

Chapter 3

Analytical Tomographic Image Reconstruction Methods

ch, tomo

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s,tomo,intro

3.1 Introduction (s,tomo,intro)

The primary focus of this book is on statistical methods for tomographic image reconstruction using reasonably realistic physical models. Nevertheless, analytical image reconstruction methods, even though based on somewhat unrealistic simplified models, are important when computation time is so limited that an approximate solution is tolerable. Analytical methods are also useful for developing intuition, and for initializing iterative algorithms associated with statistical reconstruction methods. This chapter¹ reviews classical analytical tomographic reconstruction methods. (Other names are **Fourier reconstruction methods** and **direct reconstruction methods**, because these methods are noniterative.) Entire books have been devoted to this subject [2–6], whereas this chapter highlights only a few results. Many readers will be familiar with much of this material except perhaps for the angularly weighted backprojection that is described in §3.3. This weighted backprojector is introduced here to facilitate analysis of **weighted least-squares (WLS)** formulations in Chapter 4.

There are several limitations of analytical reconstruction methods that impair their performance. Analytical methods generally ignore measurement noise in the problem formulation and treat noise-related problems as an “afterthought” by post-filtering operations. Analytical formulations usually assume continuous measurements and provide integral-form solutions. Sampling issues are treated by discretizing these solutions “after the fact.” Analytical methods require certain standard geometries (*e.g.*, parallel rays and complete sampling in radial and angular coordinates). Statistical methods for image reconstruction can overcome all of these limitations.

s,tomo,radon

3.2 Radon transform in 2D (s,tomo,radon)

The foundation of analytical reconstruction methods is the **Radon transform** that relates a 2D function $f(x, y)$ to the collection of **line integrals** of that function [7–9]. (We focus initially on the 2D case.) Emission and transmission tomography systems acquire measurements that are something like blurred line integrals, so the line-integral model represents an idealization of such systems. Fig. 3.2.1 illustrates the geometry of the line integrals associated with the (ideal) 2D Radon transform.

3.2.1 Definition

Let $\mathcal{L}(r, \varphi)$ denote the line in the Euclidean plane at angle φ counter-clockwise from the y axis and at a signed distance r from the origin:

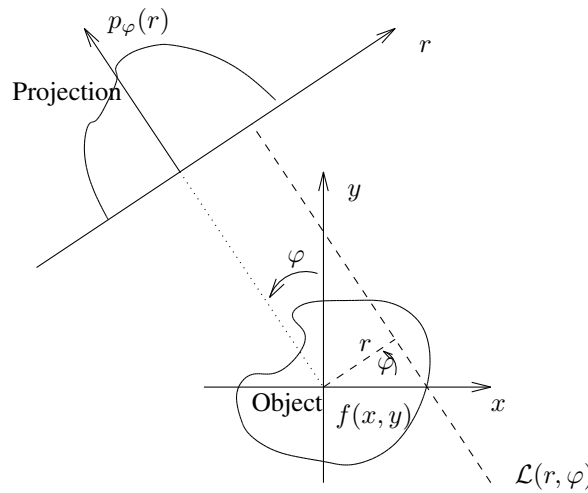
$$\mathcal{L}(r, \varphi) = \{(x, y) \in \mathbb{R}^2 : x \cos \varphi + y \sin \varphi = r\} \quad (3.2.1)$$

$$= \{(x, y) \in \mathbb{R}^2 : (x, y) \cdot (\cos \varphi, \sin \varphi) = r\} \quad (3.2.2)$$

$$= \{(r \cos \varphi - \ell \sin \varphi, r \sin \varphi + \ell \cos \varphi) : \ell \in \mathbb{R}\}. \quad (3.2.3)$$

e,tomo,ray

¹Substantial portions of this chapter appeared in [1].



fig,tomo,geom

Figure 3.2.1: Geometry of the line integrals associated with the Radon transform.

Let $p_\varphi(r)$ denote the line integral through $f(x, y)$ along the line $\mathcal{L}(r, \varphi)$. There are several equivalent ways to express this line integral, each of which has its uses:

$$p_\varphi(r) = \int_{\mathcal{L}(r, \varphi)} f(x, y) \, d\ell \tag{3.2.4}$$

$$= \int_{-\infty}^{\infty} f(r \cos \varphi - \ell \sin \varphi, r \sin \varphi + \ell \cos \varphi) \, d\ell \tag{3.2.4}$$

e,tomo,line,l

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r' \cos \varphi - \ell \sin \varphi, r' \sin \varphi + \ell \cos \varphi) \delta(r' - r) \, dr' \, d\ell \tag{3.2.5}$$

e,tomo,radon,r',l

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \varphi + y \sin \varphi - r) \, dx \, dy \tag{3.2.6}$$

e,tomo,line,dirac

$$= \begin{cases} \frac{1}{|\cos \varphi|} \int_{-\infty}^{\infty} f\left(\frac{r - t \sin \varphi}{\cos \varphi}, t\right) \, dt, & \cos \varphi \neq 0 \\ \frac{1}{|\sin \varphi|} \int_{-\infty}^{\infty} f\left(t, \frac{r - t \cos \varphi}{\sin \varphi}\right) \, dt, & \sin \varphi \neq 0, \end{cases} \tag{3.2.7}$$

e,tomo,line,xy

where $\delta(\cdot)$ denotes the 1D **Dirac impulse**. (The last form came from [10].) The step between (3.2.5) and (3.2.6) uses the following change of variables:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} r' \\ \ell \end{bmatrix}. \tag{3.2.8}$$

e,tomo,radon,change

The **Radon transform** of f is the complete collection of line integrals²

$$f \xleftrightarrow{\text{Radon}} \{p_\varphi(r) : \varphi \in [0, \pi], r \in (-\infty, \infty)\}. \tag{3.2.9}$$

e,tomo,radon

The function $p_\varphi(\cdot)$ is called the **projection**³ of f at angle φ . As discussed in §4.2, we often write $p = \mathcal{P} f$.

In its most idealized form, the **2D image reconstruction problem** is to recover $f(x, y)$ from its projections $\{p_\varphi(\cdot)\}$. To do this one must somehow return the data in projection space back to object space, as described in §3.4.

x,tomo,proj,disk

Example 3.2.1 Consider the centered **uniform disk** object with radius r_0 :

$$f(x, y) = \alpha \text{rect}\left(\frac{r}{2r_0}\right), \quad \text{rect}(t) \triangleq \mathbb{I}_{\{|t| \leq 1/2\}} = \begin{cases} 1, & |t| \leq 1/2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.10}$$

e,rect

²Sometimes one refers to values of φ outside of the domain given in (3.2.9); this is possible using the “periodic extension” described in (3.2.29). Of course a practical system has a finite maximum radius that defines its circular **field of view**.

³The term “projection” also has a meaning in the context of convex sets in Hilbert spaces §29.9.2. The two uses of “projection” can be reconciled; see Problem 4.1.

Using (3.2.4), the Radon transform of this object is:

$$p_\varphi(r) = \int_{-\infty}^{\infty} f(r \cos\varphi - \ell \sin\varphi, r \sin\varphi + \ell \cos\varphi) d\ell \quad (3.2.11)$$

$$= \int_{-\infty}^{\infty} \alpha \operatorname{rect}\left(\frac{\sqrt{(r \cos\varphi - \ell \sin\varphi)^2 + (r \sin\varphi + \ell \cos\varphi)^2}}{2r_0}\right) d\ell \quad (3.2.12)$$

$$= \alpha \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{\sqrt{r^2 + \ell^2}}{2r_0}\right) d\ell = \alpha \int_{\{\ell : r^2 + \ell^2 \leq r_0^2\}} d\ell = \alpha \int_{-\sqrt{r_0^2 - r^2}}^{+\sqrt{r_0^2 - r^2}} d\ell \quad (3.2.13)$$

$$= 2\alpha\sqrt{r_0^2 - r^2} \operatorname{rect}\left(\frac{r}{2r_0}\right), \quad (3.2.14)$$

e, tomo, proj, disk

which is a semi-circle function as shown in Fig. 3.2.2. These projections are independent of φ due to the circular symmetry of $f(x, y)$.

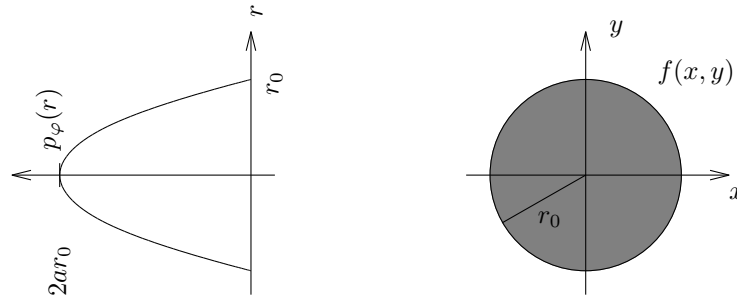


Figure 3.2.2: Projection of a centered uniform disk object, illustrated at $\varphi = \pi/2$.

fig, tomo, disk

3.2.2 Signed polar forms (s,tomo,radon,polar)

It can be useful to have a form of the Radon transform when f is represented in a polar form. Throughout this chapter, we use a “signed polar form” $f_\circ(r, \varphi) = f(r \cos\varphi, r \sin\varphi)$, in which the radial argument r can be both positive and negative. Usually we abuse notation slightly and write $f(r, \varphi)$ without the subscript.

For making changes of variables between Cartesian coordinates and signed polar coordinates, we define

$$r_\pm(a, b) \triangleq \begin{cases} \sqrt{a^2 + b^2}, & \{b > 0\} \text{ or } \{b = 0 \ \& \ a \geq 0\} \\ -\sqrt{a^2 + b^2}, & \{b < 0\} \text{ or } \{b = 0 \ \& \ a < 0\} \end{cases} \quad (3.2.15)$$

e, tomo, srad

$$\angle_\pi(a, b) \triangleq \begin{cases} \tan^{-1}\left(\frac{b}{a}\right), & ab > 0 \\ 0, & b = 0 \\ \pi/2, & a = 0, b \neq 0 \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi, & ab < 0. \end{cases} \quad (3.2.16)$$

e, tomo, angpi

These functions obey the following natural properties:

$$\angle_\pi(a, b) \in [0, \pi) \quad (3.2.17)$$

$$\angle_\pi(b, a) = \begin{cases} 0, & a = 0 \ \& \ b = 0 \\ (\pi/2 - \angle_\pi(a, b)) \bmod \pi, & \text{else} \end{cases} \quad (3.2.18)$$

$$|r_\pm(a, b)| = \sqrt{a^2 + b^2} \quad (3.2.19)$$

$$r_\pm(\alpha a, \alpha b) = \alpha r_\pm(a, b) \quad (3.2.20)$$

$$\angle_\pi(\alpha a, \alpha b) = \angle_\pi(a, b), \quad \alpha \neq 0 \quad (3.2.21)$$

$$\cos \angle_\pi(a, b) = \begin{cases} 1, & b = 0 \\ a \operatorname{sgn}(b) / \sqrt{a^2 + b^2}, & b \neq 0 \end{cases} \quad (3.2.22)$$

$$\sin \angle_\pi(a, b) = \begin{cases} 0, & b = 0 \\ |b| / \sqrt{a^2 + b^2}, & b \neq 0 \end{cases} \quad (3.2.23)$$

$$r_{\pm}(a, b) \cos \angle_{\pi}(a, b) = a \quad (3.2.24)$$

$$r_{\pm}(a, b) \sin \angle_{\pi}(a, b) = b. \quad (3.2.25)$$

Making a change of variables $r = r_{\pm}(x, y)$ and $\varphi = \angle_{\pi}(x, y)$ leads to the following integral relationship:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_0^{\pi} \int_{-\infty}^{\infty} f_{\circ}(r, \varphi) |r| dr d\varphi. \quad (3.2.26)$$

In particular, substituting $r' = r_{\pm}(x, y)$ and $\varphi' = \angle_{\pi}(x, y)$ into the Radon transform expression (3.2.6) leads to the following Radon transform in polar coordinates:

$$p_{\varphi}(r) = \int_0^{\pi} \int_{-\infty}^{\infty} f_{\circ}(r', \varphi') \delta(r' \cos(\varphi - \varphi') - r) |r'| dr' d\varphi' \quad (3.2.27)$$

The properties (3.2.25) arise in several of the subsequent derivations.

3.2.3 Radon transform properties (s,tomo,radon,prop)

The following list shows a few of the many properties of the Radon transform. This list is far from exhaustive; indeed new properties continue to be found, e.g., [11, 12]. Throughout this list, we assume $f(x, y) \stackrel{\text{Radon}}{\leftrightarrow} p_{\varphi}(r)$.

- **Linearity**

If $g(x, y) \stackrel{\text{Radon}}{\leftrightarrow} q_{\varphi}(r)$, then

$$\alpha f + \beta g \stackrel{\text{Radon}}{\leftrightarrow} \alpha p + \beta q.$$

- **Shift / translation**

$$f(x - x_0, y - y_0) \stackrel{\text{Radon}}{\leftrightarrow} p_{\varphi}(r - x_0 \cos \varphi - y_0 \sin \varphi) \quad (3.2.28)$$

- **Rotation**

$$f(x \cos \varphi' + y \sin \varphi', -x \sin \varphi' + y \cos \varphi') \stackrel{\text{Radon}}{\leftrightarrow} p_{\varphi - \varphi'}(r)$$

- **Circular symmetry**

$$f_{\circ}(r, \varphi) = f_{\circ}(r, 0) \quad \forall \varphi \implies p_{\varphi} = p_0 \quad \forall \varphi$$

- **Symmetry/periodicity**

$$p_{\varphi}(r) = p_{\varphi \pm \pi}(-r) = p_{\varphi \pm k\pi}((-1)^k r), \quad \forall k \in \mathbb{Z} \quad (3.2.29)$$

- **Affine scaling**

$$f(\alpha x, \beta y) \stackrel{\text{Radon}}{\leftrightarrow} \frac{p_{\angle_{\pi}(\beta \cos \varphi, \alpha \sin \varphi)} \left(\frac{r|\alpha|\beta}{\sqrt{(\beta \cos \varphi)^2 + (\alpha \sin \varphi)^2}} \right)}{\sqrt{(\beta \cos \varphi)^2 + (\alpha \sin \varphi)^2}}, \quad (3.2.30)$$

for $\alpha, \beta \neq 0$, where r_{\pm} and \angle_{π} were defined in §3.2.2. For a more general **affine skewing** property see [13].

The following two properties are special cases of the affine scaling property.

- **Magnification/minification**

$$f(\alpha x, \alpha y) \stackrel{\text{Radon}}{\leftrightarrow} \frac{1}{|\alpha|} p_{\varphi}(\alpha r), \quad \alpha \neq 0$$

- **Flips**

$$f(x, -y) \stackrel{\text{Radon}}{\leftrightarrow} p_{\pi - \varphi}(-r)$$

$$f(-x, y) \stackrel{\text{Radon}}{\leftrightarrow} p_{\pi - \varphi}(r)$$

- **Laplacian**

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(x, y) \stackrel{\text{Radon}}{\leftrightarrow} \frac{\partial^2}{\partial r^2} p_{\varphi}(r) \quad (3.2.31)$$

(This is a consequence of the **Fourier-slice theorem** (3.2.36) below; see Problem 3.4.)

- **The projection integral theorem**

For a scalar function $h : \mathbb{R} \rightarrow \mathbb{R}$:

$$\int p_\varphi(r) h(r) dr = \int \left(\int f(r \cos\varphi - \ell \sin\varphi, r \sin\varphi + \ell \cos\varphi) d\ell \right) h(r) dr \quad (3.2.32)$$

$$= \iint f(x, y) h(x \cos\varphi + y \sin\varphi) dx dy, \quad (3.2.33)$$

by making the orthonormal coordinate rotation: $x = r \cos\varphi - \ell \sin\varphi$, $y = r \sin\varphi + \ell \cos\varphi$.

- **Volume conservation (DC value)**

$$F(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_{-\infty}^{\infty} p_\varphi(r) dr, \quad \forall \varphi. \quad (3.2.34)$$

This is a corollary to the projection integral theorem for $h(r) = 1$. The volume conservation property is one of many **consistency conditions** of the Radon transform [4].

The following example serves to illustrate some of these properties.

Example 3.2.2 Determine the Radon transform of $f(x, y) = \text{rect}\left(\frac{1}{2}\sqrt{(x/r_x)^2 + (y/r_y)^2}\right)$, an ellipse object centered at the origin having major axes of half lengths r_x and r_y , where the function is unity within the ellipse and zero outside. Using (3.2.14) and the affine scaling property (3.2.30) with $\alpha = 1/r_x$ and $\beta = 1/r_y$:

$$p_\varphi(r) = \frac{r_x r_y}{\sqrt{(r_x \cos\varphi)^2 + (r_y \sin\varphi)^2}} g\left(\frac{r}{\sqrt{(r_x \cos\varphi)^2 + (r_y \sin\varphi)^2}}\right),$$

where $g(t) = 2\sqrt{1-t^2}\mathbb{1}_{\{|t|<1\}}$ denotes the projection of a circle of unity radius. See also Problem 3.33.

MIRT See ellipse_sino.m.

Example 3.2.3 Consider the object $f(x, y) = \delta_2(x - x_0, y - y_0)$, the **2D Dirac impulse** centered at (x_0, y_0) . Informally, we can think of this object as a disk function centered at (x_0, y_0) of radius r_0 and height $1/(\pi r_0^2)$ (so that volume is unity) in the limit as $r_0 \rightarrow 0$.

Let $C_{r_0}(r) = 2\sqrt{r_0^2 - r^2} \text{rect}\left(\frac{r}{2r_0}\right)$ denote the projection of centered uniform disk with radius r_0 as derived in (3.2.14) in Example 3.2.1. Then by the shift property (3.2.28), the projections of a disk centered at (x_0, y_0) are:

$$p_\varphi(r) = C_{r_0}(r - [x_0 \cos\varphi + y_0 \sin\varphi]).$$

(See Fig. 3.2.4 below.) Thus the projections of the 2D Dirac impulse are found as follows:

$$p_\varphi(r) = \frac{1}{\pi r_0^2} C_{r_0}(r - [x_0 \cos\varphi + y_0 \sin\varphi]) \rightarrow \delta(r - [x_0 \cos\varphi + y_0 \sin\varphi]) \text{ as } r_0 \rightarrow 0.$$

An alternative derivation uses (3.2.6). In summary, for a 2D Dirac impulse object located at (x_0, y_0) , the projection at angle φ is a 1D Dirac impulse located at $r = x_0 \cos\varphi + y_0 \sin\varphi$. See Fig. 3.2.3.

3.2.4 Sinogram

Because $p_\varphi(r)$ is a function of two arguments, we can display $p_\varphi(r)$ as a 2D grayscale picture where usually r and φ are the horizontal and vertical axes respectively. If we make such a display of the projections $p_\varphi(r)$ of a 2D Dirac impulse, then the picture looks like a sinusoid corresponding to the function $r = x_0 \cos\varphi + y_0 \sin\varphi$. Hence this 2D function is called a **sinogram** and (when sampled) represents the raw data available for image reconstruction. So the goal of tomographic reconstruction is to estimate the object $f(x, y)$ from a measured sinogram.

Each point (x, y) in object space contributes a unique sinusoid to the sinogram, with the “amplitude” of the sinusoid being $\sqrt{x^2 + y^2}$, the distance of the point from the origin, and the “phase” of the sinusoid depending on $\angle_\pi(x, y)$. A sinogram of an object $f(x, y)$ is the superposition of all of these sinusoids, each one weighted by the value $f(x, y)$. Hence it seems plausible that there could be enough information in the sinogram to recover the object f , if we can unscramble all of those sinusoids.

Example 3.2.4 Fig. 3.2.3 illustrates these concepts for the object $f(x, y) = \delta_2(x, y) + \delta_2(x - 1, y) + \delta_2(x - 1, y - 1)$ with corresponding projections $p_\varphi(r) = \delta(r) + \delta(r - \cos\varphi) + \delta(r - \cos\varphi - \sin\varphi)$.

Example 3.2.5 Fig. 3.2.4 shows the sinogram for a disk of radius $r_0 = 20$ centered at position $(x_0, y_0) = (40, 0)$.

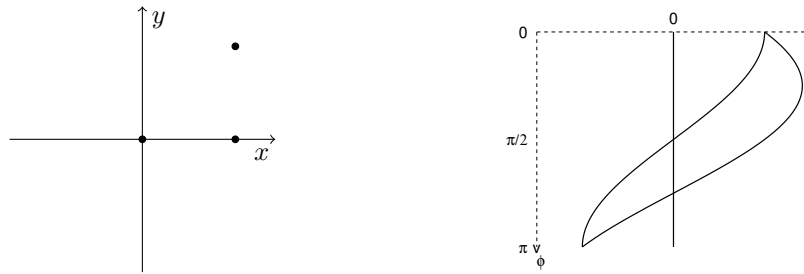


Figure 3.2.3: Left: cross-section of 2D object containing three Dirac impulses. Right: the corresponding sinogram, consisting of three sinusoidal impulse ridges. fig_tomo_sino_points

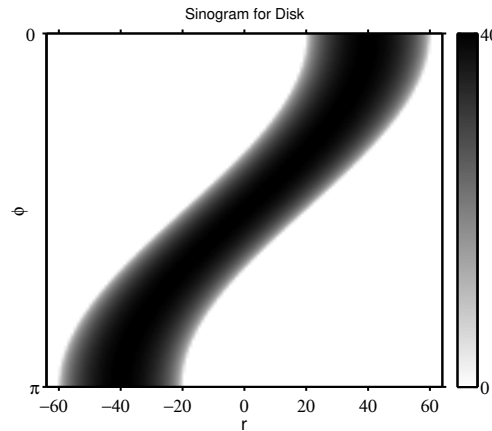


Figure 3.2.4: Sinogram for a disk object of radius $r_0 = 20$ centered at $(x_0, y_0) = (40, 0)$. fig_tomo_disk_sino

s, tomo, radon, fst **3.2.5 Fourier-slice theorem** (s,tomo,radon,fst)

The most important corollary of the projection-integral theorem (3.2.33) is the **Fourier-slice theorem**, also known as the **central-slice theorem** or **central-section theorem** or **projection-slice theorem**. In words, the statement of this theorem is as follows⁴. If $p_\varphi(r)$ denotes the Radon transform of $f(x, y)$, then the 1D Fourier transform of $p_\varphi(\cdot)$ equals the slice at angle φ through the 2D Fourier transform of $f(x, y)$.

Let $P_\varphi(\nu)$ denote the 1D Fourier transform⁵ of $p_\varphi(r)$, *i.e.*,

$$P_\varphi(\nu) = \int_{-\infty}^{\infty} p_\varphi(r) e^{-i2\pi\nu r} dr.$$

Let $F(u, v)$ denote the 2D Fourier transform of $f(x, y)$, *i.e.*,

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(ux+vy)} dx dy. \tag{3.2.35}$$

Then in mathematical notation, the **Fourier-slice theorem** is simply:

$$P_\varphi(\nu) = F(\nu \cos\varphi, \nu \sin\varphi) = F_\circ(\nu, \varphi), \quad \forall \nu \in \mathbb{R}, \quad \forall \varphi \in \mathbb{R}, \tag{3.2.36}$$

where $F_\circ(\rho, \Phi) = F(\rho \cos\Phi, \rho \sin\Phi)$ denotes the polar form of $F(u, v)$. (Again we will frequently recycle notation and omit the subscript.) The proof of the Fourier-slice theorem is remarkably simple: merely set $h(r) = \exp(-i2\pi\nu r)$ in the projection-integral theorem (3.2.33).

It follows immediately from the Fourier-slice theorem that the Radon transform (3.2.9) describes completely any (Fourier transformable) object $f(x, y)$, because there is a one-to-one correspondence between the Radon transform

⁴Apparently the first publication of this result was in Bracewell's 1956 paper [14]. However, at a symposium on 2004-7-17 held at Stanford University to celebrate the 75th birthday of Albert Macovski, Ron Bracewell stated that he believed that the theorem was "well known" to other radio astronomers at the time.

⁵Being an engineer, I simply assume existence of the Fourier transforms of all functions of interest here.

and the 2D Fourier transform $F(u, v)$, and from $F(u, v)$ we can recover $f(x, y)$ by an inverse 2D Fourier transform. (See §3.4.1.)

Example 3.2.6 For the circularly symmetric Bessel object $f(x, y) = f(r) = (\pi/2)J_0(\pi r)$, from a table of Hankel transforms $F(\rho) = \frac{1}{2} \delta(|\rho| - \frac{1}{2})$. (So $F(\rho)$ is an impulse-ring of radius $1/2$.) Thus $P_\varphi(\nu) = F(\nu) = \frac{1}{2} \delta(|\nu| - \frac{1}{2}) = \frac{1}{2} \delta(\nu - \frac{1}{2}) + \frac{1}{2} \delta(\nu + \frac{1}{2})$, so $p_\varphi(r) = \cos(\pi r)$. So the projections of Bessel objects are sinusoids.

Example 3.2.7 The 2D the uniform rectangle object and its Fourier transform are

$$f(x, y) = \text{rect}\left(\frac{x}{a}\right) \text{rect}\left(\frac{y}{b}\right) \xrightarrow{2\text{D FT}} F(u, v) = a \text{sinc}(au) b \text{sinc}(bv),$$

so in polar form: $F_\circ(\rho, \Phi) = a \text{sinc}(a \rho \cos \Phi) b \text{sinc}(b \rho \sin \Phi)$. By the Fourier slice theorem, the 1D **Fourier transform (FT)** of its projections are given by

$$P_\varphi(\nu) = F_\circ(\nu, \varphi) = a \text{sinc}(\nu a \cos \varphi) b \text{sinc}(\nu b \sin \varphi). \quad (3.2.37)$$

Thus, by the **convolution property** of the FT (29.2.3), each projection is the convolution of two rect functions:

$$p_\varphi(r) = \frac{1}{|\cos \varphi|} \text{rect}\left(\frac{r}{a \cos \varphi}\right) * \frac{1}{|\sin \varphi|} \text{rect}\left(\frac{r}{b \sin \varphi}\right), \quad (3.2.38)$$

where “*” denotes 1D convolution with respect to r . This is a **trapezoid** in general, as illustrated Fig. 3.2.5. Specifically, defining a generic trapezoid by

$$\text{trap}(t; \tau_1, \tau_2, \tau_3, \tau_4) \triangleq \begin{cases} \frac{t-\tau_1}{\tau_2-\tau_1} & \tau_1 < t < \tau_2 \\ 1 & \tau_2 \leq t \leq \tau_3 \\ \frac{\tau_4-t}{\tau_4-\tau_3} & \tau_3 < t < \tau_4 \\ 0, & \text{otherwise,} \end{cases} \quad (3.2.39)$$

the Radon transform of a rectangle object are given by

$$p_\varphi(r) = l_{\max}(\varphi) \text{trap}(r; -d_{\max}(\varphi), -d_{\text{break}}(\varphi), d_{\text{break}}(\varphi), d_{\max}(\varphi)) \quad (3.2.40)$$

$$= \begin{cases} \sqrt{a^2 + b^2} \text{tri}\left(\frac{r}{ab/\sqrt{a^2+b^2}}\right), & |a \cos \varphi| = |b \sin \varphi| \\ b \text{rect}\left(\frac{r}{a}\right), & \varphi = 0, \pm\pi, \dots \\ a \text{rect}\left(\frac{r}{b}\right), & \varphi = \pm\pi/2, \pm 3\pi/2, \dots \\ \frac{1}{|\cos \varphi \sin \varphi|} \left[d_{\max}(\varphi) \text{tri}\left(\frac{r}{d_{\max}(\varphi)}\right) - d_{\text{break}}(\varphi) \text{tri}\left(\frac{r}{d_{\text{break}}(\varphi)}\right) \right], & \text{otherwise,} \end{cases} \quad (3.2.41)$$

where the **unit triangle function** is defined by

$$\text{tri}(x) = \text{trap}(x; -1, 0, 0, 1) = (1 - |x|) \text{rect}\left(\frac{x}{2}\right) = (1 - |x|) \mathbb{I}_{\{|x| < 1\}}, \quad (3.2.42)$$

and we define

$$d_{\max}(\varphi) = \frac{|a \cos \varphi| + |b \sin \varphi|}{2} \quad (3.2.43)$$

$$d_{\text{break}}(\varphi) = \frac{||a \cos \varphi| - |b \sin \varphi||}{2} \quad (3.2.44)$$

$$l_{\max}(\varphi) = \frac{|ab|}{\max(|a \cos \varphi|, |b \sin \varphi|)}. \quad (3.2.45)$$

At angles φ that are multiples of $\pi/2$, the trapezoid degenerates to a rectangle, and at angles where $|a \cos \varphi| = |b \sin \varphi|$ the trapezoid degenerates to a triangle.

MIRT See `rect_sino.m`.

Example 3.2.8 The 2D FT of a uniform disk object $f(x, y) = \text{rect}\left(\frac{r}{2r_0}\right)$ is $F(\rho) = r_0^2 \text{jinc}(r_0 \rho)$.

Thus $P_\varphi(\nu) = r_0^2 \text{jinc}(r_0 \nu) = r_0^2 \frac{J_1(\pi r_0 \nu)}{2r_0 \nu}$, where J_1 denotes the 1st-order **Bessel function** of the first kind. Because $J_1(2\pi\nu)/(2\nu)$ and $\sqrt{1-t^2} \text{rect}(t/2)$ are 1D Fourier transform pairs [15, p. 337], we see that the projections of a uniform disk are given by $p_\varphi(r) = 2\sqrt{r_0^2 - r^2} \text{rect}\left(\frac{r}{2r_0}\right)$. This agrees with the result shown in (3.2.14) by integration.

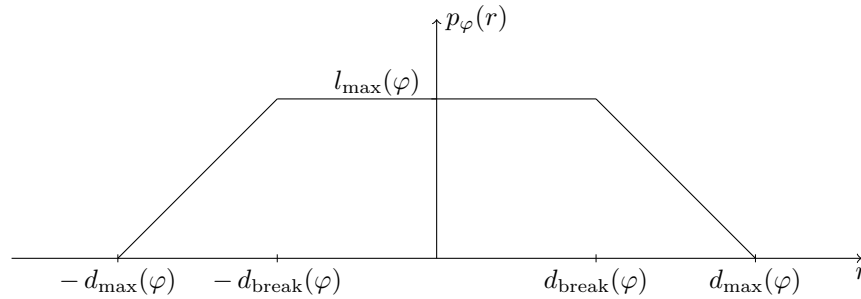


Figure 3.2.5: The trapezoidal projection at angle φ of a rectangular object.

fig,tomo,trap

Example 3.2.9 Consider the 2D gaussian object $f(x, y) = f(r) = \frac{1}{w^2} \exp(-\pi (r/w)^2)$, with corresponding 2D FT $F(\rho) = \exp(-\pi (w\rho)^2)$. By the Fourier-slice theorem: $P_\varphi(\nu) = \exp(-\pi (w\nu)^2)$, the inverse 1D Fourier transform of which is $p_\varphi(r) = \frac{1}{w} \exp(-\pi (r/w)^2)$. (Note the slight change in the leading constant.) Thus the projections of a gaussian object are gaussian, which is a particularly important relationship. This property is related to the fact that two jointly gaussian random variables have gaussian marginal distributions.

The following corollary follows directly from the Fourier-slice theorem.

Corollary 3.2.10 (Convolution property.) If $f \xleftrightarrow{\text{Radon}} p$ and $g \xleftrightarrow{\text{Radon}} q$ then

$$f(x, y) ** g(x, y) \xleftrightarrow{\text{Radon}} p_\varphi(r) * q_\varphi(r). \tag{3.2.46}$$

Example 3.2.11 In particular, it follows from Example 3.2.9 that 2D gaussian smoothing of an object is equivalent to 1D radial gaussian smoothing of each projection⁶:

$$f(x, y) ** \frac{1}{w^2} e^{-\pi (r/w)^2} \xleftrightarrow{\text{Radon}} p_\varphi(r) * \frac{1}{w} e^{-\pi (r/w)^2}.$$

3.3 Backprojection (s,tomo,back)

The Radon transform maps a 2D object $f(x, y)$ into a sinogram $p_\varphi(r)$ consisting of line integrals through the object. One approach to try to recover the object from $p_\varphi(r)$ would be to take each sinogram value and “smear” it back into object space along the corresponding ray, as illustrated in Fig. 3.3.1. (Early versions used film exposure summation for this operation [16].) This type of operation is called **backprojection** and is fundamental to tomographic image reconstruction. Unfortunately in its simplest form this procedure does not recover the object $f(x, y)$, but instead yields a blurred version of the object $f_b(x, y)$. This blurred version $f_b(x, y)$ is called a **laminogram** or **layergram** [17].

Recall from Example 3.2.3 that the projection of an impulse object centered at (x_0, y_0) is the “sinusoidal impulse” along $r = x_0 \cos\varphi + y_0 \sin\varphi$. Because each object point (x_0, y_0) contributes its own sinusoid to the sinogram, it is natural to “sum along the sinusoid” to attempt to find $f(x_0, y_0)$. (There are analogous image formation methods in other modalities such as ultrasound beamforming by delay and sum.)

When the sinogram of an asymmetric object is corrupted by noise, it is conceivable that different views will have different signal to noise ratios, so it may be useful to weight the views accordingly⁷ while “summing along the sinusoid.” Therefore, we analyze the following **angularly-weighted backprojection** operation:

$$f_b(x, y) = \int_0^\pi w(\varphi) p_\varphi(x \cos\varphi + y \sin\varphi) d\varphi, \tag{3.3.1}$$

where $w(\varphi)$ denotes the user-chosen weight for angle φ . In the usual case where $w(\varphi) = 1$, this operation is the **adjoint** of the Radon transform (see §3.4.4).

⁶Expressions of the form $f(x, y) ** h(r)$ should be interpreted as 2D convolution in Cartesian coordinates as follows: $g(x, y) = f(x, y) ** h(r) = \iint f(x-s, y-t) h(\sqrt{s^2+t^2}) ds dt$.

⁷It could also be useful to weight each ray differently, but such weighting is more difficult to analyze. Most readers should probably consider $w(\varphi) = 1$ on a first pass anyway. See [18] for related analysis of tomography with arbitrary view angles and view-dependent filters. See [19] for a noise-weighted FBP algorithm.

x,tomo,slice,gauss

c,tomo,radon,conv

e,tomo,radon,conv

x,tomo,radon,conv

s,tomo,back

e,tomo,back

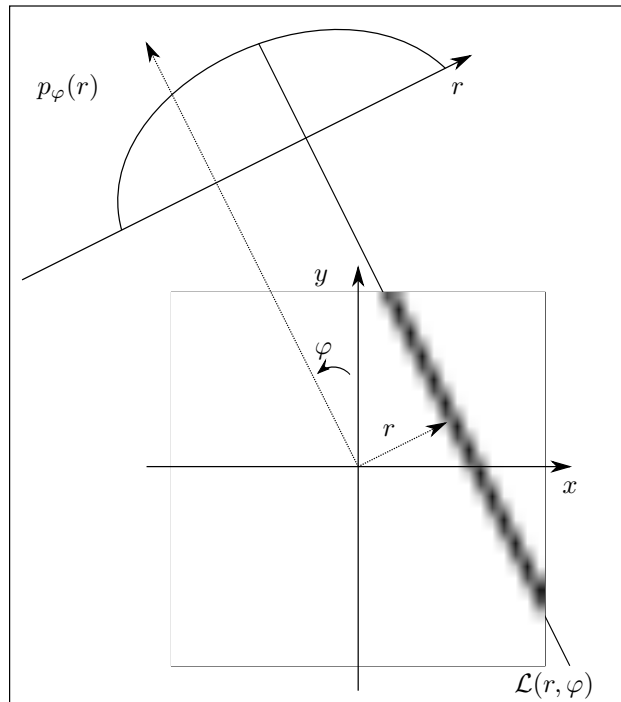


Figure 3.3.1: Illustration of backprojection operation for a single ray in a single projection view.

fig_tomo_back2

3.3.1 Image-domain analysis

The following theorem shows that the laminogram $f_b(x, y)$ is a severely blurred version of the original object $f(x, y)$.

t,tomo,1/r

Theorem 3.3.1 *If $p_\varphi(r)$ denotes the Radon transform of $f(x, y)$ in (3.2.4), and $f_b(x, y)$ denotes the angularly-weighted backprojection of $p_\varphi(r)$ as given by (3.3.1), then*

$$f_b(x, y) = h(r, \varphi) ** f(x, y), \quad \text{where} \quad h(r, \varphi) = \frac{w((\varphi + \pi/2) \bmod \pi)}{|r|}, \quad (3.3.2)$$

e,tomo,1r

for $\varphi \in [0, \pi]$ and $r \in \mathbb{R}$.

Proof:

It is clear from (3.2.4) and (3.3.1) that the operation $f(x, y) \rightarrow p_\varphi(r) \rightarrow f_b(x, y)$ is linear. Furthermore, this operation is shift invariant because

$$\begin{aligned} f_b(x - c, y - d) &= \int_0^\pi w(\varphi) p_\varphi((x - c) \cos \varphi + (y - d) \sin \varphi) d\varphi \\ &= \int_0^\pi w(\varphi) q_\varphi(x \cos \varphi + y \sin \varphi) d\varphi, \end{aligned}$$

where, using the **shift property** (3.2.28), the projections $q_\varphi(r) \triangleq p_\varphi(r - c \cos \varphi - d \sin \varphi)$ denote the Radon transform of $f(x - c, y - d)$.

Due to this shift-invariance, it suffices to examine the behavior of $f_b(x, y)$ at a single location, such as the center. Using (3.2.4):

$$f_b(0, 0) = \int_0^\pi w(\varphi') p_{\varphi'}(0) d\varphi' \quad (3.3.3)$$

$$= \int_0^\pi w(\varphi') \left[\int_{-\infty}^\infty f(0 \cos \varphi' - l \sin \varphi', 0 \sin \varphi' + l \cos \varphi') dl \right] d\varphi' \quad (3.3.4)$$

$$= \int_0^\pi \int_{-\infty}^\infty \frac{w((\varphi + \pi/2) \bmod \pi)}{|r|} f(0 - r \cos \varphi, 0 - r \sin \varphi) |r| dr d\varphi, \quad (3.3.5)$$

e,tomo,back,b00

making the variable changes $\varphi' = (\varphi + \pi/2) \bmod \pi$ and $\ell = \begin{cases} r, & \varphi' \in [\pi/2, \pi] \\ -r, & \varphi' \in [0, \pi/2). \end{cases}$ Thus, using the shift-invariance property noted above:

$$f_b(x, y) = \int_0^\pi \int_{-\infty}^\infty \frac{w((\varphi + \pi/2) \bmod \pi)}{|r|} f(x - r \cos \varphi, y - r \sin \varphi) |r| dr d\varphi, \quad (3.3.6)$$

which is the convolution integral (3.3.2) in (signed) polar coordinates. \square

An alternative proof uses the projection and backprojection of a centered Dirac impulse based on Example 3.2.3.

In the usual case where $w(\varphi) = 1$, we see from (3.3.2) that unmodified backprojection yields a result that is the original object blurred by the $1/r$ function. This PSF has very heavy tails, so the laminogram is nearly useless for visual interpretation. Fig. 3.3.2 illustrates the $1/r$ function.

Thus far we have focused on the parallel ray geometry implicit in (3.2.4). For a broad family of other geometries, there exist pixel-dependent weighted-backprojection operations that also yield the original object convolved with $1/r$ [20]. So the nature of (3.3.2) is fairly general.

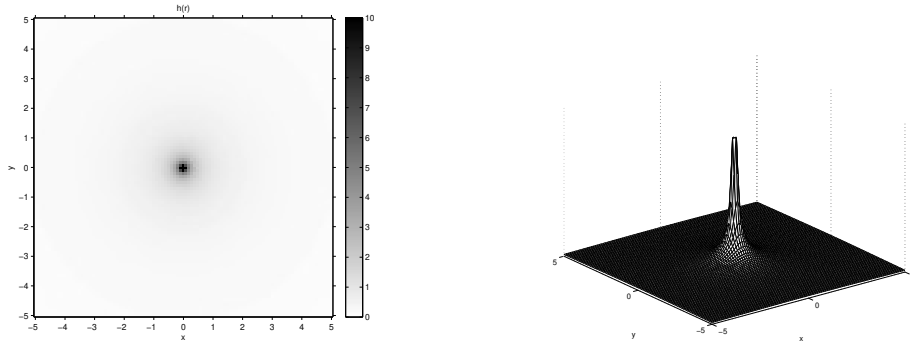


Figure 3.3.2: Illustrations of $1/r$ function and its “heavy tails.”

3.3.2 Frequency-domain analysis

Because the laminogram $f_b(x, y)$ is the object $f(x, y)$ convolved with the PSF $h(r, \varphi)$ in (3.3.2), it follows that in the frequency domain we have

$$F_b(\rho, \Phi) = H(\rho, \Phi) F_o(\rho, \Phi),$$

where $H(\rho, \Phi)$ denotes the polar form of the 2D FT of $h(r, \varphi)$.

It is well known that $1/|r|$ and $1/|\rho|$ are 2D FT pairs [15, p. 338]. The following theorem generalizes that result to the angularly weighted case.

Theorem 3.3.2 *The PSF given in (3.3.2) has the following 2DFT for⁸ $\Phi \in [0, \pi]$ and $\rho \in \mathbb{R}$:*

$$h(r, \varphi) = \frac{1}{|r|} w((\varphi + \pi/2) \bmod \pi) \xleftrightarrow{2D \text{ FT}} H(\rho, \Phi) = \frac{1}{|\rho|} w(\Phi). \quad (3.3.7)$$

Proof:

Evaluate the 2D FT of h :

$$\begin{aligned} H(\rho, \Phi) &= \int_0^\pi \int_{-\infty}^\infty h(r, \varphi) e^{-i2\pi r \rho \cos(\varphi - \Phi)} |r| dr d\varphi \\ &= \int_0^\pi w((\varphi + \pi/2) \bmod \pi) \left[\int_{-\infty}^\infty e^{-i2\pi r \rho \cos(\varphi - \Phi)} dr \right] d\varphi \\ &= \int_0^\pi w((\varphi + \pi/2) \bmod \pi) \delta(\rho \cos(\varphi - \Phi)) d\varphi \\ &= \frac{1}{|\rho|} \int_0^\pi w(\varphi') \delta(\sin(\varphi' - \Phi)) d\varphi' = \frac{1}{|\rho|} w(\Phi), \end{aligned}$$

⁸Alternatively we could write $H(\rho, \Phi) = \frac{1}{|\rho|} w(\Phi \bmod \pi)$ for $\Phi \in [0, 2\pi)$ and $\rho \geq 0$.

letting $\varphi' = (\varphi + \pi/2) \bmod \pi$ and using⁹ the following **Dirac impulse** property [15, p. 100]

$$\delta(f(t)) = \sum_{s: f(s)=0} \frac{\delta(t-s)}{|f'(s)|}. \tag{3.3.8}$$

In particular,

$$\delta(\sin(t)) = \sum_k \delta(t + \pi k).$$

Thus, the 2D FT of $h(r, \varphi)$ in (3.3.2) is $H(\rho, \Phi) = w(\Phi) / |\rho|$. □

So the frequency-space relationship between the laminogram and the original object is

$$F_b(\rho, \Phi) = \frac{w(\Phi)}{|\rho|} F_o(\rho, \Phi). \tag{3.3.9}$$

High spatial frequencies are severely attenuated by the $1/|\rho|$ term, so the laminogram is very blurry. However, the relationship (3.3.9) immediately suggests a “deconvolution” method for recovering $f(x, y)$ from $f_b(x, y)$, as described in the next section.

More generally, if $q_\varphi(r)$ is an arbitrary sinogram to which we apply a weighted backprojection of the form (3.3.1), then the Fourier transform of the resulting image is

$$F_b(\rho, \Phi) = \frac{w(\Phi)}{|\rho|} Q_\varphi(\nu) \Big|_{\nu=\rho, \varphi=\Phi} = \frac{w(\Phi)}{|\rho|} \begin{cases} Q_\Phi(\rho), & \Phi \in [0, \pi) \\ Q_{\Phi-\pi}(-\rho), & \Phi \in [\pi, 2\pi), \end{cases} \tag{3.3.10}$$

where $Q_\varphi(\nu)$ is the 1D FT of $q_\varphi(r)$ along r . (See Problem 3.15.) The special case (3.3.9) follows from the Fourier-slice theorem.

3.3.3 Summary *

Fig. 3.3.3 summarizes the various Fourier-transform relationships described above, as well as the Fourier-slice theorem, and the projection and backprojection operations.

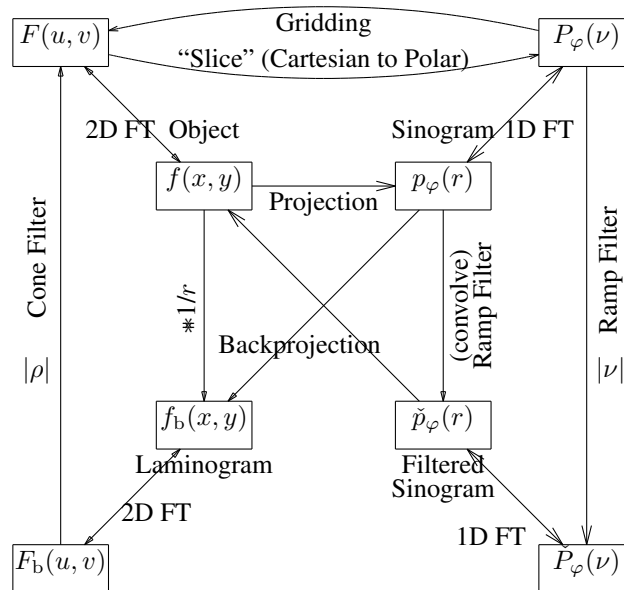


Figure 3.3.3: Relationships between a 2D object $f(x, y)$ and its projections and transforms. Left side of the figure is image domain, right side is projection domain. Inner ring is space domain, outer ring is frequency domain.

Example 3.3.3 Fig. 3.3.4 shows an object $f(x, y)$ consisting of two squares, the larger of which has several small holes in it. Also shown is the sinogram $p_\varphi(r)$ of this object. The laminogram $f_b(x, y)$ is so severely blurred that the small holes are not visible.

⁹This is not a rigorous proof because the function $1/|r|$ is not square integrable (in 2D) so its 2D FT exists only in the sense of distributions, as used for other signals like iDirac impulses, sinusoids, step functions, etc.

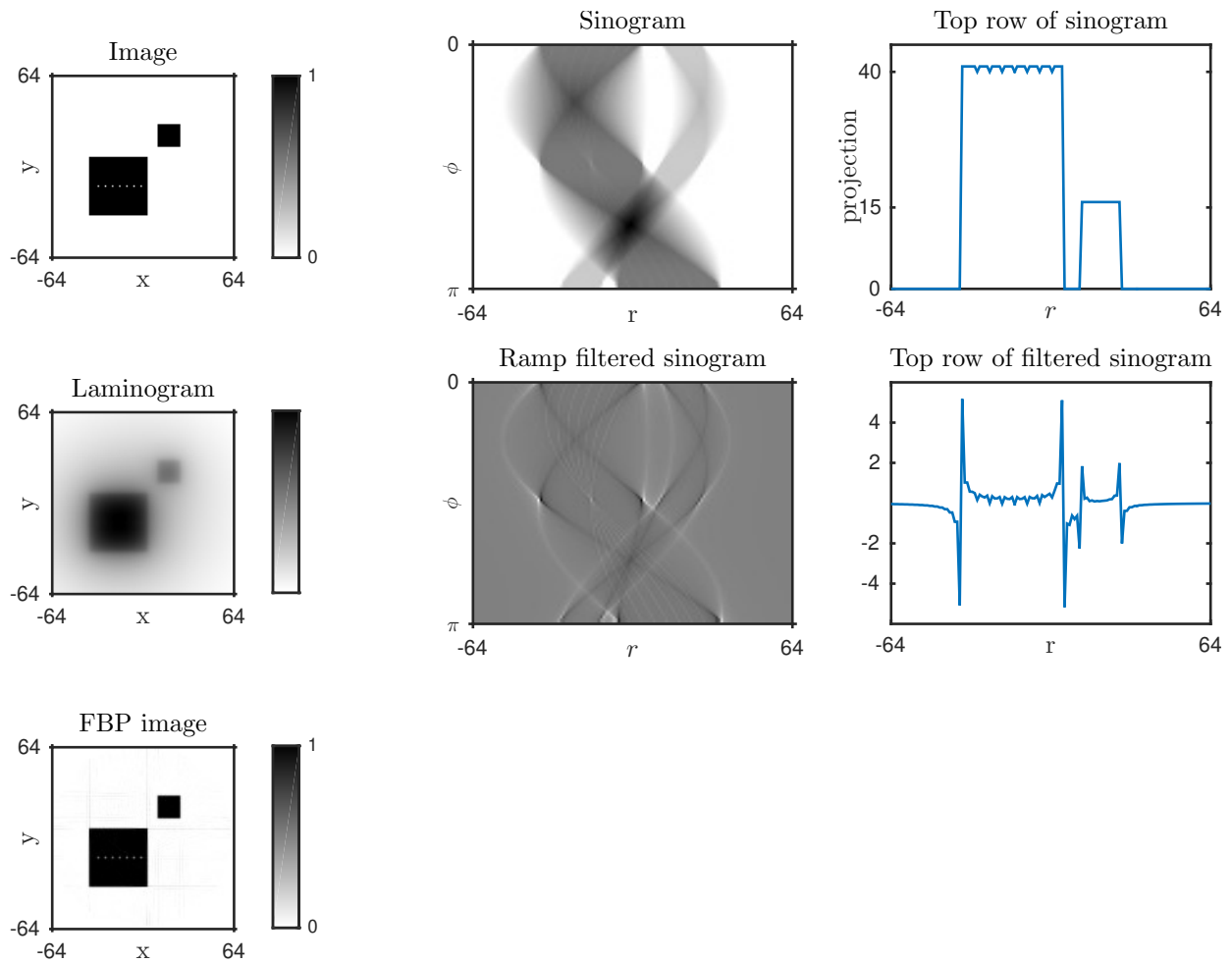
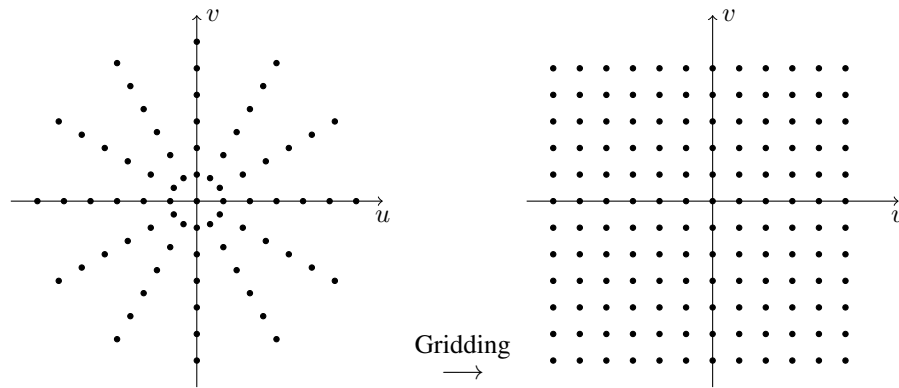


Figure 3.3.4: Illustration of FBP method. Top row: a “square” object $f(x, y)$, its sinogram $p_\varphi(r)$, its top row $p_0(r)$, and its laminogram $f_b(x, y)$. Bottom row: the ramp-filtered sinogram $\check{p}_\varphi(r)$, its top row $\check{p}_0(r)$, and filter-backproject image $\hat{f}(x, y)$. Because of the ramp filtering described in §3.4.3, the small details are recovered.



fig_tomo_gridding

Figure 3.4.1: Illustration of polar samples of $F_o(\rho, \varphi) = P_\varphi(\rho)$ that one must interpolate onto Cartesian samples of $F(u, v)$ for the direct Fourier reconstruction method.

3.4 Radon transform inversion (s,tomo,iradon)

By manipulating the expressions derived in the preceding sections, one can find several methods for inverting the Radon transform, *i.e.*, for recovering an object $f(x, y)$ from its projections $\{p_\varphi(r)\}$. This section describes three alternatives: **direct Fourier reconstruction** based on the Fourier-slice theorem, the **backproject-filter** (BPF) method based on the laminogram, and finally the **convolve-backproject** (CBP) method, also called the filter-backproject (FBP) method. Each of these methods uses some of the relationships shown in Fig. 3.3.3.

In this section we continue to treat the idealized version of the tomography problem in which the entire continuum of projections $\{p_\varphi(r)\}$ is available. In practical tomography systems, only a discrete set of projections and rays are available; these sampling considerations will be addressed in §4.3 and §3.5.

3.4.1 Direct Fourier reconstruction (s,tomo,iradon,dfr)

The **direct Fourier reconstruction** method is based on the Fourier-slice theorem (3.2.36). To reconstruct $f(x, y)$ from $\{p_\varphi(r)\}$ by the direct Fourier method, one performs the following steps.

- Take the 1D FT of each $p_\varphi(\cdot)$ to get $P_\varphi(\cdot)$ for each φ .
- Create a polar representation $F_o(\rho, \Phi)$ of the 2D FT of object $F(u, v)$ using the Fourier-slice relationship:

$$F_o(\rho, \varphi) = P_\varphi(\rho).$$

- Convert from polar representation $F_o(\rho, \Phi)$ to Cartesian coordinates $F(u, v)$. This approach, first proposed in [21], was “the first applicable method for reconstructing pictures from their projections” [22].

For sampled data, this polar to Cartesian step, often called **gridding**, requires very careful interpolation. Fig. 3.4.1 illustrates the process. Numerous papers have considered this step in detail, *e.g.*, [10, 14, 23–46]. Of these, the **nonuniform FFT (NUFFT)** methods with good interpolation kernels are particularly appealing, *e.g.*, [36, 45, 47]. See Chapter 6 for related problems.

- Take the inverse 2D FT of $F(u, v)$ to get $f(x, y)$.

In practice this is implemented using the 2D inverse FFT, which requires Cartesian samples, whereas the relationship $F_o(\rho, \varphi) = P_\varphi(\rho)$ is intrinsically polar. Hence the need for interpolation.

This method would work perfectly if given noiseless, continuous projections $p_\varphi(r)$. Practical disadvantages of this method are that it requires 2D FTs, and gridding can cause interpolation artifacts. An alternative approach uses a **Hankel transform** rather than Fourier transforms [48]; this method also uses interpolation.

Example 3.4.1 Consider the sinogram described by

$$p_\varphi(r) = \text{rect}\left(\frac{r - x_0 \cos\varphi - y_0 \sin\varphi}{w}\right).$$

What is the object $f(x, y)$ that has these projections?

First, taking the 1D FT yields

$$P_\varphi(\nu) = w \text{sinc}(w\nu) e^{-i2\pi\nu(x_0 \cos\varphi + y_0 \sin\varphi)},$$

s,tomo,iradon

s,tomo,iradon,dfr

x,tomo,direct

so by the Fourier-slice theorem the spectrum of $f(x, y)$ is given by

$$F_{\circ}(\rho, \Phi) = w \operatorname{sinc}(w\rho) e^{-i2\pi\rho(x_0 \cos\Phi + y_0 \sin\Phi)}$$

or equivalently

$$F(u, v) = w \operatorname{sinc}\left(w\sqrt{u^2 + v^2}\right) e^{-i2\pi(x_0 u + y_0 v)}.$$

Because [15, p. 338], $w \operatorname{sinc}(w\rho) \xleftrightarrow{2D \text{ FT}} \frac{1}{\pi} \frac{\operatorname{rect}(r/w)}{\sqrt{(w/2)^2 - r^2}}$, the corresponding object is

$$f(x, y) = \operatorname{rect}\left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{w}\right) \frac{1}{\pi \sqrt{(w/2)^2 - (x - x_0)^2 - (y - y_0)^2}}.$$

In other words, the object that has “flat” projections has a circular singularity. Using this relationship, one can analyze the deficiencies of simplistic pixel-driven forward projection [49].

3.4.2 The backproject-filter (BPF) method (s,tomo,bpf)

Another reconstruction method is suggested by the Fourier relationship (3.3.9) between the laminogram and the original object. Solving (3.3.9) for the 2D FT of the object yields

$$\hat{F}(u, v) = \frac{\sqrt{u^2 + v^2}}{w(\angle_{\pi}(u, v))} F_b(u, v), \quad (3.4.1)$$

where $F_b(u, v)$ denotes the 2D FT of the laminogram, and \angle_{π} was defined in (3.2.16). The filter with frequency response $|\rho| = \sqrt{u^2 + v^2}$ is called the **cone filter** due to its shape. (This method is also called the **ρ -filtered layergram** approach [17] [4, p. 153].)

The above relationship suggests the following reconstruction method.

- Choose a *nonzero* angular weighting function $w(\varphi)$ (typically unity).
- Perform angularly-weighted backprojection of the sinogram $p_{\varphi}(r)$ to form the laminogram $f_b(x, y)$ using (3.3.1).
- Take the 2D FT of $f_b(x, y)$ to get $F_b(u, v)$.
- Apply the angularly-modulated cone filter in the Fourier domain using (3.4.1).
- The cone filter nulls the **DC component** of $f(x, y)$. This component can be recovered using the **volume conservation property** (3.2.34) of the Radon transform. For noisy sinogram data, one can compute such an integral for all projections and take the average value to estimate the DC component: $\hat{F}(0, 0) = \frac{1}{\pi} \int_0^{\pi} [\int p_{\varphi}(r) dr] d\varphi$.
- Take the inverse 2D FT of $\hat{F}(u, v)$ to get $\hat{f}(x, y)$.

This approach is called the **backproject-filter (BPF)** method because we first backproject the sinograms, and then apply the cone filter to “deconvolve” the $1/|\rho|$ effect of the backprojection.

In practice, using the cone-filter without modification would excessively amplify high-frequency noise. To control noise, the cone-filter is usually apodized in the frequency domain with a windowing function. Specifically, we replace (3.4.1) by

$$\hat{F}(u, v) = A(u, v) \frac{\sqrt{u^2 + v^2}}{w(\angle_{\pi}(u, v))} F_b(u, v),$$

where $A(u, v)$ is an apodizing lowpass filter. In the absence of noise, the resulting reconstructed image satisfies

$$\hat{f}(x, y) = a(x, y) ** f(x, y),$$

where $a(x, y)$ is the inverse 2D FT of $A(u, v)$. (See [50, 51] for early 3D versions of BPF.)

One practical difficulty with the BPF reconstruction method is that the laminogram $f_b(x, y)$ has unbounded spatial support (even for a finite-support object f) due to the tails of the $1/|r|$ response in (3.3.2). In practice the support of $f_b(x, y)$ must be truncated to a finite size for computer storage, and such truncation of tails can cause problems with the deconvolution step. Furthermore, using 2D FFTs to apply the cone filter results in **periodic convolution** which can cause wrap-around effects due to the high-pass nature of the cone filter. To minimize artifacts due to spatial truncation and periodic convolution, one must evaluate $f_b(x, y)$ numerically using a sampling grid that is considerably larger than the support of the object $f(x, y)$. A large grid increases the computational costs of both the backprojection step and the 2D FFT operations used for the cone filter. The FBP reconstruction method, described next, largely overcomes this limitation. The FBP method has the added benefit of only requiring 1D Fourier transforms, whereas the direct Fourier and BPF methods require 2D transforms.

Example 3.4.2 For theoretical analysis, a convenient choice for the apodizer is $A(\rho) = e^{-a\rho}$. Using Hankel transforms [15, p. 338] and the following Laplacian property (see (29.2.8)):

$$-4\pi^2 r^2 h(r) \xleftrightarrow{\text{Hankel}} \frac{1}{\rho} \frac{d}{d\rho} H(\rho) + \frac{d^2}{d\rho^2} H(\rho),$$

one can show that the corresponding impulse response of the apodized cone filter is given by

$$h(r) = 4\pi \frac{a^2 - 2\pi^2 r^2}{(a^2 + 4\pi^2 r^2)^{5/2}} \xleftrightarrow{\text{Hankel}} \rho e^{-a\rho}.$$

Taking the limit as $a \rightarrow 0$ shows that $h(r) = -1/(4\pi^2 r^3)$ for $r \neq 0$, and that $h(r)$ has a singularity at $r = 0$.

Fig. 3.4.2 illustrates this impulse response for the case $a = 1$.

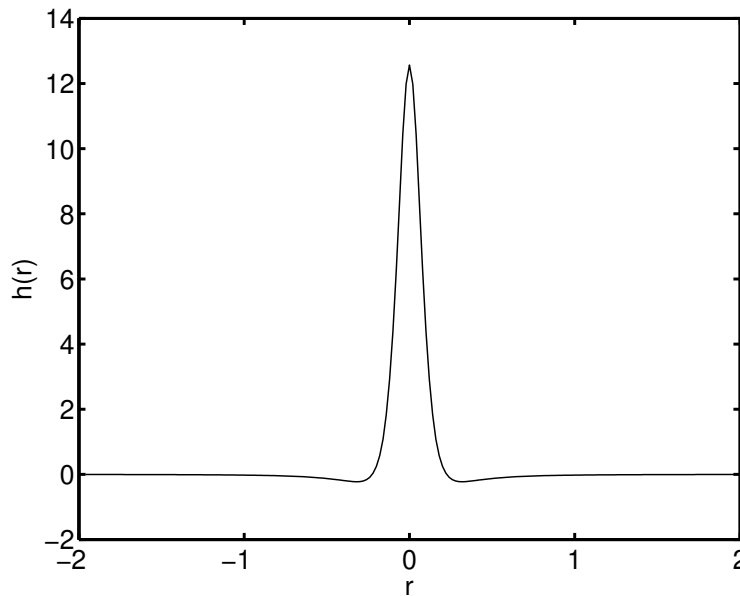


Figure 3.4.2: Impulse response of cone filter that is apodized by an exponential.

Example 3.4.3 If we band-limit the cone filter by choosing $A(\rho) = \text{rect}\left(\frac{\rho}{2\rho_{\max}}\right)$, then the resulting impulse response $h(r)$ has a complicated expression that depends on both Bessel functions and the **Struve function** [52]. Fig. 3.4.3 illustrates this impulse response for the case $\rho_{\max} = 1$.

3.4.3 The filter-backproject (FBP) method (s,tomo,fbp)

We have seen that an unfiltered backprojection yields a blurry laminogram that must be deconvolved by a cone filter to yield the original image. The steps involved look like the following:

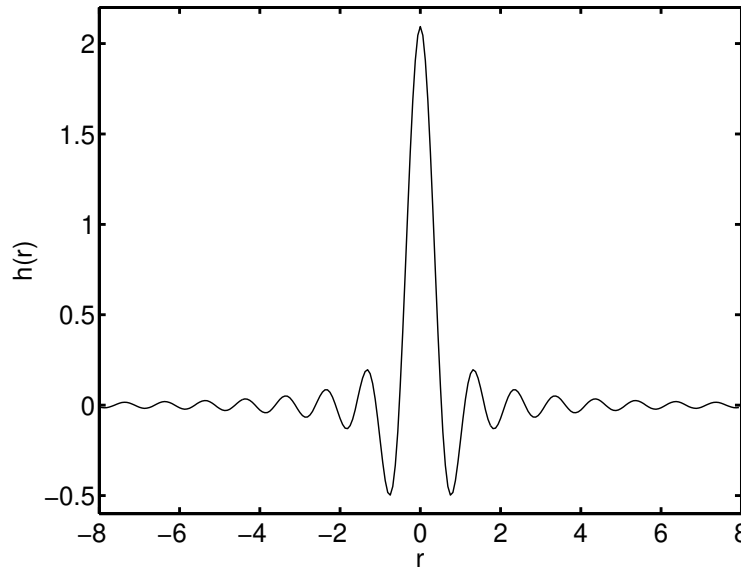
$$f(x, y) \rightarrow \underbrace{\text{Projection} \rightarrow p_\varphi(r) \rightarrow \text{Backprojection}}_{\text{convolution with } 1/|r|} \rightarrow f_b(x, y) \rightarrow \text{Cone filter} \rightarrow f(x, y).$$

Because the cascade of the first two operations is linear and shift invariant, as shown in §3.3, in principle we could move the cone filter to be the first step to obtain the same overall result:

$$f(x, y) \rightarrow \text{Cone filter} \rightarrow \check{f}(x, y) \rightarrow \text{Projection} \rightarrow \check{p}_\varphi(r) \rightarrow \text{Backprojection} \rightarrow f(x, y),$$

where, assuming $w(\varphi) = 1$ hereafter, the filtered object $\check{f}(x, y)$ has the following the spectrum:

$$\check{F}(\rho, \Phi) = |\rho| F_\circ(\rho, \Phi).$$



fig_tomo_bpf_cone_rect

Figure 3.4.3: Impulse response of band-limited cone filter.

Of course in practice we cannot filter the object before acquiring its projections. However, applying the Fourier-slice theorem to the scenario above, we see that each projection $\check{p}_\varphi(r)$ has the following 1D FT:

$$\check{p}_\varphi(r) \xleftrightarrow{\text{FT}} \check{P}_\varphi(\nu) = \check{F}(\rho, \varphi) \Big|_{\rho=\nu} = |\rho| F_o(\rho, \varphi) \Big|_{\rho=\nu} = |\nu| F_o(\nu, \varphi) = |\nu| P_\varphi(\nu).$$

This relationship implies that we can replace the cone filter above with a set of 1D filters with frequency response $|\nu|$ applied to each projection $p_\varphi(\cdot)$. This filter is called the **ramp filter** due to its shape. The block diagram above becomes:

$$f(x, y) \rightarrow \boxed{\text{Projection}} \rightarrow p_\varphi(r) \rightarrow \boxed{\text{Ramp filters}} \rightarrow \check{p}_\varphi(r) \rightarrow \boxed{\text{Backprojection}} \rightarrow f(x, y).$$

This reconstruction approach is called the filter-backproject (**FBP**) method, and is used the most widely in tomography.

A formal derivation of the FBP method uses the Fourier-slice theorem as follows:

$$\begin{aligned} f(x, y) &= \iint F(u, v) e^{i2\pi(xu+yv)} du dv \\ &= \int_0^\pi \int_{-\infty}^\infty F(\nu \cos\varphi, \nu \sin\varphi) e^{i2\pi\nu(x \cos\varphi + y \sin\varphi)} |\nu| d\nu d\varphi \\ &= \int_0^\pi \int_{-\infty}^\infty P_\varphi(\nu) e^{i2\pi\nu(x \cos\varphi + y \sin\varphi)} |\nu| d\nu d\varphi \\ &= \int_0^\pi \check{p}_\varphi(x \cos\varphi + y \sin\varphi) d\varphi, \end{aligned}$$

where we define the filtered projection $\check{p}_\varphi(r)$ as follows:

$$\check{p}_\varphi(r) = \int_{-\infty}^\infty P_\varphi(\nu) |\nu| e^{i2\pi\nu r} d\nu. \quad (3.4.2)$$

e, tomo, fbp, ramp

The steps of the FBP method are summarized as follows.

- For each projection angle φ , compute the 1D FT of the projection $p_\varphi(\cdot)$ to form $P_\varphi(\nu)$.
- Multiply $P_\varphi(\nu)$ by $|\nu|$ (ramp filtering) to get $\check{P}_\varphi(\nu) = |\nu| P_\varphi(\nu)$.
- For each φ , compute the inverse 1D FT of $\check{P}_\varphi(\nu)$ to get the filtered projection $\check{p}_\varphi(r)$ in (3.4.2). In practice this filtering is often done using an FFT, which yields **periodic convolution**. Because the space-domain kernel corresponding to $|\nu|$ is not space limited (see Fig. 3.4.5), periodic convolution can cause “wrap-around” artifacts. With care, these artifacts can be avoided by zero padding the sinogram. Sampling the ramp filter can also cause aliasing artifacts. See Example 3.4.6 below for a preferable approach.

- The ramp filter nulls the DC component of each projection. If desired, this can be restored using the **volume conservation property** (3.2.34). The approach of Example 3.4.6 avoids the need for any such DC correction. Discretizing the integrals carefully avoids the need for empirical scale factors.
- Backproject the filtered sinogram $\{\check{p}_\varphi(r)\}$ using (3.3.1) to get $\hat{f}(x, y)$, i.e.

$$\hat{f}(x, y) = \int_0^\pi \check{p}_\varphi(x \cos \varphi + y \sin \varphi) d\varphi. \quad (3.4.3)$$

In practice usually the **pixel-driven backprojection** approach of §3.5.3 is used. With some hindsight, the existence of such an approach seems natural because the Fourier-slice theorem provides a relationship between the 2D FT in object domain and the 1D FT in projection domain. Apparently [53] FBP was first applied to medical tomography by Chesler in 1972 [54]. For weighted versions of FBP, see [55, 56].

3.4.4 Ramp filters and Hilbert transforms

It can be useful to relate the ramp filter $|\nu|$ to a combination of differentiation and a Hilbert transform.

The **Hilbert transform** of a 1D function $f(t)$ is defined (using **Cauchy principal values**) by¹⁰ [58, p. 248]:

$$f_{\text{Hilbert}}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-s} f(s) ds = \frac{1}{\pi t} * f(t). \quad (3.4.4)$$

Note that this “transform” returns another function of t . The corresponding relationship in the frequency domain is

$$F_{\text{Hilbert}}(\nu) = -i \operatorname{sgn}(\nu) F(\nu). \quad (3.4.5)$$

Example 3.4.4 The Hilbert transform of the **rect function** $\operatorname{rect}(t) = \mathbb{I}_{\{|t| \leq 1/2\}}$ is [58, p. 249]: $\frac{1}{\pi} \log \left| \frac{t+1/2}{t-1/2} \right|$.

Using the Hilbert transform frequency response (3.4.5), we rewrite the ramp filter $|\nu|$ in (3.4.2) as follows:

$$|\nu| = \frac{1}{2\pi} (i2\pi\nu) (-i \operatorname{sgn}(\nu)).$$

The term $(i2\pi\nu)$ corresponds to differentiation, by the differentiation property of the Fourier transform. Therefore, another expression for the FBP method (3.4.3) is

$$\hat{f}(x, y) = \frac{1}{2\pi} \int_0^\pi \frac{d}{dr} p_{\text{Hilbert}}(r, \varphi) \Big|_{r=x \cos \varphi + y \sin \varphi} d\varphi, \quad (3.4.6)$$

where $p_{\text{Hilbert}}(r, \varphi)$ denotes the Hilbert transform of $p_\varphi(r)$ with respect to r . Combining (3.4.4) and (3.4.6) yields

$$\hat{f}(x, y) = \frac{1}{2\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial r} p_\varphi(r)}{x \cos \varphi + y \sin \varphi - r} dr d\varphi. \quad (3.4.7)$$

This form is closer to **Radon’s inversion formula** [4, p. 21] [7, 9].

Example 3.4.5 Continuing Example 3.2.7, the spectrum of the projection at angle φ of a rectangle object is given by (3.2.37), so its ramp-filtered projections are given by (for $\sin \varphi \neq 0$):

$$\begin{aligned} \check{P}_\varphi(\nu) &= |\nu| a \operatorname{sinc}(\nu a \cos \varphi) b \operatorname{sinc}(\nu b \sin \varphi) \\ &= \frac{1}{\pi \cos \varphi \sin \varphi} \sin(\pi \nu a \cos \varphi) \operatorname{sgn}(\nu) (b \sin \varphi) \operatorname{sinc}(\nu b \sin \varphi) \\ &= \frac{1}{2\pi \cos \varphi \sin \varphi} (e^{-i\pi \nu a \cos \varphi} - e^{i\pi \nu a \cos \varphi}) [i \operatorname{sgn}(\nu) (b \sin \varphi) \operatorname{sinc}(\nu b \sin \varphi)]. \end{aligned}$$

Using the **Hilbert transform** in Example 3.4.4, the inverse 1D FT of the bracketed term is $\frac{1}{\pi b \sin \varphi} \log \left| \frac{x - \frac{1}{2} b \sin \varphi}{x + \frac{1}{2} b \sin \varphi} \right|$, so by the **shift property of the FT**, the filtered projections are:

$$\check{p}_\varphi(r) = \frac{1}{2\pi^2 \cos \varphi \sin \varphi} \log \left| \frac{r^2 - \left(\frac{a \cos \varphi + b \sin \varphi}{2} \right)^2}{r^2 - \left(\frac{a \cos \varphi - b \sin \varphi}{2} \right)^2} \right|,$$

cf. [59, eqn. (14)]. Fig. 3.4.4 shows an example of the projection $p_\varphi(r)$ of a unit square and its filtered version $\check{p}_\varphi(r)$. The ramp filter causes singularities at each of the points of discontinuity in the projections. (Compare with Fig. 3.3.4.) For the case $\sin \varphi = 0$, see Problem 3.24.

¹⁰Note that some texts use the opposite sign, e.g., [15, p. 359] [57, p. 194].

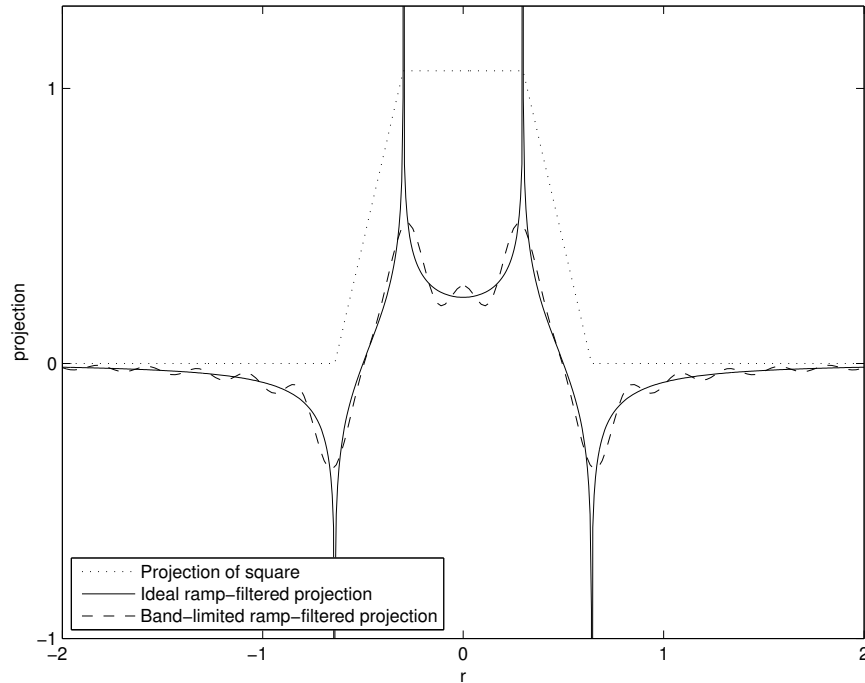


Figure 3.4.4: Projection $p_\varphi(r)$ of a unit square at angle $\varphi = \pi/9$ and its filtered versions $\check{p}_\varphi(r)$ both for ideal ramp filter $|\nu|$ and a band-limited ramp filter with cutoff frequency $\nu_0 = 4$.

3.4.5 Filtered versus unfiltered backprojection

Recall that an unfiltered backprojection of a sinogram gives an image blurred by $1/|r|$. This blurring is due to the fact that the (all nonnegative) projection values “pile up” in the laminogram, and there is no destructive interference. In contrast, after filtering with the ramp filter, the projections have both positive and negative values, so destructive interference can occur, which is desirable for the parts of the image that are supposed to be zero for example. Fig. 3.3.4 illustrates these concepts.

3.4.6 The convolve-backproject (CBP) method

The ramp filter amplifies high frequency noise, so in practice one must apodize it by a 1D lowpass filter $A(\nu)$, in which case (3.4.2) is replaced by

$$\check{p}_\varphi(r) = \int_{-\infty}^{\infty} P_\varphi(\nu) A(\nu) |\nu| e^{i2\pi\nu r} d\nu. \quad (3.4.8)$$

Alternatively, one can perform this filtering operation in the spatial domain by radial convolution:

$$\check{p}_\varphi(r) = p_\varphi(r) * h_A(r) = \int p_\varphi(r') h_A(r - r') dr', \quad (3.4.9)$$

where the filter kernel $h_A(r)$ is the inverse FT of $H_A(\nu) = A(\nu) |\nu|$, i.e.,

$$h_A(r) = \int_{-\infty}^{\infty} A(\nu) |\nu| e^{i2\pi\nu r} d\nu. \quad (3.4.10)$$

Combining with (3.4.3) and (3.4.9) leads to the following **convolve-backproject** method:

$$\hat{f}(x, y) = \int_0^\pi (p_\varphi * h_A)(x \cos\varphi + y \sin\varphi) d\varphi = \int_0^\pi \int p_\varphi(r) h_A(x \cos\varphi + y \sin\varphi - r) dr d\varphi. \quad (3.4.11)$$

Although the convolution kernel $h_A(r)$ usually is not space-limited, the object (and hence its projections) are space limited, so space-domain convolution is feasible. On the other hand, the space-domain convolution requires more computation than a frequency-space implementation using the FFT method, so the FBP approach is often more attractive than the CBP approach.

Example 3.4.6 As a concrete example, consider the case of a rectangular band-limiting window

$$A(\nu) = \text{rect}\left(\frac{\nu}{2\nu_0}\right), \quad (3.4.12)$$

which is a logical choice when the object (and hence its projections) are band limited to a maximum spatial frequency ν_0 . In this case, the **band-limited ramp filter** has the frequency response shown in Fig. 3.4.5. This is called the Ram-Lak filter [6, p. 83] after [60].

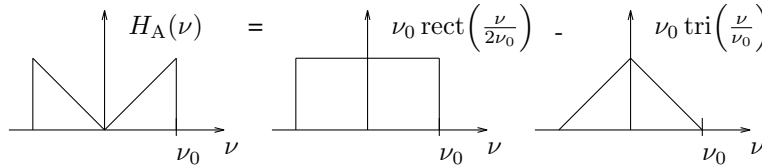


Figure 3.4.5: Frequency response of band-limited ramp filter.

To determine the corresponding convolution kernel, observe that

$$|\nu| \text{rect}\left(\frac{\nu}{2\nu_0}\right) = \nu_0 \text{rect}\left(\frac{\nu}{2\nu_0}\right) - \nu_0 \text{tri}\left(\frac{\nu}{\nu_0}\right)$$

where $\text{tri}(\cdot)$ was defined in (3.2.42). Thus the convolution kernel is [61]

$$h_A(r) = 2\nu_0^2 \text{sinc}(2\nu_0 r) - \nu_0^2 \text{sinc}^2(\nu_0 r), \quad (3.4.13)$$

as shown in Fig. 3.4.6. The ringing is due to the implicit assumption that the object is band limited. In practice one usually uses an apodization filter $A(\nu)$ that goes to zero gradually to reduce ringing; see §3.4.7.

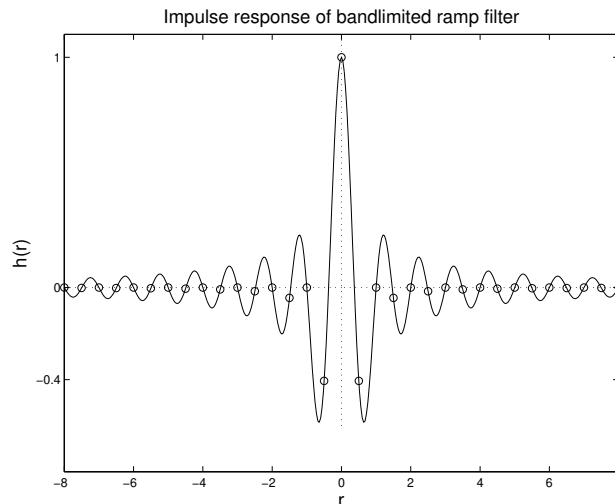


Figure 3.4.6: Convolution kernel for band-limited ramp filter $h_A(r)$ with $\nu_0 = 1$, and the sample values $h[n] \stackrel{\text{fig_tomo_ramp_bandlimit}}{=} h_A(n/(2\nu_0))$.

In practice, one uses samples of this impulse response. Sampling it using the Nyquist rate $2\nu_0$ yields [5, p. 72]:

$$h[n] = h_A\left(\frac{n}{2\nu_0}\right) = 2\nu_0^2 \text{sinc}(n) - \nu_0^2 \text{sinc}^2(n/2) = \nu_0^2 \begin{cases} 1, & n = 0 \\ 0, & n \text{ even} \\ -1/(\pi n/2)^2, & n \text{ odd.} \end{cases} \quad (3.4.14)$$

Rarely is ν_0 given in practice, so one assumes that the sampling is adequate, i.e., $\nu_0 = 1/(2\Delta_R)$, where Δ_R is the radial sample spacing. This approach is preferable to sampling the ramp filter directly in the frequency domain [5, p. 69] [62].

Although the filter (3.4.14) is infinitely long, given a sinogram with a finite number n_R of radial samples, we need only to evaluate the filtered sinogram $\tilde{p}_\varphi(r)$ at those same radial sample locations, so it suffices to compute $h[n]$ for $n = -n_R, \dots, n_R - 1$ and to zero pad the sinogram radially with n_R zeros before computing the FFTs to perform the filtering. When using this discrete-space filter $h[n]$ to approximate the convolution (3.4.9), one should include a scaling factor Δ_R to account for dr in that integral.

Example 3.4.7 In Example 3.4.5, Fig. 3.4.4 showed the projections of a square after filtering with an ideal ramp filter. Fig. 3.4.4 also shows those same projections when filtered with the rectangularly apodized ramp filter described in Example 3.4.6.

Example 3.4.8 For theoretical analysis, an alternative to the rectangular apodization considered in Example 3.4.6 is to use exponential apodization $A(\nu) = e^{-\varepsilon|\nu|}$, for some small $\varepsilon > 0$. One can verify the following 1D FT pair [63, p. 127]:

$$h_A(r) = \frac{2(\varepsilon^2 - 4\pi^2 r^2)}{(\varepsilon^2 + 4\pi^2 r^2)^2} \xleftrightarrow{\text{FT}} H_A(\nu) = e^{-\varepsilon|\nu|} |\nu|.$$

Fig. 3.4.7 shows examples of this impulse response. Taking the limit as $\varepsilon \rightarrow 0$, yields the following expression for the **ramp filter** for $r \neq 0$:

$$h_*(r) = \frac{-1}{2\pi^2 r^2}, \quad (3.4.15)$$

and a singularity at $r = 0$. (See [64] for rigorous treatment of Fourier transforms of such functions.) This ramp filter satisfies the following **scaling property**:

$$h_*(r) = \alpha^2 h_*(\alpha r). \quad (3.4.16)$$

This is also known as the **homogeneity property** [65].

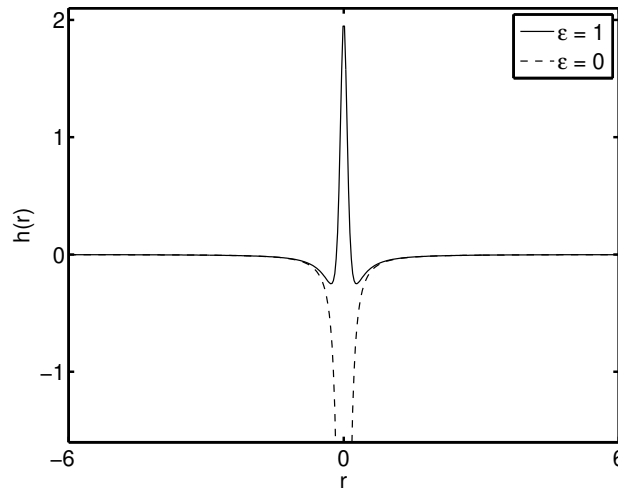


Figure 3.4.7: Impulse response $h_A(r)$ of ramp filter with exponential apodization.

MIRT See fbp2.m.

3.4.7 PSF of the FBP method (s,tomo,fbp,psf)

Apodizing the ramp filter will reduce amplification of high frequency noise, but will also degrade spatial resolution in the reconstructed object. To analyze the effects of apodization, we again turn to the Fourier-slice theorem (3.2.36). By that theorem, multiplying the 1D FT $P_\varphi(\nu)$ of each projection by $A(\nu)$ is equivalent¹¹ to pre-multiplying the object spectrum by $A(\rho)$, i.e.,

$$F(u, v) \mapsto A(\sqrt{u^2 + v^2}) F(u, v).$$

¹¹Note that for this simple relationship to hold, it is essential that the same apodizer be used for every projection angle φ .

Thus, the reconstructed object $\hat{f}(x, y)$ is a blurred version of the original:

$$\hat{f}(x, y) = f(x, y) ** h(x, y), \quad (3.4.17)$$

where

$$h(x, y) \xleftrightarrow{2D \text{ FT}} H(u, v) \triangleq A(\sqrt{u^2 + v^2}). \quad (3.4.18)$$

Because H is circularly symmetric, so is h ; thus $h(r)$ is simply the **Hankel transform** of $H(\rho) = A(\rho)$.

Example 3.4.9 For the rectangular apodizing window $A(\nu) = \text{rect}\left(\frac{\nu}{2\nu_0}\right)$, the corresponding PSF in the image domain would be

$$h(r) = \nu_0^2 \text{jinc}(\nu_0 r). \quad (3.4.19)$$

Thus the image would be blurred by a **jinc** function, which has large sidelobes that would cause undesirable “ringing.”

Example 3.4.10 A popular choice in nuclear medicine is a gaussian window: $A(\nu) = \exp(-\pi(\nu/\nu_0)^2)$. The half-amplitude cutoff frequency $\nu_{1/2}$ for this window, i.e., the point where $A(\nu_{1/2}) = A(0)/2$, is $\nu_{1/2} = \nu_0 \sqrt{\frac{\log 2}{\pi}} \approx \frac{\nu_0}{2} 0.9394 \approx \frac{\nu_0}{2}$. Because the Hankel transform of a gaussian is gaussian, in the image domain the PSF is

$$h(r) = \nu_0^2 \exp(-\pi(\nu_0 r)^2).$$

To find the FWHM of this gaussian, find r such that $h(r) = h(0)/2$, or $\exp(-\pi(\nu_0 r)^2) = 1/2$ so $\pi(\nu_0 r)^2 = \log 2$. Thus

$$\text{FWHM} = \frac{2}{\nu_0} \sqrt{\frac{\log 2}{\pi}} \approx \frac{0.9394}{\nu_0} \approx \frac{1}{\nu_0} \approx \frac{1}{2\nu_{1/2}}.$$

So for a 5mm FWHM PSF, we would use $\nu_0 = 1/5 = 0.2$ cycles/cm.

Example 3.4.11 Other popular window functions include the following.

- **Hann or Hanning:** $A(\nu) = \left[\frac{1}{2} + \frac{1}{2} \cos(\pi\nu/\nu_0)\right] \text{rect}\left(\frac{\nu}{2\nu_0}\right)$
- **Hamming:** $A(\nu) = [0.54 + 0.46 \cos(\pi\nu/\nu_0)] \text{rect}\left(\frac{\nu}{2\nu_0}\right)$
- **Generalized Hamming:** $A(\nu) = [\alpha + (1 - \alpha) \cos(\pi\nu/\nu_0)] \text{rect}\left(\frac{\nu}{2\nu_0}\right)$, for $\alpha \in [0, 1]$
- **Butterworth:** $A(\nu) = \frac{1}{\sqrt{1 + (\nu/\nu_0)^{2n}}}$, for $n \geq 0$
- **Parzen:** $A(\nu) = \begin{cases} 1 - 6(\nu/\nu_0)^2(1 - |\nu|/\nu_0) & |\nu| \leq \nu_0/2 \\ 2(1 - |\nu|/\nu_0)^3 & \nu_0/2 \leq |\nu| \leq \nu_0 \\ 0, & \text{otherwise} \end{cases}$
- **Shepp Logan [66]:** $A(\nu) = \left|\text{sinc}\left(\frac{\nu}{2\nu_0}\right)\right|$ or $\left|\text{sinc}\left(\frac{\nu}{2\nu_0}\right)\right|^3$
- **Modified Shepp Logan:** $A(\nu) = \text{sinc}\left(\frac{\nu}{2\nu_0}\right)[0.4 - 0.6 \cos(\pi\nu/\nu_0)]$

It is not always easy to find a closed-form expression for the PSF h that results from apodization. But the general rule of thumb, $\text{FWHM} \approx 1/(2\nu_{1/2})$, is usually pretty close.

In light of the result (3.4.17), one might wonder why we apply the window $A(\nu)$ to the projections rather than just smooth (post-filter) the reconstructed image? The main reason is that we must apply the ramp filter anyway, so we can include $A(\nu)$ essentially for free. In contrast, post-smoothing would require either an “expensive” convolution or a pair of 2D FFTs. However, if one wants to experiment with several different amounts of smoothing, then it is preferable to smooth after a (ramp-filtered) backprojection so that only one backprojection operation is needed.

3.4.8 Summary *

We have described three methods for inverting the Radon transform, i.e., for reconstructing a 2D object $f(x, y)$ from its projections $\{p_\varphi(r)\}$:

- direct Fourier reconstruction (gridding),
- the backproject-filter (BPF) method (cone filter),

- the filter-backproject (FBP) method (ramp filter), and its cousin the convolve-backproject (CBP) method.

The derivations of these methods all used the Fourier-slice theorem. These methods would yield identical results for noiseless continuous-space data, but are based on different manipulations of the formulas so they lead to different ways of discretizing and implementing the equations, yielding very different numerical algorithms in practice.

We also analyzed the PSF due to windowing the ramp filter. In practice one must choose the apodizing window to make a suitable compromise between spatial resolution and noise. Most implementations assume that the object is band-limited, often leading to the kind of ringing seen in (3.4.19). See [67] for an alternative formulation that assumes the object is piecewise constant.

Recently, other inversion formulas for the 2D Radon transform have been discovered for objects with compact support, *e.g.*, [68]. These methods include user-selectable parameters that allow one to avoid corrupted or missing regions of the sinogram. An interesting *open problem* is to determine whether the methods could be extended to include some type of statistical weighting.

3.5 Practical backprojection (s,tomo,prac)

The preceding sections considered the idealized case where there is a continuum of projection views. In practice, sinograms have only finite angular samples, so each of the reconstruction methods described in §3.4 requires modification for practical implementations. (There are also alternative methods for solving the problem for a finite number of views, such as the **minimum-norm** approach [69, 70].)

A critical step in both the BPF and FBP reconstruction methods is the **backprojection** operation (3.3.1). Given only a finite number n_φ of projection angles, we must approximate the integral in (3.3.1). Usually the projection angles are uniformly spaced over the interval $[0, \pi)$, *i.e.*,

$$\varphi_i = \left(\frac{i-1}{n_\varphi} \right) \pi, \quad i = 1, \dots, n_\varphi.$$

In such cases, the usual approach is to use the following Riemann sum approximation to (3.3.1):

$$f_b(x, y) \approx \frac{\pi}{n_\varphi} \sum_{i=1}^{n_\varphi} p_{\varphi_i}(x \cos \varphi_i + y \sin \varphi_i). \quad (3.5.1)$$

Whether more sophisticated approximations to this integral would be beneficial is an *open problem*.

There are at least three distinct approaches to implementing (3.5.1): **rotation-based backprojection**, **ray-driven backprojection**, and **pixel-driven backprojection**. If the available projections were continuous functions of the radial argument, then these formulations would be identical. In practice, not only are the projection angles discrete, but also we have only discrete radial samples of $p_\varphi(r)$. (See (4.3.1).) Ignoring noise and blur, we are given the discrete sinogram

$$y_i[n] = p_{\varphi_i}(r) \Big|_{\varphi=\varphi_i, r=r_c[n]}, \quad i = 1, \dots, n_\varphi, \quad n = 0, \dots, n_R - 1, \quad (3.5.2)$$

where the radial sample locations are given by

$$r_c[n] = (n - n_0) \Delta_R \quad (3.5.3)$$

and typically $n_0 = n_R/2$ or $n_0 = (n_R - 1)/2$. For such sinograms, the various backprojection methods can produce different results because they differ in how the equations are discretized.

If the true object f^{true} can be assumed to be appropriately band limited, then its projections will also be band limited (by the Fourier slice theorem), so in principle we could recover p_{φ_i} from $\{y_i[\cdot]\}$ using sinc interpolation:

$$p_{\varphi_i}(r) = \sum_{n=-\infty}^{\infty} y_i[n] \operatorname{sinc} \left(\frac{r - r_c[n]}{\Delta_R} \right).$$

In practice this interpolation is inappropriate because: real objects are space limited so they cannot be band limited, sinc interpolation expects an infinite number of samples whereas practical sinograms have only a finite number of samples, and sinc interpolation is computationally impractical. Thus, simpler interpolation methods are used in practice, such as linear interpolation or spline interpolation [71], perhaps combined with oversampling of the FFT used for the ramp filter.

3.5.1 Rotation-based backprojection

We can rewrite the backprojection formula (3.5.1) as follows:

$$f_b(x, y) = \frac{\pi}{n_\varphi} \sum_{i=1}^{n_\varphi} b_i(x, y), \quad (3.5.4)$$

where the backprojection of the i th view is given by:

$$b_i(x, y) = p_{\varphi_i}(x \cos \varphi_i + y \sin \varphi_i). \quad (3.5.5)$$

We can also write $b_i = \mathcal{P}_{\varphi_i}^* p_{\varphi_i}$, where $\mathcal{P}_{\varphi_i}^*$ is the adjoint operator defined in (4.2.4). This operator maps the i th 1D projection back into a 2D image by “smearing” that projection along the angle φ_i . In this approach, we form temporary images by backprojecting each view and accumulating the sum of those temporary images.

To better understand $b_i(x, y)$, note that when $i = 1$ we have $\varphi = 0$, so

$$b_1(x, y) = p_0(x), \quad (3.5.6)$$

which is just a 2D version of the function $p_0(x)$.

For sampled sinograms, implementing (3.5.6) is trivial¹², simply replicate the first row of the sinogram (a vector) to make a matrix. For other angles, perform the following steps to implement (3.5.5).

- Replicate the i th sinogram row to make an image, as if it were the $\varphi = 0$ case.

Mat Use `repmat`.

- Rotate that image counter clockwise by φ . This rotation will require an interpolation method, such as bilinear interpolation or a more precise spline approach [72].

Mat Use `imrotate`.

- Accumulate these rotated images over all angles, as described in (3.5.4).

In this approach, the “outer loop” is over projection angles. The first step (replication) inherently “accounts” for discrete radial samples.

The rotation approach is easily implemented but can be somewhat slow because high quality rotation is a fairly expensive operation. One of the faster methods uses three 1D passes [72].

3.5.2 Ray-driven backprojection

For **ray-driven backprojection**, one loops through all the rays and for each ray one interpolates $y_i[n]$ onto the pixels whose centers are nearest to the ray $\mathcal{L}(r_c[n], \varphi_i)$, as defined in (3.2.3). Although this approach is somewhat popular for **forward projection**, it can produce significant artifacts when used for backprojection, so will not be considered further here. Fig. 3.3.1 somewhat illustrates the approach.

When radial sample spacing equals image sample spacing, ray-driven backprojection is equivalent to rotation-based backprojection [73].

For a $N \times N$ image, ray-driven back-projection requires $O(Nn_\varphi n_R)$ operations. Usually $n_\varphi \approx N$ and $n_R \approx N$ so we say 2D ray-driven back-projection is $O(N^3)$.

3.5.3 Pixel-driven backprojection

For image display, we compute $f_b(x, y)$ on a finite grid of pixel coordinate pairs $\{(x_j, y_j) : j = 1, \dots, n_p\}$. For **pixel-driven backprojection**, we loop over the (x_j, y_j) pairs of interest and evaluate (3.5.1) for each of the grid points, thereby filling up an image array. To implement, the outer loop is over pixel index j and the inner loop is over angles φ_i . In essence, for each pixel we are summing along the corresponding sinusoid (illustrated in Fig. 3.2.3) in the sinogram.

However, the radial argument $x_j \cos \varphi_i + y_j \sin \varphi_i$ in (3.5.1) rarely exactly equals one of the radial sample locations $r_c[n]$ shown in (3.5.2). Therefore, radial interpolation is required for pixel-driven backprojection. The usual approach is **linear interpolation** which is equivalent mathematically to the following approximation:

$$p_{\varphi_i}(r) \approx \sum_n y_i[n] \operatorname{tri}\left(\frac{r - r_c[n]}{\Delta_R}\right), \quad (3.5.7)$$

¹²This method assumes that $n_0 = (n_R - 1)/2$, i.e., that the center of the image projects onto the center of each projection. It furthermore assumes that the desired pixel size equals Δ_R . Otherwise a more complicated approach is needed.

where the **unit triangle function** is denoted:

$$\text{tri}(t) = \begin{cases} 1 - |t|, & |t| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Although (3.5.7) is a mathematically correct expression for linear interpolation and is useful for theoretical analysis, it poorly conveys how one would implement linear interpolation in practice. Because support of the function $\text{tri}(t)$ is two sample units, for any given r , only two terms in the sum in (3.5.7) are possibly nonzero. An alternative expression is

$$\begin{aligned} p_{\varphi_i}(r) &\approx y_i[n(r)] \text{tri}\left(\frac{r - r_c[n(r)]}{\Delta_R}\right) + y_i[n(r) + 1] \text{tri}\left(\frac{r - r_c[n(r) + 1]}{\Delta_R}\right) \\ &= y_i[n(r)] \left(1 - \frac{r - r_c[n(r)]}{\Delta_R}\right) + y_i[n(r) + 1] \left(\frac{r - r_c[n(r)]}{\Delta_R}\right), \end{aligned}$$

where we define

$$n(r) \triangleq \lfloor r / \Delta_R + n_0 \rfloor.$$

Other interpolators, such as an oversampled FFT or spline functions are also used [71].

For a $N \times N$ image, pixel-driven back-projection requires $O(N^2 n_\varphi)$ operations. Usually $n_\varphi \approx N$ so we say 2D pixel-driven back-projection is $O(N^3)$. Hierarchical methods requiring $O(N^2 \log N)$ operations have also been developed [74, 75].

MIRT See `fbp2_back.m`.

3.5.4 Interpolation effects

Generalizing (3.5.7), suppose that we use an interpolation method of the form:

$$\hat{p}_{\varphi_i}(r) = \sum_n y_i[n] h\left(\frac{r - r_c[n]}{\Delta_R}\right)$$

for some interpolation kernel $h(\cdot)$. Suppose furthermore that $p_\varphi(r)$ is band limited with maximum frequency less than $\frac{1}{2\Delta_R}$. Then it follows from (3.5.2) and the sampling theorem that (ignoring noise):

$$\hat{P}_{\varphi_i}(\nu) = \begin{cases} P_{\varphi_i}(\nu)H(\nu), & |\nu| < \frac{1}{2\Delta_R} \\ 0, & \text{otherwise.} \end{cases}$$

For example, when h is the linear interpolator in (3.5.7), we have $H(\nu) = \Delta_R \text{sinc}^2(\Delta_R \nu)$, which is strictly positive for $|\nu| < \frac{1}{2\Delta_R}$. Therefore, while we are applying the ramp filter $|\nu|$ in the discretized version of (3.4.2), we can also apply the **inverse filter** $1/H(\nu)$ to compensate for the effects of interpolation [76, eqn. (45)] [71, 77].

MIRT See `fbp2_sino_filter.m`.

3.5.5 Summary *

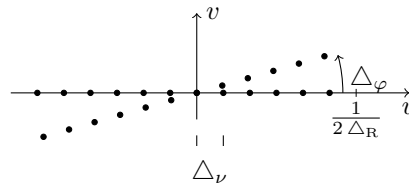
Pixel-driven, rotation-based, and ray-driven backprojection are all used in practice, depending on number of samples, sample spacing, etc. The formulations are exactly identical in continuous space, but can yield slightly different results when discretized.

3.6 Sinogram restoration (s,tomo,restore)

Because a sinogram $p_\varphi(r)$ has two coordinates (r and φ), one can display it as a 2D picture or even treat it as a 2D “image” and apply any number of image processing methods to it. Numerous linear and nonlinear filters have been applied to sinograms in an attempt to reduce noise [78–100] to extrapolate missing data [101–104] and to compensate for detector blur [105–112] and/or SPECT attenuation [113–116]. Some of these methods can even be called “statistical” methods because they include measurement noise models. Problem 3.12 explores an approach based on B-splines.

A typical linear approach for a system with shift-invariant blur having frequency response $B(\nu)$ would be to use a **Wiener filter** as the apodizing filter $A(\nu)$ in (3.4.8) as follows

$$A(\nu) = \frac{B^*(\nu)}{|B(\nu)|^2 + R(\nu)}, \tag{3.6.1}$$



fig, tomo, ang, sample

Figure 3.7.1: Angular sampling considerations.

where the “**regularizer**” $R(\nu)$ is a model for the ratio of the power spectral density of the noise over that of $p_\varphi(r)$ under the (questionable) assumption that both are **wide sense stationary (WSS)** random processes.

Nonlinear sinogram preprocessing methods, including classical methods based on view-adaptive Wiener filters [117] and contemporary approaches like wavelet-based denoising [118], have the potential to reduce noise more than linear methods with less degradation of spatial resolution. However, when a nonlinear sinogram filtering method is combined with the linear FBP reconstruction method, the resulting spatial resolution properties can be quite unusual. Modern methods typically apply nonlinear processing in the image domain, *e.g.*, by nonquadratic edge-preserving regularization, instead of in the sinogram domain.

3.7 Sampling considerations (s,tomo,samp)

In practice one can acquire only finite radial and angular samples, due to constraints such as cost and time. This section describes considerations in choosing the radial and angular sampling.

3.7.1 Radial sampling

The radial sample spacing, Δ_R , should be determined by the spatial resolution (in the radial direction) of the tomographic scanning instrument. The FWHM of the system radial resolution is a function of the detector width, the source size in X-ray imaging, etc. The radial detector response (*e.g.*, a rectangular function for square detector elements) generally is not exactly band-limited, so Nyquist sampling theory can provide only general guidance. A practical rule-of-thumb is to choose (if possible): $\Delta_R = \text{FWHM}/2$. Then the number of radial samples should be determined to cover the desired FOV by choosing: $n_R = \text{FOV}/\Delta_R$. See §4.3.9 for Fourier analysis of aliasing due to radial sampling.

3.7.2 Angular sampling

For a given FOV and radial sampling, Nyquist sampling theory can help determine the angular sampling Δ_φ . If we have n_R radial samples spaced by Δ_R , then in the Fourier domain (of the 2D DFT), the corresponding spatial frequencies are spaced by $\Delta_\nu = 1/(n_R \Delta_R)$. It is natural to choose the angular sampling so as to ensure that all samples in the 2D Fourier domain are separated by no more than this amount.

Considering Fig. 3.7.1, the appropriate angular spacing is $\Delta_\varphi = \frac{\Delta_\nu}{1/(2\Delta_R)} = 2/n_R$. So the total number of angles over 180° should be

$$n_\varphi = \frac{\pi}{\Delta_\varphi} = \frac{\pi}{2} n_R.$$

In practice often somewhat fewer angular samples are used, but usually $n_\varphi \approx n_R$. The reason for using fewer than $\pi n_R/2$ angular samples is that often in real systems there is blur in the radial direction so the spatial resolution is somewhat lower than that implied by just the radial sampling, *i.e.*, the radial sampling may be a little finer than necessary from a strict Nyquist perspective. However, we rarely use Nyquist (sinc) reconstruction, but rather only linear interpolation, so some “oversampling” is reasonable. Inadequate angular sampling can lead to significant aliasing artifacts. On the other hand, for the FBP reconstruction method, the computation time is directly proportional to the number of angles.

Example 3.7.1 Fig. 3.7.2 illustrates the effects of angular undersampling.

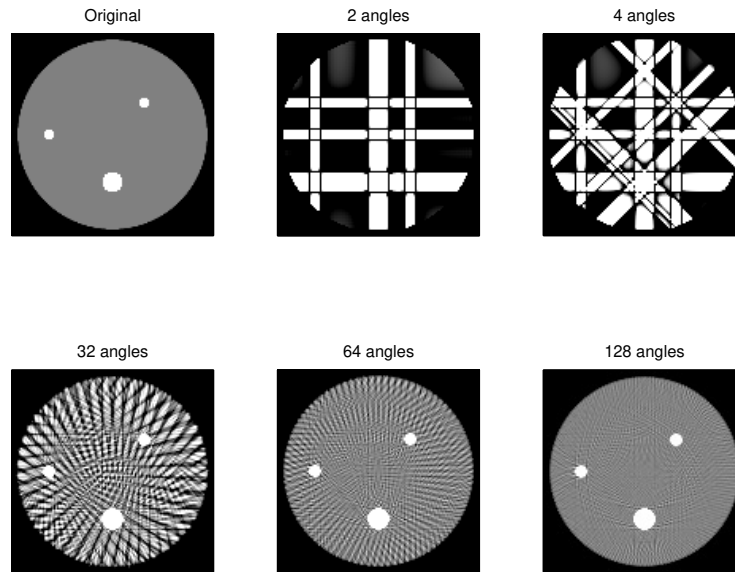


Figure 3.7.2: Illustration of the effects of angular undersampling on image quality for FBP reconstruction. The image is 128×128 , and the true values of the digital phantom are 1 in the background disk and 8 in the small disks. The grayscale display is windowed from 0.5 to 1.5 to enhance the visibility of the artifacts.

fig_tomo_sample

3.8 Linogram reconstruction (s,tomo,lino)

For tomographic imaging systems with certain geometries involving flat detectors, it can be convenient to use an alternative sinogram coordinate system of the form

$$p^{\text{EW}}(s, \beta) \triangleq \frac{1}{\sqrt{1 + \beta^2}} p_{\arctan(\beta)} \left(\frac{s}{\sqrt{1 + \beta^2}} \right) \quad (3.8.1)$$

$$p^{\text{NS}}(s, \beta) \triangleq \frac{1}{\sqrt{1 + \beta^2}} p_{\pi/2 + \arctan(\beta)} \left(\frac{s}{\sqrt{1 + \beta^2}} \right), \quad (3.8.2)$$

for $|\beta| \leq 1$. This is called a **linogram** [10, 44, 119–121], because in this coordinate system the projection of a point source is a straight line.

Taking the 1D Fourier transform of $p^{\text{EW}}(s, \beta)$ and $p^{\text{NS}}(s, \beta)$ along s and applying the **Fourier slice theorem** yields the Fourier relationships:

$$P_{\varphi}^{\text{EW}}(\nu) = \sqrt{1 + \beta^2} P_{\arctan(\beta)}(\sqrt{1 + \beta^2}\nu) = F(\nu, \beta\nu) \quad (3.8.3)$$

$$P_{\varphi}^{\text{NS}}(\nu) = \sqrt{1 + \beta^2} P_{\pi/2 + \arctan(\beta)}(\sqrt{1 + \beta^2}\nu) = F(-\beta\nu, \nu). \quad (3.8.4)$$

So the 1D Fourier transform of linogram data corresponds to samples of the object spectrum $F(u, v)$ along lines with slope β or $1/\beta$. In particular, for a projection at slope β , if the linogram data has N_s equally spaced samples along s with spacing Δ_s , then the corresponding samples of the object spectrum $F(u, v)$ in 2D Fourier space along the line at slope β are spaced by $\sqrt{1 + \beta^2}/(N_s \Delta_s)$, corresponding to the **pseudo-polar** grid shown in Fig. 3.8.1. Using this sampling pattern, one can develop **direct Fourier reconstruction** methods for linogram data akin to §3.4.1, e.g., [122, 123], and iterative methods [124]. See Chapter 6 for more options. The linogram concept has been generalized to higher dimension data, called **planogram** reconstruction [125–128].

By equating φ with $\arctan(\beta)$ or $\pi/2 + \arctan(\beta)$, one can show that $\sqrt{1 + \beta^2} = \frac{1}{\max(|\cos\varphi|, |\sin\varphi|)}$. Therefore the radial sample spacing in (3.8.2) is the same as that of **Mojette sampling** described in §27.7.6 and [22, 129–138]. The primary difference between linogram and Mojette sampling is in the angular sampling.

s,tomo,lino

e,tomo,lino,def

e,tomo,lino,Pub

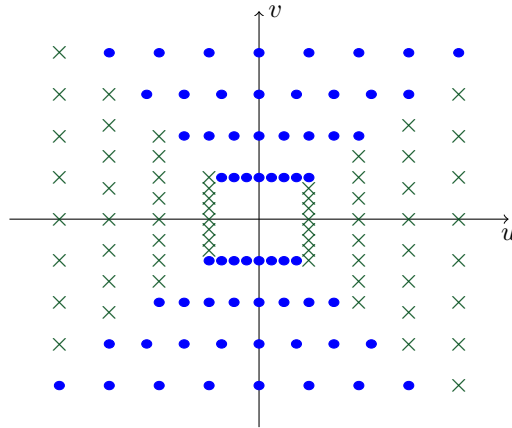


Figure 3.8.1: Illustration of pseudo-polar grid in 2D Fourier space associated with linogram sampling. The (green) fig_tomo_lino_pseudo cross marks and (blue) circles correspond to the $P_{\varphi}^{EW}(\nu)$ samples and the $P_{\varphi}^{NS}(\nu)$ samples respectively.

3.9 2D fan beam tomography (s,tomo,fan)

s,tomo,fan

The preceding sections have focused on case of 2D parallel-beam projections. Although 1st-generation X-ray CT scanners did correspond to that geometry, many contemporary tomographic imaging systems have **fan beam** geometries, including commercial X-ray CT scanners and some collimators for SPECT systems. For hypothetical continuous measurements, one could transform fan-beam projections into parallel-beam projections by a simple change of variables. For discrete, noisy measurements, **rebinning** fan-beam measurements into parallel-beam projections requires an interpolation operation that could degrade spatial resolution. To avoid such rebinning, one can derive analytical reconstruction methods directly in terms of the fan-beam coordinates, as described below.

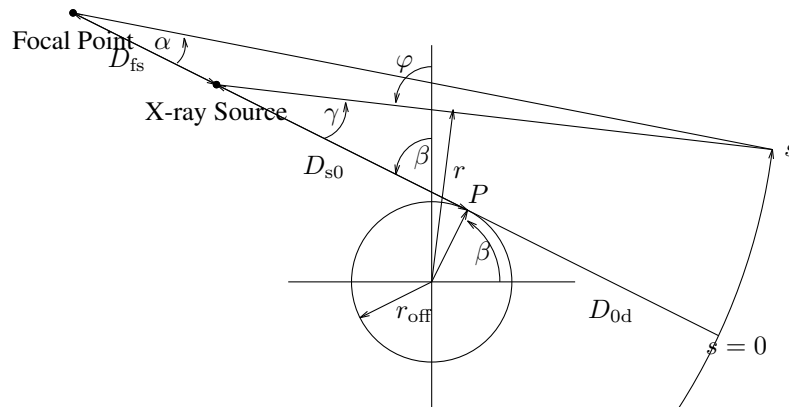


Figure 3.9.1: Illustration of fan beam geometry. fig,tomo,fan,offset

fig,tomo,fan,offset

Fig. 3.9.1 illustrates the fan-beam geometry that will be considered here. Because it can be challenging to ensure that the line between the X-ray source and the midpoint of the detector passes through the exact center of rotation, we allow an offset r_{off} between that line and the center [139]. Let P denote the point along that line that intersects the circle of radius r_{off} centered at the rotation isocenter. D_{0d} denotes the distance from the point P to the detector, D_{s0} denotes the distance from the X-ray source to P , and D_{fs} denotes the distance from the focal point of the detector arc to the X-ray source. Define $D_{sd} \triangleq D_{0d} + D_{s0}$ to be the total distance from the X-ray source to the center of the detector. This formulation allows the detector focal point to differ from the X-ray source location to encompass a variety of system configurations. For flat detectors, $D_{fs} = \infty$. For third-generation X-ray CT systems, $D_{fs} = 0$. For fourth generation X-ray CT systems, $D_{fs} = -D_{s0}$.

In our notation, the distances D_{0d} and D_{s0} are constants, rather than being functions of β , defined as the angle that the line segment between the X-ray source and the detector center makes with the y axis. Generalizations exist to allow non-circular source trajectories [140].

Let $s \in [-s_{\text{max}}, s_{\text{max}}]$ denote the (signed) **arc length** along the detector, with $s = 0$ corresponding to the detector

center. Arc length is a natural parameterization for detector elements that are spaced equally along the detector. (For a flat detector with $D_{fs} = \infty$, the arc length s is simply the position along the detector.) The various angles have the following relationships:

$$\alpha(s) = \frac{s}{D_{fd}}, \quad \gamma(s) = \arctan\left(\frac{D_{fd} \sin \alpha(s)}{D_{fd} \cos \alpha(s) - D_{fs}}\right), \quad (3.9.1)$$

where $D_{fd} \triangleq D_{fs} + D_{sd}$. The two most important cases are

$$\gamma(s) = \begin{cases} s/D_{sd}, & D_{fs} = 0 \quad (\text{equiangular}) \\ \arctan(s/D_{sd}), & D_{fs} = \infty \quad (\text{equidistant}). \end{cases} \quad (3.9.2)$$

The function $\gamma(s)$ is anti-symmetric and increasing and the (inverse) relationship between γ and s is:

$$s(\gamma) = \begin{cases} D_{sd}\gamma, & D_{fs} = 0 \\ D_{fd} \left[\gamma - \arcsin\left(\frac{D_{fs}}{D_{fd}} \sin \gamma\right) \right], & 0 \leq D_{fs} < \infty \\ D_{sd} \tan \gamma, & D_{fs} = \infty. \end{cases} \quad (3.9.3)$$

The ray corresponding to angle β and detector element s is

$$\mathcal{L}(s, \beta) = \{(x, y) : x \cos \varphi(s, \beta) + y \sin \varphi(s, \beta) = r(s)\}, \quad (3.9.4)$$

where

$$\varphi(s, \beta) \triangleq \beta + \gamma(s) \quad (3.9.5)$$

$$r(s) \triangleq D_{s0} \sin \gamma(s) + r_{\text{off}} \cos \gamma(s) \quad (3.9.6)$$

$$= \sqrt{D_{s0}^2 + r_{\text{off}}^2} \sin(\gamma(s) + \beta_{\text{off}}), \quad (3.9.7)$$

where we define $\beta_{\text{off}} \triangleq \angle(D_{s0}, r_{\text{off}})$. When $\beta = 0$, the coordinates of a point on the detector are

$$(x_d(s), y_d(s)) = \begin{cases} \left(r_{\text{off}} + D_{sd} \sin\left(\frac{s}{D_{sd}}\right), D_{s0} - D_{sd} \cos\left(\frac{s}{D_{sd}}\right) \right), & D_{fs} = 0 \quad (\text{equiangular}) \\ (r_{\text{off}} + s, -D_{0d}), & D_{fs} = \infty \quad (\text{equidistant}). \end{cases} \quad (3.9.8)$$

Unlike in our analysis of parallel-beam tomography, here the range of r is limited inherently by the position of the X-ray source and the extent of the detector:

$$|r(s)| \leq r_{\text{max}} \triangleq D_{s0} \sin \gamma_{\text{max}}, \quad (3.9.9)$$

where $\gamma_{\text{max}} \triangleq \gamma(s_{\text{max}})$ and s_{max} is half of the total arc length of the detector. The radius r_{max} defines the circular **field of view** of the imaging system: the subset of the plane that is measured completely. (We assume that $D_{0d} \geq r_{\text{max}}$, because otherwise even the detector center would limit the field of view.) The angle $2\gamma_{\text{max}}$ is called the **fan angle**. A typical clinical CT scanner has $D_{sd} \approx 95$ cm, $s_{\text{max}} \approx 45$ cm, so $\gamma_{\text{max}} \approx \pi/6$.

Fig. 3.9.2 illustrates the sampling (3.9.7) associated with the fan-beam geometry, before and after the values of (r, φ) are collapsed to the range described in (3.2.9).

The line-integral projection $p(s, \beta)$ of f along $\mathcal{L}(s, \beta)$ is¹³:

$$p(s, \beta) = \int_{\mathcal{L}(s, \beta)} f(x, y) d\ell \quad (3.9.10)$$

$$= \iint f(x, y) \delta(x \cos \varphi(s, \beta) + y \sin \varphi(s, \beta) - r(s)) dx dy, \quad (3.9.11)$$

for $|s| \leq s_{\text{max}}$ and $0 \leq \beta < \beta_{\text{max}}$. We assume $\beta_{\text{max}} \geq \pi + 2\gamma_{\text{max}}$ to ensure complete sampling. By (3.9.7), fan-beam projections satisfy the following general symmetry property:

$$p(\gamma^{-1}(-\gamma(s) - 2\beta_{\text{off}}), \beta \pm \pi + 2\gamma(s) + 2\beta_{\text{off}}) = p(s, \beta). \quad (3.9.12)$$

The corresponding rays are called **conjugate rays**. In particular, if $D_{fs} = 0$, then

$$p(-s - 2D_{sd}\beta_{\text{off}}, \beta \pm \pi + 2s/D_{sd} + 2\beta_{\text{off}}) = p(s, \beta). \quad (3.9.13)$$

Alternatively, if $\beta_{\text{off}} = 0$, then

$$p(-s, \beta \pm \pi + 2\gamma(s)) = p(s, \beta). \quad (3.9.14)$$

The reconstruction problem is to estimate f from the fan-beam projections $\{p(s, \beta)\}$.

¹³Practically speaking, the integral should be restricted to the field of view: $\sqrt{x^2 + y^2} \leq r_{\text{max}}$, but this restriction would complicate analysis by introducing a shift variance into the problem, so we ignore it.

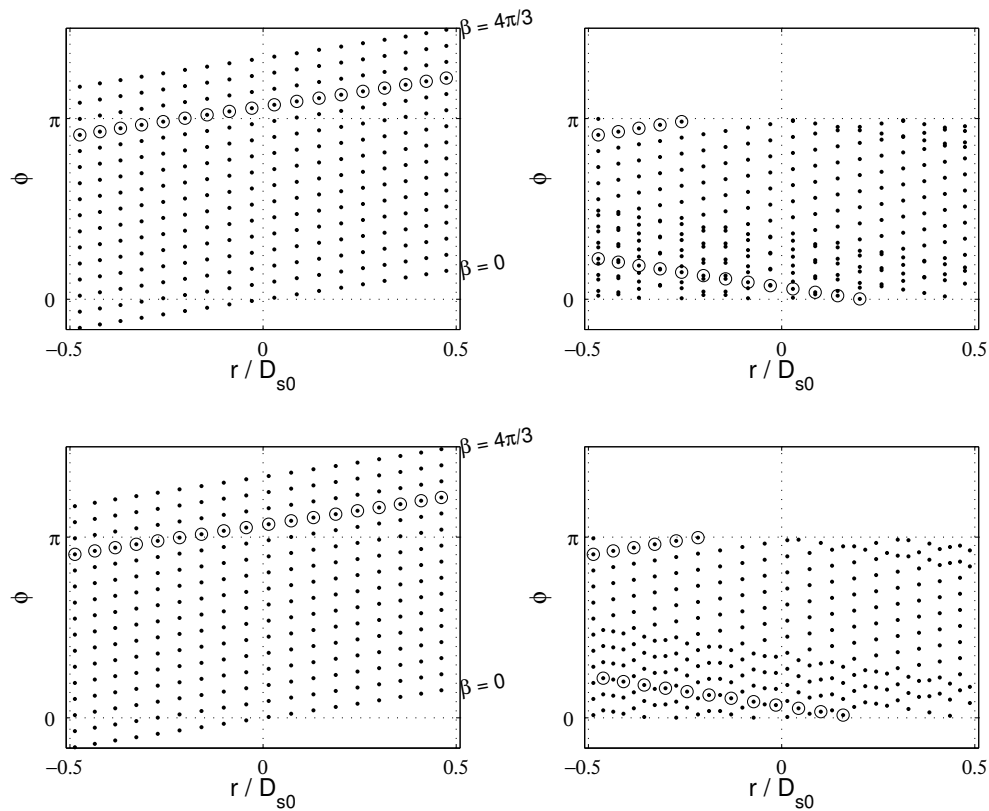


Figure 3.9.2: Left: (r, φ) coordinates for equiangular fan-beam samples based directly on (3.9.7). The fan angle is $2\gamma_{\max} = \pi/3$. Right: after converting φ to the range $[0, \pi]$ using the periodicity property (3.2.29). Top: for equally spaced samples in s symmetrical around $s = 0$. Bottom: for equally spaced samples in s with quarter-detector offset. fig_tomo_fan_sample
 The samples for one particular value of β are circled for illustration.

s,tomo,fan,rebini

3.9.1 Fan-parallel rebinning methods (s,tomo,fan,rebini)

In continuous space, to **rebin** from fan-beam to parallel-beam coordinates requires a simple change of variables based on (3.9.7). In the usual case where $r_{\text{off}} = 0$, we can express the parallel-beam projections in terms of the fan-beam projections using the following relationship:

$$p_{\varphi}(r) = p(s, \beta) \Big|_{s=s(r), \beta=\beta(r, \phi)} = p(\gamma^{-1}(\arcsin(r/D_{s0}), \phi - \arcsin(r/D_{s0})), \quad (3.9.15)$$

e,tomo,fan,rebini

where γ^{-1} is defined by (3.9.3). For sampled measurements, usually one first performs 1D interpolation along the source position using the relationship $\phi = \beta + \gamma$. Then one performs 1D interpolation along the detector by relating r and s . Fig. 3.9.3 illustrates the process. When the s samples include a **quarter detector offset**, then the radial sampling can be improved by a factor of two, at least for 360° scans, as illustrated in Fig. 3.9.2.

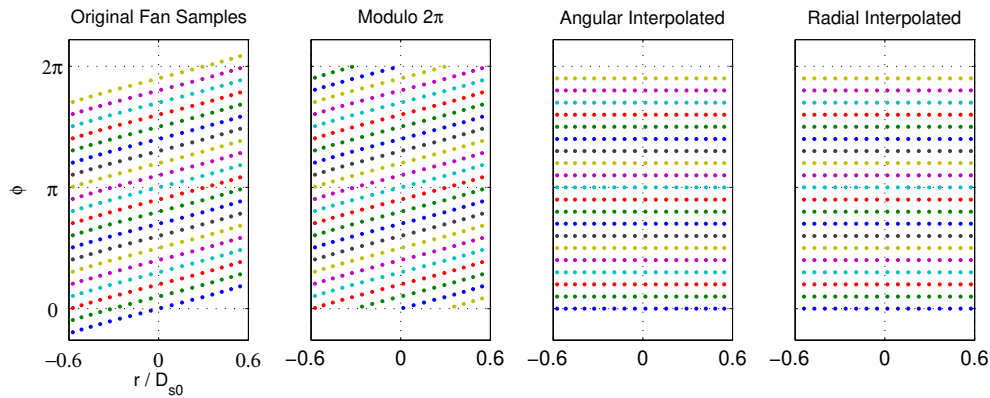


Figure 3.9.3: Illustration of fan to parallel rebinning for a 360° fan-beam scan having for equally spaced samples in s with quarter-detector offset for 3rd generation CT geometry. All sample locations are shown in the parallel-beam coordinate system (r, φ) . Left to right: original fan-beam sample locations; sample locations after consideration of the periodicity property (3.2.29); sample locations after angular interpolation; sample locations after interpolation to equally spaced radial coordinates. fig_tomo_fan_rebini1

In the presence of motion, it can be preferable to extend the data range slightly and then use transition weights similar to the **Parker weighting** described in §3.9.3 to reduce artifacts from possibly inconsistent views [141].

If one chooses the sampling coordinates appropriately, then one can minimize the amount of interpolation needed at the center of the FOV¹⁴. Specifically, using (3.9.7) when $r_{\text{off}} = 0$, it is natural to choose $\Delta_R / \Delta_S = \dot{r}(s) \Big|_{s=0} = D_{s0} \dot{\gamma}(0)$. Fig. 3.9.3 used that choice for Δ_R .

MIRT See `rebin_fan2par.m`.

s,tomo,fan,fbp

3.9.2 The filter-backproject (FBP) approach for 360° scans (s,tomo,fan,fbp)

This section derives filter-backproject methods for the fan-beam geometry shown in Fig. 3.9.1, including the case of displaced center of rotation [139]. We focus on the case of 360° rotation of the X-ray source and detector [5, p. 77].

Using the symmetry property (3.2.29) of the Radon transform, we start by rewriting the parallel-ray FBP formula (3.4.11) for the case of 360° rotation:

$$f(x, y) = \frac{1}{2} \int_0^{2\pi} \int p_{\varphi}(r) h_*(x \cos \varphi + y \sin \varphi - r) dr d\varphi, \quad (3.9.16)$$

e,tomo,fan,fbp,360,orig

where $h_*(\cdot)$ denotes the ramp filter in (3.4.15) with 1D Fourier transform $H_*(\nu) = |\nu|$. More generally we can write

$$f(x, y) = \int_0^{2\pi} w_{2\pi}(x, y; \varphi) \int p_{\varphi}(r) h_*(x \cos \varphi + y \sin \varphi - r) dr d\varphi, \quad (3.9.17)$$

e,tomo,fan,fbp,360,wxyang

¹⁴F. Noo, personal communication, 2013-06-18.

where $w_{2\pi}(x, y; \varphi)$ is any weighting function for which $w_{2\pi}(x, y; \varphi) + w_{2\pi}(x, y; \varphi + \pi) = 1$. (See Problem 3.20.) The rays corresponding to (r, φ) and $(-r, \varphi \pm \pi)$ are called **conjugate rays**. Usually $w_{2\pi}(x, y; \varphi)$ is nonnegative and independent of (x, y) and simply equals $1/2$ for 360° scans.

Now change to fan-beam coordinates by making the transformation of variables $r = r(s)$, $\varphi = \varphi(s, \beta)$, defined in (3.9.7). The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial}{\partial s} r(s) & \frac{\partial}{\partial \beta} r(s) \\ \frac{\partial}{\partial s} \varphi(s, \beta) & \frac{\partial}{\partial \beta} \varphi(s, \beta) \end{bmatrix} = \begin{bmatrix} [D_{s0} \cos \gamma(s) - r_{\text{off}} \sin \gamma(s)] \dot{\gamma}(s) & 0 \\ \dot{\gamma}(s) & 1 \end{bmatrix}, \quad (3.9.18)$$

the determinant of which is

$$J(s) \triangleq |D_{s0} \cos \gamma(s) - r_{\text{off}} \sin \gamma(s)| |\dot{\gamma}(s)|. \quad (3.9.19)$$

The reconstruction formula (3.9.17) becomes

$$\begin{aligned} f(x, y) &= \int_0^{2\pi} \int w_{2\pi}(x, y; \varphi(s, \beta)) p(s, \beta) h_*(x \cos \varphi(s, \beta) + y \sin \varphi(s, \beta) - r(s)) J(s) ds d\beta \\ &= \int_0^{2\pi} \int p(s, \beta) w_{2\pi}(x, y; \beta + \gamma(s)) J(s) \\ &\quad \cdot h_*(x \cos(\beta + \gamma(s)) + y \sin(\beta + \gamma(s)) - D_{s0} \sin \gamma(s) - r_{\text{off}} \cos \gamma(s)) ds d\beta. \end{aligned}$$

This expression fan-beam reconstruction formula is inconvenient for practical use; we prefer to manipulate it into a filter-backproject form to facilitate implementation.

Using trigonometric identities, one can simplify the argument of h_* above as follows:

$$x \cos(\beta + \gamma) + y \sin(\beta + \gamma) - D_{s0} \sin \gamma - r_{\text{off}} \cos \gamma = L_\beta(x, y) \sin(\gamma_\beta(x, y) - \gamma), \quad (3.9.20)$$

where

$$\begin{aligned} x_\beta &\triangleq x \cos \beta + y \sin \beta \\ y_\beta &\triangleq -x \sin \beta + y \cos \beta \end{aligned} \quad (3.9.21)$$

$$\begin{aligned} L_\beta(x, y) &\triangleq \sqrt{(D_{s0} - y_\beta)^2 + (x_\beta - r_{\text{off}})^2} \\ \gamma_\beta(x, y) &\triangleq \arctan\left(\frac{x_\beta - r_{\text{off}}}{D_{s0} - y_\beta}\right). \end{aligned} \quad (3.9.22)$$

Define $s_\beta(x, y)$ (or s' for short) by $\gamma_\beta(x, y) = \gamma(s')$ via (3.9.3), i.e.,

$$s_\beta(x, y) \triangleq \begin{cases} D_{sd} \gamma_\beta(x, y), & D_{fs} = 0 \\ D_{sd} \tan \gamma_\beta(x, y), & D_{fs} = \infty \end{cases} = \begin{cases} D_{sd} \arctan\left(\frac{x_\beta - r_{\text{off}}}{D_{s0} - y_\beta}\right), & D_{fs} = 0 \\ D_{sd} \frac{x_\beta - r_{\text{off}}}{D_{s0} - y_\beta}, & D_{fs} = \infty. \end{cases} \quad (3.9.23)$$

Using (3.9.20) and applying the **scaling property** of the ideal ramp filter¹⁵ (3.4.16) leads to the following form for the fan-beam reconstruction formula:

$$\begin{aligned} f(x, y) &= \int_0^{2\pi} \int p(s, \beta) w_{2\pi}(x, y; \beta + \gamma(s)) J(s) h_*(L_\beta(x, y) \sin(\gamma_\beta(x, y) - \gamma(s))) ds d\beta \\ &= \int_0^{2\pi} \int p(s, \beta) w_{2\pi}(x, y; \beta + \gamma(s)) J(s) \left(\frac{s_\beta(x, y) - s}{L_\beta(x, y) \sin(\gamma_\beta(x, y) - \gamma(s))}\right)^2 h_*(s_\beta(x, y) - s) ds d\beta. \end{aligned}$$

Hereafter, we assume that $w_{2\pi}(x, y; \varphi)$ is independent of (x, y) and define $w_{2\pi}(s, \beta) \triangleq w_{2\pi}(x, y; \beta + \gamma(s))$. For certain special fan-beam geometries, we can express the sin term in the preceding denominator as follows [140]:

$$\sin(\gamma_\beta(x, y) - \gamma(s)) = W_2(x, y, \beta) W_1(s) W_0(s_\beta(x, y) - s), \quad (3.9.24)$$

for some geometry-dependent weighting functions W_0 , W_1 , and W_2 . For such geometries, the fan-beam reconstruction formula simplifies as follows:

$$f(x, y) = \int_0^{2\pi} \frac{1}{W_2^2(x, y, \beta) L_\beta^2(x, y)} \left[\int p(s, \beta) \frac{w_{2\pi}(s, \beta) J(s)}{W_1^2(s)} g_*(s_\beta(x, y) - s) ds \right] d\beta, \quad (3.9.25)$$

¹⁵Using this property leads to nonuniform image noise variance in practice [65], and an alternative approach based on Hilbert transforms has been proposed [142].

where the inner integral is convolution with the following modified ramp filter:

$$g_*(s) \triangleq \left(\frac{s}{W_0(s)} \right)^2 h_*(s). \quad (3.9.26)$$

Thus, the fan-beam FBP method uses the following three steps.

- Step 1. Compute *weighted* projections for each β :

$$\tilde{p}(s, \beta) \triangleq p(s, \beta) \frac{w_{2\pi}(s, \beta) J(s)}{W_1^2(s)}. \quad (3.9.27)$$

- Step 2. Filter those weighted projections (along s) for each β using the modified ramp filter (3.9.26):

$$\check{p}(s, \beta) \triangleq \tilde{p}(s, \beta) * g_*(s), \quad \forall \beta.$$

- Step 3. Perform a *weighted* backprojection of those filtered projections:

$$f(x, y) = \int_0^{2\pi} \frac{1}{W_2^2(x, y, \beta) L_\beta^2(x, y)} \check{p}(s_\beta(x, y), \beta) d\beta. \quad (3.9.28)$$

This three-step procedure is quite practical and has been used routinely in commercial X-ray CT systems.

The existence of this efficient FBP approach hinges on whether a factorization of the form (3.9.24) exists. There are exactly four fan-beam geometries for which this is possible [140]. Fortunately, two of the four cases are the important ones described in (3.9.2). Somewhat unfortunately, the 4th generation CT scanner geometry is *not* one of the four cases, although most commercial CT scanners now have 3rd-generation geometries.

3.9.2.1 Equiangular case

The case where $D_{fs} = 0$ corresponds to 3rd-generation X-ray CT systems using an arc detector, called the **equiangular case** [143]. In this case, $\gamma(s) = s/D_{sd}$, so we factor the sin term in (3.9.24) as follows:

$$\sin(\gamma_\beta(x, y) - \gamma(s)) = W_2 W_0(s_\beta(x, y) - s),$$

$$\text{where } W_2 = \frac{1}{D_{sd}}, \quad W_0(s) = D_{sd} \sin\left(\frac{s}{D_{sd}}\right), \quad s_\beta(x, y) = D_{sd} \gamma_\beta(x, y),$$

with $W_1 = 1$. Thus the modified ramp filter (3.9.26) is

$$g_*(s) = \left(\frac{s/D_{sd}}{\sin(s/D_{sd})} \right)^2 h_*(s). \quad (3.9.29)$$

Because $\dot{\gamma}(s) = 1/D_{sd}$, the projection weighting ratio in (3.9.27) simplifies to

$$\frac{J(s)}{W_1^2} = J(s) = \frac{1}{D_{sd}} \left| D_{s0} \cos \frac{s}{D_{sd}} - r_{\text{off}} \sin \frac{s}{D_{sd}} \right| \approx \frac{D_{s0}}{D_{sd}} \cos \frac{s}{D_{sd}}, \quad (3.9.30)$$

and the backprojection weighting in (3.9.28) is

$$\frac{1}{W_2^2 L_\beta^2(x, y)} = \frac{D_{sd}^2}{L_\beta^2(x, y)} = \frac{D_{sd}^2}{(D_{s0} - y_\beta)^2 + (x_\beta - r_{\text{off}})^2}. \quad (3.9.31)$$

Note that as $D_{s0} \rightarrow \infty$, *i.e.*, as the rays become more parallel, the modified ramp filter approaches the usual ramp filter, and the projection and backprojection weighting ratios approach unity, as expected for the parallel-ray case.

For practical implementation the filter must be band-limited and sampled. Combining (3.9.29) with (3.4.14) for $h_A(r)$, the necessary filter samples are

$$h[n] = \begin{cases} \frac{1}{4\Delta_s^2}, & n = 0 \\ 0, & n \text{ even} \\ \frac{-1}{[\pi D_{sd} \sin(n \Delta_s / D_{sd})]^2}, & n \text{ odd}, \end{cases} \quad (3.9.32)$$

where Δ_s denotes the detector element spacing. (Note that there is an error in [5, eqn. (96)].)

There is a subtle issue in the above derivation [65, 142, 144]. We first applied the scaling property of the ramp filter, continued the derivation, and then eventually considered a band-limited ramp filter (3.9.32). If we had considered the band-limiting effect first, which might be more appropriate for sampled data, then the scaling property would not hold exactly. It has been stated that this practice “results in a nonstationary cutoff frequency in the image” [65, 142]. Nevertheless, it is used widely.

MIRT See `fbp_ramp.m`.

MIRT See `fbp_fan_arc.m`.

3.9.2.2 Equidistant case

The case where $D_{fs} = \infty$ corresponds to a flat detector, called the **equidistant case** [145]. In this case, $\gamma(s) = \arctan(s/D_{sd})$ and (with some trigonometric identities) we factor the sin term in (3.9.24) as follows:

$$\sin(\gamma' - \gamma) = \sin \gamma' \cos \gamma - \sin \gamma \cos \gamma' = \cos \gamma' \cos \gamma (\tan \gamma' - \tan \gamma) = \cos(\gamma') \frac{D_{sd}}{\sqrt{D_{sd}^2 + s^2}} \frac{s' - s}{D_{sd}},$$

where $s_\beta(x, y) = D_{sd} \tan \gamma_\beta(x, y)$, or equivalently:

$$W_2 = \frac{\cos \gamma_\beta(x, y)}{D_{sd}}, \quad W_1 = \frac{D_{sd}}{\sqrt{D_{sd}^2 + s^2}}, \quad W_0(s) = s.$$

Because $W_0(s) = s$, we can use the usual ramp filter (3.4.14) without modification. Because here

$$\dot{\gamma}(s) = (\cos^2 \gamma(s))/D_{sd} = W_1^2(s)/D_{sd},$$

the projection weighting in (3.9.27) becomes

$$w_{2\pi}(s, \beta) \frac{J(s)}{W_1^2} = \frac{w_{2\pi}(s, \beta)}{D_{sd}} |D_{s0} \cos \gamma(s) - r_{\text{off}} \sin \gamma(s)| \approx w_{2\pi}(s, \beta) \frac{D_{s0}}{\sqrt{D_{sd}^2 + s^2}}, \quad (3.9.33)$$

and the backprojection weighting in (3.9.28) is

$$\frac{1}{W_2^2 L_\beta^2(x, y)} = \frac{D_{sd}^2}{(\cos \gamma_\beta(x, y) L_\beta(x, y))^2} = \frac{D_{sd}^2}{(D_{s0} - y_\beta)^2}. \quad (3.9.34)$$

Note that as $D_{s0} \rightarrow \infty$, the projection and backprojection weighting ratios again approach unity.

MIRT See `fbp_fan_flat.m`.

The preceding derivation started with the parallel-ray FBP formula. Other formulations are possible, *e.g.*, [146].

3.9.3 FBP for short scans (s,tomo,fan,short)

The preceding analyses have assumed a full 360° rotation. For a 360° rotation, every ray is sampled exactly twice, leading to the $1/2$ factor in (3.9.16). There are also fan-beam reconstruction methods that use a **short scan** with less than a full rotation. A standard **short scan** uses $\beta_{\max} = \pi + 2\gamma_{\max}$, so that every point in the equivalent parallel-beam sinogram is sampled at least once. As illustrated in Fig. 3.9.4, some areas of the sinogram are sampled twice, so these methods require appropriate weighting, sometimes called **Parker weighting** [147], related to the $w_{2\pi}(s, \beta)$ factor in (3.9.25). In particular, by the symmetry property (3.9.14), fan-beam sinogram data where $0 < \beta < 2(\gamma_{\max} - \gamma(s))$ correspond to the same rays as $\pi - 2\gamma(s) < \beta < \pi + 2\gamma_{\max}$.

One can compensate for this sampling by defining appropriately the sinogram weighting function $w_{2\pi}(s, \beta)$ in (3.9.25) and (3.9.27), *i.e.*, by applying a weighting such as the following to the fan-beam sinogram $p(s, \beta)$ prior to filtering and backprojection:

$$w_{2\pi}(s, \beta) = \begin{cases} q\left(\frac{\beta}{2(\gamma_{\max} - \gamma)}\right), & 0 \leq \beta \leq 2(\gamma_{\max} - \gamma) \\ 1, & 2(\gamma_{\max} - \gamma) < \beta < \pi - 2\gamma \\ q\left(\frac{\pi + 2\gamma_{\max} - \beta}{2(\gamma_{\max} + \gamma)}\right), & \pi - 2\gamma \leq \beta \leq \pi + 2\gamma_{\max}, \end{cases} \quad (3.9.35)$$

where $q(x) = \sin^2\left(\frac{\pi}{2}x\right)$ and $\gamma = \gamma(s)$. Note that [147, eqn. (12)] has a typo. This weighting ensures that the relevant **conjugate rays** (*cf.* (3.9.14)) have weights that sum to unity, *i.e.*, $w_{2\pi}(s, \beta) + w_{2\pi}(-s, \beta + 2\gamma(s) \pm \pi) = 1$.

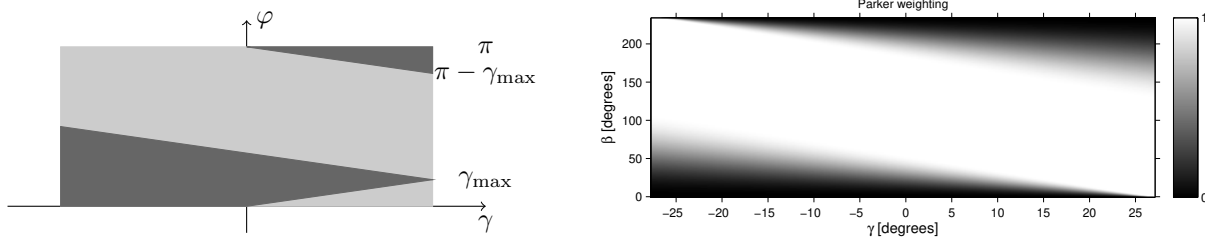


Figure 3.9.4: For short scan with $\beta_{\max} = \pi + 2\gamma_{\max}$, the left figure above illustrates (by shading) the areas of a sinogram (in ϕ, γ space) that are sampled twice. The other areas are sampled once. The right figure shows the fan-beam sinogram Parker weighting (3.9.35). fig_tomo_fan_short

MIRT See `fbp_fan_short_wt.m`.

Similar considerations apply to parallel-beam FBP with more than 180° rotation.

MIRT See `ir_fbp2_test_non180.m`.

Amazingly, one can also reconstruct images from *less* than a short scan [148]. (Interestingly, that 2D work was inspired by Katsevich’s 3D helical work [149].) Of course one can apply iterative reconstruction methods to arbitrary geometries. The relative performance of iterative methods and analytical methods for the “less than a short scan” cases is an interesting *open problem*.

s,tomo,fan,bpf

3.9.4 The backproject-filter (BPF) approach (s,tomo,fan,bpf)

Although the filter-backproject approach is used most widely for fan-beam reconstruction, it is also possible to use **backproject-filter (BPF)** methods for fan-beam tomography [20].

Given fan-beam projections $p(s, \beta)$ as defined in (3.9.11), for $\beta \in [0, \beta_{\max}]$, and given a weighting function $w_{\text{BPF}}(s, \beta)$, the weighted backprojection is given by

$$b(x, y) = \int_{-s_{\max}}^{s_{\max}} \int_0^{\beta_{\max}} \delta(x \cos \varphi(s, \beta) + y \sin \varphi(s, \beta) - r(s)) p(s, \beta) w_{\text{BPF}}(s, \beta) ds d\beta.$$

We now analyze the impulse response of the sequence of linear operations $f(x, y) \rightarrow p(s, \beta) \rightarrow b(x, y)$, by considering an object $f(x, y)$ that is a Dirac impulse at (x_0, y_0) , for which the fan-beam projections are given by $p(s, \beta) = \delta(x_0 \cos \varphi(s, \beta) + y_0 \sin \varphi(s, \beta) - r(s))$. The overall impulse response of the weighted projection / backprojection operation is thus:

$$h(x, y; x_0, y_0) = \int_0^{\beta_{\max}} \int_{-s_{\max}}^{s_{\max}} \delta(x \cos \varphi(s, \beta) + y \sin \varphi(s, \beta) - r(s)) \delta(x_0 \cos \varphi(s, \beta) + y_0 \sin \varphi(s, \beta) - r(s)) w_{\text{BPF}}(s, \beta) ds d\beta.$$

Now make the change of variables defined in (3.9.7), the Jacobian determinant of which is given in (3.9.19). Assume that the weighting term $w_{\text{BPF}}(s, \beta)$ consists of two terms:

- a term that compensates for the Jacobian determinant in (3.9.19) [20],
- a term that ensures that for any (r, φ) pair that gets contributions from more than one (s, β) pair, these (s, β) pairs are appropriately weighted [5, p. 98] [150]. (That weighting is simply 1/2 for a full rotation where $\beta_{\max} = 2\pi$.)

Assume furthermore that $\beta_{\max} \geq \pi + 2\gamma_{\max}$, so that there is complete sampling. Then after making the above change of variables, the impulse response expression simplifies to

$$\begin{aligned} h(x, y; x_0, y_0) &= \int_0^\pi \int_{-r_{\max}}^{r_{\max}} \delta(x \cos \varphi + y \sin \varphi - r) \delta(x_0 \cos \varphi + y_0 \sin \varphi - r) dr d\varphi \\ &= \int_0^\pi \delta((x - x_0) \cos \varphi + (y - y_0) \sin \varphi) \mathbb{I}_{\{|x_0 \cos \varphi + y_0 \sin \varphi| \leq r_{\max}\}} d\varphi \\ &= \int_0^\pi \delta(r_\pm(x - x_0, y - y_0) \cos(\varphi - \angle_\pi(x - x_0, y - y_0))) \mathbb{I}_{\{|x_0 \cos \varphi + y_0 \sin \varphi| \leq r_{\max}\}} d\varphi \\ &= \frac{1}{|r_\pm(x - x_0, y - y_0)|} \mathbb{I}_{\{|x_0 \cos \varphi + y_0 \sin \varphi| \leq r_{\max}\}} \Big|_{\varphi = \angle_\pi(x - x_0, y - y_0) \pm \pi/2}, \end{aligned}$$

where r_{\max} was defined in (3.9.9). Thus, *within the field of view*, the composition of projection with *appropriately weighted* backprojection is linear and shift invariant, and similarly to Theorem 3.3.1, in polar coordinates the impulse response is [151]: $h(r) = \frac{1}{|r|}$. Using this “restricted” shift invariance, one can formulate an approximate BPF reconstruction method [20]. However, the practical problems described in §3.4.2 are exacerbated here by the inherently finite field of view of the fan-beam geometry.

3.10 3D cone-beam reconstruction (s,3d,cone)

s, 3d, cone

A particularly challenging problem for analytical reconstruction methods is that of reconstructing a 3D object from cone-beam projection views. As X-ray detectors have evolved from 1D to 2D arrays, the practical importance of this problem has increased. The literature on this problem is extensive, and includes both exact and approximate methods, for various types of X-ray source trajectories and detector configurations. (See §4.6.1 for the **3D parallel-beam** case.)

For the case of a circular X-ray source trajectory (**axial scan**), the most popular method is the **Feldkamp cone-beam algorithm** or **FDK approach** [152]. This approach extends the fan-beam reconstruction formula (3.9.28) to the cone-beam case; it matches the fan-beam formula in the plane of the circular source trajectory, but is inexact (due to missing data) away from that plane. Exact 3D reconstruction for a circular trajectory (**axial scan**) is likely impossible because such a scan does not satisfy Tuy’s conditions [153].

Here, we summarize the classical FDK method for a 360° circular trajectory. The basic idea of the FDK method is to filter (a weighted version of) the measured data along each row of the detector as if it were part of a 2D fan-beam acquisition, and then to perform 3D backprojection. With this approximation, the central slice is reconstructed exactly as it would be by conventional 2D fan-beam methods, whereas the non-central slices have artifacts that increase with distance from the central slice. We focus on the case $r_{\text{off}} = 0$ for simplicity.

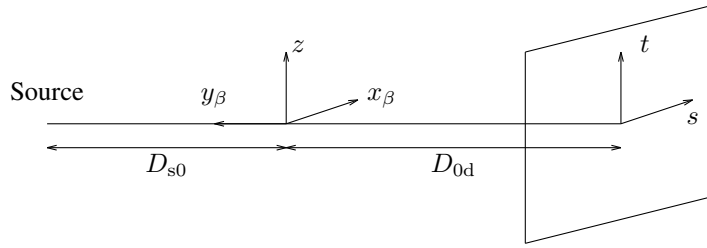


Figure 3.10.1: Cone-beam flat-detector geometry.

fig_tomo_cone

3.10.1 Equidistant case (flat detector)

Fig. 3.10.1 illustrates the flat-detector cone-beam geometry, where the rotated coordinates (x_β, y_β) were defined in (3.9.21). As in Fig. 3.9.1, let β denote the angle of the source point counter-clockwise from the y axis. The cone-beam line-integral projections are given by

$$p(s, t; \beta) \triangleq \int_{\mathcal{L}(s, t, \beta)} f(x, y, z) d\ell, \quad (3.10.1)$$

e, 3d, cone, pstb

where the locations of the source and a point on the detector and the line between them are

$$\vec{p}_0 = \vec{p}_0(\beta) \triangleq (-D_{s0} \sin \beta, D_{s0} \cos \beta, 0) \quad (3.10.2)$$

$$\vec{p}_1 = \vec{p}_1(s, t; \beta) \triangleq (s \cos \beta + D_{0d} \sin \beta, s \sin \beta - D_{0d} \cos \beta, t) \quad (3.10.3)$$

$$\mathcal{L}(s, t, \beta) \triangleq \left\{ \vec{p}_0 + \alpha \frac{\vec{p}_1 - \vec{p}_0}{\|\vec{p}_1 - \vec{p}_0\|} : \alpha \in \mathbb{R} \right\}. \quad (3.10.4)$$

e, 3d, cone, Lstb

Note that $\vec{p}_1 - \vec{p}_0 = (s \cos \beta + D_{sd} \sin \beta, s \sin \beta - D_{sd} \cos \beta, t)$ and $\|\vec{p}_1 - \vec{p}_0\| = \sqrt{D_{sd}^2 + s^2 + t^2}$.

Consider one row of the measured projection at angle β and vertical position t . That row forms a **tilted fan** with the source point, with distances “inflated” by the factor

$$\frac{\|\vec{p}_1(s, t; \beta) - \vec{p}_0\|}{\|\vec{p}_1(s, 0; \beta) - \vec{p}_0\|} = \frac{\sqrt{D_{sd}^2 + s^2 + t^2}}{\sqrt{D_{sd}^2 + s^2}}.$$

Following §3.9.2.2 we apply the projection weighting in (3.9.33), compensating for the above inflation factor, *i.e.*,

$$w_1(s, t) \triangleq \frac{D_{s0}}{\sqrt{D_{sd}^2 + s^2}} \frac{\|\vec{p}_1(s, 0; \beta) - \vec{p}_0\|}{\|\vec{p}_1(s, t; \beta) - \vec{p}_0\|} = \frac{D_{s0}}{\sqrt{D_{sd}^2 + s^2 + t^2}}. \quad (3.10.5)$$

We apply the usual ramp filter (3.4.14) to the (weighted) measurements from each row of the detector. Then we perform 3D cone-beam backprojection using the image-domain weighting given in (3.9.34). No modifications of (3.9.34) are needed. To summarize, the FDK algorithm for the equidistant case consists of the following steps.

- Step 1. Use (3.10.5) to compute weighted projections [5, p. 106, eqn. (175)]:

$$\tilde{p}(s, t; \beta) \triangleq w_1(s, t) p(s, t; \beta), \quad w_1(s, t) = \frac{D_{s0}}{\sqrt{D_{sd}^2 + s^2 + t^2}}. \quad (3.10.6)$$

- Step 2. Filter each row of those projections using the ordinary ramp filter (3.4.14):

$$\check{p}(s, t; \beta) \triangleq \tilde{p}(s, t; \beta) * h_*(s), \quad (3.10.7)$$

where $h_*(s)$ denotes the ramp filter given in (3.4.13) or (3.4.14).

- Step 3. Perform weighted cone-beam backprojection of those filtered projections:

$$\hat{f}(x, y, z) \triangleq \frac{1}{2} \int_0^{2\pi} w_2(x, y, \beta) \check{p}(s_\beta(x, y), t_\beta(x, y, z); \beta) d\beta, \quad w_2(x, y, \beta) = \frac{D_{sd}^2}{(D_{s0} - y_\beta)^2}, \quad (3.10.8)$$

where from (3.9.21) and (3.9.23)

$$s_\beta(x, y) = \frac{D_{sd}}{D_{s0} - y_\beta} x_\beta, \quad t_\beta(x, y, z) = \frac{D_{sd}}{D_{s0} - y_\beta} z. \quad (3.10.9)$$

(The factor $\frac{D_{sd}}{D_{s0} - y_\beta}$ is a magnification factor.) For view β , the ray from the source through the point (x, y, z) intersects the detector at point $(s_\beta(x, y), t_\beta(x, y, z))$.

Although the above algorithm is inexact in general, if the object is “cylinder like,” *i.e.*, if $f(x, y, z) = f(x, y, 0)$, then one can verify that (3.10.8) is exact [154]. See Problem 3.40.

3.10.2 Equiangular case (3rd generation multi-slice CT)

For a 3rd-generation multi-slice CT scanner, the FDK algorithm has the same general steps as above. Using (3.9.8), here the distances in the tilted fans are “inflated” by the factor

$$\frac{\|\vec{p}_1(s, t; \beta) - \vec{p}_0\|}{\|\vec{p}_1(s, 0; \beta) - \vec{p}_0\|} = \frac{\sqrt{D_{sd}^2 + t^2}}{D_{sd}}.$$

Thus we replace the 1D weighting in (3.10.6) by

$$w_1(s, t) \triangleq \frac{D_{s0}}{D_{sd}} \cos\left(\frac{s}{D_{sd}}\right) \frac{\|\vec{p}_1(s, 0; \beta) - \vec{p}_0\|}{\|\vec{p}_1(s, t; \beta) - \vec{p}_0\|} = \frac{D_{s0}}{\sqrt{D_{sd}^2 + t^2}} \cos\left(\frac{s}{D_{sd}}\right), \quad (3.10.10)$$

The filtering in (3.10.7) for each projection view row uses the modified ramp filter (3.9.29) in its sampled form (3.9.32). The 2D weighting in (3.10.8) is replaced by the equiangular counterpart in (3.9.31). Finally, the $s_\beta(x, y)$ term in (3.10.8) is replaced by (3.9.23). (The $t_\beta(x, y, z)$ term in (3.10.9) is unchanged.)

Again, if the object is “cylinder like,” then one can verify that (3.10.8) is exact [154]. See Problem 3.39.

MIRT See feldkamp.m.

3.10.3 Extensions (data truncation, helical scans) (s,3d,extend)

3.10.3.1 Fourier-based methods for cone-beam reconstruction

For the 2D parallel-beam geometry, this chapter described three distinct methods: BPF, FBP, and the direct Fourier method of §3.4.1. For the 3D cone-beam geometry, we have focused solely on FDK – an FBP approach. For an approach that first converts 3D cone-beam data to 3D Radon data and then uses a Fourier-based inversion, see [155].

3.10.3.2 Cone-parallel rebinning s, tomo, cp

By analogy with the fan-beam case discussed in §3.9.1, there are **cone-parallel rebinning** methods for cone-beam CT that can lead to FBP methods with simplified computation and reduced noise for both axial [154] and helical [156, 157] geometries. The simplest rebinning approach converts the fans for each detector row into cone-parallel rays by adapting (3.9.15):

$$p_{\text{CP}}(r, \varphi; t) \triangleq p(s, t; \beta) \Big|_{s=s(r), \beta=\beta(r, \phi)} = p\left(\gamma^{-1}\left(\arcsin\left(\frac{r}{D_{s0}}\right)\right), t; \phi - \arcsin\left(\frac{r}{D_{s0}}\right)\right) \quad (3.10.11)$$

where γ^{-1} is defined by (3.9.3). In particular, for a 3rd generation (equiangular) cone-beam CT geometry where $D_{\text{fs}} = 0$,

$$p_{\text{CP}}(r, \varphi; t) = p\left(D_{\text{sd}} \arcsin\left(\frac{r}{D_{s0}}\right), t; \phi - \arcsin\left(\frac{r}{D_{s0}}\right)\right).$$

3.10.3.3 Offset detectors

If the detector is too narrow (transaxially) to record the entire projection of the object, sometimes its position is offset. This is called the **offset detector** geometry. Reconstruction from such data requires appropriate weighting functions [158].

3.10.3.4 Long object problem

In practice the object axial extent usually exceeds the axial range of the detector, which is called the **long object problem**. Extrapolation techniques have been proposed for reconstructing somewhat more of the axial extent of the object from such axially truncated data [159].

3.10.3.5 Helical scans

For **helical** source trajectories, Katsevich [149, 160] developed a “theoretically exact” FBP type of reconstruction method for cone-beam reconstruction, a significant breakthrough in that field. As a result, this topic is evolving rapidly, *e.g.*, [161]. Versions of the **Fourier-slice theorem** have even been proposed for fan-beam and cone-beam geometries [162]. In practice, usually “approximate” algorithms are often used rather than the exact methods to save computation. A typical approximate algorithm involves some rebinning, ramp filtering, and weighted backprojection steps, *e.g.*, [163]. A particularly simple method is **single slice rebinning** [164].

MIRT See `rebin_helix.m`.

3.11 Summary s, tomo, summ

This chapter has reviewed analytical methods for tomographic image reconstruction. We have considered the parallel-beam geometry, fan-beam geometries for both flat and curved detectors, and both types of cone-beam geometries. For analytical methods, each geometry requires its own derivation. In contrast, for iterative image reconstruction, the basic formulations are the same for any geometry.

Interestingly, even though the topics in this chapter have been studied for decades, advances continue to be made. For example, recently the subject of reconstructing regions of interest (from truncated sinogram data) has had breakthroughs [68, 165–167]. And there is ongoing progress on image reconstruction from the exponential Radon transform [168–170]. Another interesting problem is **interior tomography** [171–178].

Because of the shift invariance of (parallel-beam) projection and backprojection, the primary tool for understanding these methods is Fourier analysis. It is something of a leap from the Fourier focus of this chapter to the linear algebra focus of subsequent chapters; the next chapter attempts to partly bridge these approaches by using operators, the continuous-space analog of matrices.

3.12 Problems s, tomo, prob

Problem 3.1 Determine the Radon transform of the 2D function [179] $f_{\circ}(r) = (1 - r^2) \text{rect}(r/2)$.

Problem 3.2 Determine the Radon transform of the 2D function $f_{\circ}(r) = 2\sqrt{1 - r^2} \mathbb{I}_{\{|r| \leq 1\}}$.

p, tomo, separable

Problem 3.3 Use the Fourier-slice theorem to show that the Radon transform of any separable function $f(x, y) = a(x)b(y)$ is given by

$$p_\varphi(r) = \frac{1}{|\cos\varphi|} a\left(\frac{r}{\cos\varphi}\right) * \frac{1}{|\sin\varphi|} b\left(\frac{r}{\sin\varphi}\right).$$

Problem 3.4 Prove the **Laplacian property** (3.2.31) of the Radon transform.

Problem 3.5 Show that the Radon transform of the function $f(x, y) = \text{sinc}(x)\text{sinc}(y)$ is given by

$$p_\varphi(r) = \frac{1}{\max(|\cos\varphi|, |\sin\varphi|)} \text{sinc}\left(\frac{r}{\max(|\cos\varphi|, |\sin\varphi|)}\right).$$

Problem 3.6 Consider the following “star function” [17]:

$$\text{star}(r, \varphi) \triangleq \frac{1}{|r|} \sum_{n=1}^N \delta((\varphi - \varphi_n) \bmod \pi).$$

Show that the 2D FT of this function is $\text{star}(\rho, \Phi \pm \pi/2)$. This relation is useful for analyzing backprojection with discrete angular samples.

Problem 3.7 Prove each of the Radon transform properties in §3.2.3.

Problem 3.8 Find a Radon transform property for an **affine transformation** of the object $f(x, y)$, i.e.,

$$f(a_{11}x + a_{12}y + b_1, a_{21}x + a_{22}y + b_2)$$

with $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Problem 3.9 Show that if $0 < a \leq b$ then

$$\frac{1}{a} \text{rect}\left(\frac{x}{a}\right) * \frac{1}{b} \text{rect}\left(\frac{x}{b}\right) = \frac{b+a}{2ba} \text{tri}\left(\frac{2x}{b+a}\right) - \frac{b-a}{2ba} \text{tri}\left(\frac{2x}{b-a}\right),$$

where tri was defined in (3.2.42).

p, radon, flat

Problem 3.10 Find a nonzero object $f(x, y)$ (other than a Dirac impulse) whose projections have the property that they depend only on φ over their support, i.e.,

$$p_\varphi(r) = \begin{cases} g_\varphi, & r_{\min}(\varphi) \leq r \leq r_{\max}(\varphi) \\ 0, & \text{otherwise,} \end{cases}$$

for some functions g_φ , $r_{\min}(\varphi)$, and $r_{\max}(\varphi)$ that depend on $f(x, y)$.

Generalize to the case of fan-beam projections, i.e., find the object $f(x, y)$ for which $p(s, \beta) = \text{rect}\left(\frac{s}{2s_0}\right)$, where $s_0 \leq s_{\max}$. (Assume $r_{\text{off}} = 0$ for simplicity.)

Problem 3.11 Let $p_\varphi(r)$ denote the Radon transform of a disk object as considered in Example 3.2.1. Determine (analytically) $\check{p}_\varphi(r)$, the corresponding ramp-filtered projections, and plot. (Need typed.)

p, tomo, spline

Problem 3.12 Given noisy, blurred, samples of the Radon transform of an object, one interesting reconstruction approach is to first fit 1D spline functions to each projection view, and then filter those continuous-space fitted projections using analytical expressions, and then backproject [181]. In other words, we make the approximation $p_\varphi(r) = \sum_{n=-\infty}^{\infty} c_n b(r/\Delta_R - n)$, where the coefficients c_n are determined from the sampled projections by filtering [182]. The ramp-filtered projections $\check{p}_\varphi(r)$ have the form $\check{p}_\varphi(r) = \sum_{n=-\infty}^{\infty} c_n \check{b}(r/\Delta_R - n)$. Find an analytical expression for $\check{b}(\cdot)$ when $b(\cdot)$ is a cubic B-spline.

p, tomo, freq, dis

Problem 3.13 An interesting property of the Radon transform is the **frequency-distance relation** or **frequency-distance principle (FDP)** that describes the characteristics of the 2D Fourier transform of a sinogram [113, 114, 183–185]. This property has been used to reduce noise in sinograms [99].

- Consider $f(x, y) = \delta(x - x_0, y - y_0)$ and take the 2D FT of its sinogram to see where it is nonzero.
- Analyze the impulse response in the sinogram domain of such an apodization.

- Analyze how much noise reduction could be achieved with such method. (Solve?)
- Take the 2D sinogram of an image (over 360° for simplicity), compute its 2D DFT, and set to zero the coefficients outside the **bowtie region** described in the papers cited above, then inverse transform and reconstruct the image by FBP. Observe the effects.

Problem 3.14 §3.3 is missing an example where the laminogram $f_b(x, y)$ can be found analytically. Find an object $f(x, y)$ whose projections ($p_\varphi(r)$ or $P_\varphi(\nu)$) and laminogram $f_b(x, y)$ have simple analytical expressions. Hint: [15, p. 338] may be useful.

Problem 3.15 Prove the general back-projection relationship (3.15).

Problem 3.16 §3.4.6 described how it is preferable to sample the (band-limited) ramp filter in the space domain for the FBP method. For the BPF method, would it be advantageous to find the impulse response $h(r)$ of the (band-limited) **cone filter** and then sample it instead of sampling the cone directly? Discuss.

Problem 3.17 Some tomographic imaging systems cannot measure the projection views at every projection angle. Such angularly incomplete data is known as **limited angle** tomography and is notoriously challenging. One way to model the effects of such missing projection views is to set $w(\varphi)$ to zero in the backprojection formula (3.3.1) for angles φ that are missing, and to unity otherwise. Suppose we apply this idea to a system that only records projection views where $\varphi \in [0, 7\pi/8]$.

Sketch the support of the spectrum of the laminogram, i.e., $F_b(u, v)$ in this case.

Explain why this is called the “missing wedge” problem (in 2D) or the “missing cone” in such 3D problems.

Explain why the BPF method cannot work in this case. (Need typed.)

Problem 3.18 Let $b_\varphi(x, y) = p_\varphi(x \cos \varphi + y \sin \varphi)$ denote the image formed by backprojection of a projection at angle φ , and let $B_\varphi(u, v)$ denote its 2D Fourier transform. Relate $B_\varphi(u, v)$ to $F(u, v)$.

Hint: relate $B_0(u, v)$ to $F(u, 0) \delta(v)$.

(Need typed.)

Problem 3.19 The set (3.9.4) is too large to be realistic for a practical system like that illustrated in Fig. 3.9.1. Find a more realistic expression for $\mathcal{L}(s, \beta)$.

Problem 3.20 Verify the 360° FBP formula (3.9.17) using (3.2.29).

Problem 3.21 Each of the images in Fig. 3.12.1 exhibits some type of artifact due to data limitations or errors. Match the figure with the corresponding problem in the following list.

1. Limited projection angular range (less than 180°).
2. Under-sampled projection view angles.
3. Single detector channel with miscalibrated gain.
4. Misaligned detector (radial shift).
5. Projection views over $[0, 180^\circ]$ instead of $[0, 180^\circ \frac{N_A - 1}{N_A}]$
6. Fan-beam data into parallel-beam reconstructor

Problem 3.22 Modify (3.3.2) and (3.3.7) to consider the case of a 360° rotation. Show that the impulse response is $h(r, \varphi) = \frac{1}{|r|} [w(\varphi + \pi/2) + w(\varphi + 3\pi/2)]$ for $\varphi \in [-\pi/2, \pi/2]$ and the frequency response is $H(\rho, \Phi) = \frac{1}{|\rho|} [w(\Phi) + w(\Phi + \pi)]$. These relations are useful for analyzing 360° SPECT scans. (Need typed.)

Problem 3.23 Find a relationship between $\check{P}_\varphi(\nu)$ and $F_b(u, v)$ or $F_b(\rho, \Phi)$ to complete the bottom link in Fig. 3.3.3.

Problem 3.24 Find $\check{P}_\varphi(\nu)$ and $\check{p}_\varphi(r)$ in Example 3.4.5 when $\varphi = 0$.

Problem 3.25 Suppose the ideal ramp filter is applied to the projections $p_\varphi(r)$ of a **gaussian object** $f(x, y) = w^{-2} e^{-\pi(r/w)^2}$. Determine analytically the resulting filtered projections $\check{p}_\varphi(r)$ and plot.

Problem 3.26 Suppose the ideal ramp filter $|\nu|$ is apodized by a gaussian $A(\nu) = e^{-\pi\nu^2}$ instead of by the discontinuous rect function (3.4.12). Determine the resulting impulse response $h_A(r)$ of the apodized ramp filter.

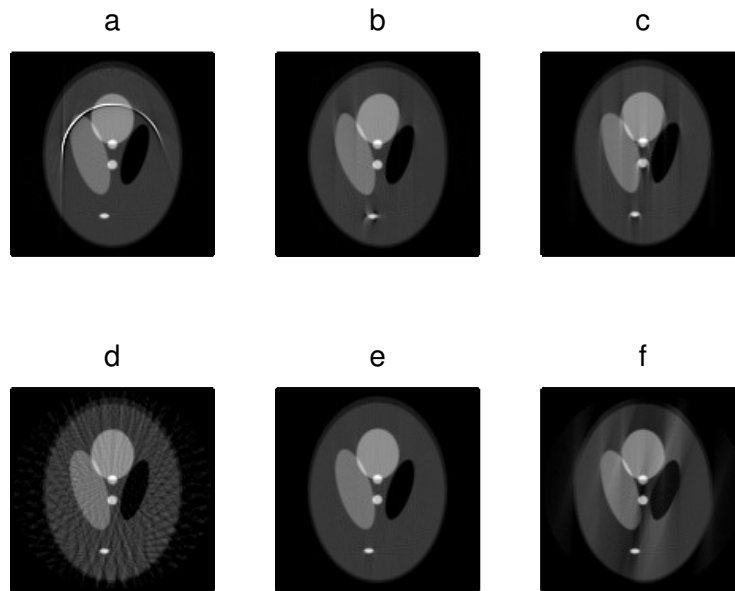


Figure 3.12.1: Illustration of common artifacts for Problem 3.21.

fig_tomo_artifact

Problem 3.27 Suppose that a **Hanning window** is applied to the ramp filter, i.e.,

$$A(\nu) = \text{rect}\left(\frac{\nu}{2\nu_0}\right) \frac{1 + \cos(\pi\nu/\nu_0)}{2}$$

in (3.4.10). Using Fourier transform properties, find analytically the impulse response $h_A(r)$ of the apodized ramp filter, and plot it. Compare to Fig. 3.4.6 and comment on the advantages and disadvantages of this choice.

Problem 3.28 Suppose the ramp filter is apodized using a triangular window function $A(\nu) = \text{tri}(\nu/\nu_c)$. Find an analytical expression for the impulse response $h_A(r)$ of the resulting apodized ramp filter using symbolic integration.

Problem 3.29 Suppose the apodizer $A(\nu) = e^{-a|\nu|}$ is used in the FBP method. Find the resulting image PSF $h(r)$.

Problem 3.30 Real tomographs can be aligned imprecisely, leading to offset projections: $q_\varphi(r) = p_\varphi(r - \tau)$ for some **offset center of rotation** τ . Determine the PSF of such an imaging system when the FBP method is applied to $q_\varphi(r)$. (Solve?)

Problem 3.31 Would the **inverse filter** approach of §3.5.4 work if we used nearest-neighbor interpolation, i.e., $h(r) = \text{rect}(r/\Delta_R)$? Explain why or why not.

Problem 3.32 A PET scanner measures line integrals along chords connecting detector pairs typically arranged around a circle. The natural measurement model would be $g(\varphi_1, \varphi_2) = \int_{\mathcal{L}(\varphi_1, \varphi_2)} f(x, y) d\ell$, where $\mathcal{L}(\varphi_1, \varphi_2)$ denotes the set of points along the chord connecting the point (r_0, φ_1) with the point (r_0, φ_2) (in polar coordinates), where r_0 is the radius of the system. Following §3.9, derive a BPF method for this geometry. It may be easier to use the transformed parameterization: $\varphi = (\varphi_1 + \varphi_2)/2$, $\gamma = (\varphi_2 - \varphi_1)/2$.

Problem 3.33 Using (3.2.14) and (3.2.30), determine the Radon transform of an ellipse object centered at (c_x, c_y) with major axes of half lengths r_x, r_y , tilted at an angle χ with respect to the horizontal axis. Assume the function is unity within the ellipse and zero outside. Generalize to the case of fan-beam projections.

Problem 3.34 The method called **lambda tomography** is a **local tomography** approach where one back-projects the second derivative $\frac{\partial^2}{\partial r^2} p_\varphi(r)$ of the projections [187–190]. Show that the frequency-domain relationship between the resulting backprojected image and the original image is

$$B(u, v) = -2\pi\Lambda(\rho) F(u, v),$$

where $\Lambda(\rho) = 2\pi|\rho|$ is the square root of $-\Delta$, the **positive Laplace operator**. In 2D, the positive Laplace operator is $-\Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$, cf. (2.4.1), which has Fourier transform $\hat{\Delta}(\rho) = (2\pi\rho)^2$.

p,tomo,hilbert

Problem 3.35 A disadvantage of the ramp filter in (3.4.2) and the Hilbert transform in (3.4.6) is that they both depend on $p_\varphi(r)$ for all $r \in \mathbb{R}$. Therefore, if the projection data are **truncated**, meaning available only for r in a finite interval $[r_{\min}, r_{\max}]$, then FBP methods based on the ramp filter will produce very large artifacts. Following [165], consider what happens if we simply differentiate each projection and then backproject:

$$b(x, y) \triangleq \frac{-1}{2\pi} \int_0^\pi \left. \frac{d}{dr} p_\varphi(r) \right|_{r=x \cos\varphi + y \sin\varphi} d\varphi. \tag{3.12.1}$$

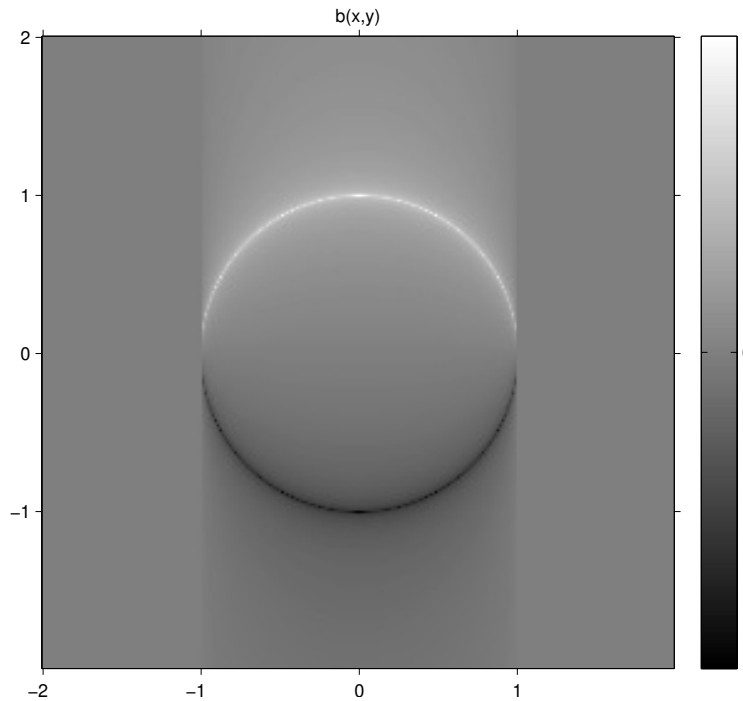
e,tomo,back,deriv

- Using (3.3.10) show that

$$B(u, v) = -i \operatorname{sgn}(v) F(u, v).$$

In other words, for each x , $b(x, y)$ is the **Hilbert transform** of $f(x, y)$ along the y -direction.

- Determine $b(x, y)$ analytically for the case $f(x, y) = \operatorname{rect}\left(\frac{r}{2r_0}\right)$. Hint: use Example 3.4.4, and see Fig. 3.12.2.



fig_tomo_prob_hilbert_disk

Figure 3.12.2: Backprojection of derivative of each projection of a disk of radius $r_0 = 1$, for Problem 3.35.

- Sample the solution in the previous part (ala Fig. 3.12.2) and apply a FFT-based inverse Hilbert transform method to recover $f[m, n]$ (approximately) from $b[m, n]$. Hint: use mirror end conditions along the y direction to avoid artifacts due to discontinuities at the image edge.
- to do b: try it in matlab
to do b: compare the FFT-based inverse Hilbert transform approach to the **finite Hilbert inverse** described in [191] [165] [192].

p,tomo,hilbert,laplace

Problem 3.36 Another way to write the ramp filter is $|\nu| = (-2\pi u^2) \left(\frac{1}{i2\pi\nu}\right) (-i \operatorname{sgn}(\nu))$. Each of the parenthesized terms is suggestive of a certain type of filtering operation.

Following [193], show that the following Hilbert/integrate/Laplacian approach to image reconstruction is valid.

- Hilbert transform each projection: $\bar{p}_\varphi(r) \xleftrightarrow{\text{FT}} \bar{P}_\varphi(\nu) \triangleq (-i \operatorname{sgn}(\nu)) P_\varphi(\nu)$
- Compute the anti-derivative of the Hilbert projections: $\hat{p}_\varphi(r) \triangleq \int_{-\infty}^r \bar{p}_\varphi(r') dr'$
- Back-project the anti-derivative projections: $b(x, y) \triangleq \int_0^\pi \hat{p}_\varphi(r) d\varphi$
- Laplacian filter the back-projected image: $f(x, y) = \frac{-1}{2\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) b(x, y)$

This method may be more robust than FBP to transaxially **truncated** projections.

p,tomo,gauss,joint

Problem 3.37 It is well known that if a pair of random variables are jointly gaussian distributed, then their marginal distributions are also gaussian. However, the converse is not true. Give an example of two random variables that

are individually gaussian distributed but their joint distribution is not gaussian. Do not use the degenerate example of $X \sim N(0, 1)$ and $Y = -X$. Hint. The relationship between the (2D) joint distribution and the (1D) marginal distributions is analogous to the Radon transform at 0° and 90° .

p, tomo, dpc

Problem 3.38 In **differential phase-contrast X-ray CT** [194], the ideal forward model is not given by (3.2.4) but rather is given by the derivative of (3.2.4) with respect to r . Because of the presence of this derivative, the filter needed for reconstruction is not the usual ramp filter in (3.4.12), but rather is a band-limited Hilbert transform-type filter with frequency response $H(\nu) = \frac{1}{i2\pi\nu} |\nu| \operatorname{rect}\left(\frac{\nu}{2\nu_0}\right) = \begin{cases} -\nu/(2\pi), & 0 \leq \nu \leq \nu_0 \\ \nu/(2\pi), & -\nu_0 \leq \nu < 0. \end{cases}$ Following Example 3.4.6, determine the sampled impulse response of this filter.

p, tomo, 3d, cyl, ea

Problem 3.39 Verify that the 3D equiangular (arc detector) FDK method in §3.10.2 is exact if the object is “cylinder like,” i.e., if $f(x, y, z) = f(x, y, 0)$.

p, tomo, 3d, cyl, ed

Problem 3.40 Verify that the 3D equidistant (flat detector) FDK method (3.10.8) is exact if the object is “cylinder like,” i.e., if $f(x, y, z) = f(x, y, 0)$.

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