Analytic Trigonometry and Trigonometric Applications

4

Unit Overview

This unit will extend your knowledge of trigonometry as you study trigonometric identities, equations, and formulas. You will explore the Law of Cosines and the Law of Sines, and apply them to solve non-right triangles.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary

- ambiguous

Math Terms

- identity
- Pythagorean identity
- trigonometric identity
- cofunction identity
- sum and difference identities
- Law of Cosines
- oblique triangle
- Law of Sines
- ambiguous case (SSA)

ESSENTIAL QUESTIONS

? How are algebraic and geometric concepts related to trigonometric identities and formulas?

? How is trigonometry used to solve real-world problems involving measure?

EMBEDDED ASSESSMENTS

These assessments, following Activities 23 and 25, will give you an opportunity to demonstrate what you have learned about trigonometry, in particular the Law of Cosines and the Law of Sines.

Getting Ready

Write your answers on notebook paper.
Show your work.

1. Factor each of the following completely.
  a. $6x^4 - 12x^3 + 3x^2$
  b. $49x^4 - 36y^2$
  c. $8x^2 - 2x - 15$

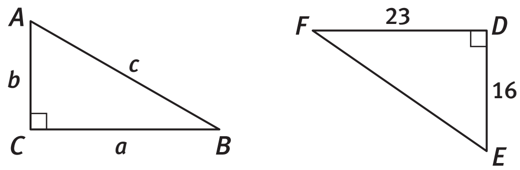
2. Simplify each of the following rational expressions.
  a. $\dfrac{2x}{x+3} + \dfrac{12}{2x+6}$
  b. $\dfrac{3}{ab^2} + \dfrac{2}{a^2b} - \dfrac{1}{a^2b^2}$
  c. $\dfrac{\dfrac{x^2}{y} - y}{\dfrac{x}{y} - 1}$
  d. $\dfrac{x^2+5x}{x^2+6x+5} \div \dfrac{x^3}{3x+3}$

3. A 32 ft ladder leans against a building, making a 75° angle with the ground. How high above the ground is the top of the ladder?

4. Tell four measures of $\angle\beta$ such that $\cos\beta = \left|\dfrac{1}{2}\right|$.

5. Sketch two angles α such that $\tan\alpha = -1$.

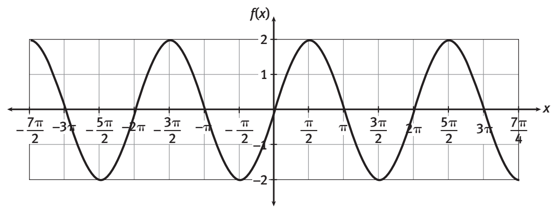
Use right triangles ABC and DEF for Items 6 and 7.



6. State the following ratios.
  a. $\sin A$ **b.** $\cos A$ **c.** $\tan A$
  d. $\cot A$ **e.** $\csc A$ **f.** $\sec A$

7. Find the measure of the following angles to the nearest tenth of a degree.
  a. $\angle E$ **b.** $\angle F$

8. Write an equation of the graph below.



Trigonometric Identities

Imagine That

Lesson 21-1 Trigonometric Identities

My Notes

Learning Targets:

- Define the reciprocal and quotient identities.
- Use and transform the Pythagorean identity.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Marking the Text, Create Representations, Look for Patterns, Graphic Organizer

The use of computer-generated imagery (CGI) in the movie industry became common with the release of blockbuster films such as *Jurassic Park* (1993) and *Toy Story* (1995). CGI animators use specialized software to render and animate 3-D images, but they also need to understand and apply mathematics, including basic trigonometry and trigonometric identities.

This activity will introduce the basic identities that are fundamental to the advanced computer graphics work done by animators and video game designers every day.

An ***identity*** is a relationship of equality that is true for all values of the variables for which each side of the equation is defined.

1. **Reason abstractly.** Which of the following statements are identities? Explain your reasoning. Be sure to use correct mathematical terms to support your reasoning and that your sentences are complete and grammatically correct.

   a. $2(x + 3) = 2x + 6$

   b. $\dfrac{2}{2x + 4} = \dfrac{1}{x + 2}$

   c. $x^2 + 1 = (x + 1)(x - 1)$

   d. $\sec\theta = \dfrac{1}{\cos\theta}$

DISCUSSION GROUP TIPS

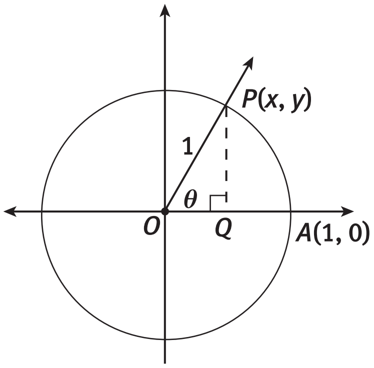
As you listen to the group discussion, take notes to aid comprehension and to help you describe your own ideas to others in your group. Ask questions to clarify ideas and to gain further understanding of key concepts.

A ***trigonometric identity*** is an identity that involves one or more trigonometric functions. Identities like the one in Item 1d are derived from the definitions of sine and cosine as coordinates of a point on a unit circle.

2. What is another identity you may already know based on the definitions of sine and cosine?

My Notes

3. Use the unit circle shown below to express each trigonometric function in terms of x and/or y.



$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

4. Make use of structure. Use the results of Item 3 to write identities that express tangent and cotangent as quotients of the sine and cosine functions.

a. $\tan\theta =$

b. $\cot\theta =$

5. Use appropriate tools strategically. How can you use your graphing calculator to support that the statements you wrote in Item 4 are identities?

6. Are there any values of θ for which the expressions in Item 4 are undefined? Explain your reasoning.

My Notes

7. Use the results of Item 3 to write identities that express each trigonometric function as a *reciprocal* of another trigonometric function.

a. $\sin\theta =$

b. $\csc\theta =$

c. $\cos\theta =$

d. $\sec\theta =$

e. $\tan\theta =$

f. $\cot\theta =$

8. Use your knowledge of special angle ratios to complete the table shown below.

θ	$60°$	$225°$	$-90°$	$-\frac{19\pi}{6}$	π	$\frac{11\pi}{6}$	$\frac{5\pi}{6}$
$\cos^2\theta$							
$\sin^2\theta$							
$\sin^2\theta + \cos^2\theta$							

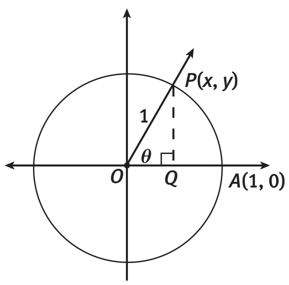
WRITING MATH

$\cos^2\theta = (\cos\theta)^2$

The exponent is written between the trigonometric function and the angle to avoid confusion with $\cos\theta^2 = \cos(\theta \cdot \theta)$.

9. Express regularity in repeated reasoning. Based on the above table, complete the identity that appears to be true for all values of θ.

10. Use the unit circle and the Pythagorean Theorem to verify that the identity you wrote in Item 9 is true for all values of θ.



My Notes

The identity in Item 9 is known as the ***Pythagorean identity***. Two other identities can be derived from the results of Item 10.

Example A

Write an identity relating the tangent and secant functions.

$\sin^2\theta + \cos^2\theta = 1$	Given
$\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$	Divide by $\cos^2\theta$.
$\tan^2\theta + 1 = \sec^2\theta$	Rewrite using basic identities.

Try These A

Complete the following to write an identity relating the cotangent and cosecant functions.

$\sin^2\theta + \cos^2\theta = 1$	Given
________________	Divide by $\sin^2\theta$.
________________	Rewrite using basic identities.

CONNECT TO AP

Identities are used frequently in AP Calculus to simplify problems and derive differentiation and integration rules.

11. Summarize your work by completing the following graphic organizer.

Quotient and Reciprocal Identities

$\sin\theta =$ ____________ $\csc\theta =$ ____________

$\cos\theta =$ ____________ $\sec\theta =$ ____________

$\tan\theta =$ ____________ $=$ ____________

$\cot\theta =$ ____________ $=$ ____________

Pythagorean Identities

____

____

____

My Notes

Check Your Understanding

12. Are there any values of θ for which the identity $\sec\theta = \dfrac{1}{\cos\theta}$ does *not* hold? Explain.

13. Construct viable arguments. Connor had difficulty remembering the three Pythagorean identities. His friend Amani said that she only memorized one of the identities and then quickly derived the others when she needed them. Explain what Amani meant by this statement.

LESSON 21-1 PRACTICE

14. Express each trigonometric function as a reciprocal to write an identity.

a. $\cos(x + 45°) =$ **b.** $\tan\left(\dfrac{\theta}{3}\right) =$ **c.** $\csc(2\theta) =$

15. Express the tangent or cotangent as a quotient to write an identity.
a. $\cot(4\theta) =$ **b.** $\tan(x - 180°) =$ **c.** $\cot(90° + x) =$

16. Explain why $\cos\theta\sec\theta = 1$.

17. Consider the function $f(x) = \sin^2 x + \cos^2 x$.
a. What is the domain of the function?
b. What is the range of the function?
c. What is $f(\sqrt{7})$?

18. Make use of structure. Suppose you know that $\sin\theta = 0.4$.
a. Using the Pythagorean identity, what can you conclude about $\cos\theta$?
b. How does your answer to part a change if you know that θ lies in Quadrant II?
c. What is the approximate value of θ, in degrees, given that θ lies in Quadrant II?

My Notes

Learning Targets:

- Simplify trigonometric expressions.
- Verify trigonometric identities.

SUGGESTED LEARNING STRATEGIES: Note Taking, Think-Pair-Share, Identify a Subtask, Simplify the Problem, Work Backward

The identities you learned in the previous lesson can be used to simplify expressions and to make certain calculations easier. There are several strategies that are helpful when simplifying trigonometric expressions.

Some key strategies are listed below.

• Substitute using an identity. • Use multiplication or factoring. • Rewrite the expression in terms of sine and cosine.	• Combine a sum or difference into a single expression. • Rewrite division as multiplication.

MATH TIP

Create an organized summary of the trigonometric and Pythagorean identities to use when simplifying trigonometric expressions.

Example A

Simplify $\sin\theta \cot\theta \sec\theta$.

$\sin\theta \cot\theta \sec\theta$

$= \dfrac{\sin\theta}{1} \bullet \dfrac{\cos\theta}{\sin\theta} \bullet \dfrac{1}{\cos\theta}$ Rewrite in terms of sine and cosine.

$= \dfrac{\cancel{\sin\theta}}{1} \bullet \dfrac{\cancel{\cos\theta}}{\cancel{\sin\theta}} \bullet \dfrac{1}{\cancel{\cos\theta}}$ Simplify.

$= 1$

Example B

Simplify $\tan^2\theta - \sec^2\theta + 1$.

$\tan^2\theta - \sec^2\theta + 1$

$= (\tan^2\theta + 1) - \sec^2\theta$ Reorder terms.

$= \sec^2\theta - \sec^2\theta$ Substitute using an identity.

$= 0$ Simplify.

MATH TIP

The Pythagorean identities can be transformed using the properties of equality and used to simplify expressions.

For example, $\sin^2\theta + \cos^2\theta = 1$ is equivalent to $\cos^2\theta = 1 - \sin^2\theta$ and $\sin^2\theta = 1 - \cos^2\theta$.

Try These A–B

Simplify each expression. The simplest form will be a single trigonometric function or a constant.

a. $\cos^2\theta + \sin^2\theta - 1$

b. $\dfrac{\cos\theta \csc\theta}{\cot\theta}$

c. $\sec\theta\,(1 - \sin^2\theta)$

My Notes

1. **Use appropriate tools strategically.** You can use your calculator to check your work when you simplify a trigonometric expression.
   a. Suppose you enter the expression from Try These A Part b in your calculator as Y1. What would you expect to see when you graph the function? Why?

   b. Enter the expression as Y1 and graph the function. Is your prediction correct?

Not every trigonometric expression simplifies to a single term or constant.

2. Fill in the strategies used to simplify the expression shown below.

$\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta + \cos\theta}$ Original expression

$\dfrac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{\sin\theta + \cos\theta}$

$\dfrac{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}}{\cancel{(\sin\theta + \cos\theta)}}$

$\sin\theta - \cos\theta$

3. **Reason abstractly.** Fill in the strategies used to simplify the expression shown below.

$\tan\theta + \dfrac{1}{\tan\theta}$ Original expression

$\dfrac{\tan\theta \cdot \tan\theta}{\tan\theta} + \dfrac{1}{\tan\theta}$

$\dfrac{\tan^2\theta + 1}{\tan\theta}$

$\dfrac{\sec^2\theta}{\tan\theta}$

$\dfrac{\frac{1}{\cos^2\theta}}{\frac{\sin\theta}{\cos\theta}}$

$\dfrac{1}{\cos^2\theta} \cdot \dfrac{\cos\theta}{\sin\theta}$

$\dfrac{1}{\sin\theta\cos\theta}$

$\csc\theta\sec\theta$

MATH TIP

Factoring patterns like the *difference of two squares* can be applied to trigonometric expressions.
$a^2 - b^2 = (a + b)(a - b)$

My Notes

4. Simplify each expression.

a. $\cos\theta\left(\frac{1}{\sin\theta}+\csc\theta\right)$

b. $\csc\theta - \tan\theta\cos\theta$

c. $\frac{\sin\theta-\csc\theta}{\cot\theta}$

Earlier in this activity, you used a table and a graphing calculator to explore and informally confirm identities. To formally verify an identity, you must use the known identities and the strategies you learned when simplifying expressions to show that one side of the identity can be transformed into the other side.

To Verify an Identity

Transform one side of the equation into the other side using simplification techniques.

Example C

Verify the identity $\cot\theta\sin\theta = \cos\theta$.

$\cot\theta\sin\theta = \cos\theta$	Original identity
$\frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1} =$	Rewrite using sine and cosine.
$\frac{\cos\theta}{\cancel{\sin\theta}}\cdot\frac{\cancel{\sin\theta}}{1} =$	Simplify by canceling common sine factors.
$\cos\theta = \cos\theta$	

Example D

Verify the identity $\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}=2\csc^2\theta$

$\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}=2\csc^2\theta$	Original identity
$\frac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)}=$	Add two fractions using LCD.
$\frac{2}{1-\cos^2\theta}=$	Simplify numerator and multiply denominator.
$\frac{2}{\sin^2\theta}=$	Pythagorean identity
$2\csc^2\theta=2\csc^2\theta$	Basic identity

CONNECT TO TECHNOLOGY

Use a graphing calculator to provide a visual confirmation of your work by graphing both sides of the identity.

Try These C–D

Verify each identity.

a. $\tan\theta\csc\theta=\sec\theta$

b. $2-\sin^2\theta-\cos^2\theta=1$

c. $\frac{\cos x}{\sin x+\cos x}-\frac{\sin x}{\sin x-\cos x}=\frac{-1}{\sin^2 x-\cos^2 x}$

My Notes

When verifying identities, you can start on either side of the equal sign, but it is often advantageous to start on the more complicated side.

Example E

Verify the identity $1 - \sin^2\theta = \dfrac{\sin\theta\cot\theta}{\sec\theta}$.

$1 - \sin^2\theta = \dfrac{\sin\theta\cot\theta}{\sec\theta}$ Original identity

$= \dfrac{\frac{\sin\theta}{1}\cdot\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}}$ Rewrite using sine and cosine.

$= \dfrac{\frac{\cancel{\sin\theta}}{1}\cdot\frac{\cos\theta}{\cancel{\sin\theta}}}{\frac{1}{\cos\theta}}$ Simplify.

$= \dfrac{\cos\theta}{1}\cdot\dfrac{\cos\theta}{1}$ Rewrite multiplication as division.

$= \cos^2\theta$

$1 - \sin^2\theta = 1 - \sin^2\theta$ Substitute using an identity.

Try These E

Verify each identity.

a. $\tan x + \cot x = \csc x \sec x$

b. $\cos^2 x(\sec^2 x - 1) = 1 - \cos^2 x$

c. $\sec\theta + 1 = \dfrac{\sin\theta\tan\theta}{1-\cos\theta}$

My Notes

Check Your Understanding

4. Describe two different ways to simplify the expression shown below. Do you get the same result with either method?

$$\frac{1}{\sin\theta} + \csc\theta\cot^2\theta$$

5. Reason abstractly. Rima was asked to verify the identity $\sin x = \dfrac{\cos x}{\sin x\csc x\cot x}$. She handed in the work shown below. When she got her paper back, there was a note from her teacher saying that her method was incorrect. Explain what her teacher meant.

$\sin x = \dfrac{\cos x}{\sin x\csc x\cot x}$	Original identity
$\sin^2 x = \dfrac{\cos x}{\csc x\cot x}$	Multiply both sides by $\sin x$.
$\csc x\cot x\sin^2 x = \cos x$	Multiply both sides by $\csc x\cot x$.
$\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}\cdot\sin^2 x = \cos x$	Rewrite using sine and cosine.
$\dfrac{1}{\cancel{\sin x}}\cdot\dfrac{\cos x}{\cancel{\sin x}}\cdot\cancel{\sin^2 x} = \cos x$	Simplify.
$\cos x = \cos x$	

My Notes

LESSON 21-2 PRACTICE

6. Simplify each expression.
  a. $\tan x \cdot \cot x$
  b. $\cos^2 \theta + 1 - \sin^2 \theta$
  c. $\csc^2 \theta - \cot^2 \theta$
  d. $\cos x \tan x \csc x$

7. Verify each identity.
  a. $\sin x = \cos x \tan x$
  b. $\dfrac{\cos x \tan x}{\sec x} = \sin x \cos x$
  c. $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$
  d. $\dfrac{\cos^2 \theta}{1 + \sin \theta} = 1 - \sin \theta$
  e. $\csc^2 \theta + 2 \cot \theta = (1 + \cot \theta)^2$
  f. $\sec^2 x = \dfrac{\cot x \sec x \tan x}{\cos x}$

8. Use trigonometric identities to help you compare the graphs of the functions $f(x)$ and $g(x)$, where $f(x) = \cos^2 x + 1$ and $g(x) = 1 - \sin^2 x$.

9. Describe two different ways you could use a graphing calculator to verify the identity $\sin^2 \theta \csc \theta \sec \theta = \tan \theta$.

10. **Critique the reasoning of others.** Nick was asked to simplify the expression $\dfrac{\sin^2 \theta}{1 - \cos \theta}$ and justify his steps. His work is shown below. Do you agree with his result? If not, explain his error and provide the correct answer.

Step 1:	$\dfrac{\sin^2 \theta}{1 - \cos \theta}$	Original expression
Step 2:	$\dfrac{1 - \cos^2 \theta}{1 - \cos \theta}$	Pythagorean identity
Step 3:	$\dfrac{(1 - \cos \theta)(1 - \cos \theta)}{1 - \cos \theta}$	Factor.
Step 4:	$\dfrac{\cancel{(1 - \cos \theta)}(1 - \cos \theta)}{\cancel{1 - \cos \theta}}$	Simplify.
Step 5:	$1 - \cos \theta$	

ACTIVITY 21 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 21-1

Express each trigonometric function as a reciprocal to write an identity.

1. $\sec(x - 30°) =$
2. $\cot(3\theta) =$
3. $\sin\left(\dfrac{\theta}{4}\right) =$

Express the tangent or cotangent as a quotient to write an identity.

4. $\tan(2\theta) =$
5. $\cot(\theta - \pi) =$
6. $\tan\left(\dfrac{\theta}{2}\right) =$
7. Consider the expressions in the table.

Column A	Column B
$\cos^2 x + \sin^2 x$	$1 + \tan^2 x$

Which of the following statements is true for every value of x?

A. The values of the two columns are equal.
B. The value of Column A is greater than the value of Column B.
C. The value of Column B is greater than or equal to the value of Column A.
D. The values of both columns are greater than 1.

8. Diego said that the identity $\sin\theta = \dfrac{1}{\csc\theta}$ holds for all values of θ. Do you agree? Explain.
9. Which expression is NOT equivalent to the other three?

A. $\dfrac{\cos\theta}{\sin\theta}$
B. $\cot\theta$
C. $\dfrac{1}{\tan\theta}$
D. $1 + \cot^2\theta$

10. Consider an angle θ such that $\cos\theta = 0.2$.
    a. What do you know about the value of $\sin\theta$? Explain how you know.
    b. Find the value of $\sin\theta$ given that $270° \le \theta \le 360°$.
    c. What is the approximate value of θ, in degrees, given that $270° \le \theta \le 360°$?
    d. Explain how you can check that your answer to part c is reasonable.
11. As θ increases from 0 to $\dfrac{\pi}{2}$, what happens to the value of $\sin\theta$? Use an identity to describe what happens to the value of $\csc\theta$ as θ increases from 0 to $\dfrac{\pi}{2}$.
12. Is there an identity that states $\sin^2\theta = \sin(\theta^2)$? If so, explain why. If not, provide a counterexample to show why the identity does not hold.

Determine whether each statement is always, sometimes, or never true.

13. The value of $\tan^2\theta + 1$ is negative.
14. If $\cos\theta = 0.5$, then $\sin\theta = \sqrt{1 - (0.5)^2}$.
15. $\cos^2\theta = 1 - \sin^2\theta$
16. For a constant k, the graph of the function $y = k + \sin^2 x + \cos^2 x$ is a straight line.

Lesson 21-2

Simplify each expression.

17. $\dfrac{\tan^2\theta}{\sec\theta - 1}$

18. $\cos\theta(\sec\theta + \tan\theta)$

19. $\dfrac{1+\cot^2\theta}{\sec^2\theta}$

20. $\cos^2 x\sin x + \sin^3 x$

Verify each identity.

21. $\dfrac{1}{1+\cos x} - \dfrac{1}{1-\cos x} = -2\csc x\cot x$

22. $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$

23. $\dfrac{-\sin\theta}{\cos\theta - 1} = \cot\theta + \csc\theta$

24. $\tan x\sec^2 x - \tan^3 x = \dfrac{\sec x}{\csc x}$

25. $\dfrac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec^3 x} = \cos x$

26. Which expression is NOT equivalent to $\dfrac{1+\cot^2\theta}{\csc\theta}$?

A. $\dfrac{\sin\theta}{\csc\theta}$ **B.** $\dfrac{1}{\sin\theta}$

C. $\csc\theta$ **D.** $\sin\theta\csc^2\theta$

27. Explain how you can find the minimum value of the function $f(x) = \tan x(\tan x + \cot x)$ without using your graphing calculator.

28. Consider the three expressions shown below.

I. $\cos^2 x\tan^2 x$

II. $\dfrac{\sec^2\theta - 1}{\sec^2\theta}$

III. $\sin^2\theta$

Which of the expressions are equivalent?

A. I and II only **B.** II and III only

C. I and III only **D.** I, II, and III

29. Given that $\dfrac{\cos^2\theta}{\sin\theta - \sin^2\theta} = \csc\theta + p$, find the value of p. Show your work.

30. Mayumi wanted to know if the equation below was a correct identity.

$$\frac{1}{\sec\theta} = \cos\theta - \cos\theta\sin^2\theta$$

She entered the left side of the equation in her calculator as Y1 and the right side as Y2. Then she viewed a table of values, as shown below.

X	Y1	Y2
3.1416	-1	-1
6.2832	1	1
9.4248	-1	-1
12.566	1	1
15.708	-1	-1
18.85	1	1
21.991	-1	-1

X=3.14159265359

Mayumi concluded that the equation was a correct identity. Do you agree? Justify your answer.

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

31. It is possible to write any of the six trigonometric functions in terms of any of the other trigonometric functions. For example, $\sin x$ can be written in terms of $\cos x$ as $\sin x = \pm\sqrt{1-\cos^2 x}$. Show how to write $\sin x$ in terms of $\tan x$. (*Hint:* Your expression should involve no trigonometric functions other than $\tan x$.)

Identities and Equations

Triangle Measure

Lesson 22-1 Cofunction Identities

My Notes

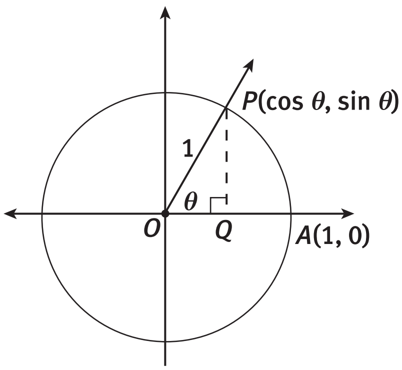
Learning Targets:

- Use the unit circle to write equivalent trigonometric expressions.
- Write cofunction identities for sine and cosine.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Marking the Text, Create Representations, Look for a Pattern, Visualization, Think-Pair-Share

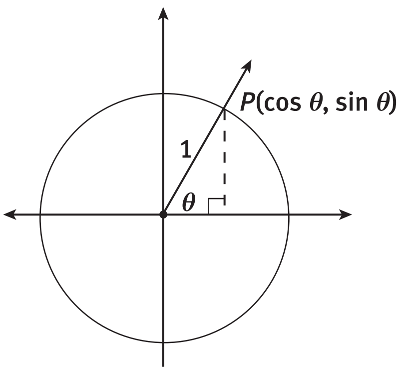
When the German mathematician Bartholomaeus Pitiscus wrote *Trigonometria* in 1595, the word *trigonometry* made its first appearance in print. However, Egyptian and Babylonian mathematicians used aspects of what we now call trigonometry as early as 1800 B.C. The word *trigonometry* comes from two Greek words: *trigon*, meaning triangle, and *metron*, meaning measure. Thus, trigonometry is the study of triangle measures. The definitions of the trigonometric functions on the unit circle are attributed to Swiss mathematician Leonhard Euler.

In this activity, you will derive and use identities based on the measures of triangles found in a unit circle.



In the diagram above, θ is an angle in standard position whose terminal side passes through the point $P(\cos\theta, \sin\theta)$ located on a unit circle.

1. On the unit circle below, draw the angle $-\theta$ in standard position, label the coordinates of the point where it passes through the unit circle in terms of sine and cosine, and then complete the identities.

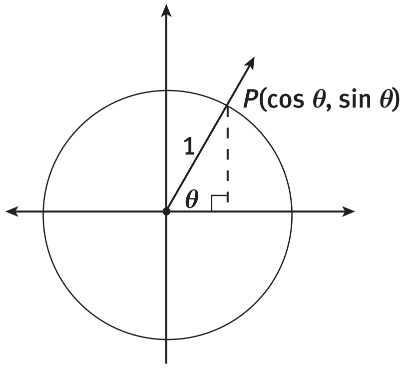


a. $\sin(-\theta) =$

b. $\cos(-\theta) =$

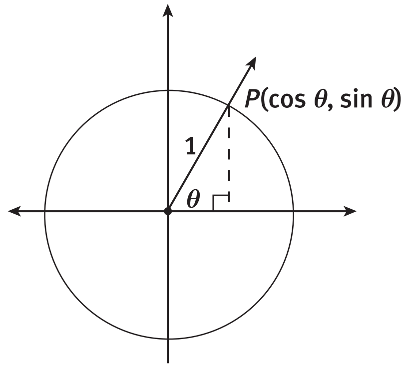
My Notes

2. On the unit circle below, draw the angles $180° + \theta$ and $180° - \theta$ in standard position, label the coordinates of the point where each angle passes through the unit circle in terms of sine θ and cosine θ, and then complete the identities.



a. $\sin(180° + \theta) =$
b. $\cos(180° + \theta) =$
c. $\tan(180° + \theta) =$
d. $\sin(180° - \theta) =$
e. $\cos(180° - \theta) =$
f. $\tan(180° - \theta) =$

3. On the unit circle below, draw the angles $360° + \theta$ and $360° - \theta$ in standard position, label the coordinates of the point where each angle passes through the unit circle in terms of sine and cosine, and then complete the identities.



a. $\sin(360° + \theta) =$
b. $\cos(360° + \theta) =$
c. $\tan(360° + \theta) =$
d. $\sin(360° - \theta) =$
e. $\cos(360° - \theta) =$
f. $\tan(360° - \theta) =$

4. Use appropriate tools strategically. Explain how a graphing calculator can be used to check your answers to Items 1 to 3. Then, check your work.

5. Express each of the identities in Items 2 and 3 using radians.

MATH TIP

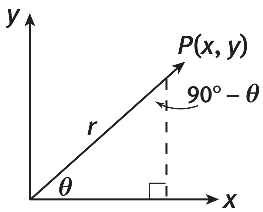
To convert degrees to radians, multiply the degree measure by $\frac{\pi}{180°}$.

My Notes

6. **Reason abstractly.** If $\cos\theta = a$ and $\sin\theta = b$, complete each statement using a and/or b with the correct sign.
   a. $\sin(-\theta) =$
   b. $\cos(360° + \theta) =$
   c. $\sin(360° - \theta) =$
   d. $\sin(180° + \theta) =$
   e. $\cos(180° - \theta) =$
   f. $\tan(180° + \theta) =$

7. Use the diagram to help you complete each statement below in terms of x, y, and r.

   $\sin\theta =$ $\quad$ $\cos(90° - \theta) =$ $\quad$ $\cos\theta =$ $\quad$ $\sin(90° - \theta) =$



8. What do you notice about the relationship between the sine of an angle and the cosine of its complement? Express this relationship as an identity.

The statement you wrote in Item 8 is called a ***cofunction identity***.

9. Cofunction identities exist for other pairs of trigonometric functions. List these identities below.

10. **Construct viable arguments.** Explain why cofunction is an appropriate description for these identities. Be certain to include an explanation of the relationship between θ and $(90° - \theta)$.

11. Given $\cos\theta = \frac{2}{3}$, what is the value of each expression?
   a. $\sin\left(\frac{\pi}{2} - \theta\right)$
   b. $\cos(\pi + \theta)$
   c. $\sin(-\theta)$

12. Given $\sin\frac{\pi}{6} = \frac{1}{2}$, what is the value of each expression?
   a. $\cos\frac{\pi}{3}$
   b. $\sin\frac{13\pi}{6}$
   c. $\csc\frac{\pi}{6}$

13. Given $\sin 63° = x$, what is the value of each expression?
   a. $\cos 27°$
   b. $\cos(-27°)$
   c. $\sin(243°)$

My Notes

Check Your Understanding

14. Suppose you are told that $\sin 57° \approx 0.8387$. What is another angle between 0° and 360° whose sine is approximately equal to 0.8387? Justify your response with an identity.

15. **Make use of structure.** Use the relationship between the value of $\cos(2\pi - \theta)$ and $\cos(-\theta)$ to write an identity. Explain your reasoning.

LESSON 22-1 PRACTICE

16. Complete each statement.

a. If $\sin(x) = a$, then $\sin(-x) =$ _______.

b. If $\cos(x) = 0.4$, then $\cos(\pi - x) =$ _______.

c. If $\sin(-x) = -0.13$, then $\sin(x + 2\pi) =$ _______.

d. If $\cos(47°) = y$, then $\sin(43°) =$ _______.

e. If $\sin\left(\frac{\pi}{5}\right) = y$, then $\csc\left(\frac{4\pi}{5}\right) =$ _______.

17. You have seen that $\sin(-\theta) = -\sin\theta$. What does this identity tell you about the relationship between the graphs of $f(x) = \sin(-x)$ and $g(x) = \sin x$? Graph the two functions on your calculator to check that your answer is reasonable.

18. Use the figure of the unit circle to find the value of each of the following.

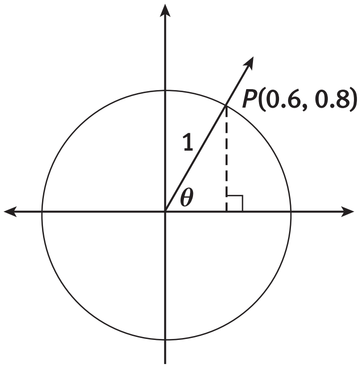
a. $\tan\theta$

b. $\cos(180° - \theta)$

c. $\sin(\pi + \theta)$

d. $\tan(360° - \theta)$

e. $\csc(-\theta)$



19. Given that $\cos\frac{\pi}{7} \approx 0.9$, name three additional angles whose cosine is approximately equal to 0.9.

20. **Construct viable arguments.** Verify the identity $\tan\left(\frac{\pi}{2} - \theta\right)\sin\theta = \cos\theta$. Be sure to justify each step of your argument with a valid reason.

My Notes

Learning Targets:

- Use trigonometric identities to solve equations.
- Solve trigonometric equations by graphing.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Identify a Subtask, Simplify the Problem

Trigonometric graphs and identities like the ones in Items 1 to 3 of Lesson 22-1 can help you find solutions to trigonometric equations.

1. Consider the equation $2\sin\theta - 1 = 0$

a. Sketch the graph of the function $f(\theta) = 2\sin\theta - 1$ over the interval $[0°, 360°]$.

b. Using the graph, explain why there must be two solutions to the equation $2\sin\theta - 1 = 0$ on the interval $[0°, 360°)$.

c. Determine the solutions to the equation $2\sin\theta - 1 = 0$ over the interval $[0°, 360°)$. Explain how these solutions relate to the identity $\sin\theta = \sin(180° - \theta)$.

d. Use an algebraic method to solve the equation $2\sin\theta - 1 = 0$. Compare the solutions to the answers in item *c*.

e. Use an algebraic method to solve $\sqrt{2}\sin\theta + 1 = 0$ over the interval $[0°, 360°)$. Use a graph to confirm your solutions.

For each trigonometric equation shown below, find the angle measures that satisfy the equation.

2. Solve for θ on $[0°, 360°)$.

a. $1 = \sqrt{3}\tan\theta$ **b.** $2\cos\theta + \sqrt{3} = 0$ **c.** $\sin\theta - 3 = 0$

3. Make use of structure. Solve each equation on the given interval by using factoring.

a. $4\sin^2 x - 1 = 0$, $[0, 2\pi)$

b. $2\tan^2\theta - \tan\theta - 3 = 0$, $[0°, 360°)$

c. $\sin^2 x = 2\sin x$, $[-\pi, \pi]$

d. $\cos\theta - 2\cos\theta\sin\theta = 0$, $[0°, 180°]$

My Notes

When there is more than one trigonometric function in a particular equation, another solution strategy is using identities to rewrite the equation in terms of a single trigonometric function.

4. Solve the equation $\cos^2\theta = 1 - \sin\theta$ on the interval $[0°, 360°)$. Start by rewriting the cosine term in terms of sine.

5. Solve the following equations on the interval $[0, 2\pi)$. Use identities to rewrite the equation in terms of a single trigonometric function.

a. $\csc x - \sin x = 0$

b. $\tan^2\theta = 1 - \sec\theta$

c. $\tan^2 x + \sec^2 x = 3$

6. Use appropriate tools strategically. Trigonometric equations can also be solved by graphing the functions represented by each side of the equation and finding their intersection point(s).

a. Solve the equation $\sin x = \cos x$ on the interval $[0, 2\pi]$ using a graphing calculator.

b. Change the graphing window to $[-2\pi, 2\pi]$. What are the solutions to this equation on this interval?

c. Change the graphing window to $[0, 4\pi]$. What are the solutions to this equation on this interval?

d. How many solutions would this equation have on the interval $(-\infty, \infty)$?

My Notes

When the solution interval is not restricted to a subset of the real numbers, you must find the *general solutions*.

7. Attend to precision. Use your work from Items 1 to 6 to write the general solutions to these equations on the interval $(-\infty, \infty)$.

a. $\sqrt{2}\sin\theta + 1 = 0$

b. $2\tan^2\theta - \tan\theta - 3 = 0$

c. $\cos^2\theta = 1 - \sin\theta$

d. $\sin x = \cos x$

8. The equation $4\sin^2 x - 1 = 0$ has the solutions $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ on $[0, 2\pi)$.

a. What are the solutions on $(-\infty, \infty)$?

b. Find a way to write your answers to part a in a more concise manner.

9. Attend to precision. Find a way to write your answers to Items 7b and 7d in a more concise manner.

MATH TIP

You can write *general solutions* by adding integral multiples of the period. Use $2\pi k$ or $360°k$ for sine and cosine. Use πk or $180°k$ for tangent.

My Notes

Check Your Understanding

10. Explain how you could rewrite each equation in terms of a single trigonometric function.

a. $\sin^2 x - 1 + \cos x = 0$

b. $1 = \sin\left(\frac{\pi}{2} - x\right)\cos x$

c. $\tan x = 3\cot x$

11. Melinda was asked to solve the equation $\sec^2 x - \tan^2 x = 1$. She looked at the equation and quickly stated that all real numbers are solutions. Explain how Melinda knew this.

12. Make use of structure. How is the process of finding all solutions to the equation $2\cos^2 x + \cos x - 1 = 0$ similar to the process of finding all solutions to the equation $2x^2 + x - 1 = 0$? How is the process different?

LESSON 22-2 PRACTICE

13. Solve each equation over the given interval. You do not need a calculator.

a. $3\csc x - 6 = 0$, $[0, 2\pi)$

b. $4\sin^2\theta - 3 = 0$, $[-180°, 180°]$

c. $\sin x\tan x - \sin x = 0$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d. $\sin^2\theta - \cos^2\theta = 1$, $(0°, 360°)$

14. Write the general solutions to Item 13 on the interval $(-\infty, \infty)$.

15. Solve each equation over the given interval. You may use a calculator.

a. $3\cos\theta - 2 = 0$, $(-\infty, \infty)$

b. $\sin^2 x - 4 = 0$, $[0, 2\pi)$

c. $\sin\theta = \sqrt{3}\cos\theta$, $[0°, 360°)$

d. $\tan^2\theta + 2\tan\theta - 3 = 0$, $(-\infty, \infty)$

16. Without using a calculator, determine the number of times the graphs of the functions $f(x) = 4\sin^2 x$ and $g(x) = 5 - 4\cos x$ intersect on the interval $[0°, 360°)$. Explain your method.

17. A student was asked to find all solutions to the equation $\sin x = 0.25x$. The student stated that the equation must have infinitely many solutions on the interval $(-\infty, \infty)$ since the sine function is periodic. Do you agree or disagree with the student's reasoning? Explain.

18. Critique the reasoning of others. Carlos solved the equation $1 - \sin^2 x = 0.5\cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$. His work is shown below. Explain his error and find the correct solution.

Step 1:	$1 - \sin^2 x = 0.5\cos x$	Original equation
Step 2:	$\cos^2 x = 0.5\cos x$	Pythagorean identity
Step 3:	$\cos x = 0.5$	Divide both sides by $\cos x$.
Step 4:	$x = \frac{\pi}{3}$	Solve for x.

19. Make use of structure. How can identities help you to understand how to find all of the solutions to trigonometric equations?

ACTIVITY 22 PRACTICE

Write your answers on notebook paper. Show your work.

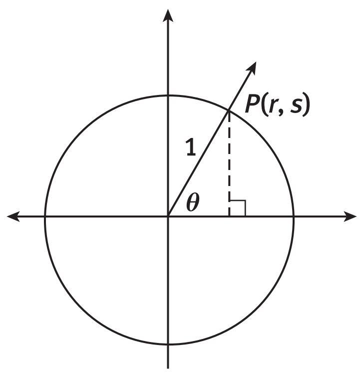
Lesson 22-1

Complete each statement.

1. If $\cos\theta = k$, then $\cos(-\theta) =$ _______.
2. If $\tan(180° - x) = 0.41$, then $\tan x =$ _______.
3. If $\sin 16° = x$, then $\cos(74°) =$ _______.
4. If $\tan x = a$, then $\tan(2\pi - x) =$ _______.
5. If $\cos\left(\frac{\pi}{11}\right) = y$, then $\sec\left(\frac{10\pi}{11}\right) =$ _______.

Given $\sin\theta = \frac{3}{5}$, what is the value of each expression?

6. $\sin(2\pi - \theta)$
7. $\cos\left(\frac{\pi}{2} - \theta\right)$
8. $\tan\theta$
9. Which of the following expressions must have the same value for any angle θ? Choose all that apply.
   A. $\cos\theta$
   B. $\sin\theta$
   C. $\sin\left(\frac{\pi}{2} - \theta\right)$
   D. $\cos(-\theta)$
   E. $\cos(\pi - \theta)$
   F. $\cos(2\pi - \theta)$
10. Verify the identity $\sec^2\theta \sin\left(\frac{\pi}{2} - \theta\right) = \sec\theta$. Be sure to justify each step of your argument with a valid reason.
11. The sentence shown below appears in Jamal's textbook. However, part of the sentence is covered by a drop of ink. Which expression could be covered by the drop of ink?
    If $\cos\theta = \frac{3}{7}$, then $\sin(?) = \frac{3}{7}$.
    A. $-\theta$ **B.** $180° - \theta$
    C. $90° - \theta$ **D.** $90° + \theta$
12. Point P lies on the unit circle, with coordinates (r, s), as shown. Find each of the following in terms of r and s.



    A. $\sin(\pi - \theta)$ **B.** $\cot\left(\frac{\pi}{2} - \theta\right)$
    C. $\cos\left(\frac{\pi}{2} - \theta\right)$ **D.** $\csc(2\pi - \theta)$
13. What is the range of the function $h(x)$, where $h(x)$ is defined as follows? Explain.
$$h(x) = \frac{\sin x}{\sin\left(\frac{\pi}{2} - x\right)} + 1$$
14. Write an expression involving sine that has the same value as $\cos(\pi - x)$ for all x. Then explain how you can use your calculator to check that your answer is reasonable.
15. If $\sin\theta > 0$, which of the following must be negative?
    A. $\sin(-\theta)$ **B.** $\cos\theta$
    C. $\sin\left(\frac{\pi}{2} - \theta\right)$ **D.** $\cos(-\theta)$

Lesson 22-2

Solve each equation over the given interval. You do not need a calculator.

16. $\sin\theta = \sqrt{3} - \sin\theta, [0°, 360°)$

17. $\sin^2 x + \cos x + 1 = 0, [0, 2\pi)$

18. $\csc x \tan x - \csc x = 0, \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

19. $2\cos^2\theta + 3\sin\theta = 3, [0°, 360°)$

20. Write the general solutions to Item 17 on the interval $(-\infty, \infty)$.

Solve each equation over the given interval. You may use a calculator.

21. $7 = 5\csc\theta + 1, (-\infty, \infty)$

22. $9\cos^2\theta - 12\cos\theta + 4 = 0, (-\infty, \infty)$

23. $1 - 2\sec x = 0, [0, 2\pi)$

24. The solution of the equation $3\cos x - p = 0$ on the interval $[0, \pi)$ is $x = \frac{\pi}{3}$. What is the value of the constant p?

25. What are the general solutions to the equation $\cos x \sin\left(\frac{\pi}{2} - x\right) = 1$?

A. $2\pi k$, where k is an integer
B. πk, where k is an integer
C. $\frac{\pi}{2} + 2\pi k$, where k is an integer
D. $\frac{\pi}{2} + \pi k$, where k is an integer

26. Give the general solutions of the equation $\sin(-x)\sin x = 1$. Explain your answer.

27. Which of the following equations has the greatest number of solutions on the interval $[0, 2\pi)$?
A. $4\cos\theta = -3$
B. $4\cos^2\theta = -3$
C. $4\cos\theta = 3$
D. $4\cos^2\theta = 3$

28. Write and solve an equation to find the exact point(s) of intersection of the graphs of $f(x) = 2 - 2\cos^2 x$ and $g(x) = 5\sin x - 2$ on the interval $[0, 2\pi)$.

Determine whether each statement is always, sometimes, or never true.

29. If c is a real number, then the equation $\sin x \sin(\pi - x) = c$ has a solution on the interval $[0, 2\pi)$.

30. The graph of $y = \tan x$ intersects the graph of $y = kx$, where k is a real number.

31. 0 is a solution of $\cos^2 x = p\cos x$ when p is a nonzero real number.

MATHEMATICAL PRACTICES
Model with Mathematics

32. The height of a car on a Ferris wheel, in feet, is modeled by the function $h(t) = 60\cos\left(\frac{\pi}{2} - t\right) + 70$, where t is the time in minutes since the ride began.
a. What is the height of the car when the ride begins? How do you know?
b. What is the maximum height of the car?
c. During the first revolution of the Ferris wheel, when is the car at a height of 100 feet? Write and solve an equation algebraically to find the exact times. Then use a calculator to find the times to the nearest hundredth of a minute.
d. How many solutions did you find for part c? Explain why this number of solutions makes sense in terms of the real-world context.

Multiple Angle Identities

ACTIVITY 23

Sounds Like Trigonometry

Lesson 23-1 Exploring Sums of Trig Functions

Learning Targets:

- Model a sound wave with a trigonometric function.
- Derive an expression for the cosine of a difference.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Look for a Pattern, Visualization, Identify a Subtask

The sound of a single musical note can be represented using the function

$$y = a \sin(2\pi ft)$$

where a is the amplitude (volume) of the sound measured in decibels (dB), f is the frequency (pitch) of the note measured in hertz (Hz), and t is time.

1. The frequency of the musical note middle C is about 262 Hz. Write the equation for a sine wave that represents middle C played at a volume of 60 dB.

2. In music, a note one octave higher vibrates twice as fast. Write an equation for a sine wave for the note one octave above middle C played at a volume of 70 dB.

When more than one musical note is played at the same time, the sound is represented by the sum or difference of two or more sine or cosine waves.

3. Use appropriate tools strategically. Use your calculator to graph the functions you wrote in Items 1 and 2. Also graph the sum of the two functions. What do you notice about the graphs?

The sum of another pair of sinusoidal curves whose periods are equal could represent two musical notes with the same frequency played at the same time.

4. Model with mathematics. Let $y_1 = \frac{1}{2}\sin x$ and $y_2 = \frac{\sqrt{3}}{2}\cos x$.

a. Use your calculator to graph the functions Y1, Y2, and Y1 + Y2 on the interval $[-2\pi, 2\pi]$ and then make a sketch in the space below.

My Notes

CONNECT TO MUSIC

Periodic functions like sine and cosine model physical phenomena that oscillate or vibrate in waves. Sound is one such physical phenomenon.

As a increases, a musical note sounds louder. As f increases, a musical note sounds higher.

My Notes

b. Describe the graph of $y_1 + y_2$.

c. Express the function $y_1 + y_2$ in the form $y = \sin(x + c)$.

d. You found the value of c so that $\sin(x + c) = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$. Does $\sin(x + c)$ also equal $\sin(x) + \sin(c)$? Explain how you determine your answer.

To further explore the properties of sums and differences of sine and cosine functions, answer the following items.

5. Reason quantitatively. Determine whether or not each statement below is true. Explain your reasoning.

a. $\sin(90° + 60°) = \sin 90° + \sin 60°$

b. $\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \cos\frac{\pi}{3} - \cos\frac{\pi}{6}$

6. Are the following statements true for all values of x and y? Explain.

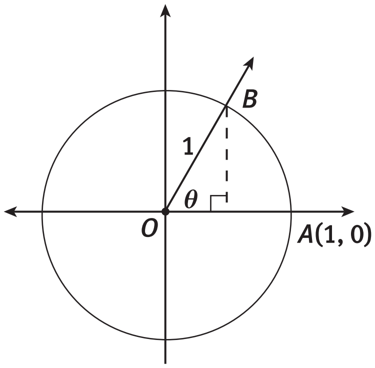
a. $\sin(x - y) = \sin x - \sin y$

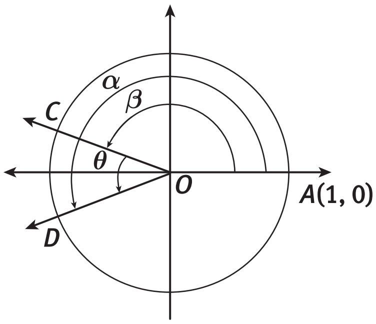
b. $\cos(x + y) = \cos x + \cos y$

My Notes

The unit circle, the distance formula, and trigonometric identities can be used to determine a formula for the sine or cosine of the sum or difference of two angles. The next items will help you derive an expression equal to $\cos(\alpha - \beta)$, where α, β, and θ are measures of three unit circle angles with $\theta = \alpha - \beta$ as shown in the diagrams.

7. Write the coordinates of points B, C, and D in terms of the sine and cosine.





8. Draw segment AB and segment CD. Explain why their lengths are equal.

9. Use the distance formula to write the lengths of segments AB and CD in terms of sine and cosine. Then solve the resulting equation for $\cos\theta$. Show your work below.

Length of segment AB = Length of segment CD

10. **Express regularity in repeated reasoning.** Recall that $\theta = \alpha - \beta$. Use your answer from Item 9 to complete the statement below.

$$\cos(\alpha - \beta) =$$

My Notes

Check Your Understanding

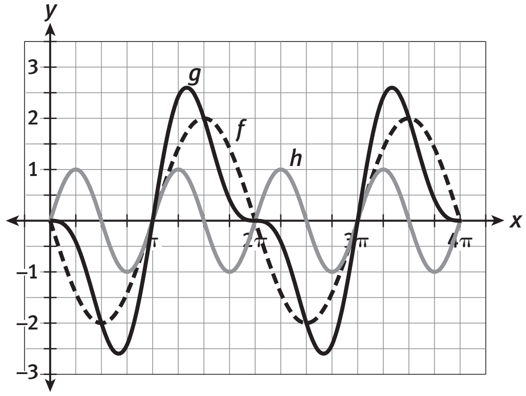
11. Explain how you wrote the coordinates for point B in terms of the sine and cosine in Item 7.

12. Explain how you used a trigonometric identity in Item 9 to help simplify the expressions on either side of the equation.

13. **Reason abstractly.** Write the formula you discovered in Item 10 in words. Be sure to use one or more complete sentences.

LESSON 23-1 PRACTICE

14. The figure shows the graphs of three functions, f, g, and h.



a. One of the functions is the sum of the other two. Which function is it? Justify your answer.
b. Write the other two functions as sine functions.
c. Write the function you identified in part a as a sum of the expressions you wrote in part b.
d. Do you think the function you wrote in part c could be written in the form $y = a\sin(bx + c) + d$? Why or why not?

15. Is the statement $\tan(x + y) = \tan x + \tan y$ true for all values of x and y? Explain.

16. Verify that the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ works when $\alpha = 150°$ and $\beta = 90°$.

17. Simplify the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ when $\alpha = \frac{\pi}{2}$. Explain what identity this proves.

18. **Critique the reasoning of others.** Wei said she could use the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to prove that $\cos 90° = 0$. Her work is shown below. Is this a correct way to prove that $\cos 90° = 0$? Why or why not?

$$\begin{aligned}\cos 90° &= \cos(90° - 0°)\\ &= \cos 90°\cos 0° + \sin 90°\sin 0°\\ &= 0 \cdot 1 + 1 \cdot 0\\ &= 0\end{aligned}$$

My Notes

Learning Targets:

- Write the sum and difference identities for sine, cosine, and tangent.
- Use sum and difference identities to find exact values of a trig function.
- Derive the double angle and half angle identities.

SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Simplify the Problem, Look for a Pattern, Identify a Subtask, Think-Pair-Share

Using the identities learned in Activity 22 and the results of Item 10 from the last lesson, you can derive the following ***sum and difference identities***.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

These identities are sometimes written in a condensed fashion as shown below for the tangent sum and difference identities.

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

MATH TIP

When reading $\pm$ say "plus or minus." When reading $\mp$ say "minus or plus."

1. Express the tangent identity as two separate identities, one for the sum and one for the difference.

2. **Make sense of problems.** In Item 4 of the last lesson you discovered that $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$. Use the sum identity for sine to algebraically verify this identity.

My Notes

The sum and difference identities are also used to find exact values of nonspecial angle measures.

Example A

Find the exact value of sin 15°.

Step 1: Rewrite sin 15°. $\sin 15° = \sin(60° - 45°)$

Step 2: Substitute using identity. $= \sin 60° \cos 45° - \sin 45° \cos 60°$

Step 3: Evaluate. Use special angle ratios. $= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

Step 4: Simplify. $= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

Solution: $\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4}$

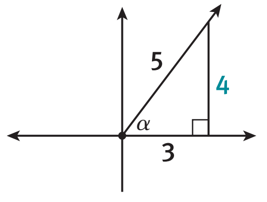
Try These A

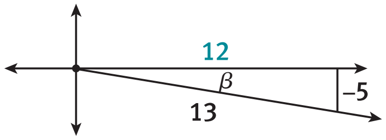
a. $\cos 105°$ **b.** $\sin \frac{7\pi}{12}$ **c.** $\tan 75°$

3. **Use appropriate tools strategically.** Explain how you could use your calculator to check that your answer to Try These Part A is correct.

MATH TIP

Draw a right triangle to help you quickly determine the other trigonometric ratios for a given angle if you know one ratio.





Here is another type of sum-and-difference problem.

Example B

Let α be an angle on the interval [0°, 90°] and β be an angle on the interval [−90°, 0°] with $\cos\alpha = \frac{3}{5}$ and $\sin\beta = -\frac{5}{13}$. Find $\cos(\alpha + \beta)$.

Step 1: Cosine sum identity $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Step 2: Substitute trig ratios from right triangle diagrams. $= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)$

Step 3: Simplify. $= \frac{36}{65} + \frac{20}{65}$

Solution: $\cos(\alpha + \beta) = \frac{56}{65}$

My Notes

Try These B

Reason quantitatively. Let α terminate in Quadrant I and β terminate in Quadrant II.

a. If $\sin\alpha = \frac{3}{5}$ and $\tan\beta = -\frac{8}{15}$, find $\sin(\alpha - \beta)$ and $\tan(\alpha + \beta)$.

b. If $\sin\alpha = \frac{1}{3}$ and $\sin\beta = \frac{3}{4}$, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

From the sum and difference identities, you can derive double angle identities.

4. Use the fact that $2\theta = \theta + \theta$ and the appropriate sum identity to verify each double-angle identity.

a. $\sin 2\theta = 2\sin\theta\cos\theta$

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

c. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

5. Use the identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin^2\theta + \cos^2\theta = 1$ to derive an identity for $\cos 2\theta$ in terms of cosine only.

6. Use the identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin^2\theta + \cos^2\theta = 1$ to derive an identity for $\cos 2\theta$ in terms of sine only.

My Notes

Example C

Given $\sin\theta = \frac{1}{4}$ with $\frac{\pi}{2} < \theta < \pi$, what is the exact value of $\cos\theta$, $\sin 2\theta$, and $\cos 2\theta$?

$\cos\theta = -\frac{\sqrt{4^2 - 1^1}}{4} = -\frac{\sqrt{15}}{4}$ Use a triangle, cosine is negative, θ terminates in Quadrant II.

$\sin 2\theta = 2\left(\frac{1}{4}\right)\left(-\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{15}}{8}$ Use $\sin 2\theta = 2\sin\theta\cos\theta$, substituting sine and cosine values.

$\cos 2\theta = \left(-\frac{\sqrt{15}}{4}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{14}{16} = \frac{7}{8}$ Use $\cos 2\theta = \cos^2\theta - \sin^2\theta$, substituting sine and cosine values.

Try These C

a. Given $\cos\theta = \frac{3}{5}$ with $0 < \theta < \frac{\pi}{2}$, find $\sin\theta$, $\tan\theta$, $\sin 2\theta$, and $\tan 2\theta$.

b. Given $\tan\theta = -\frac{8}{17}$ with $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos\theta$, $\cos 2\theta$, and $\tan 2\theta$.

c. Given $\sin\theta = \frac{2}{3}$ and $\cos\theta < 0$, find $\cos\theta$, $\sin 2\theta$, and $\cos 2\theta$.

7. Classify each statement as true or false.

a. $\sin 60° = 2\sin 30°\cos 30°$

b. $\cos\pi = \cos^2\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)$

c. $\tan 48° = \frac{2\tan 24°}{1 + \tan 24°}$

d. $1 - 2\sin^2\left(\frac{\pi}{5}\right) = \sin\left(\frac{\pi}{10}\right)$

My Notes

8. Reason abstractly. From the double angle identities for cosine, you can derive the **half-angle identities**.

a. Solve the identity $\cos 2\theta = 2\cos^2\theta - 1$ for $\cos\theta$.

b. Let $2\theta = \alpha$; solve for θ.

c. Write your results from part a in terms of α.

d. Derive the half-angle identity for $\sin\frac{\alpha}{2}$.

9. How could you use the half-angle identities for sine and cosine to derive this half-angle identity: $\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$?

Example D

Use a half-angle identity to find the exact value of $\sin(-15°)$.

Substitute $\theta = -30°$ and use the sine half-angle identity. Use the negative square root because the sine of an angle terminating in Quadrant IV is negative. Simplify.

$$\sin(-15°) = \sin\left(\frac{-30°}{2}\right) = -\sqrt{\frac{1-\cos(-30°)}{2}} = -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{4}$$

$$= -\frac{\sqrt{2-\sqrt{3}}}{2}$$

My Notes

Try These D

Use a half-angle identity to find the exact value of each trigonometric function.

a. $\cos\left(\frac{\pi}{8}\right)$ **b.** $\tan 15°$ **c.** $\sin\left(\frac{13\pi}{12}\right)$

Example E

If $\sin\theta = -\frac{5}{13}$ and $\cos\theta < 0$, find the exact value of $\cos\left(\frac{\theta}{2}\right)$ and $\tan 2\theta$.

a. Find the exact value of $\cos\left(\frac{\theta}{2}\right)$.

Step 1: Half-angle identity. Since θ terminates in Quadrant III, $\frac{\theta}{2}$ terminates in Quadrant II and cosine will be negative.

$$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}}$$

Step 2: Substitute ratio. Use a triangle if needed.

$$= -\sqrt{\frac{1+\left(-\frac{12}{13}\right)}{2}}$$

Step 3: Simplify.

$$= -\sqrt{\left(\frac{1}{13}\right)\left(\frac{1}{2}\right)}$$

Solution: $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1}{26}}$ or $-\frac{\sqrt{26}}{26}$

b. Find the exact value of $\tan 2\theta$.

Step 1: Double angle identity

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Step 2: Substitute ratios. Use a triangle if needed.

$$= \frac{2\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2}$$

Step 3: Simplify.

$$= 2\left(\frac{5}{12}\right) \div \left(\frac{119}{144}\right)$$

$$= 2\left(\frac{5}{12}\right)\left(\frac{144}{119}\right)$$

Solution: $\tan 2\theta = \frac{120}{119}$

Try These E

If $\tan\theta = \frac{3}{4}$ and θ terminates in Quadrant III, find $\cos\left(\frac{\theta}{2}\right)$, $\tan\left(\frac{\theta}{2}\right)$, and $\sin 2\theta$.

My Notes

10. Calculate the exact value of $\sin(105°)$ two different ways, once using a sum or difference identity and once using a half-angle identity.

Check Your Understanding

11. In Example 5, how do you use the information that $\cos\theta < 0$ to solve the problem? Explain why this is important.

12. **Make use of structure.** Is it possible to use the identities you learned in this lesson to find the exact value of $\sin 65°$? If so, explain how. If not, explain why not.

LESSON 23-2 PRACTICE

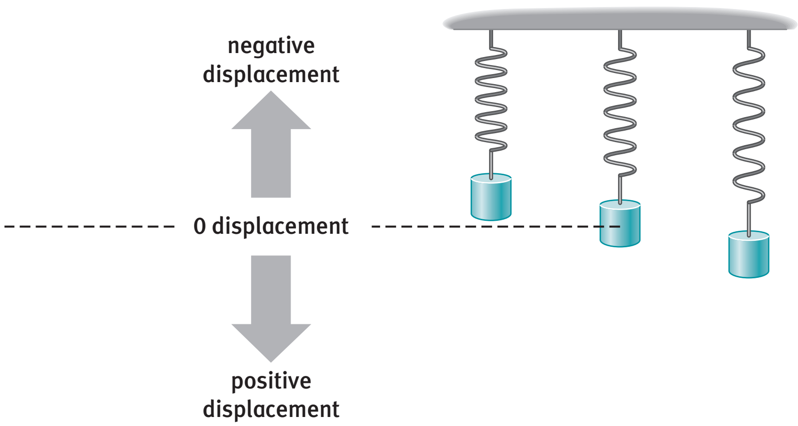
13. Find the exact value.
a. $\cos(165°)$ **b.** $\tan\left(\frac{17\pi}{12}\right)$

14. Given $\sin a = -\frac{8}{17}$ and $\tan b = -\frac{3}{4}$ with a terminating in Quadrant II and b terminating in Quadrant IV. Find the exact value of each ratio.
a. $\sin(a - b)$ **b.** $\tan(a + b)$
c. $\cos(2a)$ **d.** $\sin\left(\frac{b}{2}\right)$

15. Find the exact value.
$\tan(-67.5°)$

16. Use the sum identities for the sine and cosine to derive the sum identity for tangent: $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

17. **Reason quantitatively.** The function $y = 2.5\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)$ gives the displacement y, in centimeters, of a mass on a spring, where t is the time in seconds. As shown in the figure, when the mass is at rest, the displacement is 0 centimeters.



a. Write the displacement function using only the sine function.
b. What is the displacement after 4 seconds? 5 seconds? What do these values tell you about the position of the mass at these times?

My Notes

Learning Targets:

- Use trigonometric identities to solve equations.
- Verify trigonometric identities.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Identify a Subtask, Simplify the Problem, Note Taking

The identities introduced in this activity can be used to verify identities.

Example A

Verify the identity $\sin(\theta + 180) = -\sin\theta$.

$\sin(\theta + 180°) = -\sin\theta$	
$\sin\theta\cos 180° + \sin 180°\cos\theta =$	Sum identity for sine
$(\sin\theta)(1) + (0)(\cos\theta) =$	Evaluate special angle ratios.
$-\sin\theta + 0 =$	
$-\sin\theta = -\sin\theta$	Simplify.

Try These A

Make use of structure. Verify each identity.

a. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

b. $\frac{\sin 2x}{\tan x} = 2\cos^2 x$

c. $\cos^4\theta - \sin^4\theta = \cos 2\theta$

My Notes

The identities introduced in this activity can also be used to solve equations.

Example B

Solve $\sin 2\theta - \sin\theta = 0$ on the interval $[0°, 360°)$.

Step 1:	Double-angle formula	$\sin 2\theta - \sin\theta = 0$
Step 2:	Factor.	$2\sin\theta\cos\theta - \sin\theta = 0$
Step 3:	Set each factor equal to 0 and solve for θ on the given interval.	$\sin\theta(2\cos\theta - 1) = 0$ $\sin\theta = 0$ or $2\cos\theta - 1 = 0$ $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$
Solution:	$\theta = 0°, 60°, 180°, 300°$	

Try These B

Solve each equation on the interval $[0°, 360°)$.

a. $\cos 2\theta - \sin\theta = 0$

b. $\cos 2\theta = \cos^2\theta$

Another way to solve trigonometric equations with multiple angles is shown below.

Example C

Solve $\sin \sin 3x = \frac{1}{2}$ on the interval $[0, 2\pi)$.

Step 1:	Let $3x$ equal the solutions to $\sin\theta = \frac{1}{2}$ on the interval $[0, 6\pi)$. Multiply the interval by the x-coefficient.	$\sin 3x = \frac{1}{2}$ $3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$
Step 2:	Divide these solutions by 3 to isolate x.	$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

Try These C

Solve each equation on the interval $[0, 2\pi)$.

a. $\tan 4x = 1$

b. $4\sin^2(2x) - 3 = 0$

c. $2\sin 3x\cos 3x = 0$

My Notes

Check Your Understanding

1. Describe two different ways to verify the identity $\cos(180° - x) = -\cos x$.
2. In Example B, you saw that the solutions of the equation $\sin 2x - \sin x = 0$ on the interval $[0°, 360°)$ are 0°, 60°, 180°, and 300°. What does this tell you about the graphs of $f(x) = \sin 2x$ and $g(x) = \sin x$?
3. **Critique the reasoning of others.** As the first step in solving the equation $\cos 2x + \sin x = 0$, Miguel used a double-angle identity to substitute $2\cos^2 x - 1$ for $\cos 2x$. Critique Miguel's method.

LESSON 23-3 PRACTICE

4. Verify the identity.

   $$\sin\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}\sin x - \cos x}{2}$$

5. Verify the identity.
   a. $\cos(\pi - x) = -\cos x$
   b. $\cot 2x = \frac{1}{2}(\cot x - \tan x)$
6. Solve each equation on the interval [0°, 360°).
   a. $2\tan\theta \sin 2\theta = 1$
   b. $2\sin 3\theta = \sin^2\theta + \cos^2\theta$
7. Rewrite the expression $\sin 4\theta$ in terms of trigonometric functions of θ. Simplify and then write your result as a new identity.
8. **Express regularity in repeated reasoning.** Determine the number of solutions on the interval $[0, 2\pi)$ for each of the equations shown below.
   a. $\sin 2x = \frac{\sqrt{3}}{2}$ **b.** $\sin 3x = \frac{\sqrt{3}}{2}$ **c.** $\sin 4x = \frac{\sqrt{3}}{2}$
   d. Make a conjecture about the number of solutions on the interval $[0, 2\pi)$ for the equation $\sin nx = \frac{\sqrt{3}}{2}$, where n is a positive integer.

ACTIVITY 23 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 23-1

Determine whether or not each statement is true. Explain your reasoning.

1. $\tan(45° + 45°) = \tan 45° + \tan 45°$

2. $\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)$

3. $\sin(x + y) = \sin x + \sin y$ for all values of x and y.

4. Consider the function $f(x) = \cos\left(x - \frac{3\pi}{2}\right)$.

a. Use the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to write the function $f(x)$ in terms of the sine function only.
b. How does the graph of $f(x)$ compare to the graph of $y = \sin x$? Explain.

5. Let $y_1 = \frac{\sqrt{2}}{2}\cos x$ and $y_2 = \frac{\sqrt{2}}{2}\sin x$.

a. Graph y_1, y_2, and $y_1 + y_2$ on the interval $[-2\pi, 2\pi]$.
b. Use the graph to help you express the function $y_1 + y_2$ in the form $y = a\cos(x - b)$.
c. How can you apply the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to show that your answer to part c is correct?

6. The expression $\cos\frac{2\pi}{3}\cos\frac{\pi}{2} + \sin\frac{2\pi}{3}\sin\frac{\pi}{2}$ can be used to calculate the value of which of the following?

A. $\cos\frac{\pi}{6}$ **B.** $\sin\frac{\pi}{6}$
C. $\cos\frac{7\pi}{6}$ **D.** $\cos\frac{\pi}{3}$

7. True or false? $\cos(2\theta) = 2\cos\theta$ for all θ. Justify your answer.

Lesson 23-2

Find the exact value of each of the following.

8. $\sin\left(\frac{11\pi}{12}\right)$ **9.** $\cos(-75°)$

10. $\tan\left(\frac{5\pi}{12}\right)$ **11.** $\sin\left(\frac{\pi}{8}\right)$

12. $\cos(112.5°)$ **13.** $\tan\left(\frac{3\pi}{8}\right)$

14. Let α be an angle on the interval $[180°, 270°]$ and β be an angle on the interval $[0°, 90°]$ with $\sin\alpha = -\frac{7}{25}$ and $\cos\beta = \frac{12}{13}$.

a. Find $\cos(\alpha + \beta)$.
b. Find $\tan(\alpha + \beta)$.

15. Which of the following is a true statement?

A. $\tan 56° = \frac{\tan 28°}{1 - \tan^2 28°}$

B. $\cos\left(\frac{\pi}{8}\right) = \cos^2\left(\frac{\pi}{16}\right) - \sin^2\left(\frac{\pi}{16}\right)$

C. $\sin 31° = 2\sin 62°\cos 62°$

D. $1 + 2\sin^2\left(\frac{\pi}{10}\right) = \cos\left(\frac{\pi}{5}\right)$

16. If $\sin\theta = -\frac{24}{25}$ and θ lies in Quadrant III, which of the following has a value of $\frac{4}{5}$?

A. $\sin\left(\frac{\theta}{2}\right)$ **B.** $\cos\left(\frac{\theta}{2}\right)$
C. $\tan\left(\frac{\theta}{2}\right)$ **D.** $\cot\left(\frac{\theta}{2}\right)$

17. Given that $\sin\theta = \frac{4}{5}$ with $0 < \theta < \frac{\pi}{2}$, what is the exact value of $\tan 2\theta$?

18. Jenna solved the following problem: Given that $\cos\alpha = -\frac{4}{5}$ with $\pi < \alpha < \frac{3\pi}{2}$ and $\sin\beta = \frac{5}{13}$ with $0 < \beta < \frac{\pi}{2}$, find the value of $\cos(\alpha - \beta)$.

Jenna's work is shown below. Do you agree with her solution? If not, explain her error and find the correct solution.

$$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= \frac{-4}{5}\left(\frac{12}{13}\right) + \frac{3}{5}\left(\frac{5}{13}\right)\\ &= -\frac{33}{65}\end{aligned}$$

Lesson 23-3

Verify the identity.

19. $\cos\left(\frac{3\pi}{2}+x\right)=\sin x$

20. $\tan(\pi+x)=\tan x$

21. $(\cos\theta+\sin\theta)^2=1+\sin 2\theta$

22. Which of the following is NOT a correct way to complete the equation $\cos 80^\circ = \ldots$?
A. $2\cos^2 40^\circ - 1$
B. $1-2\sin^2 40^\circ$
C. $\sqrt{\frac{1+\cos 160^\circ}{2}}$
D. $\cos^2 160^\circ - \sin^2 160^\circ$

23. Tyrell was asked to simplify the expression $\frac{1-\cos 2\theta}{\sin 2\theta}$. His work is shown below. Find the error and provide the correct simplified expression.

$$\frac{1-\cos 2\theta}{\sin 2\theta}=\frac{1-(1+2\sin^2\theta)}{\sin 2\theta}$$
$$=\frac{-2\sin^2\theta}{2\sin\theta\cos\theta}$$
$$=\frac{-\sin\theta}{\cos\theta}$$
$$=-\tan\theta$$

Solve each equation on the interval $[0^\circ, 360^\circ)$.

24. $\cos 2\theta+3\cos\theta+2=0$

25. $2\sin^2\theta=\cos 2\theta$

26. $\sin 2\theta=-\cos\theta$

27. $\tan 3x=1$

28. $2\sin x\cos x-1=0$

29. Let $g(x)=\cos 2x$ and $h(x)=-1-\cos x$. Which of the following is a true statement about the functions?
A. The graphs of the functions intersect at 4 points on the interval $[0^\circ, 360^\circ)$.
B. $g(x)=h(x)$
C. Both graphs have the same x-intercepts.
D. $g(x)+h(x)=1$

30. Olivia simplified the expression $\frac{\cos 2\theta-1}{\sin^2\theta}$. Assuming each step of her work was correct, which of the following could have been her simplified expression?
A. -2
B. 2
C. $-\tan\theta$
D. $\tan\theta$

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

31. Let $f(x)=\frac{\cos x\sin 2x}{1+\cos 2x}$.

a. Use your calculator to graph $f(x)$. Then use the graph to help you discover and write an identity.
b. Verify the identity you wrote in part a.
c. Use the identity to help you evaluate $f\left(\frac{\pi}{6}\right)$.

Trigonometric Identities and Equations

A QUICK-START GUIDE FOR TRIGONOMETRY

Embedded Assessment 1

Use after Activity 23

Many user's manuals include a quick-start guide as a handy reference. The guide includes examples of the most common tasks or procedures. Create a quick-start guide that a precalculus student could use as a trigonometry reference. The items below describe the elements your guide must include. Be sure each of your completed examples and explanations is clear so that another student can easily understand it.

1. Explain how to use the unit circle to derive each of the following trigonometric relationships.

a. $\cos(2\pi - \theta) =$ ____ **b.** $\sin(2\pi - \theta) =$ ____ **c.** $\tan(2\pi - \theta) =$ ____

d. If you know that $\tan 38° = 0.7813$, what can you conclude about $\tan 322°$?

2. Simplify each expression.

a. $\dfrac{\cos\theta}{\sin^2\theta - 1}$ **b.** $\dfrac{1 + \tan\theta}{1 + \cot\theta}$ **c.** $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta}$

3. Develop and use sum and difference identities, as follows.

a. Is the statement $\tan(x - y) = \tan x - \tan y$ true for all values of x and y? Justify your answer.

b. Given the sum and difference identities for sine and cosine, explain how to derive an identity for $\tan(\alpha - \beta)$.

c. Show how you can use your result to derive an identity for $\tan(\pi - \theta)$.

4. Evaluate each expression without using a calculator.

a. $\sin(75°)$

b. $\cos 2x$, given $\sin x = \dfrac{1}{5}$ and $\cos x < 0$

5. Verify each identity.

a. $\cos x \cot x + \sin x = \csc x$

b. $\sin(2\pi - \theta) = -\sin\theta$

c. $\dfrac{2\cot 2x}{\cos x - \sin x} = \csc x + \sec x$

6. Solve each equation over the given interval.

a. $\tan^2 x + 2 = 3$, $[0, 2\pi)$

b. $2\cos^2\theta - 3\sin\theta = 0$, $[0°, 360°)$

c. $\sin 2x = \cos x$ for all values of x

7. How does solving an equation over an interval like $[0, 2\pi)$ differ from solving the same equation for all real values of the variable?

Trigonometric Identities and Equations

A QUICK-START GUIDE FOR TRIGONOMETRY

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 2, 3, 4, 5, 6)	• Clear and accurate understanding of trigonometric identities, particularly double angle formulae, sum and difference formulae, Pythagorean identities, and $\tan x = \frac{\sin x}{\cos x}$	• A functional understanding of trigonometric identities, applying them generally in a correct way, but with some errors	• Partial understanding of trigonometric identities	• Little or no understanding of any trigonometric identities
Problem Solving (Items 1, 6)	• An appropriate and efficient strategy that results in a correct answer when solving trigonometric equations	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 7)	• Clear and accurate understanding of representations of the unit circle and how it is used to find the ratios of sine, cosine, and tangent	• A functional understanding of representations of the unit circle and how it is used to find the ratios of sine, cosine, and tangent	• Partial understanding of representations of the trigonometric ratios and how they exist as functions and on the unit circle	• Little or no understanding of representations of the trigonometric ratios and how they exist as functions and on the unit circle
Reasoning and Communication (Items 1d, 3b, 7)	• Precise use of appropriate math terms and language to describe behavior of trigonometric ratios on the unit circle, explain how to derive identities, and state solutions to trigonometric equations	• Correct characterization of the behavior of trigonometric ratios on the unit circle, explanation of how to derive identities, and statement of differing solutions to trigonometric equations	• Misleading or confusing characterization of behavior of trigonometric ratios on the unit circle, explanation of how to derive identities, and statement of differing solutions to trigonometric equations	• Incomplete or inaccurate characterization of behavior of trigonometric ratios on the unit circle, explanation of how to derive identities, and statement of differing solutions to trigonometric equations

Law of Cosines

The Chocolate Factory

Lesson 24-1 Modeling with Trigonometric Functions

My Notes

Learning Targets:

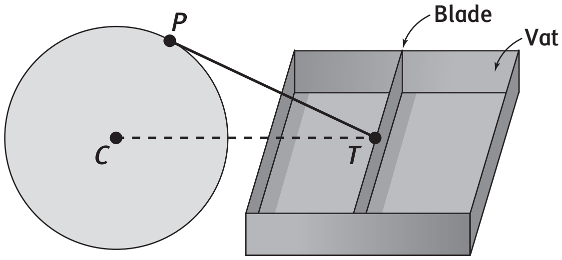
- Use trigonometry to draw and interpret diagrams for a model.
- Write a trigonometric function for a real-world situation.

SUGGESTED LEARNING STRATEGIES: Create Representations, Group Presentation, Guess and Check, Identify a Subtask, Look for a Pattern, Marking the Text, Quickwrite, Simplify the Problem, Visualization, Work Backward

At the Ghirardelli Chocolate Factory in San Francisco, California, the original equipment that stirred the milk chocolate mixture was driven by a wheel that pushed a stirrer blade back and forth across the bottom of the vat, as illustrated below. The wheel has a radius of 2 feet, and the rod connecting the wheel and the blade, represented below by segment PT, has length of 5 feet. Point T is on the same level as the center of the wheel C as shown by the dotted line.

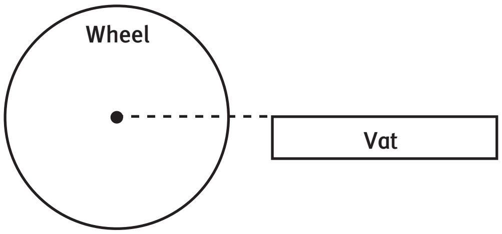
CONNECT TO ENGINEERING

In many mechanical devices, circular motion is converted to linear motion or linear motion is converted to circular motion in order to apply energy as required.

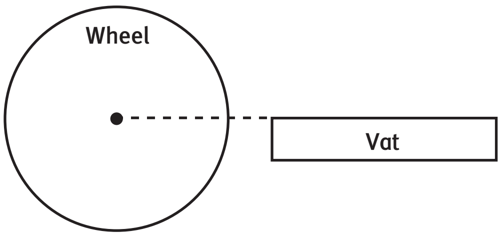


1. **Model with mathematics.** Sketch the position of the rod and blade at each angle of rotation. Assume angles are in the standard position. Then, find the distance from the center of the wheel to the stirrer blade for each angle.

   a. angle of rotation $= 0°$



   b. angle of rotation $= 180°$

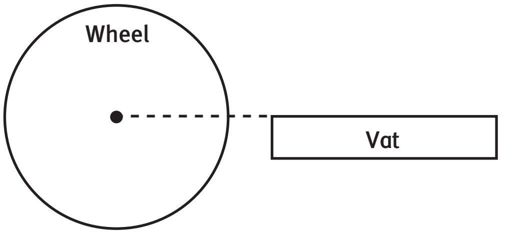


DISCUSSION GROUP TIP

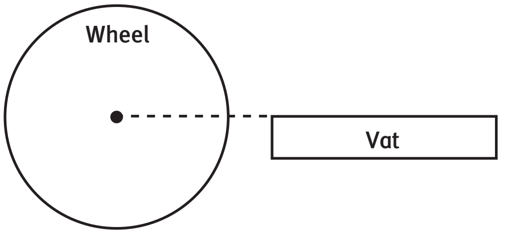
As you share ideas in your group, ask your group members or your teacher for clarification of any language, terms, or concepts that you do not understand.

My Notes

c. angle of rotation $= 30°$



d. angle of rotation $= 225°$

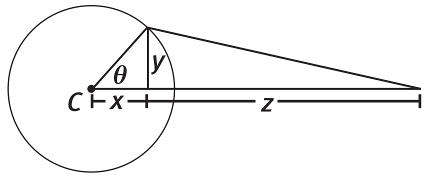


2. **Make sense of problems.** What is the length of the vat? Explain your reasoning.

3. How far away from the center of the wheel is the closest edge of the vat? Explain your reasoning.

My Notes

The stirring mechanism is shown below. Suppose point C is at the origin. The wheel turns counterclockwise and θ is the angle in standard position whose terminal side contains point $P(x, y)$.



4. Express x as function of θ.

5. Express z as a function of θ.

6. How far is the stirrer blade from the center of the wheel for each angle of rotation? Write a function d for the distance in terms of the angle θ.

Check Your Understanding

7. Your answer to Item 6 should be a sum of two terms to represent $d(\theta)$.
  a. Using the diagram above Item 4, describe what the two terms represent.
  b. Explain the steps you took to obtain an expression for each of those two terms.

8. **Reason quantitatively.** Verify that your function is correct by evaluating $d(0°)$, $d(180°)$, $d(30°)$, and $d(225°)$ and then comparing the results with your answers to Item 1.

Use the radius of 2 feet and a rod length of 5 feet to answer the following.

9. If the stirrer blade is at the midpoint of the vat, what is the measure of the angle from the horizontal ray to the attachment point P? Explain how you arrived at your answer.

10. If the stirrer blade is three-fourths of the distance from the end nearest the wheel to the far end of the vat, use algebraic methods to find the measure of the angle from the horizontal ray to the attachment point P.

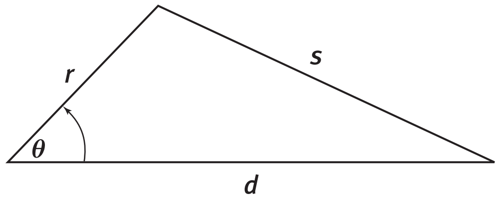
CONNECT TO AP

In this activity, you are exploring a problem using multiple representations. In AP Calculus, you will be expected to:

- Work with functions represented in a variety of ways: graphical, numerical, analytical, or verbal;
- Model a written description of a physical situation with a function, a differential equation, or an integral; and
- Use technology to help solve problems, experiment, interpret results, and support conclusions.

My Notes

11. Write the function for the distance of the stirrer blade from the center of the wheel in terms of the angle of rotation θ, radius r, and rod length s.



12. There are several ways to think about the relationship between s and r.

a. What expression contains s and r in your answer to Item 11? What relationship must exist between s and r?

b. In the diagram for Item 11, draw the ***altitude*** to the side whose length is d. Write an expression for the length of the altitude using what you know about right triangles.

MATH TERMS

The **altitude** of a triangle is the perpendicular line from the base to the opposite vertex.

c. Compare your relationships from part a and part b.

Check Your Understanding

13. For the chocolate stirrer, the situation with the wheel and stirrer blade represents a conversion between linear motion and circular motion.

a. Do both motions represent constant velocity? Explain.

b. Do both the wheel and the blade represent motions in a constant direction? Explain.

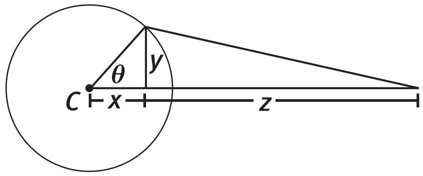
14. How is trigonometry used to translate motion along the point of a wheel into a different kind of motion?

15. Reason quantitatively. The equation from Item 11 relates three sides, r, d, and s, and the included angle between r and d. Rewrite this equation below and solve it for s^2, the side opposite the given angle. Use the Pythagorean identity to simplify your result.

My Notes

LESSON 24-1 PRACTICE

Refer to the diagram from Item 4.



16. Find the distance from the center of the wheel to the stirrer blade for each angle.
a. 45° **b.** 270°

17. **a.** At what value(s) of θ is the speed of the stirrer blade at 0 ft/sec?
b. At what value(s) of θ does the speed of the stirrer blade reach its maximum?
c. During what range for θ is the velocity of the stirrer blade increasing? During what range for θ is the velocity of the stirrer blade decreasing?

18. Suppose the stirrer blade starts at the far end of the vat and the wheel makes one revolution in 30 seconds. After how many seconds does the stirrer blade reach the middle of the vat for the *first time*? After how many seconds does it reach the middle of the vat for the *second time*?

19. Express y as a function of θ.

20. **Construct viable arguments.** Express $x^2 + y^2$ in terms of θ and simplify your expression. Relate your result to the ordered pair (x, y).

My Notes

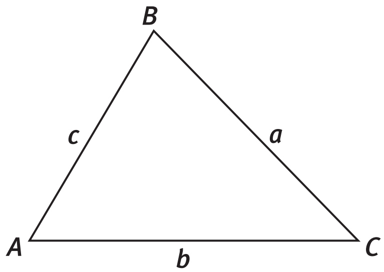
Learning Targets:

- Write equations for the Law of Cosines using a standard angle.
- Apply the Law of Cosines in real-world and mathematical situations.

SUGGESTED LEARNING STRATEGIES: Create Representations, Debriefing, Interactive Word Wall, Marking the Text, Simplify the Problem, Think-Pair-Share, Vocabulary Organizer

The equation you wrote in Item 15 of Lesson 24-1 is known as the ***Law of Cosines***.

1. Express regularity in repeated reasoning. Using the standard triangle, complete the Law of Cosines for each set of given information.



a. Given sides a and b and included angle C.

b. Given sides b and c and included angle A.

c. Given sides a and c and included angle B.

Check Your Understanding

2. Here is an expression for the Law of Cosines:

$[\ \ ]^2 = (\ \ )^2 + \{\ \ \}^2 - 2(\ \ )\{\ \ \}\cos [\ \ ]$

a. The expressions inside the square brackets are related to each other. What is that relationship?

b. The expressions inside the other grouping symbols are also related. What are those relationships?

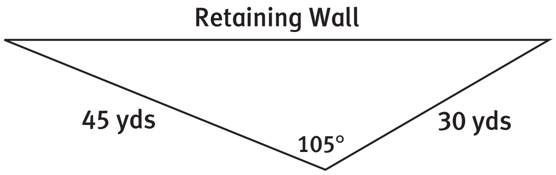
c. Reason abstractly. Write out the Law of Cosines using your own words.

3. In order to use the Law of Cosines, what is the minimum information needed about the sides and/or the angles of the triangle?

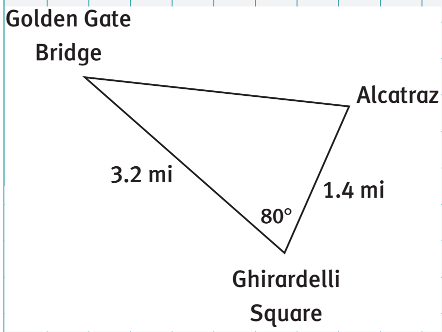
My Notes

The Law of Cosines is useful in many applications involving non-right triangles, also known as ***oblique triangles***.

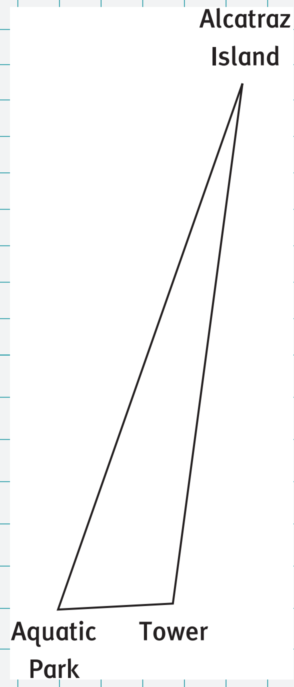
4. A new courtyard at Ghirardelli Square in San Francisco will be triangular. Find the length of the retaining wall given the diagram shown below.



5. From Ghirardelli Square in San Francisco, you can see the Golden Gate Bridge and Alcatraz Island. The angle between the sight lines to these landmarks is approximately 80°. If the approximate distance from Ghirardelli Square to the Golden Gate Bridge is 3.2 miles and to Alcatraz is 1.4 miles, how far is Alcatraz Island from the Golden Gate Bridge?



6. **Model with mathematics.** The annual Escape from Alcatraz Triathlon includes a 1.5-mile open-water swim from the island to Aquatic Park. Due to the strong currents, swimmers aim for a tower located 0.3 miles up the shore from Aquatic Park and 1.25 miles from Alcatraz Island. What is the angle between the course the swimmers set and the actual course they swim in the race?



7. Two sides of an isosceles triangle have lengths 8.5 and 10.5.

a. What are the angles of the triangle if the two congruent sides are 10.5?

b. What are the angles of the triangle if the two congruent sides are 8.5?

8. Two sides of an isosceles triangle have lengths 12.2 and 5.8.

a. Why is there only one possible triangle in this situation?

b. What are the angles of the triangle?

My Notes

9. Two ships leave a dock at the same time. They travel in straight lines so the angle between them is 52.1°. One ship travels at a rate of 15.8 miles per hour and the other travels at a rate of 12.16 miles per hour.

a. How far apart are they after four hours?

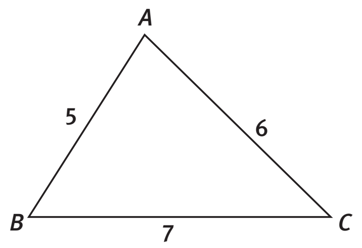
b. After four hours, they change direction and sail directly toward each other. If they maintain their same rates of travel, how long will it be until they meet?

Check Your Understanding

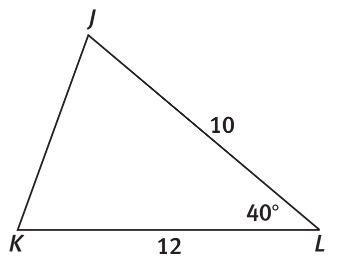
10. Model with mathematics. In Item 6, suppose you want to calculate the measure of the angle whose vertex is the Tower. Write a specific form of the Law of Cosines that will let you find that angle. Explain why you chose that form, and then find the angle.

LESSON 24-2 PRACTICE

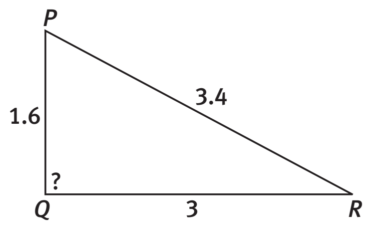
11. Find the three angles of triangle *ABC*.



12. Find the missing sides and angles.



13.



a. Use the Law of Cosines to find angle *Q*.

b. What kind of triangle is $\triangle PQR$? Verify your result.

14. When you use the Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos C$, under what circumstances is a positive number subtracted from $a^2 + b^2$? Under what circumstances is a negative number subtracted from $a^2 + b^2$?

15. Make use of structure. The equation for the Law of Cosines is similar to the equation for the Pythagorean Theorem.

Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos C$

Pythagorean Theorem

$c^2 = a^2 + b^2$

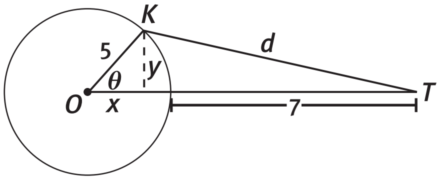
Compare the two equations. Then explain how the Law of Cosines changes based on whether the triangle is acute, right, or obtuse.

ACTIVITY 24 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 24-1

1. Kyle is riding on a merry-go-round in a counterclockwise direction. Suppose point K represents his location on the ride as the merry-go-round turns. Kyle's mom Tammy is watching her son at point T. The radius of the merry-go-round is 5 meters and Tammy is 7 meters from the edge of the ride. Kyle's distance d from his mom can be expressed as a function of the angle of rotation θ.



   a. What is d when $\theta = 0°$?
   b. What is d when $\theta = 150°$?
   c. What is d when $\theta = 180°$?
   d. What is the range of d?

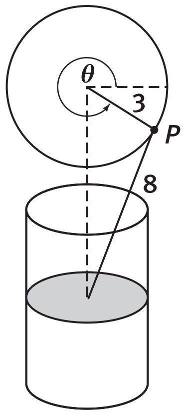
2. Write the distance d between Kyle and Tammy as a function of angle θ.

3. For one rotation of the wheel, find all values of θ such that d is equal to 10.

4. For one rotation of the wheel, find all values of θ such that d is less than 9.

5. The wheel spins so that θ changes at a constant rate. Which statement about the rate at which d changes is true?
   A. d changes at a constant rate.
   B. d changes most rapidly when θ is near 90° or near 270°.
   C. d changes most rapidly when θ is near 0° or near 180°.
   D. The value of d does not change.

6. An 8-foot rod is attached to a wheel with a 3-foot radius to produce a vertical motion used to crush cocoa shells. The crusher plate moves up and down in a cylinder containing the shells as point P rotates counterclockwise around the wheel.



   a. Write a function in terms of the angle of rotation to give the distance of the crusher plate from the center of the wheel.
   b. Find the minimum and maximum distances that the crusher plate will extend below the center of the wheel. Illustrate your answer with a diagram.
   c. Graph the function on your calculator, sketch the graph on your paper, and identify the period and the range of the function.
   d. Find the distance of the crusher plate from the center of the wheel for $\theta = 0°, 30°, 90°, 180°$, and 225°.
   e. For what angles of rotation will the crusher plate be 10 feet from the center of the wheel?

7. In Item 6, suppose the radius of the wheel is c, the length of the rod is r, and the distance between the center of the wheel and the crusher plate is g. Write a function for g in terms of θ, c, and r.

Lessons 24-2

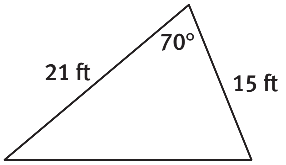
8. Use the Law of Cosines to solve for the indicated side or angle measure.

a. $a = 10$, $b = 11$, $C = 45°$. Find c.
b. $b = 9$, $c = 22$, $A = 150°$. Find a.
c. $a = 15$, $b = 62$, $c = 65$. Find B.
d. $a = 4$, $b = 10$, $C = 75°$. Find c.
e. $b = 50$, $c = 25$, $A = 150°$. Find a.
f. $a = 10$, $b = 6$, $c = 13$. Find C.

9. What is the measure of the largest angle in a triangle with sides of 3, 5, and 7?

10. Use the Law of Cosines to find the perimeter of a regular pentagon inscribed in a circle with a radius of 6.

11. Find the perimeter of Terry's triangular garden plot in the diagram below.



12. Two fishing boats leave the harbor at 6 a.m., each traveling in a straight line. The angle between their paths is 65°. The first boat averages 15 mph and the second boat averages 25 mph.

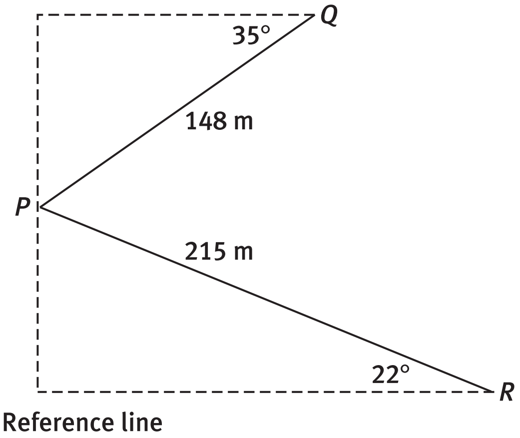
a. How far apart are the two boats after 1 hour? After 2 hours?
b. Without using the Law of Cosines, predict how far apart the boats would be after n hours if they maintain their respective speeds and directions. Explain your reasoning.

13. You CANNOT use the Law of Cosines if you know:

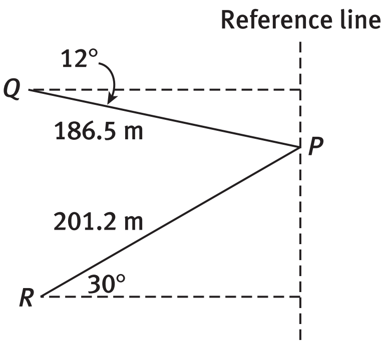
A. the three sides of a triangle
B. two sides and the included angle
C. two angles and the included side
D. two sides and the nonincluded angle

14. A surveyor establishes a reference point R and a reference line. Then the surveyor measures the distance from the reference point to two other points P and Q and measures the angles from P and Q to the reference line. Find the distance QR for each diagram below.

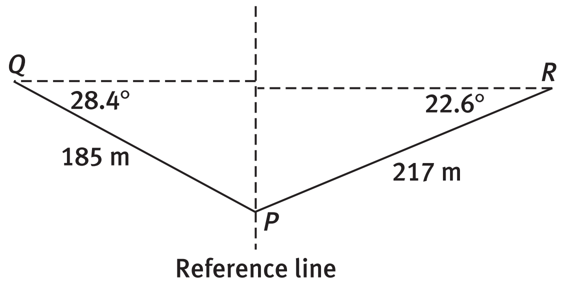
a.



b.



c.



MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

15. A student claims that all four choices in Item 13 are correct. How would you respond?

The Law of Sines

Got Lost?

Lesson 25-1 Modeling and Applying the Law of Sines

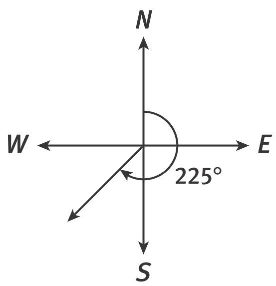
Learning Targets:

- Calculate the bearing of a flight.
- Derive and use the Law of Sines.
- Find unknown sides or angles in oblique triangles.

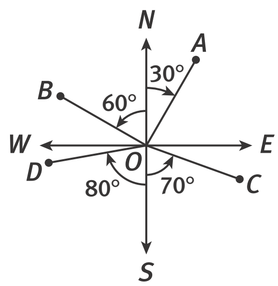
SUGGESTED LEARNING STRATEGIES: Marking the Text, Visualization, Identify a Subtask, Simplify the Problem, Create Representations, Summarizing, Paraphrasing, Work Backward, Look for a Pattern, Quickwrite, Graphic Organizer, Think-Pair-Share, Group Presentation, Guess and Check

In navigation, an object's heading indicates the direction of movement as measured by an angle rotated clockwise from north. A heading of 90° means an object is heading due east. A heading of 225° means an object is heading southwest. The *directional bearing* of a point is stated as the number of degrees east or west of the north-south line. To state the directional bearing of a point, write:

- N or S which is determined by the angle being measured
- the angle between the north or south line and the point, measured in degrees
- E or W which is determined by the location of the point relative to the north-south line



Heading



Directional Bearing

In the figure, *A* from *O* is N30°E, *B* from *O* is N60°W, *C* from *O* is S70°E, and *D* from *O* is S80°W.

International Flight 22 was on a course due north from Auckland, New Zealand, to Honolulu, Hawaii. Two thousand miles south of Honolulu, the plane encountered unexpected weather and the pilot changed bearing by 20°, as shown in the figure. The plane traveled on this new course for 1.5 hours, averaging 500 miles per hour.

1. How far did the plane travel during the 1.5 hours it was flying on its new flight path?

2. How far was the plane from Honolulu after 1.5 hours?

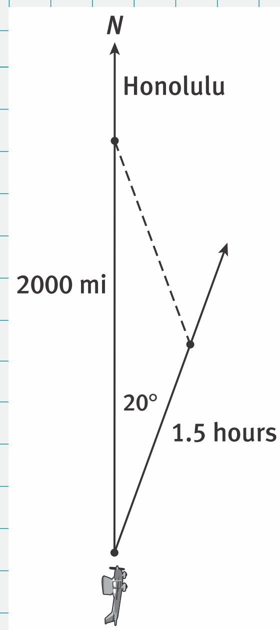
My Notes

CONNECT TO AVIATION

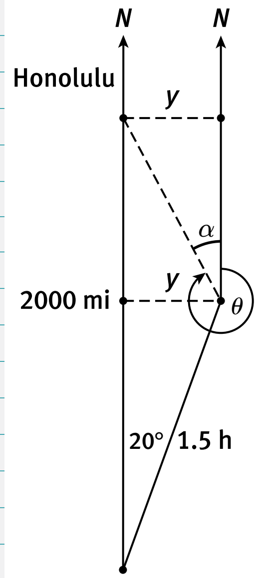
This activity measures speed in miles per hour. However, the speed of commercial jets is typically represented as a Mach number, a percentage of the speed of sound. Mach speed varies as temperature and altitude change.

CONNECT TO NAVIGATION

Bearing is the direction an aircraft is pointing, but the course is the actual direction in which the plane is moving when wind is taken into account. The heading is the clockwise angle in degrees between an aircraft's destination and north.



My Notes



3. To adjust the flight path, the pilot changed the course by α degrees, as shown in the figure.

a. Find the length y of the horizontal dotted lines in the diagram. Then use that length and your answer to Item 2 to find the value of α. What is the directional bearing of the plane?

b. Use the value of α to determine the bearing θ of the plane.

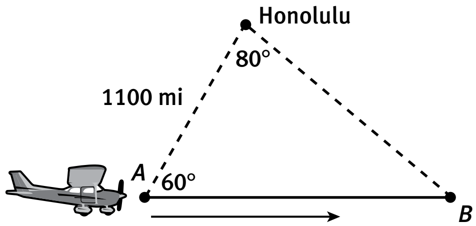
Another flight, Flight 33, was 1100 miles southwest of Honolulu when the plane sped up and headed due east. Radio contact with Flight 33 occurred 2.5 hours later, and air traffic controllers placed it somewhere over the Cook Islands at point B.

4. Draw the altitude of the triangle from the point representing Honolulu to the horizontal flight path.

a. What is length of the altitude?

b. How far did the plane travel in 2.5 hours?

c. How fast was the plane traveling along the path from point A to point B?

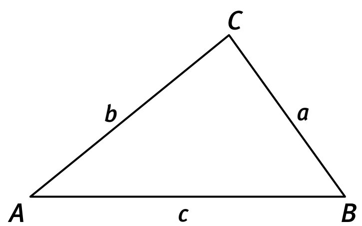


The ***Law of Sines***, shown below, could also be used to solve problems like Items 3 and 4.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The next series of items will show you how your work from the previous items can be generalized to derive the Law of Sines.

5. Model with mathematics. Use the triangle shown below to answer the questions.



a. Draw the altitude h from vertex C to side c.

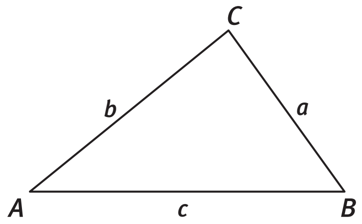
b. What is the measure of h in terms of A and b?

c. What is the measure of h in terms of B and a?

d. Use your work from parts b and c to write an equation relating A, B, a, and b.

My Notes

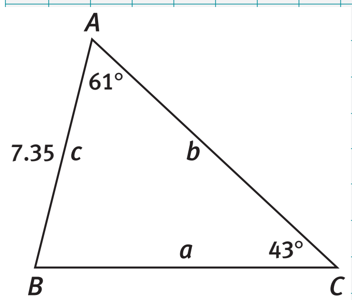
6. Express regularity in repeated reasoning. Repeat the process used in Item 5, but draw the altitude from vertex B to side b.



7. How do the equations you wrote in Items 5 and 6 compare to the Law of Sines written before Item 5?

Example A

Use the Law of Cosines and the Law of Sines to find the missing parts of this triangle. Explain which law you used.



Step 1: There are two known angles and one known side. We know c and C, and we know A, so we can use the Law of Sines to find a.

$$\frac{c}{\sin C} = \frac{a}{\sin A};\ a = \frac{(7.35)(\sin 61^\circ)}{\sin 43^\circ} = 9.43$$

Step 2: Find angle B: $B = 180^\circ - (61^\circ + 43^\circ) = 76^\circ$

Step 3: To find the third side, we can use the Law of Cosines. We can check the answer using the Law of Sines.

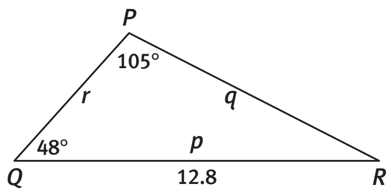
$b^2 = a^2 + c^2 - 2ac\cos B = 9.43^2 + 7.35^2 - 2(9.43)(7.35)\cos 76^\circ = 109.41$

$b = 10.46$

Check: $\dfrac{b}{\sin B} = \dfrac{c}{\sin C};\ b = \dfrac{(7.35)(\sin 76^\circ)}{\sin 43^\circ} = 10.46$

Try These A

a. Use the Law of Sines to find q.

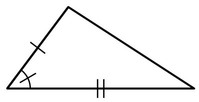


b. Use the Law of Cosines to find r. Check your value for r using the Law of Sines.

My Notes

Like the Law of Cosines, the Law of Sines relates the sides and angles in an ***oblique triangle***, and these can be used to find unknown sides or angles given at least three known measures that are not all angle measures.

8. The following table summarizes when each rule should be used to find missing measures in an oblique triangle. For each abbreviation, complete the given information and illustrate the given information by drawing and marking a triangle.

Rule	**Given information**	**Illustration**
Law of Cosines	SAS side, included angle, side SSS	
Law of Sines	ASA AAS	

Check Your Understanding

9. What is the minimum information needed in order to use the Law of Sines?

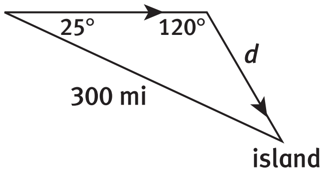
10. **Make use of structure.** What information about a triangle is enough to apply the Law of Sines but not the Law of Cosines? What information about a triangle is enough to apply the Law of Cosines but not the Law of Sines?

11. In a board game, a plane is flying from the origin of a coordinate plane toward the point (3, 4). What is the directional bearing of the plane? (Assume that north is the direction of the positive y-axis.)

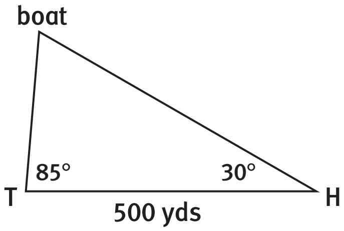
My Notes

Use the Law of Sines to solve the following problems.

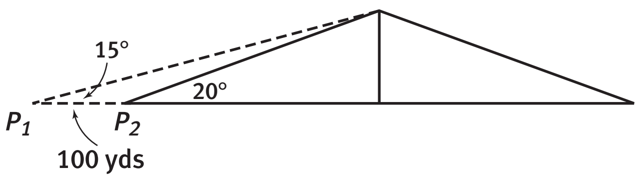
12. The pilots of Flight 33 spotted a deserted island 300 miles from their current location but continued on their course knowing there was another island 500 miles ahead on their current course. After a while they experienced engine trouble and turned to head for the deserted island, hoping to land safely. After making the turn, they estimated the plane could travel another 200 miles. They landed the plane knowing they were very lucky. How many additional miles could they have flown? Explain your reasoning.



13. Survivors Taylor and Hank are on the beach of the deserted island, 500 yards apart. They spot a boat out at sea and estimate the angles between their positions and the boat as shown below. How far is the boat from Hank? How far is the boat from Taylor?



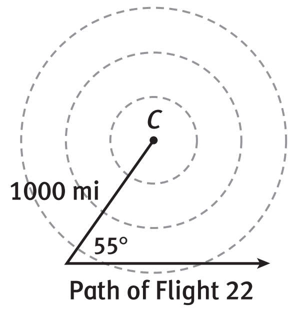
14. Reason quantitatively. Survivors Tariq and Jess were trying to estimate the distance to the top of a mountain they hoped to hike up to get a better view of the landscape. They measured the angles of elevation at points P_1 and P_2 located 100 yards apart. If the two survivors started at point P_2, how far would they have to walk to get to the top of the mountain?



My Notes

Check Your Understanding

The last known location of Transcontinental Flight 22 was 1000 miles from a tracking station located at point *C*. The plane was flying due east at a 55° angle with the tracking station.



15. Suppose Flight 22 continued to fly due east.
   a. If a search plane is sent out from the tracking station and searches a 600-mile radius, will the search plane intercept the flight path? Show your work to explain your reasoning.
   b. Would the search plane intercept the flight path if the search radius were 1000 miles? Explain.
   c. Would the search plane intercept the flight path if the search radius were 900 miles? Explain.

16. What are some other possible search radii that would intercept the flight path in exactly one point?

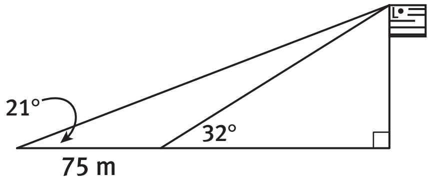
LESSON 25-1 PRACTICE

17. A flight has a heading of 32°, as measured clockwise from north. What is its new heading under the following changes?
   a. a quarter turn counterclockwise
   b. a quarter turn clockwise
   c. a rotation of 55° counterclockwise

18. In triangle *ABC*, angle *A* is 54°, angle *C* is 96°, and $AC = 275$. Find angle *B*, *AB*, and *BC*.

19. In triangle *PQR*, $PR = 14$, $QR = 15$, and $PQ = 13$. Find the three angles of the triangle.

20. In the diagram below, two sightings of the top of a flagpole are taken 75 meters apart on level ground. The two sightings are 21° and 32°. What is the height of the flagpole?



21. Attend to precision. Two groups of students were given a copy of triangle *MNP*. Group 1 measured sides and angles of the triangle and found that $MN = 21$ cm, angle $N = 47°$, and $MP = 37$ cm. Group 2 found that $MN = 20.8$ cm, angle $N = 47.42°$, and $MP = 37.4$ cm. Using the Law of Sines, what values will Group 1 and Group 2 get for angle *P*, for angle *M*, and for *NP*?

My Notes

Learning Targets:

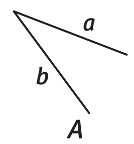
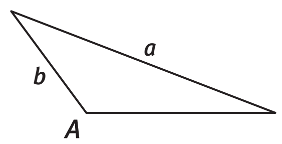
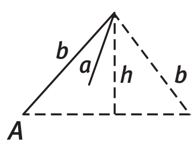
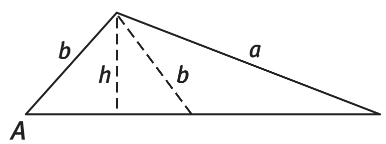
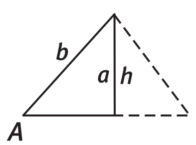
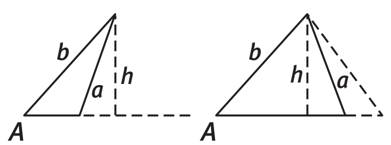
- Determine the number of distinct triangles given certain criteria.
- Use the Law of Sines to solve triangles with unknown sides or angles.

SUGGESTED LEARNING STRATEGIES: Note Taking, Interactive Word Wall, Graphic Organizer, Quick Write, Look for a Pattern, Think-Pair-Share, Create Representations, Guess and Check, Role Play, Identify a Subtask, Simplify the Problem, Group Presentation

When you are given two sides and the opposite angle, it is possible to have zero, one, or two distinct triangles depending on the given information. This situation is known as the ***ambiguous case (SSA)*** and is summarized below.

The Ambiguous Case (SSA)

Given a, b, and A with $h = b \sin A$, where b is adjacent to and a is opposite angle A, and h is the altitude of the potential triangle.

A is obtuse, $a \le b$  no triangle	A is obtuse, $a > b$  one triangle
A is acute, $a < h$  no triangle	A is acute, $a \ge b$  one triangle
A is acute, $a = h$  one triangle	A is acute, $h < a < b$  two triangles

1. How can the value of $b \sin A$ help you determine the number of solutions given the ambiguous case?

2. How would you use this table to interpret the number of possible triangles for the ambiguous case if you were given angle C, side b, and side c?

ACADEMIC VOCABULARY

The word ***ambiguous*** is used for things that are open to interpretation, or can have two meanings.

My Notes

3. **Construct viable arguments.** Determine how many triangles are possible with the given information. Draw a sketch and show any calculations you used.

   a. $A = 30°, b = 10, a = 5$

   b. $C = 75°, c = 18, a = 7$

   c. $B = 100°, b = 50, c = 75$

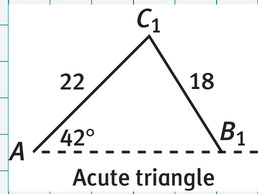
   d. $C = 40°, b = 25, c = 21$

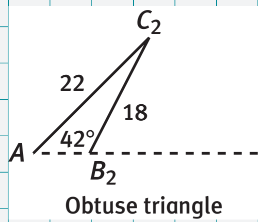
   e. $A = 63°, a = 10, c = 45$

4. The Ambiguous Case Game 1: Use the information given to you by your teacher. With your classmates, organize into groups so you have the three groups with one solution and two groups with two solutions. Record your results.

5. The Ambiguous Case Game 2: Use the information given to you by your teacher. With your classmates, organize into groups so that every group of three people forms no triangle. Record your results.

My Notes





Example A

Solving the two-solution SSA situation: Use the Law of Sines to solve a triangle given $A = 42°$, $a = 18$, $b = 22$.

Step 1: Determine the number of solutions. There are two solutions since the measure of side a is between $b \sin A$ and b.

$b \sin A = 22 \sin 42° \approx 14.720 < 18$

Step 2: Solve the first acute triangle using the Law of Sines.

$$\frac{18}{\sin 42°} = \frac{22}{\sin B_1}$$

$$\sin B_1 = \frac{22 \sin 42°}{18} \approx 0.8178$$

Find B_1.

$$B_1 = \sin^{-1}\left(\frac{22 \sin 42°}{18}\right) \approx 54.9°$$

Find C_1.

$$C_1 = 180° - (42° + 54.9°) = 83.1°$$

Find c_1.

$$c_1 = \frac{18 \sin(83.1°)}{\sin 42°} \approx 26.7$$

Step 3: Solve the second obtuse triangle. The angles B_1 and B_2 opposite the given adjacent side are supplementary.

$$B_2 = 180 - B = 180 - 54.9 = 125.1$$

$$C_2 = 180 - (42 + 125.1) = 12.9$$

$$c_2 = \frac{18 \sin 12.9}{\sin 42} \approx 6.0$$

Try These A

Each of these triangles has two possible solutions. Find them both.

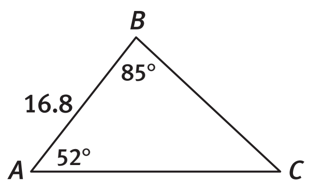
a. $A = 55°$, $b = 40$, $a = 35$

b. $C = 20°$, $c = 6$, $b = 12$

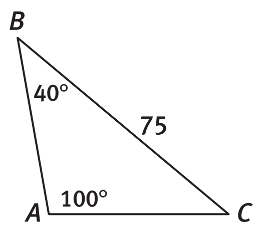
Check Your Understanding

6. Solve each triangle using the Law of Sines.

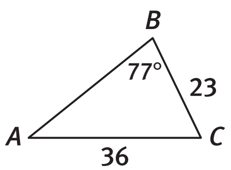
a. $A = 52°$, $B = 85°$, $c = 16.8$



b. $A = 100°$, $B = 40°$, $a = 75$



c. $B = 77°$, $a = 23$, $b = 36$



My Notes

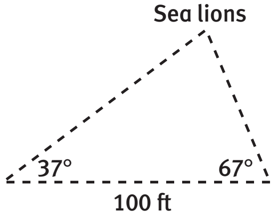
7. A lookout tower, firefighters located 25 miles from the tower, and a forest fire form three vertices of a triangle. At the lookout tower, the angle between the forest fire and the firefighters is 35°. At the firefighters' location, the angle between the lookout tower and the fire is 100°. How far are the firefighters from the fire?

8. **Make sense of problems.** Determine the number of possible triangles for each given situation.
a. $A = 45°, c = 100, a = 25$
b. $B = 70°, c = 90, b = 85$
c. $C = 100°, c = 6, a = 7.5$
d. $A = 60°, b = 4, a = 2\sqrt{3}$

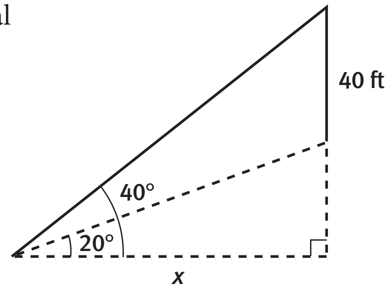
9. Why is the term *ambiguous case* used in this lesson? Explain how you know the situation is ambiguous. Describe how to solve an "ambiguous case" situation without using a formula.

LESSON 25-2 PRACTICE

10. Two marine biologists spotted some sea lions in the bay. The biologists were located on a beach about 100 feet apart. The angle between the shore and the sea lions for each biologist is shown below. How far were the sea lions from each biologist?



11. A billboard is 40 feet tall. At a horizontal distance x feet from the billboard, the angle of elevation to the bottom of the sign is 20° and the angle of elevation to the top of the sign is 40°. How far away is the billboard?



12. Solve the two-solution ambiguous case situation given $C = 50°, b = 120, c = 100$.

13. Solve each triangle using the Law of Sines.
a. $A = 22°, B = 35°, c = 43$
b. $A = 110°, B = 30°, a = 8$
c. $B = 57°, a = 13, b = 30$

14. The angle of elevation from a point 50 yards from a tree to the top of the tree is 23°. The tree leans 4° away from vertical in the direction opposite the point 50 yards away. How tall is the tree?

15. Joaquin is fencing in a triangular pasture. Two posts are located 300 yards apart, and the angles from the posts to the third one are 75° and 68°, respectively. About how much fencing does Joaquin need?

16. **Use appropriate tools strategically.** Use a ruler and protractor to construct triangle ABC with $AB = 12.5$ cm, angle $A = 42°$, and angle $B = 40°$. Use your ruler and protractor to measure AC, CB, and angle C. Then calculate the size of angle C and use the Law of Sines to find AC and CB. How close were your measurements to your calculated values?

ACTIVITY 25 PRACTICE

Write your answers on notebook paper.
Show your work.

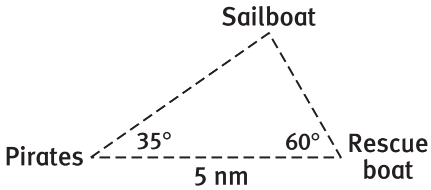
Lesson 25-1

1. Use the Law of Sines to solve triangle ABC with the following measures.

   angle $A = 150°$, angle $C = 20°$, $a = 200$

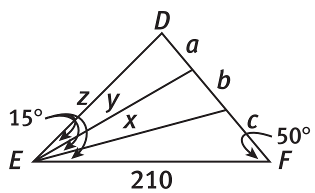
2. Two points, A and B, are 6 miles apart on level ground. An airplane is flying between A and B. The angle of elevation to the plane from point A is 51° and from point B is 68°. What is the altitude of the airplane?

3. A rescue boat and a pirate ship located 5 nautical miles apart both spotted a stranded sailboat at the same time. The rescue boat had a maximum speed of 18 knots (nautical miles per hour), and the pirate ship was capable of 22 knots. The angle between boats is shown below. If both ships set off at their top speed, which one will get to the stranded sailboat first, and how long will it take?

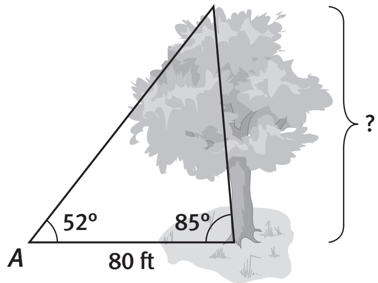


4. The angle of elevation from a point P 65 yards from a tree to the top of the tree is 31°. The tree leans 7 degrees away from P. How tall is the tree?

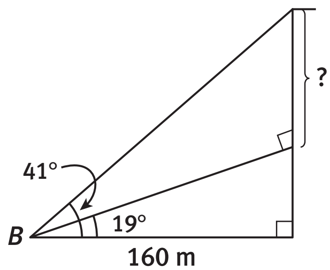
5. In triangle DEF below, angle DEF is divided into three angles, each of 15°, and angle F is 50°. If $XZ = 210$, find the values of x, y, z, a, b, and c.



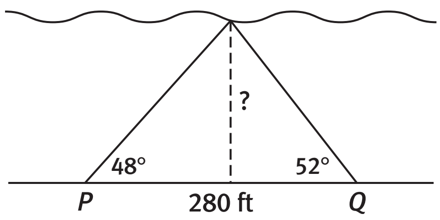
6. Joanna is interested in determining the height of a tree. She is at a point A, 80 feet from the base of the tree, and she notices that the angle of elevation to the top of the tree is 52°. The tree is leaning toward her and is growing at an angle of 85° with respect to the ground. What is the height of the tree?



7. From a point B on the ground that is level with the base of a building and is 160 meters from the building, the angle of elevation to the top of the building is 41°. From point B, the angle of elevation to a ledge on the side of the building is 19°. What is the distance between the ledge and the top of the building?



8. An explorer wants to know the width of a river. She starts by establishing two points, P and Q, on one side of the river that are 280 feet apart. She notices that a particular tree on the far side of the river forms an angle of 48° with side PQ when sighted from point P, and forms an angle of 52° with side PQ when sighted from point Q. How wide is the river? Show your work.



9. Which of the following statements is NOT true?
A. You can use the Law of Sines if you know any two angles and any one side of a triangle.
B. You can use the Law of Sines if you know the three sides of a triangle.
C. You can use the Law of Cosines if you know any two sides and any one angle of a triangle.
D. You can use the Law of Cosines if you know the three sides of a triangle.

Lesson 25-2

10. Determine the number of possible triangles for each situation:
a. $A = 30°, c = 10, a = 5$
b. $B = 63°, c = 90, b = 75$
c. $C = 110°, c = 60, a = 47$
d. $A = 60°, b = 9, a = 9$

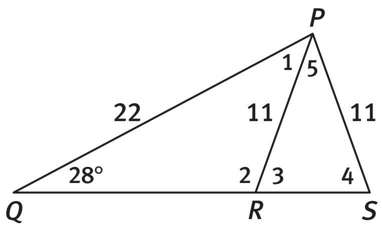
11. Solve the two-solution ambiguous case situation.
a. $B = 52°, a = 9, b = 8$
b. $C = 30°, b = 20, c = 12$

12. Explain why only one triangle ABC is possible if $a = 20$, $b = 16$, and angle $A = 30°$.

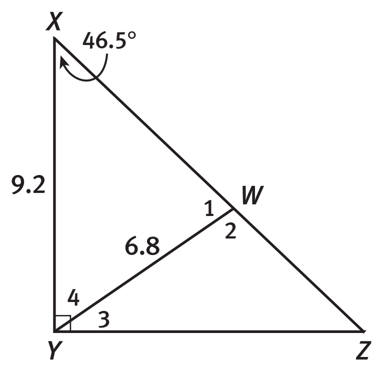
13. Explain why no triangle MNP is possible if $m = 7$, $p = 16$, and angle $M = 30°$.

14. Explain why two triangles are possible if $x = 10$, $y = 16$, and angle $X = 30°$.

15. For the figure below, find angles 1, 2, 3, 4, and 5, and find QR, RS, and QS.



16. For the figure below, find angles Z, 1, 2, 3, and 4, and find XW, WX, YZ, and XZ. If necessary, round values to the nearest tenth.



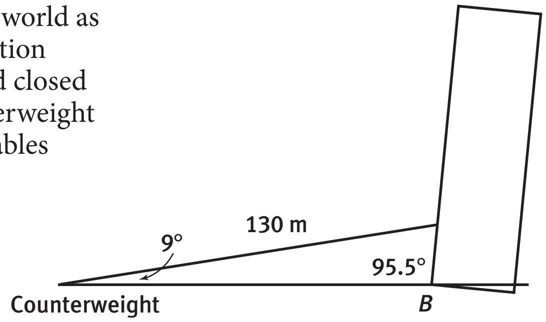
MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

17. A student claims that the ambiguous case means you cannot tell whether 0, 1, or 2 triangles are possible given information about the triangle. Is that statement correct? Explain.

Right and Oblique Triangles, Area

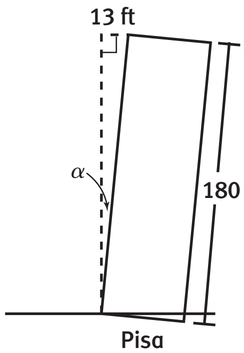
TILTING TOWERS AND TRIANGLES

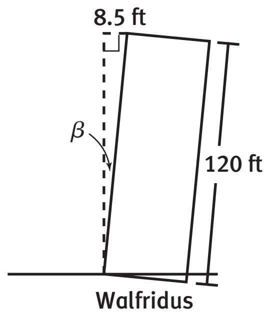
The bell tower of the cathedral in Pisa, Italy, is known to most of the world as the Leaning Tower of Pisa. The tower began leaning during construction around 1275. In 1990, Italian officials feared it would topple over and closed the site to the public. Engineers used steel cables attached to a counterweight to stabilize the structure during repairs. At the time the 130-meter cables were attached, the tower leaned about 5.5° past vertical.



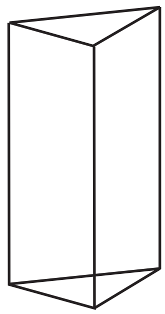
1. Why is the measure of angle B equal to 95.5°?
2. How far from the base of the tower were the counterweights?

The Leaning Tower of Pisa is not the only leaning bell tower in Europe. In 2008, a newspaper reported that since the repair of the Leaning Tower of Pisa, the 120 ft Tower of Walfridus in Bedum, Netherlands, was now the most tilted tower in Europe. It leaned 8.5 feet off center while the postrepair Tower of Pisa, at 180 ft tall, leaned only 13 feet off center.





3. What are the angles α and β at which each tower is leaning?
4. Is the newspaper's report valid? Explain your reasoning.
5. The Leaning Tower of Pisa is a *cylindrical* building. Architects design buildings in all shapes to fit various sites. The Flatiron Building in New York City is probably the most famous *triangular* building. The sides of the building are 173 feet along 5th Ave, 190 feet along Broadway, and 87 feet along 22nd St. Follow the steps below to find the area of the roof.
   a. Draw triangle ABC for the roof, with $AB = 173$, $BC = 87$, and $AC = 190$. Use the Law of Cosines to find angle A.
   b. Draw the altitude h from vertex B to side AC. Find an expression for h in terms of angle A and side c.
   c. A formula for the area of a triangle is Area $= \frac{1}{2}bh$, where b is a side of the triangle and h is the altitude to that side. Restate that formula using your results from part b. Then use your restated formula to find the area of triangle ABC.
   d. For any triangle XYZ, use the ideas in parts b and c to write three formulas for the area of the triangle in terms of two sides and one angle.



6. Two airplanes leave an airport at the same time. One flies with a bearing of 24° at an average speed of 475 mph, and the other flies with a bearing of 285° at an average speed of 450 mph. How far apart are the two planes after two hours?

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 2, 3, 4, 5, 6)	• Clear and accurate understanding of the Law of Cosines, Law of Sines, and general right triangle trigonometry	• A functional understanding of the Law of Cosines, Law of Sines, and general right triangle trigonometry in a mostly correct way, but with some errors	• Partial understanding of some of the following: Law of Cosines, Law of Sines, and general right triangle trigonometry	• Little or no understanding of the Law of Cosines, Law of Sines, and general right triangle trigonometry
Problem Solving (Items 1, 2, 3, 4, 5, 6)	• An appropriate and efficient strategy that results in a correct answer when solving triangles using Law of Cosines, Law of Sines or Heron's Area Theorem	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 2, 3, 4, 5)	• Clear and accurate understanding of the Law of Cosines, Law of Sines, and right triangle trigonometry within the context of the problem	• A functional understanding of representations of the Law of Cosines, Law of Sines, and right triangle trigonometry within the context of the problem	• Partial understanding of representations of the Law of Cosines, Law of Sines, and right triangle trigonometry within the context of the problem	• Little or no understanding of representations of the Law of Cosines, Law of Sines, and right triangle trigonometry within the context of the problem
Reasoning and Communication (Items 1, 4)	• Precise use of appropriate math terms and language to explain the application and resulting solutions of the Law of Cosines, Law of Sines, and right triangle trigonometry	• Correct characterization of the terms and language to explain the application and resulting solutions of the Law of Cosines, Law of Sines, and right triangle trigonometry	• Misleading or confusing terms and language to explain the application and resulting solutions of the Law of Cosines, Law of Sines, and right triangle trigonometry	• Incomplete or inaccurate characterization and language to explain the application and resulting solutions of the Law of Cosines, Law of Sines, and right triangle trigonometry