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Analytic Topology

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PREFACE

The material here presented represents an elaboration on my Colloquium Lectures delivered before the American Mathematical Society at its September, 1940 meeting at Dartmouth College. The results of some of my own efforts together with a selection of those of other mathematicians relative to the subject chosen are presented in what is intended to be a coherent and approximately self-contained exposition, framed in the familiar topology of separable metric spaces. No attempt is made at comprehensive coverage either of the known work embraced by the title of the book or of the work of others, or even myself, which may be closely related to that included.

I wish to express my appreciation to the American Mathematical Society for the opportunity of delivering the lectures and publishing in its Colloquium Series. Thanks are due also to the Waverly Press for its careful and sympathetic handling of the manuscript.

On the personal side, it has been my privilege and good fortune to stand in the middle ground between distinguished teachers on the one hand and a group of distinguished students and associates on the other and receive stimulus and inspiration from both. Of the former, the influence of R. L. Moore will be apparent and his invaluable contribution in this way is gratefully though inadequately acknowledged. Of the other group, too numerous to mention, Hubert A. Arnold, M. Garcia and Paul A. White have helped directly by reading and correcting parts of the manuscript and proof. Finally, I wish to acknowledge the generous assistance rendered by my wife, Lucille Whyburn, who contributed materially to the content and organization of the lectures and manuscript and assisted greatly in the preparation of both and whose unfailing encouragement transcends all attempts at evaluation.

G. T. WHYBURN

CHARLOTTESVILLE, VA.
November, 1941

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INTRODUCTION

As used here the term "Analytic Topology" is meant to cover those phases of topology which are being developed advantageously by methods in which continuous transformations play the essential role. In the process of evolving, coming of age and assuming more stable form, topology, through interaction with other branches of mathematics, not only is leaving its mark on them but is itself adopting more and more the language and symbolism of the older fields. Thus, for example, we have not only a topological function theory giving the results of analysis which are essentially topological in character, but also a function-theoretic topology dealing with topological situations with the aid and principal use of some of the basic tools of analysis. Without drawing the lines too sharply or giving too clear cut a definition, let us say in a general way that analytic topology deals with topological situations with the aid of analytical language and tools, and to some extent conversely, just as analytic geometry handles geometric situations by analytic methods. I hope this concept will be made clearer as the treatment progresses and actual examples are given illustrating the type of relationship which has been so vaguely defined.

The major questions to be dealt with are, first, the existence of transformations of various sorts from a space A to the same or another space B and, second, the analysis of the action of these transformations on A to produce B. Since thus we are dealing with the transition from A to itself or to something else possibly quite different topologically, our subject exhibits kinship with earlier work on dynamics in the Colloquium Series. This is especially true of the final chapter on periodicity which connects directly with many of the concepts of this subject as discussed by G. D. Birkhoff. However, the even closer kinship with other purely topological treatises, notably that of R. L. Moore in the Colloquium Series and that of K. Menger on "Kurventheorie", will be too obvious to require comment.

The book divides roughly into two parts, corresponding to the first six and last six chapters, respectively. In the first part there is developed the necessary topological machinery and framework for the latter part, which is devoted to pure analytic topology. Even in the second chapter, however, notably in §§3, 4, there emerge some of the fruits of the application of analytic or transformation methods to topological situations. For here a variety of results, some classic and others quite recent, are brought together in what seems their proper relationship and derived in a simple and novel way from one central mapping theorem.

The book is meant to be largely self-contained, at least in so far as topological developments are concerned. In the later stages some use is made of a few notions of combinatorial topology and of the theory of groups without any attempt at adequate introduction. Since these appear largely in end-results and applications, there seems little need or justification for taking the space to develop them here.

At the beginning we assume once for all a set of axioms sufficient to make all spaces considered separable and metrizable. Once the metric is introduced, however, attention is no longer focused on the axioms, but rather on the (equivalent) standpoint that we are operating always in a given separable metric space.

Cross references are given in brackets, with the roman numeral for the chapter followed by the section and number of the theorem, lemma, or corollary referred to, e.g., [IV, (3.2)] refers to result (3.2) of §3 in Chapter IV, which would be the *second* main result in this section. If only the number in parenthesis is given, as (3.2) for example, the reference is to the result of that number in the *present chapter*, i.e., the one being read at the time. To assist in locating results referred to, the chapter number and section number appear at the heading of each double page.

References to the literature in the main are held to a minimum. For convenience these are made at the ends of the chapters in the form of author's name followed by numerals in brackets referring to his books or papers by that number in the bibliography at the close of the book. In some cases one or more authors' names have been used in connection with a theorem though by no means in all where this might well, or possibly should, be done. A considerable amount of the material in the first part is of such a classical nature and so well known that specific citations to sources in the literature are not made. Later on more attempt is made to cite the original author and source. In some cases, also, closely related material not actually covered is mentioned in the references.

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NOTES

1. The conclusions in (7.3) on p. 147 hold without any restriction on A provided R is assumed conditionally compact. The thus broadened (7.3) then yields the further corollary.

(7.32) *Local compactness is invariant under interior transformations.*

Local compactness of A also should be assumed in (7.31).

2. The theorem in (3.1) on p. 189 may be strengthened by allowing A to be any *generalized continuum* instead of restricting it to *continua*. This improved form of (3.1) applies more directly in the proof of (3.42) on page 191 and facilitates the reading of that proof.

3. In the proof of (ii) on pgs. 194–195, there is no advantage in having the set M connected; and some simplification results if it is taken merely to be compact and locally connected.

4. In the proof of (iv) on p. 195, the points r and s may and should be taken on the arc uqv in the order u, r, q, s, v.

5. The proof of (v) on p. 195 is sketchy and difficult to follow. It should be replaced by the following detailed argument:

By (iv), every point of K is of order $\leqq 2$. Suppose, first, that some $x \in K$ is of order 1 in K. Let V be an $\epsilon/2$-neighborhood of x in B such that $F(V) \cdot K = b \in K$. By (ii) and (iv) there exists an $\epsilon/2$-neighborhood W of b in B whose boundary is a simple arc rs with $rs \cdot K = r + s$ and so that $\bar{W}$ does not contain x. Since x is of order 1 in K, it is clear that for ϵ sufficiently small one end point of rs, say r, is in V and the other in $B - \bar{V}$, as otherwise we could obtain from V and W an ϵ-neighborhood of x with boundary disjoint from K. Let t be an interior point of rs such that $rt \subset V$. Then if T denotes the sum of a finite number of locally connected subcontinua of B so chosen that $T \supset F(V) - W \cdot F(V)$, $T \cdot (K + x) = 0$, $\delta(T + V) < 2\epsilon/3$ and $M = T + ts$, then M is a compact locally connected set ϵ-separating x in B and such that $M \cdot K = s$. But then the reasoning given under (ii) and (iv) shows that M contains a simple arc X *both* of whose end points belong to K and this is impossible.

Now in case some $x \in K$ were of order 0, we need only choose V as above so that $F(V) \cdot K = 0$ and let M be a compact locally connected set in B containing $F(V)$, disjoint from $K + x$ and so that $\delta(V + M) < \epsilon$. Then by the reasoning under (ii) and (iv) we would obtain in M a simple closed curve X which, by (iii), would show that x is a regular point, contrary to $x \in K$.

6. The set $f^{-1}(K)$ used under (viii) on p. 196 is necessarily a finite graph as there stated because it contains only a finite number of simple closed curves and has no end points.

7. In the proof of (4.6) on p. 249, it is tacitly assumed that the mapping f^n is pointwise almost periodic at p. This can be verified fairly easily under the assumptions made in (4.6). Also, it is a consequence of more general propositions along this line established later by W. H. Gottschalk. See, for example, Bulletin of the American Mathematical Society, vol. 50 (1944), p. 223.

8. In line 3 of the proof of (5.2), p. 71, to see that X is separated between a and b, let $X = X_1 + X_2$ be any separation where $b \in X_2$. If $a \in X_2$, then since X_1 is nonempty so that not all points of $E(a, b)$ can be in X_2, there is a point c of $E(a, b)$ in X_1. Then if Q is the component of $M - c$ containing b, the set $X_b = Q \cdot X_2$ is both open and closed in X and does not contain a.

9. The author is indebted to Dr. R. J. Bean for pointing out that the result (7.2), p. 136, in earlier editions of this book is incorrectly worded. The words *for each* $b \in B$ should be inserted between "that" and "there" in the second line of the statement of this result in the earlier editions.

10. In the proof of (3.4), p. 216, to see that M is not unicoherent about J, we reason as follows. For each point p of the intersection of A (resp. B) with the open arc ayb let R_p be a region in M about p of diameter $< 1/3$ the distance from p to $axb + B$ (resp. A) and let R_p^n be a finite union of the R_p covering $\overline{ayb} \cdot (A + B) - V_{1/n}(a + b)$. Then the sets $H' = H - \sum_{n=1}^{\infty} R_p^n$, $K' = K + \sum_{n=1}^{\infty} \overline{R_p^n}$, are closed and we have

$$H' \cdot J = axb, \ K' \cdot J = ayb, \ M = H' + K', \ H' \cdot K' \subset V_{d/3}(H \cdot K)$$

where $d = \rho(A, B)$. Thus a and b lie in different components of $H' \cdot K'$.

ERRATA

p. 19. Substitute the following for the final paragraph of the proof of (13.2).

Consider first the case of compact M. Now let C_1 and D_1 be components of $M_1 = M - V_{\epsilon/2}(x)$ intersecting both $F[V_{\epsilon/2}(x)]$ and $M - V_\epsilon(x)$ and let

$$M_1 = C_1^0 + D_1^0$$

be a separation of M_1 between C_1 and D_1. Then for at least one of the sets $C_1^0 + \overline{V_{\epsilon/2}(x)}$ and $D_1^0 + \overline{V_{\epsilon/2}(x)}$, say X_1, it must be true that for every $i > 2$, infinitely many components of $X_1 - V_{\epsilon/i}(x)$ intersect $M - V_\epsilon(X)$. We suppose this holds for $X_1 = C_1^0 + V_{\epsilon/2}(x)$ and let K_1 be a component of $D_1 - D_1 \cdot V_{3\epsilon/4}(x)$ intersecting $M - V_\epsilon(x)$.

Then let C_2 and D_2 be components of $M_2 = X_1 - V_{\epsilon/3}(x)$ intersecting both $F[V_{\epsilon/3}(x)]$ and $M - V_\epsilon(x)$ and let

$$M_2 = C_2^0 + D_2^0$$

be a separation of M_2 between C_2 and D_2. Proceeding as before we find $X_2 = C_2^0 + \overline{V_{\epsilon/3}(x)}$ [or $= D_2^0 + \overline{V_{\epsilon/3}(x)}$] so that infinitely many components of $X_2 - V_{\epsilon/i}(x)$, $i > 3$, intersect $M - V_\epsilon(x)$ and let K_2 be a component of $D_2 - D_2 \cdot V_{2\epsilon/3}(x)$ intersecting $M - V_\epsilon(x)$. Continuing this process indefinitely we obtain an infinite sequence $K_1, K_2, \cdots$ of disjoint continua each intersecting $M - V_\epsilon(x)$, with $K_n \cdot F[V_{\frac{2\epsilon}{n+1}}(x)] \neq 0$ and

$$K_n \subset D_n, \; \sum_{n+1}^{\infty} K_i \subset C_n^0.$$

Thus if $[K_{n_i}]$ is a convergent subsequence of $[K_n]$ with limit K, we have $K \subset \prod C_n^0$ so that $K \cdot K_n = 0$ for all n; and thus K is a continuum of convergence of M containing x.

Next suppose, contrary to our theorem, that x is on no continuum of convergence of M. Then M is locally connected at x and, accordingly, for any $\epsilon > 0$, x is interior to the component C_ϵ of $\overline{V_\epsilon(x)}$ containing x. Suppose ϵ chosen so that C_ϵ is compact. By the compact case treated above, C_ϵ must be semi-locally connected at x. Thus we can find an arbitrarily small $\sigma < \epsilon$ such that $V_\sigma(x) \subset C_\epsilon$ and so that $C_\epsilon - V_\sigma(x) = N$ has only a finite number k of components. But then $M - V_\sigma(x) = N + M - C_\epsilon$ can have at most k components since $M - C_\epsilon$ and $V_\sigma(x)$ are separated and $M - N = (M - C_\epsilon) + V_\sigma(x)$. Whence, M is s.l.c. at x contrary to hypothesis.

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