = 3$.

Also, $c^2 = a^2 - b^2 = 25 - 9 = 16$ so that $c = 4$.

Center: $(0, 0)$

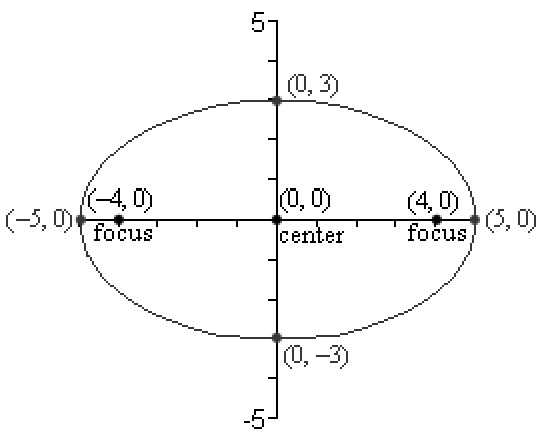
Foci: $(-c, 0) = (-4, 0)$ and $(c, 0) = (4, 0)$

Length of Major Axis: $2a = 10$

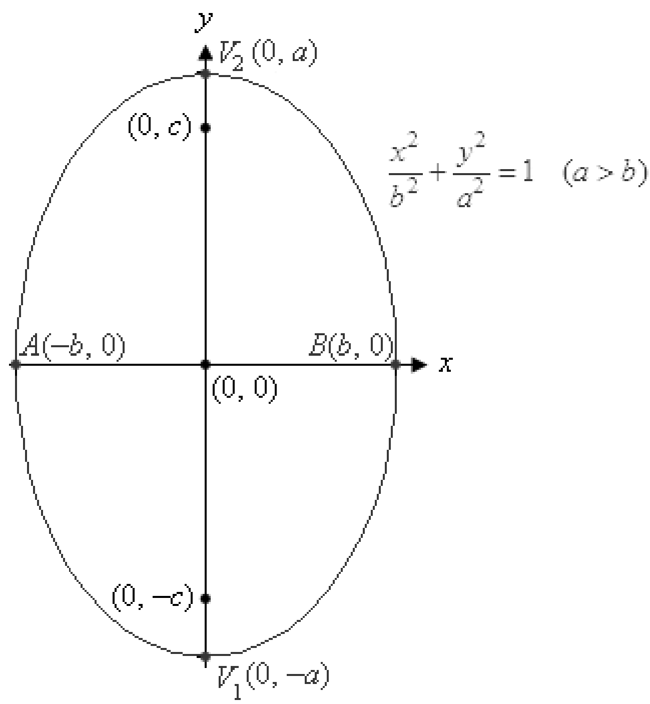
Length of Minor Axis: $2b = 6$

Eccentricity: $e = \dfrac{c}{a} = \dfrac{4}{5}$

The graph is shown below.



In the derivation of the equation of an ellipse, we assumed that the foci were located on the x-axis at the points $(-c, 0)$ and $(c, 0)$. We could go through a similar derivation for an ellipse where the foci are located on the y-axis at the points $(0, -c)$ and $(0, c)$ to obtain the equation $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \quad (a>b)$.



Center: $(0, 0)$

Foci: $(0, -c)$ and $(0, c)$, where $c^2 = a^2 - b^2$

Vertices: $V_1(0, -a)$ and $V_2(0, a)$

Major Axis: $\overline{V_1V_2}$ Length of Major Axis: $2a$

Minor Axis: $\overline{AB}$ Length of Minor Axis: $2b$

Eccentricity: $e=\frac{c}{a}$

Example:

Graph the ellipse $\frac{x^2}{2}+\frac{y^2}{3}=1$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

The equation is of the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where $a=\sqrt{3}$ and $b=\sqrt{2}$.

Also, $c^2=a^2-b^2=3-2=1$ so that $c=1$.

Center: $(0,0)$

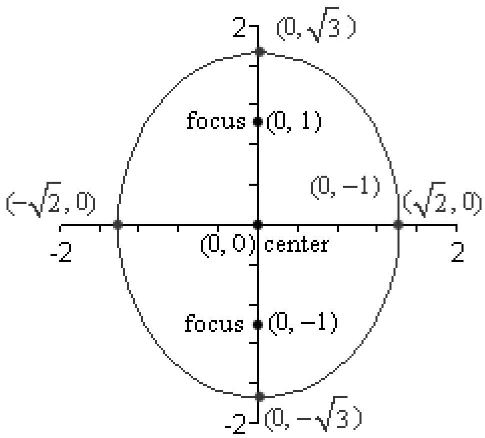
Foci: $(0,-c,)=(0,-1)$ and $(0,c)=(0.1)$

Length of Major Axis: $2a=2\sqrt{3}$

Length of Minor Axis: $2b=2\sqrt{2}$

Eccentricity: $e=\frac{c}{a}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}\approx 0.58$

The graph is shown below.



The Standard Form for the Equation of an Ellipse:

The center of the types of ellipses considered above is the origin. Equations can be written for ellipses whose center has been translated to the point (h,k).

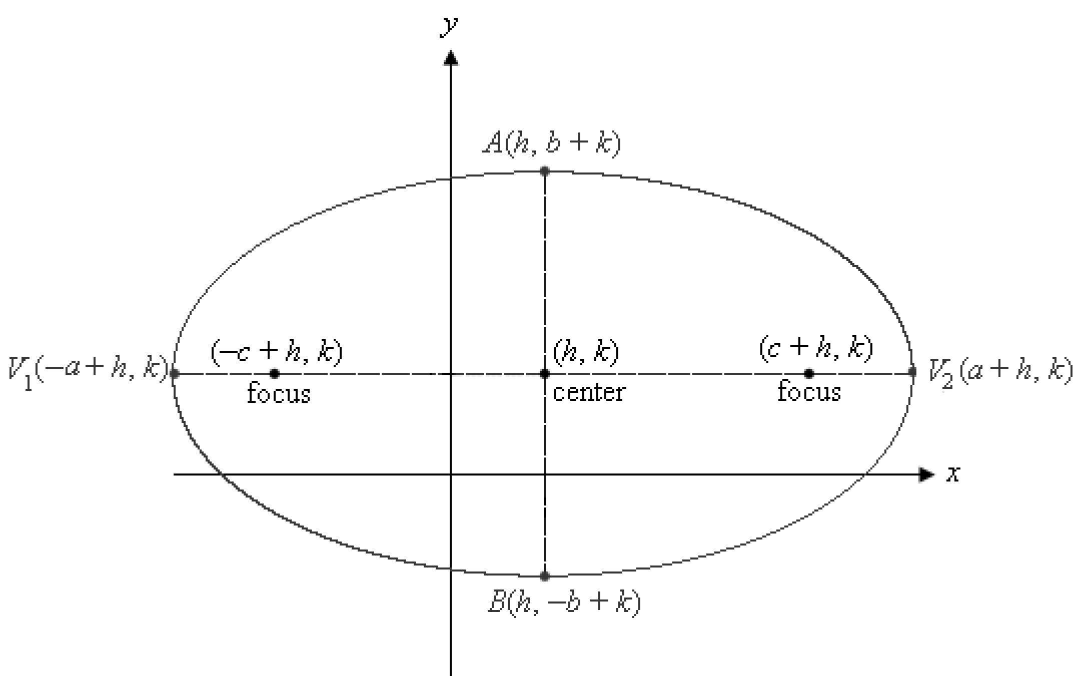
The standard form of an ellipse is represented by one of the following equations:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \qquad (a>b)$$

To graph the ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1 \quad (a>b)$, shift the graph of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ horizontally $|h|$ units and vertically $|k|$ units. If $h>0$, the horizontal shift is to the right. If $h<0$, the horizontal shift is to the left. If $k>0$, the vertical shift is upward. If $k<0$, the vertical shift is downward.

The following list reflects the changes in translating the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ to the ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1 \quad (a>b)$:

Center: The point $(0,0)$ changes to the point (h,k).
Foci: The foci change from the points $(-c,0)$ and $(c,0)$ to the points $(-c+h,k)$ and $(c+h,k)$, where $c^2=a^2-b^2$.
Vertices: The vertices change from the points $(-a,0)$ and $(a,0)$ to the points $(-a+h,k)$ and $(a+h,k)$.
Major Axis: $\overline{V_1V_2}$ Length of Major Axis: $2a$
Minor Axis: $\overline{AB}$ Length of Minor Axis: $2b$



Example:

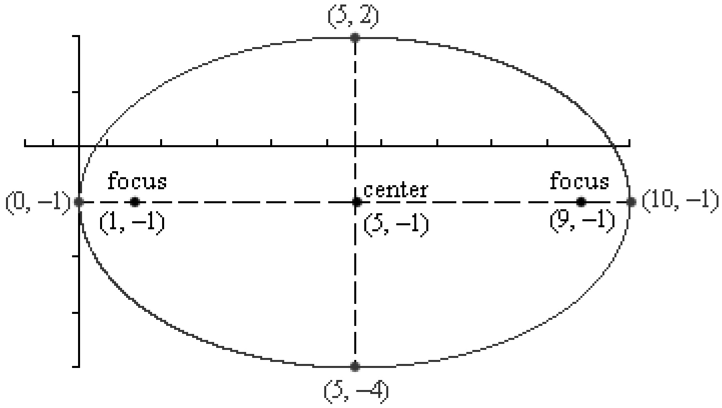
Graph the ellipse $\frac{(x-5)^2}{25}+\frac{(y+1)^2}{9}=1$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

The equation is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a=5$, $b=3$, $h=5$, and $k=-1$. Also, $c^2=a^2-b^2=25-9=16$ so that $c=4$.

Center: $(h,k)=(5,-1)$.
Foci: $(-c+h,k)=(1,-1)$ and $(c+h,k)=(9,-1)$
Length of Major Axis: $2a=10$
Length of Minor Axis: $2b=6$
Eccentricity: $e=\frac{c}{a}=\frac{4}{5}$

The graph is shown below.



To graph the ellipse $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \quad (a>b)$, shift the graph of the ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ horizontally $|h|$ units and vertically $|k|$ units. If $h>0$, the horizontal shift is to the right. If $h<0$, the horizontal shift is to the left. If $k>0$, the vertical shift is upward. If $k<0$, the vertical shift is downward.

The following list reflects the changes in translating the ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ to the ellipse $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \quad (a>b)$:

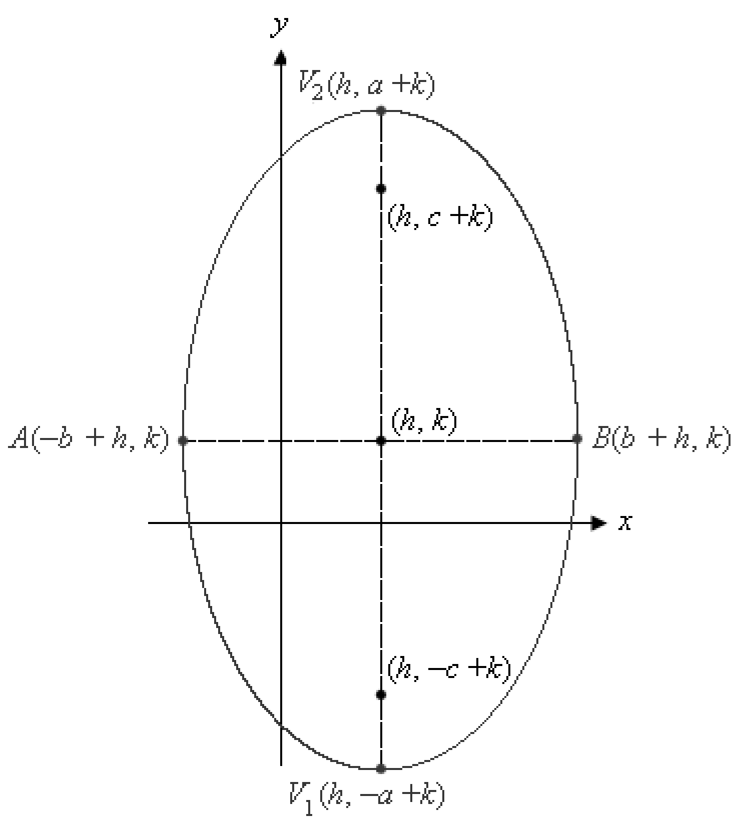
Center: The point $(0,0)$ changes to the point (h,k).

Foci: The foci change from the points $(0,-c)$ and $(0,c)$ to the points $(h,-c+k)$ and $(h,c+k)$, where $c^2=a^2-b^2$.

Vertices: The vertices change from the points $(0,-a)$ and $(0,a)$ to the points $(h,-a+k)$ and $(h,a+k)$.

Major Axis: $\overline{V_1V_2}$ Length of Major Axis: $2a$

Minor Axis: $\overline{AB}$ Length of Minor Axis: $2b$



Example:

Write an equation for the ellipse in standard form with foci at the points $(-1,1)$ and $(7,1)$ and eccentricity $\frac{1}{2}$.

Solution:

It is clear from the given information that the major axis is horizontal. The standard form for an ellipse in this position is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ with $a>b$.

Since the foci are given as $(-1,1)$ and $(7,1)$, we can determine by inspection that the center is given by $(h,\ k)=(3,1)$ with $c=4$. Substituting $h=3$ and $k=1$ into the standard form, we now have $\frac{(x-3)^2}{a^2}+\frac{(y-1)^2}{b^2}=1$.

The eccentricity is given as $\frac{1}{2}$. Thus, $e=\frac{c}{a}=\frac{1}{2}$. Since $c=4$, then $\frac{c}{a}=\frac{1}{2}\Rightarrow a=8$.
Thus, $a^2=64$.

Also,

$$\begin{aligned} c^2 &= a^2-b^2 \\ 16 &= 64-b^2 \\ -48 &= -b^2 \\ 48 &= b^2 \end{aligned}$$

Substituting $a^2=64$ and $b^2=48$ into the equation $\frac{(x-3)^2}{a^2}+\frac{(y-1)^2}{b^2}=1$,
we obtain the desired result.

$$\frac{(x-3)^2}{64}+\frac{(y-1)^2}{48}=1$$

Special Case:

The circle with center $(h,\ k)$ is a special case of the ellipse, where we have $a^2=b^2$ in the equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. In this case, we set $a=b=r$ so that the equation becomes $(x-h)^2+(y-k)^2=r^2$, the equation of a circle with center at $(h,\ k)$ and radius r.

In the case of a circle, the eccentricity e is equal to 0. The shapes of ellipses become increasingly circular as their eccentricites get closer to 0.

Example:

Write an equation for the circle in standard form with center at $(-4, 5)$ and passing through the point $(6,1)$.

Solution:

The standard form for the equation of a circle with center at (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$. We are given that $(h,k)=(-4,5)$. Substituting $h=-4$ and $k=5$ into the standard form, we now have $(x+4)^2+(y-5)^2=r^2$.

To determine r, we can use the fact that the circle passes through the point $(6,1)$. We substitute $x=6$ and $y=1$ into the equation $(x+4)^2+(y-5)^2=r^2$.

$$\begin{aligned}(x+4)^2+(y-5)^2&=r^2\\(6+4)^2+(1-5)^2&=r^2\\100+16&=r^2\\116&=r^2\end{aligned}$$

Substituting $r^2=116$ into the equation $(x+4)^2+(y-5)^2=r^2$, we obtain the desired result.

$$(x+4)^2+(y-5)^2=116$$

Additional Example 1:

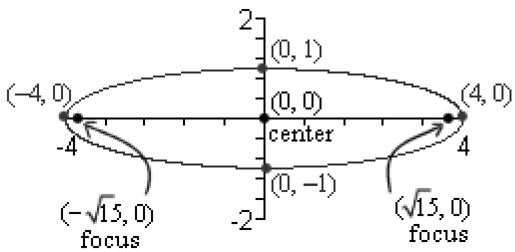
Graph the ellipse $x^2+16y^2=16$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

Divide both sides of the equation by 16 to obtain $\frac{x^2}{16}+\frac{y^2}{1}=1$. The equation is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a=4$ and $b=1$. Also, $c^2=a^2-b^2=16-1=15$ so that $c=\sqrt{15}$.

Center: $(0,0)$
Foci: $(-c,0)=(-\sqrt{15},0)$ and $(c,0)=(\sqrt{15},0)$
Length of Major Axis: $2a=8$
Length of Minor Axis: $2b=2$
Eccentricity: $e=\dfrac{c}{a}=\dfrac{\sqrt{15}}{4}\approx .97$

The graph is shown below.



Additional Example 2:

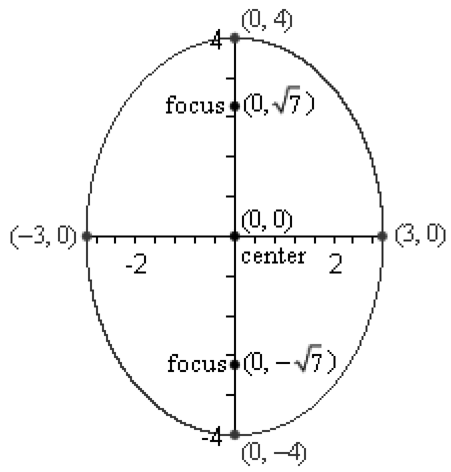
Graph the ellipse $16x^2+9y^2=144$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

Divide both sides of the equation by 144 to obtain $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$. The equation is of the form $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$, where $a=4$ and $b=3$. Also, $c^2=a^2-b^2=16-9=7$ so that $c=\sqrt{7}$.

Center: $(0,0)$
Foci: $(0,-c,)=(0,-\sqrt{7})$ and $(0,c)=(0,\sqrt{7})$
Length of Major Axis: $2a=8$
Length of Minor Axis: $2b=6$
Eccentricity: $e=\dfrac{c}{a}=\dfrac{\sqrt{7}}{4}\approx 0.66$

The graph is shown below.



Additional Example 3:

Graph the ellipse $3x^2 + 4y^2 - 6x + 16y + 7 = 0$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

We will use the technique of completing the square in order to locate the center of the ellipse.

$$3x^2 + 4y^2 - 6x + 16y + 7 = 0$$
$$3x^2 - 6x + 4y^2 + 16y = -7$$
$$3(x^2 - 2x) + 4(y^2 + 4y) = -7$$
$$3(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = -7 + 3 + 16$$
$$3(x-1)^2 + 4(y+2)^2 = 12$$
$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{3} = 1$$

The equation is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a = 2$, $b = \sqrt{3}$, $h = 1$, and $k = -2$. Also, $c^2 = a^2 - b^2 = 4 - 3 = 1$ so that $c = 1$.

Center: $(h, k) = (1, -2)$.

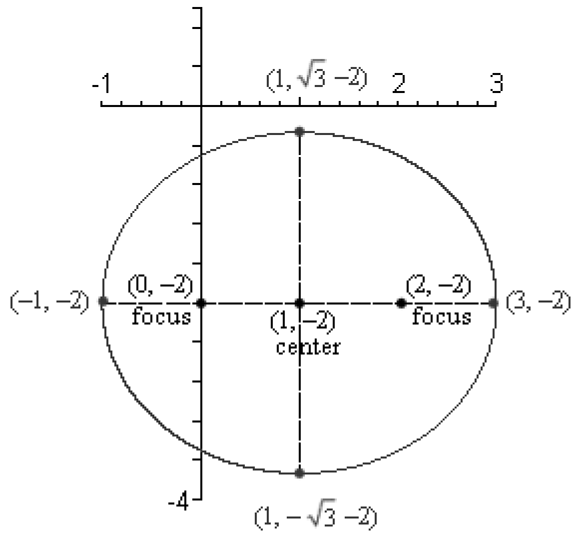
Foci: $(-c + h, k) = (0, -2)$ and $(c + h, k) = (2, -2)$

Length of Major Axis: $2a = 4$

Length of Minor Axis: $2b = 2\sqrt{3}$

Eccentricity: $e = \frac{c}{a} = \frac{1}{2}$

The graph is shown below.



Additional Example 4:

Graph the ellipse $4x^2 + y^2 - 8x - 4y + 4 = 0$. Specify the center, the lengths of the major and minor axes, the foci, and the eccentricity.

Solution:

We will use the technique of completing the square in order to locate the center of the ellipse.

$$4x^2 + y^2 - 8x - 4y + 4 = 0$$
$$4x^2 - 8x + y^2 - 4y = -4$$
$$4(x^2 - 2x) + (y^2 - 4y) = -4$$
$$4(x^2 - 2x + 1) + (y^2 - 4y + 4) = -4 + 4 + 4$$
$$4(x-1)^2 + (y-2)^2 = 4$$
$$\frac{(x-1)^2}{1} + \frac{(y-2)^2}{4} = 1$$

The equation is of the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $a=2$, $b=1$, $h=1$, and $k=2$. Also, $c^2=a^2-b^2=4-1=3$ so that $c=\sqrt{3}$.

Center: $(h,k)=(1,2)$.

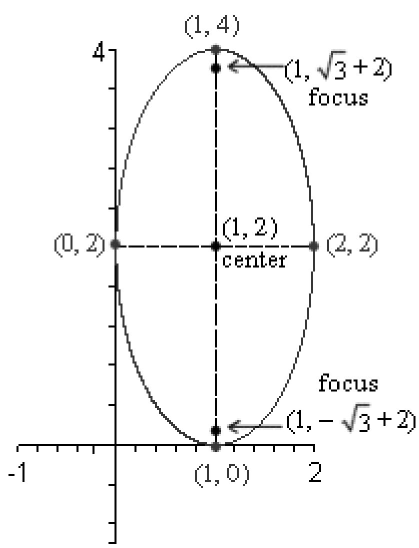
Foci: $(h,-c+k)=(1,-\sqrt{3}+2)$ and $(h,c+k)=(1,\sqrt{3}+2)$

Length of Major Axis: $2a=4$

Length of Minor Axis: $2b=2$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{3}}{2}\approx .87$

The graph is shown below.



Exercise Set 8.2: Ellipses

Write each of the following equations in the standard form for the equation of an ellipse, where the standard form is represented by one of the following equations:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

1. $25x^2+4y^2-100=0$

2. $9x^2+16y^2-144=0$

3. $9x^2-36x+4y^2-32y+64=0$

4. $4x^2+24x+16y^2-32y-12=0$

5. $3x^2+2y^2-30x-12y=-87$

6. $x^2+8y^2+113=14x+48y$

7. $16x^2-16x-64=-8y^2-24y+42$

8. $18x^2+9y^2=153-24x+6y$

Answer the following.

9. **(a)** What is the equation for the eccentricity, e, of an ellipse?
(b) As e approaches 1, the ellipse appears to become more (choose one):
elongated   circular
(c) If $e=0$, the ellipse is a __________.

10. The sum of the focal radii of an ellipse is always equal to __________.

Answer the following for each ellipse. For answers involving radicals, give exact answers and then round to the nearest tenth.

(a) **Write the given equation in the standard form for the equation of an ellipse. (Some equations may already be given in standard form.)**

It may be helpful to begin sketching the graph for part (g) as a visual aid to answer the questions below.

(b) **State the coordinates of the center.**

(c) **State the coordinates of the vertices of the major axis, and then state the length of the major axis.**

(d) **State the coordinates of the vertices of the minor axis, and then state the length of the minor axis.**

(e) **State the coordinates of the foci.**

(f) **State the eccentricity.**

(g) **Sketch a graph of the ellipse which includes the features from (b)-(e). Label the center C, and the foci F_1 and F_2.**

11. $\frac{x^2}{9}+\frac{y^2}{49}=1$

12. $\frac{x^2}{36}+\frac{y^2}{4}=1$

13. $\frac{(x-2)^2}{16}+\frac{y^2}{4}=1$

14. $\frac{x^2}{9}+\frac{(y+1)^2}{5}=1$

15. $\frac{(x-2)^2}{25}+\frac{(y+3)^2}{16}=1$

16. $\frac{(x+5)^2}{16}+\frac{(y+2)^2}{25}=1$

17. $\frac{(x+4)^2}{9}+\frac{(y-3)^2}{1}=1$

18. $\frac{(x+2)^2}{36}+\frac{(y+3)^2}{16}=1$

19. $\frac{(x-2)^2}{11}+\frac{(y+4)^2}{36}=1$

20. $\frac{(x+3)^2}{20}+\frac{(y-5)^2}{4}=1$

21. $4x^2+9y^2-36=0$

22. $4x^2+y^2=1$

23. $25x^2+16y^2-311=50x-64y$

24. $16x^2+25y^2=150y+175$

25. $16x^2-32x+4y^2-40y+52=0$

26. $25x^2+9y^2-100x+54y-44=0$

27. $16x^2+7y^2+64x-42y+15=0$

28. $4x^2+3y^2-16x+6y-29=0$

Exercise Set 8.2: Ellipses

Use the given features of each of the the following ellipses to write an equation for the ellipse in standard form.

29. Center: $(0, 0)$
$a = 8$
$b = 5$
Horizontal Major Axis

30. Center: $(0, 0)$
$a = 7$
$b = 3$
Vertical Major Axis

31. Center: $(-4, 7)$
$a = 5$
$b = 3$
Vertical Major Axis

32. Center: $(2, -4)$
$a = 5$
$b = 2$
Horizontal Major Axis

33. Center: $(-3, -5)$
Length of major axis $= 6$
Length of minor axis $= 4$
Horizontal Major Axis

34. Center: $(2, 1)$
Length of major axis $= 10$
Length of minor axis $= 2$
Vertical Major Axis

35. Foci: $(2, 5)$ and $(2, -5)$
$a = 9$

36. Foci: $(4, -3)$ and $(-4, -3)$
$a = 7$

37. Foci: $(-8, 1)$ and $(2, 1)$
$a = 6$

38. Foci: $(-2, -3)$ and $(-2, 5)$
$a = 8$

39. Foci: $(-1, 2)$ and $(7, 2)$
Passes through the point $(3, 5)$

40. Foci: $(-3, 4)$ and $(7, 4)$
Passes through the point $(2, 1)$

41. Center: $(-5, 2)$
$a = 8$
$e = \frac{3}{4}$
Vertical major axis

42. Center: $(-4, -2)$
$a = 6$
$e = \frac{2}{3}$
Horizontal major axis

43. Foci: $(0, 4)$ and $(0, 8)$
$e = \frac{1}{3}$

44. Foci: $(1, 5)$ and $(1, -3)$
$e = \frac{1}{2}$

45. Foci: $(2, 3)$ and $(6, 3)$
$e = 0.4$

46. Foci: $(2, 1)$ and $(10, 1)$
$e = 0.8$

47. Foci: $(3, 0)$ and $(-3, 0)$
Sum of the focal radii $= 8$

48. Foci: $(0, \sqrt{11})$ and $(0, -\sqrt{11})$
Sum of the focal radii $= 12$

Exercise Set 8.2: Ellipses

A circle is a special case of an ellipse where $a = b$. It then follows that $c^2 = a^2 - b^2 = a^2 - a^2 = 0$, so $c = 0$. (Therefore, the foci are at the center of the circle, and this is simply labeled as the center and not the focus.)

The standard form for the equation of a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

(Note: If each term in the above equation were divided by r^2, it would look like the standard form for an ellipse, with $a = b = r$.)

Using the above information, use the given features of each of the following circles to write an equation for the circle in standard form.

49. Center: $(0, 0)$
Radius: 9

50. Center: $(0, 0)$
Radius: $\sqrt{5}$

51. Center: $(7, -2)$
Radius: 10

52. Center: $(-2, 5)$
Radius: 7

53. Center: $(-3, -4)$
Radius: $3\sqrt{2}$

54. Center: $(-8, 0)$
Radius: $2\sqrt{5}$

55. Center: $(2, -5)$
Passes through the point $(7, -6)$

56. Center: $(-6, -3)$
Passes through the point $(-8, -2)$

57. Endpoints of diameter: $(-4, -6)$ and $(-2, 0)$

58. Endpoints of diameter: $(3, 0)$ and $(7, 10)$

59. Center: $(-3, 5)$
Circle is tangent to the *x*-axis

60. Center: $(-3, 5)$
Circle is tangent to the *y*-axis

Answer the following for each circle. For answers involving radicals, give exact answers and then round to the nearest tenth.

(a) **Write the given equation in the standard form for the equation of a circle. (Some equations may already be given in standard form.)**

(b) **State the coordinates of the center.**

(c) **State the length of the radius.**

(d) **Sketch a graph of the circle which includes the features from (b) and (c). Label the center C and show four points on the circle itself (these four points are equivalent to the vertices of the major and minor axes for an ellipse).**

61. $x^2 + y^2 - 36 = 0$

62. $x^2 + y^2 - 8 = 0$

63. $\dfrac{(x-3)^2}{16} + \dfrac{(y+2)^2}{16} = 1$

64. $\dfrac{x^2}{9} + \dfrac{(y+2)^2}{9} = 1$

65. $(x+5)^2 + (y-2)^2 = 4$

66. $(x-1)^2 + (y-4)^2 = 36$

67. $(x+4)^2 + (y+3)^2 = 12$

68. $(x-1)^2 + (y-5)^2 = 7$

69. $x^2 + y^2 + 2x - 10y + 17 = 0$

70. $x^2 + y^2 + 6x - 2y - 6 = 0$

71. $x^2 + y^2 + 10x - 8y + 36 = 0$

72. $x^2 + y^2 - 4x - 14y + 50 = 0$

73. $3x^2 + 3y^2 + 18x - 24y + 63 = 0$

74. $2x^2 + 2y^2 + 10x + 12y + 52 = 10$

Exercise Set 8.2: Ellipses

Answer the following.

75. A circle passes through the points $(7, 6)$, $(7, -2)$ and $(1, 6)$. Write the equation of the circle in standard form.

76. A circle passes through the points $(-4, 3)$, $(-2, 3)$ and $(-2, -1)$. Write the equation of the circle in standard form.

77. A circle passes through the points $(2, 1)$, $(2, -3)$ and $(8, 1)$. Write the equation of the circle in standard form.

78. A circle passes through the points $(-7, -8)$, $(-7, 2)$ and $(-1, 2)$. Write the equation of the circle in standard form.

Section 8.3: Hyperbolas

- Equations of Hyperbolas with Center at the Origin

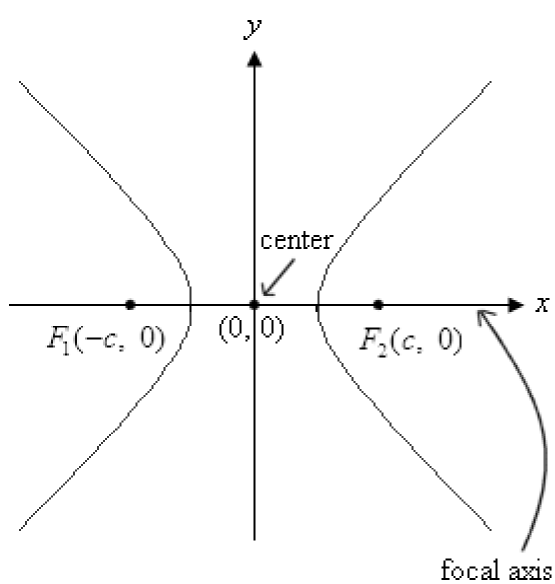
Equations of Hyperbolas with Center at the Origin

Definition of a Hyperbola:

A hyperbola is the set of all points in the plane so that for every point on the hyperbola, the difference of its distances from two fixed points is a positive constant. The fixed points are called the foci of the hyperbola.

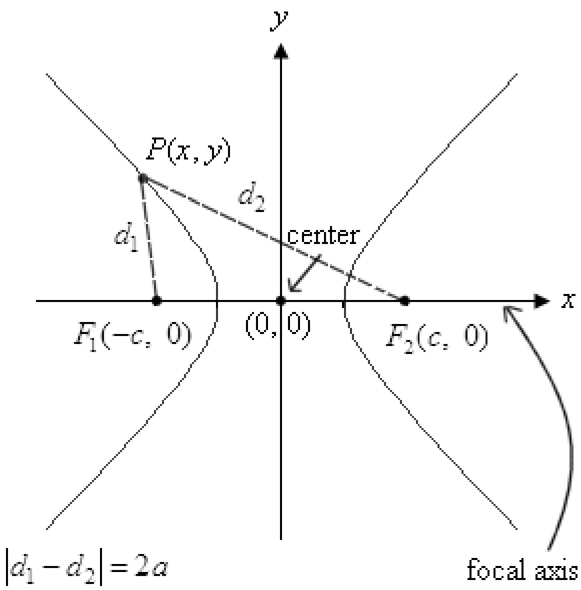
Equations of Hyperbolas with Center at the Origin:

To see how to derive the equation of a hyperbola, we consider a special case where the foci are the points $F_1(-c,\ 0)$ and $F_2(c,\ 0)$ and the constant distance is $2a$. (It is assumed that a and c are positive numbers.)



The line that passes through the foci is called the focal axis. The center of the hyperbola is the point that is midway between the foci.

Now, let (x, y) be any point on the hyperbola. By definition, the absolute value of the difference of the distances from the point (x, y) to the foci is equal to $2a$. Thus, $|d_1 - d_2| = 2a$. (See the figure below.)



We have the following:

$$|d_1 - d_2| = 2a$$

$$\left|\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}\right| = 2a$$

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a$$

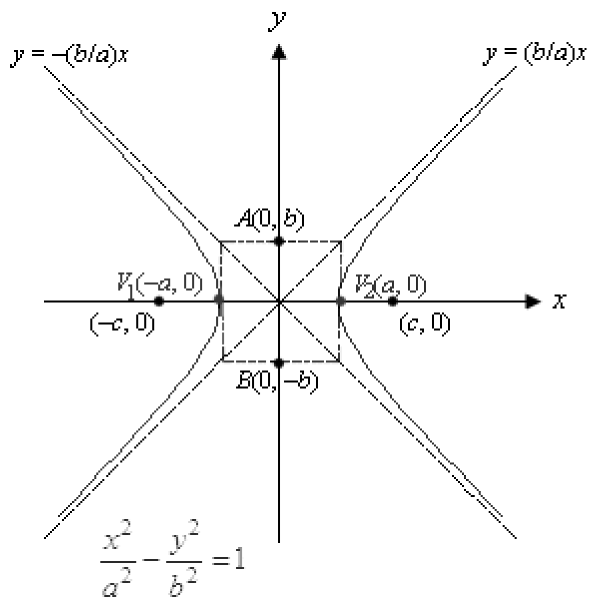
The technique for simplifying the above expression is similar to that used in working with the ellipse in the previous section 8.2. After simplifying the expression in this manner we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

Now, define the positive number b by $b^2 = c^2 - a^2$. Substituting b^2 for $c^2 - a^2$ in the equation derived above, we have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Letting $y = 0$, we obtain the x-intercepts: $\frac{x^2}{a^2} = 1 \Rightarrow x = \pm a$. The points $(-a, 0)$ and $(a, 0)$ are the points of intersection of the focal axis and the hyperbola. These points are called the vertices of the hyperbola. The line segment joining the vertices is called the transverse axis.

The conjugate axis of the hyperbola is the line segment through the center of the hyperbola and perpendicular to the transverse axis with endpoints $(0, -b)$ and $(0, b)$.



Center: $(0, 0)$

Foci: $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 + b^2$

Vertices: $V_1(-a, 0)$ and $V_2(a, 0)$

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

The eccentricity of a hyperbola is given by the formula $e = \frac{c}{a}$.

The lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are slant asymptotes for the hyperbola since it can be shown that as $|x|$ becomes large, $y \to \pm \frac{b}{a}x$.

The rectangle shown in the figure above with vertices $(-a, b)$, (a, b), $(a, -b)$, and $(-a, -b)$ is called the central rectangle and is a helpful guide in graphing the hyperbola since the slant asympotes pass through the corners of this rectangle.

Example:

Graph the hyberbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Specify the center, the foci, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

The equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a = 4$ and $b = 3$.

Also, $c^2 = a^2 + b^2 = 16 + 9 = 25$ so that $c = 5$.

Center: $(0, 0)$

Foci: $(-c, 0) = (-5, 0)$ and $(c, 0) = (5, 0)$

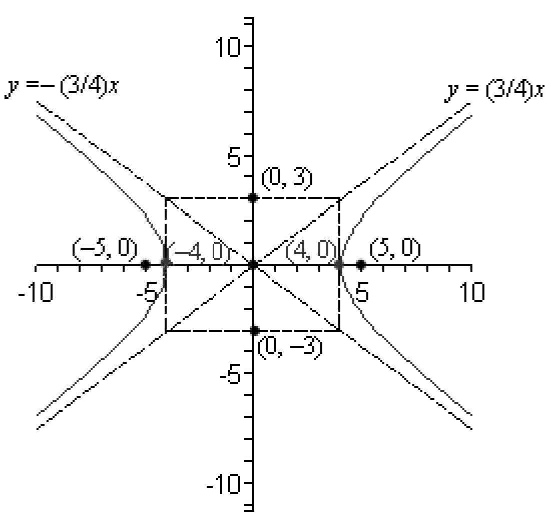
Length of Transverse Axis: $2a = 8$

Length of Conjugate Axis: $2b = 6$

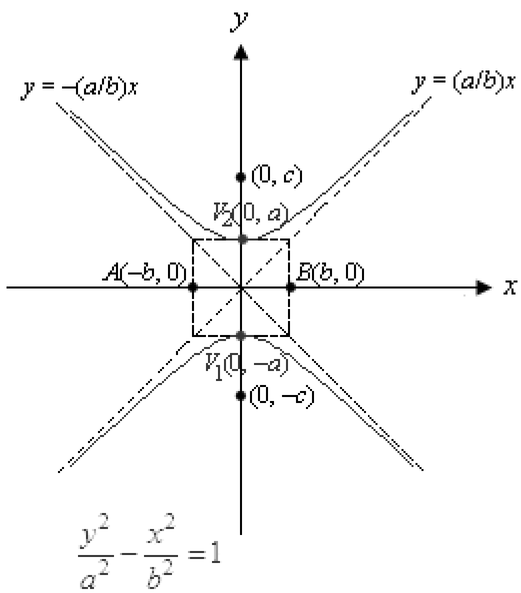
Eccentricity: $e = \frac{c}{a} = \frac{5}{4}$

Equations of the Asymptotes: $y = \frac{b}{a}x \Rightarrow y = \frac{3}{4}x$ and $y = -\frac{b}{a}x \Rightarrow y = -\frac{3}{4}x$

The graph is shown below.



In the derivation of the equation of a hyperbola, we assumed that the foci were located on the x-axis at the points $(-c, 0)$ and $(c, 0)$. We could go through a similar derivation for a hyperbola where the foci are located on the y-axis at the points $(0, -c)$ and $(0, c)$ to obtain the equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.



Center: $(0, 0)$

Foci: $(0, -c)$ and $(0, c)$, where $c^2 = a^2 + b^2$

Vertices: $V_1(0, -a)$ and $V_2(0, a)$

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

Eccentricity: $e = \frac{c}{a}$

Equations of the Asymptotes: $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$

Example:

Graph the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$. Specify the center, the foci, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

The equation is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, where $a = 3$ and $b = 4$.

Also, $c^2 = a^2 + b^2 = 9 + 16 = 25$ so that $c = 5$.

Center: $(0, 0)$

Foci: $(0, -c) = (0, -5)$ and $(0, c) = (0, 5)$

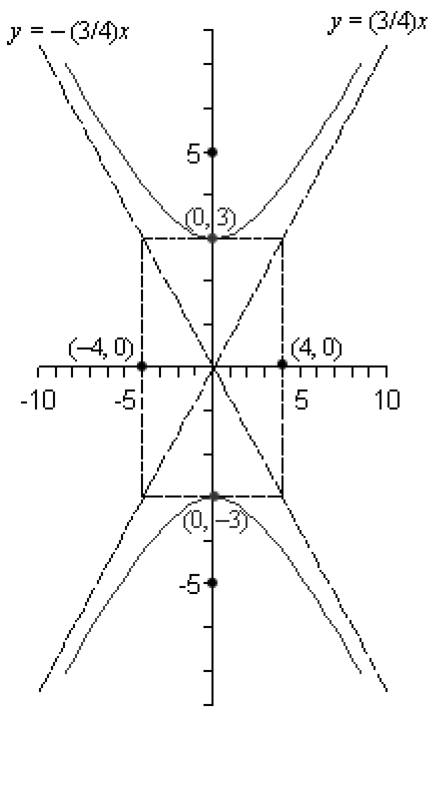
Length of Transverse Axis: $2a = 6$

Length of Conjugate Axis: $2b = 8$

Eccentricity: $e = \frac{c}{a} = \frac{5}{3}$

Equations of the Asymptotes: $y = \frac{a}{b}x \Rightarrow y = \frac{3}{4}x$ and $y = -\frac{a}{b}x \Rightarrow y = -\frac{3}{4}x$

The graph is shown below.



The Standard Form for the Equation of a Hyperbola:

The center of the types of hyperbolas considered above is the origin. Equations can be written for hyperbolas whose center has been translated to the point (h, k).

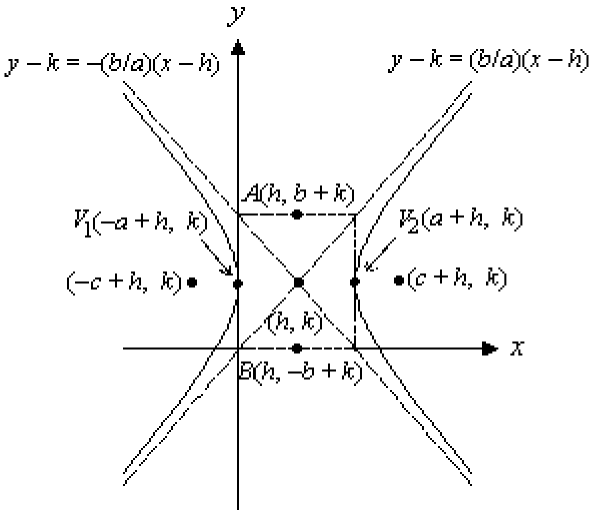
The standard form of a hyperbola is represented by one of the following equations:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

To graph the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, shift the graph of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ horizontally $|h|$ units and vertically $|k|$ units. If $h>0$, the horizontal shift is to the right. If $h<0$, the horizontal shift is to the left. If $k>0$, the vertical shift is upward. If $k<0$, the vertical shift is downward.

The following list reflects the changes in translating the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$:

Center: The point $(0, 0)$ changes to the point (h, k).
Foci: The foci change from the points $(-c, 0)$ and $(c, 0)$ to the points $(-c+h, k)$ and $(c+h, k)$, where $c^2=a^2+b^2$.
Vertices: The vertices change from the points $(-a, 0)$ and $(a, 0)$ to the points $(-a+h, k)$ and $(a+h, k)$.
Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$
Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$
Equations of the Asymptotes: The lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ change to the lines $y-k=\frac{b}{a}(x-h)$ and $y-k=-\frac{b}{a}(x-h)$.



Example:

Graph the hyberbola $\frac{(x-4)^2}{16}-\frac{(y+1)^2}{9}=1$. Specify the center, the foci, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

The equation is of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, where $a=4$, $b=3$, $h=4$, and $k=-1$. Also, $c^2=a^2+b^2=16+9=25$ so that $c=5$.

Center: $(h,k)=(4,-1)$.

Foci: $(-c+h,k)=(-1,-1)$ and $(c+h,k)=(9,-1)$

Length of Transverse Axis: $2a=8$

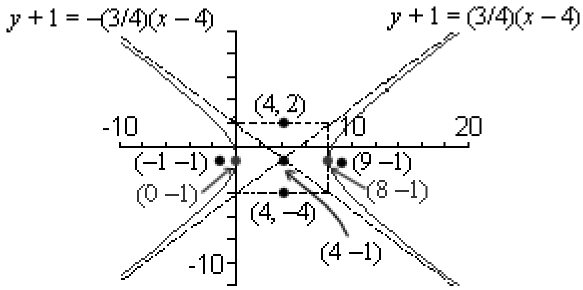
Length of Conjugate Axis: $2b=6$

Eccentricity: $e=\frac{c}{a}=\frac{5}{4}$

Equations of Asymptotes: $y-k=\frac{b}{a}(x-h)\Rightarrow\ y+1=\frac{3}{4}(x-4)$ and

$y-k=-\frac{b}{a}(x-h)\Rightarrow\ y+1=-\frac{3}{4}(x-4)$

The graph is shown below.



To graph the hyperbola $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$ shift the graph of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ horizontally $|h|$ units and vertically $|k|$ units. If $h > 0$, the horizontal shift is to the right. If $h < 0$, the horizontal shift is to the left. If $k > 0$, the vertical shift is upward. If $k < 0$, the vertical shift is downward.

The following list reflects the changes in translating the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ to the hyperbola $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$:

Center: The point $(0, 0)$ changes to the point (h, k).

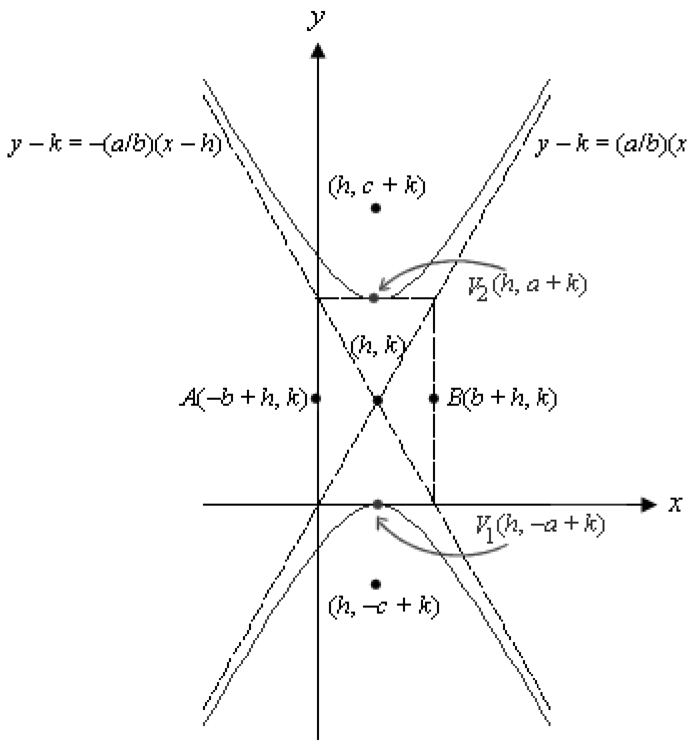
Foci: The foci change from the points $(0, -c)$ and $(0, c)$ to the points $(h, -c+k)$ and $(h, c+k)$, where $c^2 = a^2 + b^2$.

Vertices: The vertices change from the points $(0, -a)$ and $(0, a)$ to the points $(h, -a+k)$ and $(h, a+k)$.

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

Equations of the Asymptotes: The lines $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$ change to the lines $y - k = \frac{a}{b}(x-h)$ and $y - k = -\frac{a}{b}(x-h)$.



Example:

Write an equation for the hyperbola in standard form with foci at the points $(2, 6)$ and $(2, -4)$ and eccentricity $\frac{5}{3}$.

Solution:

It is clear from the given information that the transverse axis is vertical. The standard form for a hyperbola in this position is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

Since the foci are given as $(2, 6)$ and $(2, -4)$, we can determine by inspection that the center is given by $(h, k) = (2, 1)$ with $c = 5$. Substituting $h = 2$ and $k = 1$ into the standard form, we now have $\frac{(y-1)^2}{a^2} - \frac{(x-2)^2}{b^2} = 1$.

The eccentricity is given as $\frac{5}{3}$. Thus, $e = \frac{c}{a} = \frac{5}{3}$. Since $c = 5$, then $\frac{c}{a} = \frac{5}{3} \Rightarrow a = 3$.

Thus, $a^2 = 9$.

Also,

$$c^2 = a^2 + b^2$$
$$25 = 9 + b^2$$
$$16 = b^2$$

Substituting $a^2 = 9$ and $b^2 = 16$ into the equation $\frac{(y-1)^2}{a^2} - \frac{(x-2)^2}{b^2} = 1$,

we obtain the desired result.

$$\frac{(y-1)^2}{9} - \frac{(x-2)^2}{16} = 1$$

Example:

Identify the type of conic section (parabola, ellipse, circle, or hyperbola) represented by each of the following equations. In the case of a circle, identify the conic section as a circle rather than an ellipse. Do not complete the square and write the equation in standard form; these questions can be answered by looking at the signs of the quadratic terms.

(a) $x^2 - 4x + 2y = 20 - y^2$

(b) $x^2 + 4x + 5 = y$

(c) $x^2 - 2x + 4y = 12 + y^2$

(d) $4y^2 - 2x - 16y = -13 - x^2$

Solution:

Part (a):

The equation can be written as $x^2 + y^2 - 4x + 2y - 20 = 0$. There are two quadratic terms whose coefficients are equal and have the same sign. In this case, the graph of the equation is a circle.

Part (b):

The equation can be written as $x^2 + 4x - y + 5 = 0$. There is only one quadratic term. In this case, the graph of the equation is a parabola.

Part (c):

The equation can be written as $x^2 - y^2 - 2x + 4y - 12 = 0$. There are two quadratic terms whose coefficients have opposite signs. In this case, the graph of the equation is a hyperbola.

Part (d):

The equation can be written as $x^2 + 4y^2 - 2x - 16y + 13 = 0$. There are two quadratic terms whose coefficients are unequal and have the same sign. In this case, the graph of the equation is an ellipse.

Additional Example 1:

Graph the hyperbola $16x^2 - 25y^2 = 400$. Specify the center, the foci, the vertices, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

Divide both sides of the equation by 400 to obtain $\frac{x^2}{25} - \frac{y^2}{16} = 1$. The equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a = 5$ and $b = 4$. Also, $c^2 = a^2 + b^2 = 25 + 16 = 41$ so that $c = \sqrt{41}$.

Center: $(0, 0)$

Foci: $(-c, 0) = (-\sqrt{41}, 0)$ and $(c, 0) = (\sqrt{41}, 0)$

Vertices: $(-a, 0) = (-5, 0)$ and $(a, 0) = (5, 0)$

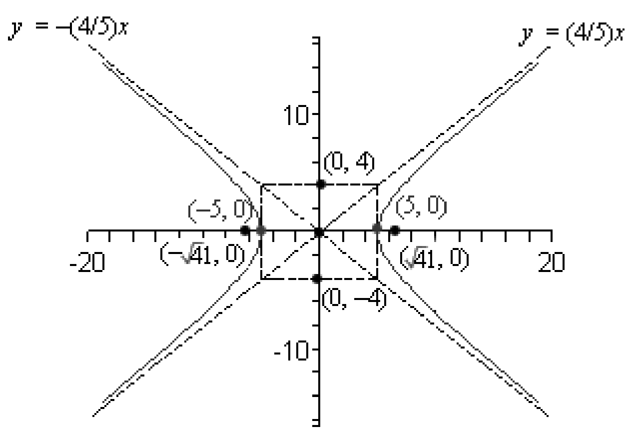
Length of Transverse Axis: $2a = 10$

Length of Conjugate Axis: $2b = 8$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{41}}{5} \approx 1.28$

Equations of the Asymptotes: $y = \frac{b}{a}x \Rightarrow y = \frac{4}{5}x$ and $y = -\frac{b}{a}x \Rightarrow y = -\frac{4}{5}x$

The graph is shown below.



Additional Example 2:

Graph the hyberbola $y^2 - 4x^2 = 4$. Specify the center, the foci, the vertices, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

Divide both sides of the equation by 4 to obtain $\frac{y^2}{4} - \frac{x^2}{1} = 1$. The equation is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, where $a = 2$ and $b = 1$. Also, $c^2 = a^2 + b^2 = 4 + 1 = 5$ so that $c = \sqrt{5}$.

Center: $(0, 0)$

Foci: $(0, -c) = (0, -\sqrt{5})$ and $(0, c) = (0, \sqrt{5})$

Vertices: $(0, -a) = (0, -2)$ and $(0, a) = (0, 2)$

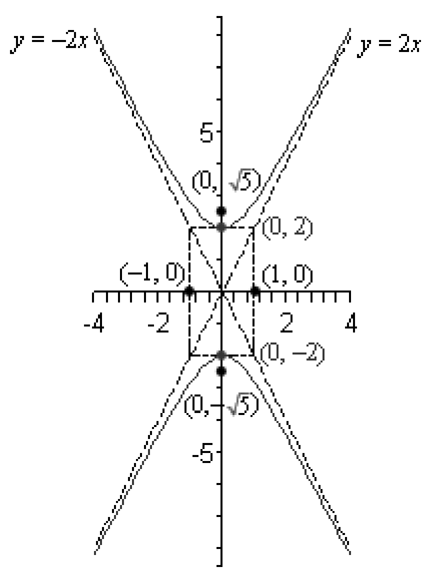
Length of Transverse Axis: $2a = 4$

Length of Conjugate Axis: $2b = 2$

Eccentricity: $e = \frac{c}{a} = \frac{\sqrt{5}}{2} \approx 1.12$

Equations of the Asymptotes: $y = \frac{a}{b}x \Rightarrow y = 2x$ and $y = -\frac{a}{b}x \Rightarrow y = -2x$

The graph is shown below.



Additional Example 3:

Graph the hyberbola $x^2 - y^2 + 2y - 5 = 0$. Specify the center, the foci, the vertices, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

We will use the technique of completing the square in order to locate the center of the hyperbola.

$$\begin{aligned} x^2 - y^2 + 2y - 5 &= 0 \\ x^2 - y^2 + 2y &= 5 \\ x^2 - (y^2 - 2y) &= 5 \\ x^2 - (y^2 - 2y + 1) &= 5 - 1 \\ x^2 - (y-1)^2 &= 4 \\ \frac{x^2}{4} - \frac{(y-1)^2}{4} &= 1 \end{aligned}$$

The equation is of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a = 2$, $b = 2$, $h = 0$, and $k = 1$. Also, $c^2 = a^2 + b^2 = 4 + 4 = 8$ so that $c = 2\sqrt{2}$.

Center: $(h, k) = (0, 1)$.

Foci: $(-c+h, k) = (-2\sqrt{2}, 1)$ and $(c+h, k) = (2\sqrt{2}, 1)$

Vertices: $(-a+h, k) = (-2, 1)$ and $(a+h, k) = (2, 1)$

Length of Transverse Axis: $2a = 4$

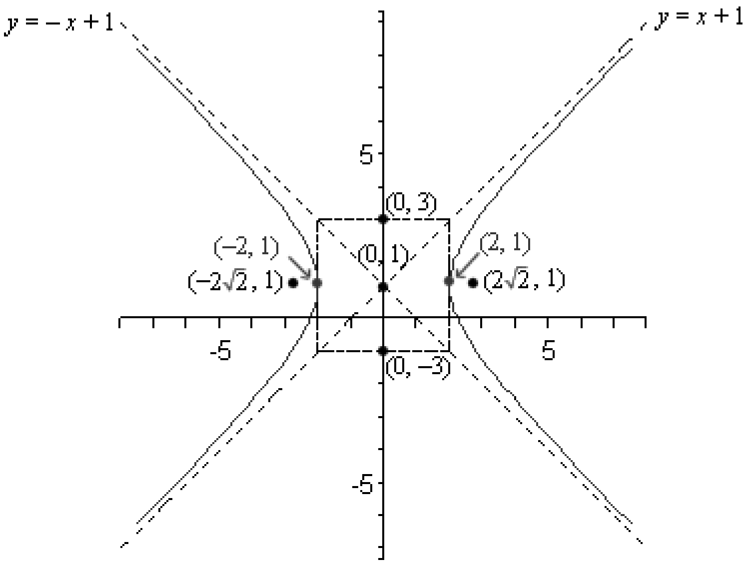
Length of Conjugate Axis: $2b = 4$

Eccentricity: $e = \frac{c}{a} = \frac{2\sqrt{2}}{2} = \sqrt{2} \approx 1.41$

Equations of Asymptotes: $y - k = \frac{b}{a}(x-h) \Rightarrow y - 1 = x$ and

$y - k = -\frac{b}{a}(x-h) \Rightarrow y - 1 = -x$

The graph is shown below.



Additional Example 4:

Graph the hyberbola $y^2 - 25x^2 + 8y - 9 = 0$. Specify the center, the foci, the vertices, the lengths of the transverse and conjugate axes, the eccentricity, and the equations of the asymptotes.

Solution:

We will use the technique of completing the square in order to locate the center of the hyperbola.

$$\begin{aligned} y^2 - 25x^2 + 8y - 9 &= 0 \\ y^2 + 8y - 25x^2 &= 9 \\ (y^2 + 8y + 16) - 25x^2 &= 9 + 16 \\ (y+4)^2 - 25x^2 &= 25 \\ \frac{(y+4)^2}{25} - \frac{x^2}{1} &= 1 \end{aligned}$$

The equation is of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, where $a=5$, $b=1$, $h=0$, and $k=-4$. Also, $c^2=a^2+b^2=25+1=26$ so that $c=\sqrt{26}$.

Center: $(h,k)=(0,-4)$.

Foci: $(h,-c+k)=(0,-\sqrt{26}-4)$ and $(h,c+k)=(0,\sqrt{26}-4)$

Vertices: $(h,-a+k)=(0,-9)$ and $(h,a+k)=(0,1)$

Length of Transverse Axis: $2a=10$

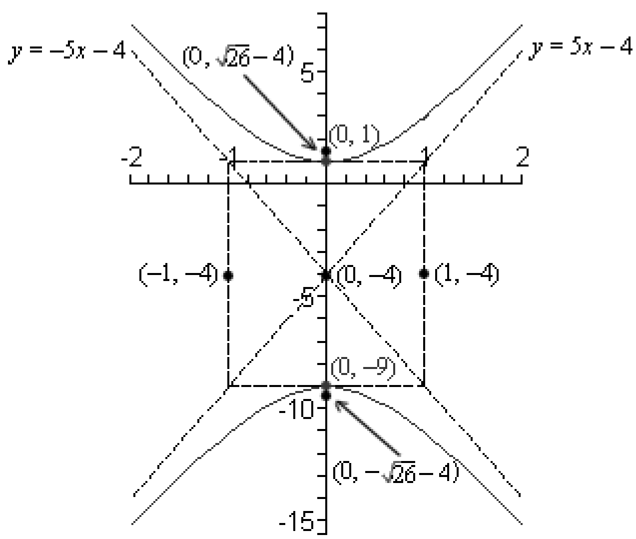
Length of Conjugate Axis: $2b=2$

Eccentricity: $e=\frac{c}{a}=\frac{\sqrt{26}}{5}\approx 1.02$

Equations of Asymptotes: $y-k=\frac{a}{b}(x-h)\Rightarrow\ y+4=5x$ and

$y-k=-\frac{a}{b}(x-h)\Rightarrow\ y+4=-5x$

The graph is shown below.



Exercise Set 8.3: Hyperbolas

Identify the type of conic section (parabola, ellipse, circle, or hyperbola) represented by each of the following equations. (In the case of a circle, identify the conic section as a circle rather than an ellipse.) Do NOT write the equations in standard form; these questions can instead be answered by looking at the signs of the quadratic terms.

1. $2y + x^2 + 9x = 0$

2. $14x^2 + 7x - 12y = -6y^2 + 95$

3. $7x^2 - 3y^2 = 5x - y + 40$

4. $y^2 + 9 = 9y - x$

5. $3x^2 - 7x + 3y^2 = -12y + 13$

6. $x^2 + 10x = -2y - y^2 + 5$

7. $4y^2 + 2x^2 = 8y - 6x + 9$

8. $8y^2 + 24x = 8x^2 + 30$

Write each of the following equations in the standard form for the equation of a hyperbola, where the standard form is represented by one of the following equations:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

9. $y^2 - 8x^2 - 8 = 0$

10. $3x^2 - 10y^2 - 30 = 0$

11. $x^2 - y^2 - 6x = -2y - 3$

12. $9x^2 - 3y^2 = 48y + 192$

13. $7x^2 - 5y^2 + 14x + 20y - 48 = 0$

14. $9y^2 - 2x^2 + 90y + 16x + 175 = 0$

Answer the following.

15. The length of the transverse axis of a hyperbola is __________.

16. The length of the conjugate axis of a hyperbola is __________.

17. The following questions establish the formulas for the slant asymptotes of

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

(a) State the point-slope equation for a line.

(b) Substitute the center of the hyperbola, (h, k) into the equation from part (a).

(c) Recall that the formula for slope is represented by $\frac{\text{rise}}{\text{run}}$. In the equation $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, what is the "rise" of each slant asymptote from the center? What is the "run" of each slant asymptote from the center?

(d) Based on the answers to part (c), what is the slope of each of the asymptotes for the graph of $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$? (Remember that there are two slant asymptotes passing through the center of the hyperbola, one having positive slope and one having negative slope.)

(e) Substitute the slopes from part (d) into the equation from part (b) to obtain the equations of the slant asymptotes.

18. The following questions establish the formulas for the slant asymptotes of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

(a) State the point-slope equation for a line.

(b) Substitute the center of the hyperbola, (h, k) into the equation from part (a).

(c) Recall that the formula for slope is represented by $\frac{\text{rise}}{\text{run}}$. In the equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, what is the "rise" of each slant asymptote from the center? What is the "run" of each slant asymptote from the center?

(d) Based on the answers to part (c), what is the slope of each of the asymptotes for the graph of $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$? (Remember that there are two slant asymptotes passing through the center of the hyperbola, one having positive slope and one having negative slope.)

(e) Substitute the slopes from part (d) into the equation from part (b) to obtain the equations of the slant asymptotes.

19. In the standard form for the equation of a hyperbola, a^2 represents (choose one):

the larger denominator
the denominator of the first term

20. In the standard form for the equation of a hyperbola, b^2 represents (choose one):

the smaller denominator
the denominator of the second term

Answer the following for each hyperbola. For answers involving radicals, give exact answers and then round to the nearest tenth.

(a) **Write the given equation in the standard form for the equation of a hyperbola. (Some equations may already be given in standard form.)**

It may be helpful to begin sketching the graph for part (h) as a visual aid to answer the questions below.

(b) **State the coordinates of the center.**

(c) **State the coordinates of the vertices, and then state the length of the transverse axis.**

(d) **State the coordinates of the endpoints of the conjugate axis, and then state the length of the conjugate axis.**

(e) **State the coordinates of the foci.**

(f) **State the equations of the asymptotes. (Answers may be left in point-slope form.)**

(g) **State the eccentricity.**

(h) **Sketch a graph of the hyperbola which includes the features from (b)-(f), along with the central rectangle. Label the center C, the vertices V_1 and V_2, and the foci F_1 and F_2.**

21. $\frac{y^2}{9}-\frac{x^2}{49}=1$

22. $\frac{x^2}{36}-\frac{y^2}{25}=1$

23. $9x^2-25y^2-225=0$

24. $16y^2-x^2-16=0$

25. $\frac{(x+1)^2}{16}-\frac{(y-5)^2}{9}=1$

26. $\frac{(x+5)^2}{4}-\frac{(y+2)^2}{16}=1$

27. $\frac{(y-3)^2}{25}-\frac{(x-1)^2}{36}=1$

28. $\frac{(y-6)^2}{64}-\frac{(x+4)^2}{36}=1$

29. $x^2-25y^2+8x-150y-234=0$

30. $4y^2-81x^2=-162x+405$

31. $64x^2-9y^2+18y=521-128x$

32. $16x^2-9y^2-64x-18y-89=0$

33. $5y^2-4x^2-50y-24x+69=0$

34. $7x^2-9y^2-72y=32-70x$

35. $x^2-3y^2=18x+27$

36. $4y^2-21x^2-8y-42x-89=0$

Use the given features of each of the following hyperbolas to write an equation for the hyperbola in standard form.

37. Center: $(0,0)$
$a=8$
$b=5$
Horizontal Transverse Axis

38. Center: $(0,0)$
$a=7$
$b=3$
Vertical Transverse Axis

39. Center: $(-2,-5)$
$a=2$
$b=10$
Vertical Transverse Axis

Exercise Set 8.3: Hyperbolas

40. Center: $(3,-4)$
$a=1$
$b=6$
Horizontal Transverse Axis

41. Center: $(-6,1)$
Length of transverse axis: 10
Length of conjugate axis: 8
Vertical Transverse Axis

42. Center: $(2,5)$
Length of transverse axis: 6
Length of conjugate axis: 14
Horizontal Transverse Axis

43. Foci: $(0,9)$ and $(0,-9)$
Length of transverse axis: 6

44. Foci: $(5,0)$ and $(-5,0)$
Length of conjugate axis: 4

45. Foci: $(-2,3)$ and $(10,3)$
Length of conjugate axis: 10

46. Foci: $(-3,8)$ and $(-3,-6)$
Length of transverse axis: 8

47. Vertices: $(4,-7)$ and $(4,9)$
$b=4$

48. Vertices: $(1,6)$ and $(7,6)$
$b=7$

49. Center: $(5,3)$
One focus is at $(5,8)$
One vertex is at $(5,6)$

50. Center: $(-3,-4)$
One focus is at $(7,-4)$
One vertex is at $(5,-4)$

51. Center: $(-1,2)$
Vertex: $(3,2)$
Equation of one asymptote:
$7x-4y=-15$

52. Center: $(-1,2)$
Vertex: $(3,2)$
Equation of one asymptote:
$6x+5y=-7$

53. Vertices: $(-1,4)$ and $(7,4)$
$e=3$

54. Vertices: $(2,6)$ and $(2,-1)$
$e=\frac{7}{3}$

55. Center: $(4,-3)$
One focus is at $(4,6)$
$e=\frac{3}{2}$

56. Center: $(-1,-2)$
One focus is at $(9,-2)$
$e=\frac{5}{2}$

57. Foci: $(3,0)$ and $(3,8)$
$e=\frac{4}{3}$

58. Foci: $(-6,-5)$ and $(-6,7)$
$e=2$