INTRODUCTION TO

ANALYTIC GEOMETRY

BY

PERCEY F. SMITH, PH.D.
PROFESSOR OF MATHEMATICS IN THE SHEFFIELD SCIENTIFIC SCHOOL
YALE UNIVERSITY

AND

ARTHUR SULLIVAN GALE, PH.D.
ASSISTANT PROFESSOR OF MATHEMATICS IN
THE UNIVERSITY OF ROCHESTER

GINN & COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON

65.8

The Athenæum Press
GINN & COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE

In preparing this volume the authors have endeavored to write a drill book for beginners which presents, in a manner conforming with modern ideas, the fundamental concepts of the subject. The subject-matter is slightly more than the minimum required for the calculus, but only as much more as is necessary to permit of some choice on the part of the teacher. It is believed that the text is complete for students finishing their study of mathematics with a course in Analytic Geometry.

The authors have intentionally avoided giving the book the form of a treatise on conic sections. Conic sections naturally appear, but chiefly as illustrative of general analytic methods.

Attention is called to the method of treatment. The subject is developed after the Euclidean method of definition and theorem, without, however, adhering to formal presentation. The advantage is obvious, for the student is made sure of the exact nature of each acquisition. Again, each method is summarized in a rule stated in consecutive steps. This is a gain in clearness. Many illustrative examples are worked out in the text.

Emphasis has everywhere been put upon the analytic side, that is, the student is taught to start from the equation. He is shown how to work with the figure as a guide, but is warned not to use it in any other way. Chapter III may be referred to in this connection.

The object of the two short chapters on Solid Analytic Geometry is merely to acquaint the student with coördinates in space and with the relations between surfaces, curves, and equations in three variables.

Acknowledgments are due to Dr. W. A. Granville for many helpful suggestions, and to Professor E. H. Lockwood for suggestions regarding some of the drawings.

New Haven, Connecticut
January, 1905

CONTENTS

CHAPTER I

REVIEW OF ALGEBRA AND TRIGONOMETRY

CHAPTER II

CARTESIAN COÖRDINATES

CHAPTER III

THE CURVE AND THE EQUATION

CHAPTER IV

THE STRAIGHT LINE AND THE GENERAL EQUATION OF THE FIRST DEGREE

CHAPTER V

THE CIRCLE AND THE EQUATION $x^2 + y^2 + Dx + Ey + F = 0$

CHAPTER VI

POLAR COÖRDINATES

CHAPTER VII

TRANSFORMATION OF COÖRDINATES

CHAPTER VIII

CONIC SECTIONS AND EQUATIONS OF THE SECOND DEGREE

CHAPTER IX

TANGENTS AND NORMALS

CHAPTER X

CARTESIAN COÖRDINATES IN SPACE

CHAPTER XI

SURFACES, CURVES, AND EQUATIONS

ANALYTIC GEOMETRY

CHAPTER I

REVIEW OF ALGEBRA AND TRIGONOMETRY

1. Numbers. The numbers arising in carrying out the operations of Algebra are of two kinds, *real* and *imaginary*.

A *real number* is a number whose square is a *positive* number. Zero also is a real number.

A *pure imaginary number* is a number whose square is a *negative* number. Every such number reduces to the square root of a negative number, and hence has the form $b\sqrt{-1}$, where b is a real number, and $(\sqrt{-1})^2 = -1$.

An *imaginary* or *complex number* is a number which may be written in the form $a + b\sqrt{-1}$, where a and b are real numbers, and b is not zero. Evidently the square of an imaginary number is in general also an imaginary number, since

$$(a + b\sqrt{-1})^2 = a^2 - b^2 + 2\,ab\sqrt{-1},$$

which is imaginary if a is not equal to zero.

2. Constants. A quantity whose value remains unchanged is called a *constant*.

Numerical or *absolute constants* retain the same values in all problems, as $2, -3, \sqrt{7}, \pi$, etc.

Arbitrary constants, or *parameters*, are constants to which any one of an unlimited set of numerical values may be assigned, and these assigned values are retained throughout the investigation.

Arbitrary constants are denoted by letters, usually by letters from the first part of the alphabet. In order to increase the number of symbols at our disposal, it is convenient to use *primes* (*accents*) or *subscripts* or both. For example:

Using primes,

a' (read "a prime or a first"), a'' (read "a double prime or a second"), a''' (read "a third"), are all different constants.

Using subscripts,

b_1 (read "b one"), b_2 (read "b two"), are different constants.

Using both,

c_1' (read "c one prime"), c_3'' (read "c three double prime"), are different constants.

3. The quadratic. Typical form. Any quadratic equation may by transposing and collecting the terms be written in the Typical Form

(1) $$Ax^2 + Bx + C = 0,$$

in which the *unknown* is denoted by x. The coefficients A, B, C are arbitrary constants, and may have any values whatever, except that A cannot equal zero, since in that case the equation would be no longer of the second degree. C is called the **constant term.**

The left-hand member

(2) $$Ax^2 + Bx + C$$

is called a **quadratic**, and any quadratic may be written in this Typical Form, in which the letter x represents the unknown. The quantity $B^2 - 4AC$ is called the **discriminant** of either (1) or (2), and is denoted by Δ.

That is, the discriminant Δ of a quadratic or quadratic equation in the Typical Form is equal to the square of the coefficient of the first power of the unknown diminished by four times the product of the coefficient of the second power of the unknown by the constant term.

The **roots** of a quadratic are those numbers which make the quadratic equal to zero when substituted for the unknown.

The roots of the quadratic (2) are also said to be roots of the quadratic equation (1). A root of a quadratic equation is said to **satisfy** that equation.

In Algebra it is shown that (2) or (1) has two roots, x_1 and x_2, obtained by solving (1), namely,

(3)
$$\begin{cases} x_1 = -\dfrac{B}{2A} + \dfrac{1}{2A}\sqrt{B^2 - 4AC}, \\ x_2 = -\dfrac{B}{2A} - \dfrac{1}{2A}\sqrt{B^2 - 4AC}. \end{cases}$$

Adding these values, we have

(4)
$$x_1 + x_2 = -\frac{B}{A}.$$

Multiplying gives

(5)
$$x_1 x_2 = \frac{C}{A}.$$

Hence

Theorem I. *The sum of the roots of a quadratic is equal to the coefficient of the first power of the unknown with its sign changed divided by the coefficient of the second power.*

The product of the roots equals the constant term divided by the coefficient of the second power.

The quadratic (2) may be written in the form

(6)
$$Ax^2 + Bx + C \equiv^* A(x - x_1)(x - x_2),$$

as may be readily shown by multiplying out the right-hand member and substituting from (4) and (5).

For example, since the roots of $3x^2 - 4x + 1 = 0$ are 1 and $\frac{1}{3}$, we have identically $3x^2 - 4x + 1 \equiv 3(x - 1)(x - \frac{1}{3})$.

The character of the roots x_1 and x_2 as *numbers* (§ 1) when the coefficients A, B, C are real numbers evidently depends entirely upon the discriminant. This dependence is stated in

Theorem II. *If the coefficients of a quadratic are real numbers, and if the discriminant be denoted by* Δ, *then*

when Δ *is positive the roots are real and unequal;*
when Δ *is zero the roots are real and equal;*
when Δ *is negative the roots are imaginary.*

* The sign $\equiv$ is read "is identical with," and means that the two expressions connected by this sign differ only in *form*.

In the three cases distinguished by Theorem II the quadratic may be written in three forms in which only *real numbers* appear. These are

$$(7)\quad \begin{cases} Ax^2+Bx+C \equiv A(x-x_1)(x-x_2), \text{ from (6), } \textit{if } \Delta \textit{ is positive;} \\ Ax^2+Bx+C \equiv A(x-x_1)^2, \text{ from (6), } \textit{if } \Delta \textit{ is zero;} \\ Ax^2+Bx+C \equiv A\left[\left(x+\dfrac{B}{2A}\right)^2+\dfrac{4AC-B^2}{4A^2}\right], \textit{ if } \Delta \textit{ is negative.} \end{cases}$$

The last identity is proved thus:

$$Ax^2+Bx+C \equiv A\left(x^2+\frac{B}{A}x+\frac{C}{A}\right)$$

$$\equiv A\left(x^2+\frac{B}{A}x+\frac{B^2}{4A^2}+\frac{C}{A}-\frac{B^2}{4A^2}\right),$$

adding and subtracting $\dfrac{B^2}{4A^2}$ within the parenthesis.

$$\therefore\ Ax^2+Bx+C \equiv A\left[\left(x+\frac{B}{2A}\right)^2+\frac{4AC-B^2}{4A^2}\right]. \qquad \text{Q.E.D.}$$

4. Special quadratics. If one or both of the coefficients B and C in (1), p. 2, is zero, the quadratic is said to be **special.**

CASE I. $C = 0$.

Equation (1) now becomes, by factoring,

$$(1)\qquad Ax^2+Bx \equiv x(Ax+B) = 0.$$

Hence the roots are $x_1 = 0$, $x_2 = -\dfrac{B}{A}\cdot$ Therefore one root of a quadratic equation is zero if the constant term of that equation is zero. And *conversely*, if zero is a root of a quadratic, the constant term must disappear. For if $x = 0$ satisfies (1), p. 2, by substitution we have $C = 0$.

CASE II. $B = 0$.

Equation (1), p. 2, now becomes

$$(2)\qquad Ax^2+C = 0.$$

From Theorem I, p. 3, $x_1 + x_2 = 0$, that is,

$$(3)\qquad x_1 = -x_2.$$

Therefore, if the coefficient of the first power of the unknown in a quadratic equation is zero, the roots are equal numerically but have opposite signs. Conversely, if the roots of a quadratic equation are numerically equal but opposite in sign, then the coefficient of the first power of the unknown must disappear. For, since the sum of the roots is zero, we must have, by Theorem I, $B = 0$.

CASE III. $B = C = 0$.

Equation (1), p. 2, now becomes

$$Ax^2 = 0. \tag{4}$$

Hence the roots are both equal to zero, since this equation requires that $x^2 = 0$, the coefficient A being, by hypothesis, always different from zero.

5. Cases when the roots of a quadratic are not independent. If a relation exists between the roots x_1 and x_2 of the Typical Form

$$Ax^2 + Bx + C = 0,$$

then this relation imposes a *condition* upon the coefficients A, B, and C, which is expressed by an equation involving these constants.

For example, if the roots are equal, that is, if $x_1 = x_2$, then $B^2 - 4AC = 0$, by Theorem II, p. 3.

Again, if one root is zero, then $x_1x_2 = 0$; hence $C = 0$, by Theorem I, p. 3.

This correspondence may be stated in parallel columns thus:

Quadratic in Typical Form

Relation between the roots	*Equation of condition satisfied by the coefficients*

In many problems the coefficients involve one or more arbitrary constants, and it is often required to find the equation of condition satisfied by the latter when a given relation exists between the roots. Several examples of this kind will now be worked out.

Ex. 1. What must be the value of the parameter k if zero is a root of the equation

(1) $$2x^2 - 6x + k^2 - 3k - 4 = 0?$$

Solution. Here $A = 2$, $B = -6$, $C = k^2 - 3k - 4$. By Case I, p. 4, zero is a root when, and only when, $C = 0$.

$$\therefore\ k^2 - 3k - 4 = 0.$$

Solving, $$k = 4 \text{ or } -1. \quad \textit{Ans.}$$

Ex. 2. For what values of k are the roots of the equation

$$kx^2 + 2kx - 4x = 2 - 3k$$

real and equal?

Solution. Writing the equation in the Typical Form, we have

(2) $$kx^2 + (2k - 4)x + (3k - 2) = 0.$$

Hence, in this case,

$$A = k,\ B = 2k - 4,\ C = 3k - 2.$$

Calculating the discriminant Δ, we get

$$\Delta = (2k - 4)^2 - 4k(3k - 2)$$
$$= -8k^2 - 8k + 16 = -8(k^2 + k - 2).$$

By Theorem II, p. 3, the roots are real and equal when, and only when, $\Delta = 0$.

$$\therefore\ k^2 + k - 2 = 0.$$

Solving, $$k = -2 \text{ or } 1. \quad \textit{Ans.}$$

Verifying by substituting these answers in the given equation (2):

when $k = -2$, the equation (2) becomes $-2x^2 - 8x - 8 = 0$, or $-2(x+2)^2 = 0$;
when $k = \ \ 1$, the equation (2) becomes $x^2 - 2x + 1 = 0$, or $(x-1)^2 = 0$.

Hence, for these values of k, the left-hand member of (2) may be transformed as in (7), p. 4.

Ex. 3. What equation of condition must be satisfied by the constants a, b, k, and m if the roots of the equation

(3) $$(b^2 + a^2m^2)y^2 + 2a^2kmy + a^2k^2 - a^2b^2 = 0$$

are equal?

Solution. The equation (3) is already in the Typical Form; hence

$$A = b^2 + a^2m^2,\ B = 2a^2km,\ C = a^2k^2 - a^2b^2.$$

By Theorem II, p. 3, the discriminant Δ must vanish; hence

$$\Delta = 4a^4k^2m^2 - 4(b^2 + a^2m^2)(a^2k^2 - a^2b^2) = 0.$$

Multiplying out and reducing,

$$a^2b^2(k^2 - a^2m^2 - b^2) = 0. \quad \textit{Ans.}$$

Ex. 4. For what values of k do the common solutions of the simultaneous equations

(4) $$3x + 4y = k,$$

(5) $$x^2 + y^2 = 25$$

become identical?

Solution. Solving (4) for y, we have

(6) $$y = \tfrac{1}{4}(k - 3x).$$

Substituting in (5) and arranging in the Typical Form gives

(7) $$25x^2 - 6kx + k^2 - 400 = 0.$$

Let the roots of (7) be x_1 and x_2. Then substituting in (6) will give the corresponding values y_1 and y_2 of y, namely,

(8) $$y_1 = \tfrac{1}{4}(k - 3x_1),\quad y_2 = \tfrac{1}{4}(k - 3x_2),$$

and we shall have two common solutions (x_1, y_1) and (x_2, y_2) of (4) and (5). But, by the condition of the problem, *these solutions must be identical.* Hence we must have

(9) $$x_1 = x_2 \text{ and } y_1 = y_2.$$

If, however, the first of these is true $(x_1 = x_2)$, then from (8) y_1 and y_2 will also be equal.

Therefore the two common solutions of (4) *and* (5) *become identical when, and only when, the roots of the equation* (7) *are equal;* that is, when the discriminant Δ of (7) vanishes (Theorem II, p. 3).

$$\therefore\ \Delta = 36k^2 - 100(k^2 - 400) = 0.$$

Solving, $$k^2 = 625,$$

$$k = 25 \text{ or } -25. \quad \textit{Ans.}$$

Verification. Substituting each value of k in (7),

when $k=25$, the equation (7) becomes $x^2-6x+9=0$, or $(x-3)^2=0$; $\therefore x=3$;
when $k=-25$, the equation (7) becomes $x^2+6x+9=0$, or $(x+3)^2=0$; $\therefore x=-3$.

Then from (6), substituting corresponding values of k and x,

when $k = 25$ and $x = 3$, we have $y = \tfrac{1}{4}(25 - 9) = 4$;
when $k = -25$ and $x = -3$, we have $y = \tfrac{1}{4}(-25 + 9) = -4$.

Therefore the two common solutions of (4) and (5) are identical for each of these values of k, namely,

if $k = 25$, the common solutions reduce to $x = 3$, $y = 4$;
if $k = -25$, the common solutions reduce to $x = -3$, $y = -4$.

Q.E.D.

PROBLEMS

1. Calculate the discriminant of each of the following quadratics, determine the sum, the product, and the character of the roots, and write each quadratic in one of the forms (7), p. 4.

(a) $2x^2 - 6x + 4$.
(b) $x^2 - 9x - 10$.
(c) $1 - x - x^2$.
(d) $4x^2 - 4x + 1$.
(e) $5x^2 + 10x + 5$.
(f) $3x^2 - 5x - 22$.

2. For what real values of the parameter k will one root of each of the following equations be zero?

(a) $6x^2 + 5kx - 3k^2 + 3 = 0$. *Ans.* $k = \pm 1$.
(b) $2k - 3x^2 + 6x - k^2 + 3 = 0$. *Ans.* $k = -1$ or 3.

3. For what real values of the parameter are the roots of the following equations equal? Verify your answers.

(a) $kx^2 - 3x - 1 = 0$. *Ans.* $k = -\frac{9}{4}$.
(b) $x^2 - kx + 9 = 0$. *Ans.* $k = \pm 6$.
(c) $2kx^2 + 3kx + 12 = 0$. *Ans.* $k = \frac{32}{3}$.
(d) $2x^2 + kx - 1 = 0$. *Ans.* None.
(e) $5x^2 - 3x + 5k^2 = 0$. *Ans.* $k = \pm \frac{3}{10}$.
(f) $x^2 + kx + k^2 + 2 = 0$. *Ans.* None.
(g) $x^2 - 2kx - k - \frac{1}{4} = 0$. *Ans.* $k = -\frac{1}{2}$.

4. Derive the equation of condition in order that the roots of the following equations may be equal.

(a) $m^2x^2 + 2kmx - 2px = -k^2$. *Ans.* $p(p - 2km) = 0$.
(b) $x^2 + 2mpx + 2bp = 0$. *Ans.* $p(m^2p - 2b) = 0$.
(c) $2mx^2 + 2bx + a^2 = 0$. *Ans.* $b^2 = 2a^2m$.

5. For what real values of the parameter do the common solutions of the following pairs of simultaneous equations become identical?

(a) $x + 2y = k$, $x^2 + y^2 = 5$. *Ans.* $k = \pm 5$.
(b) $y = mx - 1$, $x^2 = 4y$. *Ans.* $m = \pm 1$.
(c) $2x - 3y = b$, $x^2 + 2x = 3y$. *Ans.* $b = 0$.
(d) $y = mx + 10$, $x^2 + y^2 = 10$. *Ans.* $m = \pm 3$.
(e) $lx + y - 2 = 0$, $x^2 - 8y = 0$. *Ans.* None.
(f) $x + 4y = c$, $x^2 + 2y^2 = 9$. *Ans.* $c = \pm 9$.
(g) $x^2 + y^2 - x - 2y = 0$, $x + 2y = c$. *Ans.* $c = 0$ or 5.
(h) $x^2 + 4y^2 - 8x = 0$, $mx - y - 2m = 0$. *Ans.* None.

6. If the common solutions of the following pairs of simultaneous equations are to become identical, what is the corresponding equation of condition?

(a) $bx + ay = ab$, $y^2 = 2px$. *Ans.* $ap(2b^2 + ap) = 0$.
(b) $y = mx + b$, $Ax^2 + By = 0$. *Ans.* $B(m^2B - 4bA) = 0$.
(c) $y = m(x - a)$, $By^2 + Dx = 0$. *Ans.* $D(4am^2B - D) = 0$.

6. Variables. A *variable* is a quantity to which, in the same investigation, an unlimited number of values can be assigned. In a particular problem the variable may, in general, assume any value within certain limits imposed by the nature of the problem. It is convenient to indicate these limits by *inequalities.*

For example, if the variable x can assume any value between -2 and 5, that is, if x must be greater* than -2 and less than 5, the simultaneous inequalities

$$x > -2,\ x < 5,$$

are written in the more compact form

$$-2 < x < 5.$$

Similarly, if the conditions of the problem limit the values of the variable x to any negative number less than or equal to -2, and to any positive number greater than or equal to 5, the conditions

$$x < -2 \text{ or } x = -2, \text{ and } x > 5 \text{ or } x = 5$$

are abbreviated to $$x \leqq -2 \text{ and } x \geqq 5.$$

Write inequalities to express that the variable

(a) x has any value from 0 to 5 inclusive.
(b) y has any value less than -2 or greater than -1.
(c) x has any value not less than -8 nor greater than 2.

7. Equations in several variables. In Analytic Geometry we are concerned chiefly with equations in two or more variables.

An equation is *said to be satisfied* by any given set of values of the variables if the equation reduces to a numerical equality when these values are substituted for the variables.

For example, $x = 2$, $y = -3$ satisfy the equation

$$2x^2 + 3y^2 = 35,$$

since $$2(2)^2 + 3(-3)^2 = 35.$$

Similarly, $x = -1$, $y = 0$, $z = -4$ satisfy the equation

$$2x^2 - 3y^2 + z^2 - 18 = 0,$$

since $$2(-1)^2 - 3 \times 0 + (-4)^2 - 18 = 0.$$

* The meaning of *greater* and *less* for *real* numbers (§ 1) is defined as follows: a is greater than b when $a - b$ is a positive number, and a is less than b when $a - b$ is negative. Hence any negative number is less than any positive number; and if a and b are both negative, then a is *greater* than b when the *numerical* value of a is *less* than the numerical value of b.

Thus $3 < 5$, but $-3 > -5$. *Therefore changing signs throughout an inequality reverses the inequality sign.*

An equation is said to be **algebraic** in any number of variables, for example x, y, z, if it can be transformed into an equation each of whose members is a sum of terms of the form $ax^m y^n z^p$, where a is a constant and m, n, p are positive integers or zero.

Thus the equations
$$x^4 + x^2y^2 - z^3 + 2x - 5 = 0,$$
$$x^5y + 2x^2y^2 = -y^3 + 5x^2 + 2 - x$$
are algebraic.

The equation
$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$$
is algebraic.

For, squaring, we get $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = a.$

Transposing, $2x^{\frac{1}{2}}y^{\frac{1}{2}} = a - x - y.$

Squaring, $4xy = a^2 + x^2 + y^2 - 2ax - 2ay + 2xy.$

Transposing, $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0.$ Q.E.D.

The **degree** of an algebraic equation is equal to the highest degree of any of its terms.* An algebraic equation is said to be **arranged** *with respect to the variables* when all its terms are transposed to the left-hand side and written in the order of descending degrees.

For example, to *arrange* the equation
$$2x'^2 + 3y' + 6x' - 2x'y' - 2 + x'^3 = x'^2y' - y'^2$$
with respect to the variables x', y', we transpose and rewrite the terms in the order
$$x'^3 - x'^2y' + 2x'^2 - 2x'y' + y'^2 + 6x' + 3y' - 2 = 0.$$
This equation is of the third degree.

An equation which is not algebraic is said to be **transcendental.**

Examples of transcendental equations are
$$y = \sin x,\ \ y = 2^x,\ \ \log y = 3x.$$

PROBLEMS

1. Show that each of the following equations is algebraic; arrange the terms according to the variables x, y, or x, y, z, and determine the degree.

(a) $x^2 + \sqrt{y - 5} + 2x = 0.$
(b) $x^{\frac{3}{2}} + y + 3x = 0.$
(c) $xy + 3x^4 + 6x^2y - 7xy^3 + 5x - 6 + 8y = 2xy^2.$
(d) $x + y + z + x^2z - 3xy - 2z^2 = 5.$
(e) $y = 2 + \sqrt{x^2 - 2x - 5}.$

* The degree of any term is the sum of the exponents of the variables in that term.

(f) $y = x + 5 + \sqrt{2x^2 - 6x + 3}.$

(g) $x = -\frac{1}{2}D + \sqrt{\frac{D^2}{4} - F - Ey - y^2}.$

(h) $y = Ax + B + \sqrt{Lx^2 + Mx + N}.$

2. Show that the homogeneous quadratic *

$$Ax^2 + Bxy + Cy^2$$

may be written in one of the three forms below analogous to (7), p. 4, if the discriminant $\Delta \equiv B^2 - 4AC$ satisfies the condition given:

CASE I. $Ax^2 + Bxy + Cy^2 \equiv A(x - l_1 y)(x - l_2 y)$, if $\Delta > 0$;

CASE II. $Ax^2 + Bxy + Cy^2 \equiv A(x - l_1 y)^2$, if $\Delta = 0$;

CASE III. $Ax^2 + Bxy + Cy^2 \equiv A\left[\left(x + \frac{B}{2A}y\right)^2 + \frac{4AC - B^2}{4A^2}y^2\right]$, if $\Delta < 0$.

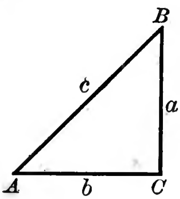
8. Functions of an angle in a right triangle. In any right triangle one of whose acute angles is A, the functions of A are defined as follows:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \csc A = \frac{\text{hypotenuse}}{\text{opposite side}},$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

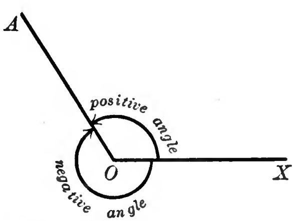
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}}, \qquad \cot A = \frac{\text{adjacent side}}{\text{opposite side}}.$$

From the above the theorem is easily derived:



In a right triangle a side is equal to the product of the hypotenuse and the sine of the angle opposite to that side, or of the hypotenuse and the cosine of the angle adjacent to that side.

9. Angles in general. In Trigonometry an angle XOA is considered as generated by the line OA rotating from an initial position OX. The angle is positive when OA rotates from OX *counter-clockwise*, and negative when the direction of rotation of OA is *clockwise*.



* The coefficients A, B, C and the numbers l_1, l_2 are supposed real.

The fixed line OX is called the *initial line*, the line OA the *terminal line*.

Measurement of angles. There are two important methods of measuring angular magnitude, that is, there are two unit angles.

Degree measure. The unit angle is $\frac{1}{360}$ of a complete revolution, and is called a *degree*.

Circular measure. The unit angle is an angle whose subtending arc is equal to the radius of that arc, and is called a *radian*.

The fundamental relation between the unit angles is given by the equation

$$180 \text{ degrees} = \pi \text{ radians } (\pi = 3.14159 \cdots).$$

Or also, by solving this,

$$1 \text{ degree} = \frac{\pi}{180} = .0174 \cdots \text{ radians},$$

$$1 \text{ radian} = \frac{180}{\pi} = 57.29 \cdots \text{ degrees}.$$

These equations enable us to change from one measurement to another. In the higher mathematics circular measure is always used, and will be adopted in this book.

The generating line is conceived of as rotating around O through as many revolutions as we choose. Hence the important result:

Any real number is the circular measure of some angle, and conversely, any angle is measured by a real number.

10. Formulas and theorems from Trigonometry.

1. $\cot x = \dfrac{1}{\tan x}$; $\sec x = \dfrac{1}{\cos x}$; $\csc x = \dfrac{1}{\sin x}$.

2. $\tan x = \dfrac{\sin x}{\cos x}$; $\cot x = \dfrac{\cos x}{\sin x}$.

3. $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$.

4. $\sin(-x) = -\sin x$; $\csc(-x) = -\csc x$;
   $\cos(-x) = \cos x$; $\sec(-x) = \sec x$;
   $\tan(-x) = -\tan x$; $\cot(-x) = -\cot x$.

5. $\sin(\pi - x) = \sin x$; $\sin(\pi + x) = -\sin x$;
$\cos(\pi - x) = -\cos x$; $\cos(\pi + x) = -\cos x$;
$\tan(\pi - x) = -\tan x$; $\tan(\pi + x) = \tan x$;

6. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$; $\sin\left(\frac{\pi}{2} + x\right) = \cos x$;
$\cos\left(\frac{\pi}{2} - x\right) = \sin x$; $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$;
$\tan\left(\frac{\pi}{2} - x\right) = \cot x$; $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$.

7. $\sin(2\pi - x) = \sin(-x) = -\sin x$, etc.

8. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

9. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

10. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

11. $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

12. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$. 13. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.

14. $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$; $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

15. $\sin\frac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$; $\cos\frac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$; $\tan\frac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$.

16. *Theorem. Law of sines.* In any triangle the sides are proportional to the sines of the opposite angles;

that is, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

17. *Theorem. Law of cosines.* In any triangle the square of a side equals the sum of the squares of the two other sides diminished by twice the product of those sides by the cosine of their included angle;

that is, $$a^2 = b^2 + c^2 - 2bc \cos A.$$

18. *Theorem. Area of a triangle.* The area of any triangle equals one half the product of two sides by the sine of their included angle;

that is, $$\textit{area} = \tfrac{1}{2} ab \sin C = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ca \sin B.$$

11. Natural values of trigonometric functions.

Angle in Radians	Angle in Degrees	Sin	Cos	Tan	Cot		
.0000	0°	.0000	1.0000	.0000	∞	90°	1.5708
.0873	5°	.0872	.9962	.0875	11.430	85°	1.4835
.1745	10°	.1736	.9848	.1763	5.671	80°	1.3963
.2618	15°	.2588	.9659	.2679	3.732	75°	1.3090
.3491	20°	.3420	.9397	.3640	2.747	70°	1.2217
.4363	25°	.4226	.9063	.4663	2.145	65°	1.1345
.5236	30°	.5000	.8660	.5774	1.732	60°	1.0472
.6109	35°	.5736	.8192	.7002	1.428	55°	.9599
.6981	40°	.6428	.7660	.8391	1.192	50°	.8727
.7854	45°	.7071	.7071	1.0000	1.000	45°	.7854
		Cos	Sin	Cot	Tan	Angle in Degrees	Angle in Radians

Angle in Radians	Angle in Degrees	Sin	Cos	Tan	Cot	Sec	Csc
0	0°	0	1	0	∞	1	∞
$\frac{\pi}{2}$	90°	1	0	∞	0	∞	1
π	180°	0	– 1	0	∞	– 1	∞
$\frac{3\pi}{2}$	270°	– 1	0	∞	0	∞	– 1
2π	360°	0	1	0	∞	1	∞

Angle in Radians	Angle in Degrees	Sin	Cos	Tan	Cot	Sec	Csc
0	0°	0	1	0	∞	1	∞
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	90°	1	0	∞	0	∞	1

12. Rules for signs.

Quadrant	Sin	Cos	Tan	Cot	Sec	Csc
First	+	+	+	+	+	+
Second . . .	+	−	−	−	−	+
Third	−	−	+	+	−	−
Fourth . . .	−	+	−	−	+	−

13. Greek alphabet.

Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ ς	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega