ANALYTICAL SOLUTION OF BEAM ON ELASTIC FOUNDATION BY SINGULARITY FUNCTIONS

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The paper deals with a new manner of obtaining a closed-form analytical solution of the problem of bending of a beam on an elastic foundation. The basic equations are obtained by a variational formulation based on the minimum of the total potential energy functional. The basic methods for solving the governing equations are considered and their advantages and disadvantages are analyzed. The author proposes a felicitous approach for solving the equilibrium equation and applying the boundary conditions by transformation of the loading using singularity functions. This approach, combined with the resources of the modern computational algebra systems, allows a reliable and effective analysis of beams on an elastic foundation. The numerical examples show the applicability and efficiency of the approach for the solution of classical problems of soil-structure interaction.

Keywords: *beam on elastic foundation, soil-structure interaction, singularity functions*

1. Introduction

The computational model of a beam or a plate on an elastic foundation is often used to describe a lot of engineering problems and has application in geotechnics, road, railroad and marine engineering and bio-mechanics. The key issue in the analysis is modelling the contact between the structural elements- the beam and the soil bed. In most cases the contact is presented by replacing the elastic foundation with simple models, usually spring elements, because the main task is considered to be the analysis of the beam not the soil bed. The spring's stiffness describes the behaviour of the elastic foundation. A lot of methods are developed for determination of the spring stiffness and reduction of the 3-D problem to 2-D or 1-D, see [4], [24], [8].

2. Elastic foundation models

The elastic modelling of the soil bed is based on an assumption for the behaviour of the subgrade reaction under loading. The most popular relation between forces and deformations is linear because of the simplicity of the equations' solution. The elastic subgrade reaction is represented by:

– One-, two- or three-parameter models;
– Continuum models;
– Mixed models.

The parameter models are briefly presented in the next section.

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2.1. One-parameter model

The one-parameter model developed by Winkler in [26] assumes that the vertical displacement of a point of the elastic foundation is proportional to the pressure at that point and does not depend on the pressure at the adjacent points. The Winkler model can be interpreted as a system of mutually independent vertical springs with stiffness k. The strain energy of the elastic foundation is

$$U_{\mathrm{f}} = \int_0^\ell \frac{1}{2}\, k\, b\, w^2 \,\mathrm{d}x \;, \tag{1}$$

where b and ℓ are the width and the length of the deformed zone and w is the vertical displacement. The Winkler soil model assumes that the displacement appears only in the loaded zone. Outside this zone the deflections are zero. This assumption leads to a discontinuous displacement field and this is the main disadvantage of the Winkler model.

2.2. Two-parameter models

Two-parameter soil models restore the continuity of the elastic foundation by introducing a second parameter. The two-parameter models of Filonenko-Borodich [9], Hetenyi [11] and Pasternak [17] provide the continuity of the soil medium by adding a second spring which interacts with the first one. In [12] Kerr generalizes the Pasternak model by including a third spring in vertical direction. The models of Reissner [18] and Vlasov–Leontiev [25] make simplifying assumptions by introducing functions for distribution of the displacements or the stresses in the soil medium. The general expression for the strain energy in two-parameter models is

$$U_{\mathrm{f}} = \int_0^\ell \frac{1}{2}\, k\, b\, w^2 \,\mathrm{d}x + \int_0^\ell \frac{1}{2}\, G\, b \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \;. \tag{2}$$

The second integral in (2) includes the second parameter G which represents the stiffness of a generalized rotation spring. Different interpretations exist of the physical meaning of G and the relation with the first parameter k:

- Filonenko-Borodich model – the G parameter is presented as an internal tension force in a virtual elastic string placed on the transversal springs which constrains the vertical displacements of the springs;
- Hetenyi model – constrains the vertical displacements by adding an imaginary beam in bending. The second parameter represents the beam's stiffness;
- Pasternak model – the G parameter represents a shear modulus of a virtual layer that integrates the vertical spring elements;
- Vlasov-Leontiev model – the k and G parameters are obtained on the basis of the elastic continuum approach by making assumptions for the displacement field.

3. Problem formulation

The derivation of the field equations is based on the variation of the total potential energy functional. This method is widely used in [13], [25], [16] and [1] for the problem formulation or the derivation of the subgrade parameters. Many authors use different models to represent the behaviour of the beam: the classic beam theory is presented in [11], [22] and [19]; the first-order shear deformation theory of beams is used in [2], [28] and [29]. The assumptions and the equations of the classical beam theory are used in the presented paper:

- The beam and the soil materials are linearly elastic, homogeneous and isotropic;
- The displacements are small compared to the beam's thickness;
- The axial strains are small compared to unity;
- The transversal normal strains and the shear stresses are negligibly small;
- The cross-sections are plane and perpendicular to the longitudinal axis before and after a deformation (Bernoulli hypothesis).

The positive directions of the loading, the displacements and the internal forces coincide with the positive coordinates at fig. 1.

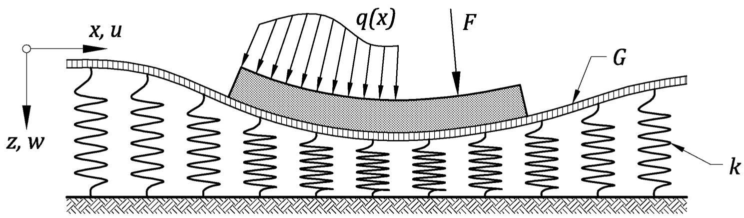

Fig.1: Beam on two-parameter elastic foundation

The following relations and equations are based on the previous assumptions:

- A displacement field

$$u(x,y,z) = -z\,\frac{\partial w}{\partial x}\;, \qquad w(x,y,z) = w(x,z)\;; \tag{3}$$

- A compatibility equation

$$\kappa \approx \frac{\partial^2 w}{\partial x^2}\;, \tag{4}$$

where κ is a curvature of the beam;

- A strain field

$$\varepsilon_{\mathrm{xx}} = -z\,\kappa\;, \qquad \gamma_{\mathrm{xz}} = 0\;; \tag{5}$$

- Stress-strain relation

$$\sigma_{\mathrm{xx}} = -z\,E\,\kappa\;, \tag{6}$$

where E is an elastic modulus of the beam.

The beam's strain energy is

$$U_{\mathrm{b}} = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\,\mathrm{d}V = \int_0^{\ell} \frac{1}{2}\,E\,I\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x\;, \tag{7}$$

where I is the moment of inertia of the beam section and ℓ is the beam length.

The strain energy of the elastic foundation is

$$U_\mathrm{f} = \int\limits_0^\ell \frac{1}{2}\, k\, b\, w^2 \,\mathrm{d}x + \int\limits_0^\ell \frac{1}{2}\, G\, b \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x \ . \tag{8}$$

The considered 1-D problem requires the width of the deformed foundation zone b to be equal to the beam width.

The total strain energy of the coupled system is

$$U = U_\mathrm{b} + U_\mathrm{f} = \frac{1}{2}\int\limits_0^\ell \left[E\, I \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 + G\, b \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + k\, b\, w^2\right] \mathrm{d}x \ . \tag{9}$$

The load potential is

$$W = \int\limits_o^\ell q\, b\, w \,\mathrm{d}x \ , \tag{10}$$

where q is the load intensity.

The total potential energy functional is

$$\Pi(w) = U - W = \frac{1}{2}\int\limits_0^\ell \left[E\, I \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 + G\, b \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + k\, b\, w^2\right] \mathrm{d}x - \int\limits_o^\ell q\, b\, w \,\mathrm{d}x \ . \tag{11}$$

The first variation of (11) is

$$\begin{aligned}\delta\Pi(w) = \int\limits_0^\ell E\, I\, \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\, \delta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) \mathrm{d}x + \int\limits_0^\ell G\, b\, \frac{\mathrm{d}w}{\mathrm{d}x}\, \delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right) \mathrm{d}x \,+ \\ + \int\limits_0^\ell k\, b\, w\, \delta w \,\mathrm{d}x - \int\limits_0^\ell q\, b\, \delta w \,\mathrm{d}x \ .\end{aligned} \tag{12}$$

The simplification of (12) gives

$$\begin{aligned}\delta\Pi(w) = \left[E\, I\, \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\, \delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right) - E\, I\, \frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\, \delta w + G\, b\, \frac{\mathrm{d}w}{\mathrm{d}x}\, \delta w\right]_0^\ell + \\ + \int\limits_0^\ell \left(E\, I\, \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - G\, b\, \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k\, b\, w - q\, b\right) \delta w \,\mathrm{d}x \ .\end{aligned} \tag{13}$$

The extremum condition of (13) is $\delta\Pi(w) = 0$ or

$$\begin{aligned}\left[E\, I\, \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\, \delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\right]_0^\ell + \left[\left(-E\, I\, \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + G\, b\, \frac{\mathrm{d}w}{\mathrm{d}x}\right) \delta w\right]_0^\ell + \\ + \int\limits_0^\ell \left(E\, I\, \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - G\, b\, \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k\, b\, w - q\, b\right) \delta w \,\mathrm{d}x = 0 \ .\end{aligned} \tag{14}$$

The first part of (14) describes:
– The essential boundary conditions- $\delta[w]_0^\ell$ and $\delta[\mathrm{d}w/\mathrm{d}x]_0^\ell$;
– The natural boundary conditions- $[E\,I\,\mathrm{d}^2w/\mathrm{d}x^2]_0^\ell$ and $[-E\,I\,\mathrm{d}^3w/\mathrm{d}x^3 + G\,b\,\mathrm{d}w/\mathrm{d}x]_0^\ell$.

An arbitrary variation of the displacement $\delta w \neq 0$ gives a nontrivial solution of the extremum problem which is the equilibrium equation of a beam on an elastic foundation

$$E\,I\,\frac{\mathrm{d}^4w}{\mathrm{d}x^4} - G\,b\,\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + k\,b\,w - q\,b = 0\ . \tag{15}$$

The variational formulation of the above boundary-value problem can be used for an analytical, a numerical or an approximate solution of the problem.

4. Solution of the boundary-value problem

The equilibrium equation (15) is an ordinary differential equation and has a solution as

$$\begin{aligned} w(x) = {} & \mathrm{e}^{\frac{\sqrt{\frac{b\,G-\sqrt{b}\,\sqrt{b\,G^2-4EIk}}{EI}}}{\sqrt{2}}\,x}\,C_1 + \mathrm{e}^{-\frac{\sqrt{\frac{b\,G-\sqrt{b}\,\sqrt{b\,G^2-4EIk}}{EI}}}{\sqrt{2}}\,x}\,C_2 + \\ & + \mathrm{e}^{\frac{\sqrt{\frac{b\,G+\sqrt{b}\,\sqrt{b\,G^2-4EIk}}{EI}}}{\sqrt{2}}\,x}\,C_3 + \mathrm{e}^{-\frac{\sqrt{\frac{b\,G+\sqrt{b}\,\sqrt{b\,G^2-4EIk}}{EI}}}{\sqrt{2}}\,x}\,C_4 + \bar{w}(x)\ , \end{aligned} \tag{16}$$

where C_1, C_2, C_3 and C_4 are constants of integration that depend on the boundary conditions; $\bar{w}(x)$ is a particular solution which is determined by the loading.

The solution (16) describes the deflection curve of a continuous region of the beam. The constants of integration provide continuity of the deformations and equilibrium of the beam. The existence of multiple loads requires dividing the beam into continuous regions. At the ends of these regions the continuity and the equilibrium conditions must be satisfied. These conditions will provide the data to obtain the constants of integration for each of these regions.

From a mathematical point of view it is not a problem to solve a *system of ordinary differential equations* each corresponding to the one of the continuous regions. The application of the boundary, the equilibrium and the continuity conditions gives a system of linear equations with the constants of integration as unknowns. This method leads to the need to solve a large system of equations in the presence of multiple loads. The disadvantage of this method is complicated mathematics, but the method allows to solve problems with structural discontinuities such as: intermediate supports or hinges, different beam cross-sections or subgrade moduli, etc.

The *method of initial parameters* presented in [14] avoids solving big systems of equations to obtain the constants of integration. The method assumes as knowns two of the boundary conditions at the left end of the beam. Then we proceed to the next loading point. This load affects the solution for the next point. Adding the influence of the load to the general solution gives the deflected curve between the considered load and the next load. This method leads to solving a system of only two equations [19].

The above mentioned methods lead to a solution of a system of equations to obtain the constants of integration. The application of these methods for practical problems gives a complicated solution.

The *method of superposition* presented in [11] avoids these complications. The method uses solutions of simple problems of infinitely long beams with different simple loads to construct the final solution of an arbitrary beam, loads and supports.

In [27] the authors consider the generalized solutions of Euler-Bernoulli and Timoshenko beams with jump discontinuities on an elastic foundation. This method gives the theory to obtain the solutions of many complicated problems with structural discontinuities such as intermediate supports or hinges, different beam cross-sections or subgrade moduli, etc.

A *Laplace transformation* is used in [3] for a solution of the equilibrium equation. The author used it together with the resources of the Maple mathematical computer program which lead to a simplified solution of the boundary-value problem.

The *approximated and the numerical methods* such as Ritz and Galerkin methods, finite differences, finite elements and differential quadratures methods are widely used for solving the basic equations by many scientists [21], [20], [16], [13], [6]. All of these methods are superior in comparison with the analytical methods for solving complex problems with various boundary conditions or loads. The numerical methods also have disadvantages such as: it is difficult to study the influence of the problem's parameters on the solution; a special computer program is needed to obtain a solution of the problem. A few commercial finite elements programs can be found with an implemented element for a beam on an elastic foundation. The implementation of a special element or the development of a computer code requires particular skills from the user.

5. Transformation by singularity functions

The previously described analytical solutions involve a lot of complications due to complex loading. A significant decrease of the mathematical work can be achieved by a presentation of the loading as a sum of singularity functions. Thus the loading is presented by a single function and the functions of the internal forces and deflections can be evaluated by an integration of the load function. Hence the division of the structure into separate regions can be omitted. The idea of this method is presented by Clebsch [7] and made popular by Macaulay [15]. This method is briefly discussed in the undergraduate class of Strength of materials in engineering education [10].

The representation of the load by a single function can be performed by the following discontinuous and singularity functions:

– Discontinuous functions – represent distributed loads

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{when} \quad x < a \\ (x-a)^n & \text{when} \quad x \geq a \end{cases} \qquad n = 0, 1, 2, \dots\ , \tag{17}$$

where a is the distance along the x axis to the beginning of the discontinuous function. When $n = 0$ the function represents an uniform load, when $n = 1$ it is a linear load and so on. The particular case of an unit step function when $n = 0$ is known as a Heaviside function and it is denoted by $H(x-a)$.

– Singularity functions – represent forces and couples

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{when} \quad x \neq a \\ \pm\infty & \text{when} \quad x = a \end{cases} \qquad n = -1, -2, -3, \dots\ . \tag{18}$$

The case of $n = -1$ is a Dirac delta function and it is denoted by $\delta(x-a)$.

The discontinuous functions obey the same rules for integration and differentiation as ordinary functions

$$\int \langle x-a\rangle^n \,\mathrm{d}x = \frac{1}{1+n}\,\langle x-a\rangle^{n+1} \quad \text{for} \quad n \geq 0 \;, \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle x-a\rangle^n = n\,\langle x-a\rangle^{n-1} \;. \tag{20}$$

Using the above functions a various set of loading can be presented by a single function. The end loads can be applied as natural boundary conditions.

6. Application in computer algebra system

The programs for computational algebra such as *MATLAB*, *Mathcad*, *Maple* and *Mathematica* are high-level language program systems with built-in functions for algebraic computations including manipulations with discontinuous and singularity functions.

The suggested approach for an analytical solution is a combination of the transformation of the loading as singularity functions with the resources of the modern computer algebra systems for a solution of ordinary differential equations. The procedure for structural analysis of a beam on an elastic foundation is as follows:

- Transformation of the loads as a sum of discontinuous and singularity functions;
- Computation of the general solution of the given boundary-value problem;
- Definition of the boundary conditions at the beam's ends and calculation of the constants of integration.

6.1. Numerical examples

The presented examples of the solutions of the classical problems show the efficiency of the proposed approach. The solutions of the problems are obtained by using the computer algebra system "Mathematica".

6.1.1. Two-parameter beam on elastic foundation

The considered problem is given in [23] and the subgrade moduli are obtained by the modified Vlasov method as $k = 4803.14\,\mathrm{kN/m^3}$ and $G = 13032.44\,\mathrm{kN/m}$. The beam is 20 m long, 0.5 m wide and has bending stiffness of $E\,I = 1\,125\,000\,\mathrm{kNm^2}$. It is loaded at mid-span with a force of $F = 1000\,\mathrm{kN/m}$.

The load is presented by a singularity function as

$$q(x) = F\,\delta(x-a) \;, \tag{21}$$

where a is the distance from the force F to the left end of the beam. The equilibrium equation is defined as

$$E\,I\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - G\,b\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k\,b\,w = b\,F\,\delta(x-a) \;. \tag{22}$$

The natural (force) boundary conditions used for the calculation of the constants of integration are

$$
\begin{aligned}
-E\,I\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(0) &= 0\ ,\\
-E\,I\,\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}(0) + G\,b\,\frac{\mathrm{d}w}{\mathrm{d}x}(0) &= 0\ ,\\
-E\,I\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(\ell) &= 0\ ,\\
-E\,I\,\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}(\ell) + G\,b\,\frac{\mathrm{d}w}{\mathrm{d}x}(\ell) &= 0\ .
\end{aligned}
\tag{23}
$$

According to the basic equations and the classical beam theory the functions of the internal forces and the subgrade reaction are

$$
\begin{aligned}
&R(x) = k\,b\,w(x) &&\rightarrow \quad \text{Subgrade reaction,}\\
&S(x) = G\,b\,\frac{\mathrm{d}w(x)}{\mathrm{d}x} &&\rightarrow \quad \text{Shear force at soil bed,}\\
&M(x) = -E\,I\,\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} &&\rightarrow \quad \text{Beam's bending moment,}\\
&V(x) = -E\,I\,\frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} &&\rightarrow \quad \text{Beam's shear force.}
\end{aligned}
\tag{24}
$$

The command line for the solution of the differential equation with application of the boundary conditions is

```
DSolve[{1125000*w''''[x]-6516.22*w''[x]+2401.57*w[x]==500*DiracDelta[x-10],
      -1125000*w''[0]==0, -1125000*w'''[0]+6516.22*w'[0]==0,
      -1125000*w''[20]==0, -1125000*w'''[20]+6516.22*w'[20]==0}, w[x], x].
```

The functions of the internal forces can be obtained from (24). The diagrams of the subgrade reaction and the internal forces are shown at figs. 2, 3 and 4.

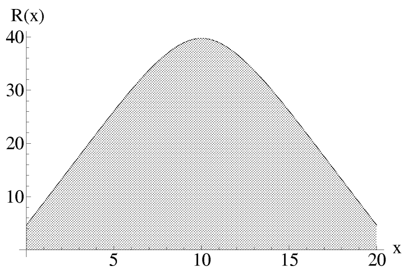

Fig.2: Subgrade reaction, kN/m

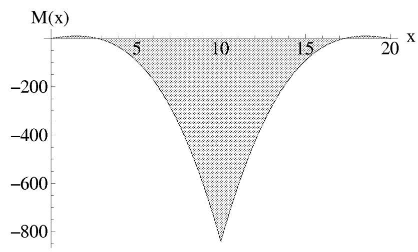

Fig.3: Beam's bending moment, kNm

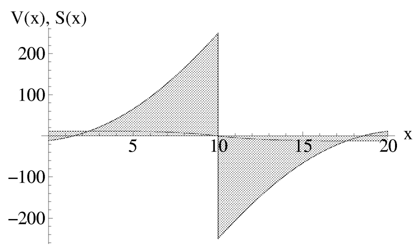

Fig.4: Beam's shear force and shear force at soil bed, kN

6.1.2. Beam on Winkler foundation

The example shows a classical solution of a beam on a Winkler foundation with various loading ([11], pp. 47). The beam has a rectangular cross-section with dimentions of 10/8 in. The beam's material has a modulus of elasticity $E = 15\times10^5$ lbs/in^2. The foundation parameter is $k = 200$ lbs/in^3. The beam's dimensions and loading are shown at fig. 5.

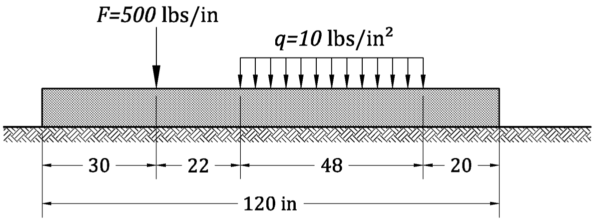

Fig.5: Beam loading

The loads can be presented by singularity functions as

$$q(x) = F\,\delta(x-a) + q\,H(x-b) - q\,H(x-c)\ , \tag{25}$$

where a is the distance from the force F to the left end of the beam; b is the distance from the beam's left end to the beginning of the uniform load q; c is the distance from the beam's left end to the uniform load's end. The differential equation can be obtained from (15) by applying $G = 0$

$$E\,I\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + k\,b\,w = b\,(F\,\delta(x-a) + q\,H(x-b) - q\,H(x-c))\ . \tag{26}$$

The force boundary conditions and the internal forces are the same as in the previous example.

The computer algebra system script is

```
DSolve[{640000000*w''''[x]+2000*w[x]==5000*DiracDelta[x-30]
       +100*(HeavisideTheta[x-52])-100*(HeavisideTheta[x-100]),
       -640000000*w''[0]==0, -640000000*w'''[0]==0,
       -640000000*w''[120]==0, -640000000*w'''[120]==0}, w[x], x].
```

The bending moments and the shear forces are shown at figs. 6 and 7.

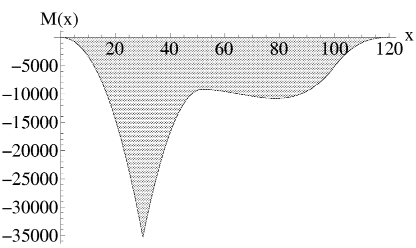

Fig.6: Beam's bending moments, kNm

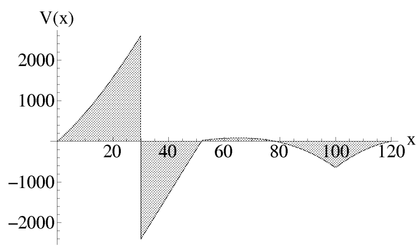

Fig.7: Beam's shear forces, kN

7. Laterally loaded pile

The structural analysis of vertical piles with lateral loads gives almost the same boundary-value problem as a beam on an elastic foundation. The differences are in the loading which is concentrated at the pile's top and in the elastic foundation model. The pile foundation model includes variable spring stiffness while the beam model has constant stiffness of the bed's springs. The difficulties that are involved in solving a differential equation with variable coefficients require the application of numerical methods such as finite differences or finite elements [5].

The following example is solved in [5] on pp. 953 using the finite elements method. The pile is a steel section HP 360×174, 378 mm wide and 19 m long and has bending stiffness of $EI = 101600\,\mathrm{kNm}^2$. The pile is loaded with a lateral force $F = 50.78\,\mathrm{kN}$ at the top and it is assumed that the top end is fixed in the pile cap. The soil stiffness variation is assumed as $k = 200 + 50\,x^{1/2}\,\mathrm{kN/m}^3$. The equilibrium equation is

$$E\,I\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + (200 + 50\,x^{1/2})\,b\,w = 0\;. \tag{27}$$

The pile top has mixed boundary conditions and the bottom has force boundary conditions

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}x}(0) &= 0\;,\\ -E\,I\,\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}(0) &= -F\;,\\ -E\,I\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(\ell) &= 0\;,\\ -E\,I\,\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}(\ell) &= 0\;. \end{aligned} \tag{28}$$

The command line for the solution of the above problem is

```
NDSolve[{101600*w''''[x]+(75.6+18.9*x^(1/2))*w[x]==0, w'[0]==0,
        -101600*w'''[0]==-50.78, -101600*w''[19]==0, -101600*w'''[19]==0},
        w[x], {x, 0, 19}].
```

The figs. 8, 9 and 10 show the displacement, bending moments and subgrade reaction diagrams.

The following results are calculated at the top of the pile : displacement – $w = 0.0622\,\mathrm{m}$; bending moment – $M = 208.152\,\mathrm{kNm}$ and soil pressure – $R = 12.4466\,\mathrm{kPa}$, which fully correspond to those obtained in [5].

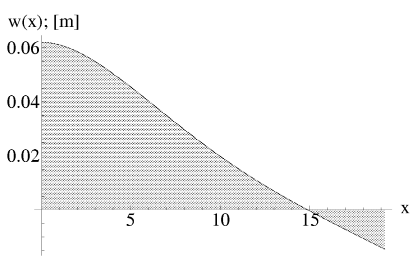

Fig.8: Pile displacements

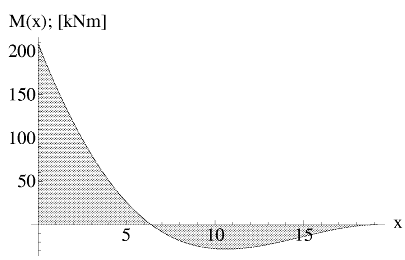

Fig.9: Bending moments

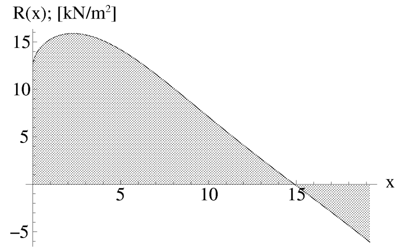

Fig.10: Soil pressure

The presented examples are also solved by commercial finite element computer programs SAP2000 and ANSYS. The finite element method is based on approximation. When the finite element mesh is refined the obtained results are sufficiently close to the results obtained by using the analytical solution. Even with approximated results, the finite element solutions have a lot of advantages as non-linear modelling of the soil medium, presentation of jump discontinuities in foundation and beam stiffness, spatial description of the soil-structure interaction problem and etc.

8. Conclusions

The presented examples show some of the advantages of the suggested approach for an analytical solution of a beam on an elastic foundation. It gives opportunities for :

- Application of various loads at an arbitrary point or a region on the beam;
- Application of force or displacement boundary conditions at the beam's ends;
- The approach can be performed on a short, a medium or a long beam;
- The beam and the soil medium stiffness can vary smoothly along the beam's length;
- The implementation of the approach into a computer algebra system is simple and do not require special programing skills from the user;
- The presented solution can be implemented in every mathematical system that can solve ordinary differential equations;
- The obtained analytical solution can be used to analyze the influence of the different problem parameters on the structural behaviour.

The presented approach has also some disadvantages such as :

- The application of various loads and complex variations of the beam and the foundation stiffness makes it difficult to obtain a computer solution;
- The structural discontinuities such as jumps in beam or foundation stiffness, hinges and etc. can not be applied;
- It is not possible to assign displacement values to points along the beam's length.

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