

Multiplicative number theory

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2011

The Snowbird version
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*This is an early draft of the book we are writing on the subject.
You will find that the first eight or so chapters are written reasonably carefully.*

*After that, well, it's varied.
Hopefully the ideas are understandable,
even if the writing is not as clean as it might be.*

*This edition of the notes includes open problems;
you are invited to solve some of them!
We hope that the theory will be a lot cleaner by the time
that the book comes to print thanks to the
efforts of some of the participants here at Snowbird.*

*If you do solve some of the questions here, you are welcome to go
ahead and publish your results, of course. However please do not
reference this version of the book since it will soon be outdated.*

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PREFACE

Riemann's seminal 1860 memoir showed how questions on the distribution of prime numbers are more-or-less equivalent to questions on the distribution of zeros of the Riemann zeta function. This was the starting point for the beautiful theory which is at the heart of analytic number theory. Heretofore there has been no other coherent approach that was capable of addressing all of the central issues of analytic number theory.

In this book we present the pretentious view of analytic number theory; allowing us to recover the basic results of prime number theory *without* use of zeros of the Riemann zeta-function and related L -functions, and to improve various results in the literature. This approach is certainly more flexible than the classical approach since it allows one to work on many questions for which L -function methods are not suited. However there is no beautiful explicit formula that promises to obtain the strongest believable results (which is the sort of thing one obtains from the Riemann zeta-function). So why *pretentious*?

- It is an intellectual challenge to see how much of the classical theory one can reprove without recourse to the more subtle L -function methodology (For a long time, top experts had believed that it is impossible to prove the prime number theorem without an analysis of zeros of analytic continuations. Selberg and Erdős refuted this prejudice but until now, such methods had seemed *ad hoc*, rather than part of a coherent theory).

- Selberg showed how sieve bounds can be obtained by optimizing values over a wide class of combinatorial objects, making them a very flexible tool. Pretentious methods allow us to introduce analogous flexibility into many problems where the issue is not the properties of a very specific function, but rather of a broad class of functions.

- This flexibility allows us to go further in many problems than classical methods alone, as we shall see in the latter chapters of this book.

The Riemann zeta-function $\zeta(s)$ is defined when $\operatorname{Re}(s) > 1$; and then it is given a value for each $s \in \mathbb{C}$ by the theory of analytic continuation. Riemann pointed to the study of the zeros of $\zeta(s)$ on the line where $\operatorname{Re}(s) = 1/2$. However we have few methods that truly allow us to say much so far away from the original domain of definition. Indeed almost all of the unconditional results in the literature are about understanding zeros with $\operatorname{Re}(s)$ very close to 1. Usually the methods used to do so, can be viewed as an extrapolation of our strong understanding of $\zeta(s)$ when $\operatorname{Re}(s) > 1$. This suggests that, in proving these results, one can perhaps dispense with an analysis of the values of $\zeta(s)$ with $\operatorname{Re}(s) \leq 1$, which is, in effect, what we do.

Our original goal in the first part of this book was to recover all the main results of Davenport's *Multiplicative Number Theory* [21] by pretentious methods, and then to prove as much as possible of the result of classical literature, such as the results in [7]. It turns out that pretentious methods yield a much easier proof of Linnik's Theorem, and quantitatively yield much the same quality of results throughout the subject.

However Siegel's Theorem, giving a lower bound on $|L(1, \chi)|$, is one result that we have little hope of addressing without considering zeros of L -functions. The difficulty is that all proofs of his lower bound run as follows: Either the Generalized Riemann Hypothesis (GRH) is true, in which case we have a good lower bound, or the GRH is false, in which case we have a lower bound in terms of the first counterexample to GRH. Classically this explains the inexplicit constants in analytic number theory (evidently Siegel's lower bound cannot be made explicit unless another proof is found, or GRH is resolved) and, without a fundamentally different proof, we have little hope of avoiding zeros. Instead we give a proof, due to Pintz, that is formulated in terms of multiplicative functions and a putative zero.

Although this is the first coherent account of this theory, our work rests on ideas that have been around for some time, and the contributions of many authors. The central role in our development belongs to Halász's Theorem. Much is based on the results and perspectives of Paul Erdős and Atle Selberg. Other early authors include Wirsing, Halász, Daboussi and Delange. More recent influential authors include Elliott, Hall, Hildebrand, Iwaniec, Montgomery and Vaughan, Pintz, and Tenenbaum. In addition, Tenenbaum's book [101] gives beautiful insight into multiplicative functions, often from a classical perspective.

Our own thinking has developed in part thanks to conversations with our collaborators John Friedlander, Régis de la Bréteche and Antal Balog. We are particularly grateful to Dimitris Koukoulopoulos who has been working with us while we have worked on this book, and proved several results that we needed, when we needed them!

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THE PRIME NUMBER THEOREM

As a boy Gauss determined that the density of primes around x is $1/\log x$, leading him to conjecture that the number of primes up to x is well-approximated by the estimate

$$\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x}. \quad (1.1) \quad \boxed{\text{PNT}}$$

It may seem less intuitive, but in fact it is simpler to weight each prime with $\log p$; and, as we have seen, it is natural to throw the prime powers into this sum, which has little impact on the size. Thus we define the von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^m, \text{ where } p \text{ is prime, and } m \geq 1 \\ 0 & \text{otherwise,} \end{cases} \quad (1.2) \quad \boxed{\text{vM}}$$

and then, in place of $\boxed{\text{PNT}}$, we conjecture that

$$\psi(x) := \sum_{n \leq x} \Lambda(n) \sim x. \quad (1.3) \quad \boxed{\text{PNT2}}$$

The equivalent estimates $\boxed{\text{PNT}}$ and $\boxed{\text{PNT2}}$, known as the *prime number theorem*, are difficult to prove. In this chapter we show how the prime number theorem is equivalent to understanding the mean value of the Möbius function. This will motivate our study of multiplicative functions in general, and provide new ways of looking at many of the classical questions in analytic number theory.

1.1 Partial Summation

We begin with a useful technique known as *Abel's partial summation*. Let a_n be a sequence of complex numbers, and let $f : \mathbb{R} \rightarrow \mathbb{C}$ be some function. Set $S(t) = \sum_{k \leq t} a_k$, and our goal is to understand

$$\sum_{n=A+1}^B a_n f(n)$$

in terms of the partial sums $S(t)$. Let us first assume that $A < B$ are non-negative integers. Since $a_n = S(n) - S(n-1)$ we may write

$$\sum_{n=A+1}^B a_n f(n) = \sum_{n=A+1}^B f(n)(S(n) - S(n-1)),$$

and with a little rearranging we obtain

$$\sum_{n=A+1}^B a_n f(n) = S(B)f(B) - S(A)f(A) - \sum_{n=A}^{B-1} S(n)(f(n+1) - f(n)). \quad (1.4) \quad \boxed{\text{PS1}}$$

If now we suppose that f is continuously differentiable on $[A, B]$ then we may write the above as

$$\sum_{A < n \leq B} a_n f(n) = S(B)f(B) - S(A)f(A) - \int_A^B S(t)f'(t)dt. \quad (1.5) \quad \boxed{\text{PS2}}$$

We leave to the reader to check that $\boxed{\text{PS2}}$ continues to hold for all non-negative real numbers $A < B$. If we think of $\sum_{A < n \leq B} a_n f(n)$ as the Riemann-Stieltjes integral $\int_{A+}^{B+} f(t)d(S(t))$ then $\boxed{\text{PS2}}$ amounts to integration by parts.

Exercise 1.1 Using partial summation show that $\boxed{\text{PNT}}$ and $\boxed{\text{PNT2}}$ are equivalent, and that both are equivalent to

$$\theta(x) = \sum_{p \leq x} \log p = x + o(x). \quad (1.6) \quad \boxed{\text{PNT3}}$$

ex:harmonic

Exercise 1.2 Using partial summation, prove that for any integer $N \geq 1$

$$\sum_{n=1}^N \frac{1}{n} = \log N + 1 - \int_1^N \frac{\{t\}}{t^2} dt,$$

where throughout we write $[t]$ for the integer part of t , and $\{t\}$ for its fractional part (so that $t = [t] + \{t\}$). Deduce that for any real $x \geq 1$

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where γ is the Euler-Mascheroni constant

$$\gamma := \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \log N \right) = 1 - \int_1^{\infty} \frac{\{t\}}{t^2} dt.$$

ex:stirling

Exercise 1.3 For an integer $N \geq 1$ show that

$$\log N! = N \log N - N + 1 + \int_1^N \frac{\{t\}}{t} dt.$$

Using that $\int_1^x (\{t\} - 1/2)dt = (\{x\}^2 - \{x\})/2$ and integrating by parts, show that

$$\int_1^N \frac{\{t\}}{t} dt = \frac{1}{2} \log N - \frac{1}{2} \int_1^N \frac{\{t\} - \{t\}^2}{t^2} dt.$$

Conclude that $N! \sim C\sqrt{N}(N/e)^N$. Here one also knows that

$$C = \exp\left(1 - \frac{1}{2} \int_1^\infty \frac{\{t\} - \{t\}^2}{t^2} dt\right) = \sqrt{2\pi},$$

and the resulting asymptotic for $N!$ is known as Stirling's formula.

Recall that the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Here the Dirichlet series and the Euler product both converge absolutely in the region $\operatorname{Re}(s) > 1$.

zeta

Exercise 1.4 Prove that for $\operatorname{Re}(s) > 1$

$$\zeta(s) = s \int_1^\infty \frac{[y]}{y^{s+1}} dy = \frac{s}{s-1} - s \int_1^\infty \frac{\{y\}}{y^{s+1}} dy.$$

Observe that the right hand side above is an analytic function of s in the region $\operatorname{Re}(s) > 0$ except for a simple pole at $s = 1$ with residue 1. Thus we have an analytic continuation of $\zeta(s)$ to this larger region, and near $s = 1$ we have the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + \dots$$

Adapting the argument in Exercise 1.3 ^{ex:stirling} obtain an analytic continuation of $\zeta(s)$ to the region $\operatorname{Re}(s) > -1$. Generalize.

1.2 Chebyshev's elementary estimates

Chebyshev made significant progress on the distribution of primes by showing that there are constants $0 < c < 1 < C$ with

$$(c + o(1)) \frac{x}{\log x} \leq \pi(x) \leq (C + o(1)) \frac{x}{\log x}. \quad (1.7) \quad \text{Cheb1}$$

Moreover he showed that if

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x/\log x}$$

exists, then it must equal 1.

The key to obtaining such information is to write the prime factorization of n in the form

$$\log n = \sum_{d|n} \Lambda(d).$$

Summing both sides over n (and re-writing “ $d|n$ ” as “ $n = dk$ ”), we obtain that

$$\sum_{n \leq x} \log n = \sum_{n \leq x} \sum_{n=dk} \Lambda(d) = \sum_{k=1}^{\infty} \psi(x/k). \quad (1.8) \quad \boxed{\text{Cheb2}}$$

Using Stirling's formula, Exercise [ex:stirling](#) I.3, we deduce that

$$\sum_{k=1}^{\infty} \psi(x/k) = x \log x - x + O(\log x). \quad (1.9) \quad \boxed{\text{Cheb3}}$$

Exercise 1.5 Deduce that

$$\limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq 1 \geq \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x},$$

so that if $\lim_{x \rightarrow \infty} \psi(x)/x$ exists it must be 1.

To obtain Chebyshev's estimates ([Cheb1](#) I.7), take ([Cheb2](#) I.8) at $2x$ and subtract twice that relation taken at x . This yields

$$x \log 4 + O(\log x) = \psi(2x) - \psi(2x/2) + \psi(2x/3) - \psi(2x/4) + \dots,$$

and upper and lower estimates for the right hand side above follow upon truncating the series after an odd or even number of steps. In particular we obtain that

$$\psi(2x) \geq x \log 4 + O(\log x),$$

which gives the lower bound of ([Cheb1](#) I.7) with $c = \log 2$ a permissible value. And we also obtain that

$$\psi(2x) - \psi(x) \leq x \log 4 + O(\log x),$$

which, when used at $x/2, x/4, \dots$ and summed, leads to $\psi(x) \leq x \log 4 + O((\log x)^2)$. Thus we obtain the upper bound in ([Cheb1](#) I.7) with $C = \log 4$ a permissible value.

ex:Bertrand

Exercise 1.6 Using that $\psi(2x) - \psi(x) + \psi(2x/3) \geq x \log 4 + O(\log x)$, prove Bertrand's postulate that there is a prime between N and $2N$.

Returning to ([Cheb2](#) I.8), we may recast it as

$$\sum_{n \leq x} \log n = \sum_{d \leq x} \Lambda(d) \sum_{k \leq x/d} 1 = \sum_{d \leq x} \Lambda(d) \left(\frac{x}{d} + O(1) \right).$$

Using Stirling's formula, and the recently established $\psi(x) = O(x)$, we conclude that

$$x \log x + O(x) = x \sum_{d \leq x} \frac{\Lambda(d)}{d},$$

or in other words

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p \leq x} \frac{\log p}{p} + O(1) = \log x + O(1). \quad (1.10) \quad \boxed{\text{Pavg}}$$

Exercise 1.7 Show that (I.10) would follow from the prime number theorem and partial summation. Why does the prime number theorem not follow from (I.10) and partial summation? What stronger information on $\sum_{p \leq x} \log p/p$ would yield the prime number theorem?

Exercise 1.8 Use (I.10) and partial summation show that there is a constant c such that

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c + O\left(\frac{1}{\log x}\right).$$

Deduce Mertens' Theorem, that there exists a constant γ such that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}.$$

(In fact γ is the Euler-Mascheroni constant. There does not seem to be a straightforward, intuitive proof known that it is indeed this constant.)

1.3 Multiplicative functions and Dirichlet series

The main objects of study in this book are multiplicative functions. These are functions $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfying $f(mn) = f(m)f(n)$ for all coprime integers m and n . If the relation $f(mn) = f(m)f(n)$ holds for all integers m and n we say that f is completely multiplicative. If $n = \prod_j p_j^{\alpha_j}$ is the prime factorization of n , where the primes p_j are distinct, then $f(n) = \prod_j f(p_j^{\alpha_j})$ for multiplicative functions f . Thus a multiplicative function is specified by its values at prime powers and a completely multiplicative function is specified by its values at primes.

A handy way to study multiplicative functions is through Dirichlet series. We let

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right).$$

The product over primes above is called an Euler product, and viewed formally the equality of the Dirichlet series and the Euler product above is a restatement of the unique factorization of integers into primes. If we suppose that the multiplicative function f does not grow rapidly – for example, that $|f(n)| \ll n^A$ for some constant A – then the Dirichlet series and Euler product will converge absolutely in some half-plane with $\operatorname{Re}(s)$ suitably large.

Given any two functions f and g from $\mathbb{N} \rightarrow \mathbb{C}$ (not necessarily multiplicative), their Dirichlet convolution $f * g$ is defined by

$$(f * g)(n) = \sum_{ab=n} f(a)g(b).$$

If $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$ are the associated Dirichlet series, then the convolution $f * g$ corresponds to their product $F(s)G(s) = \sum_{n=1}^{\infty} (f * g)(n)n^{-s}$.

Here are some examples of the basic multiplicative functions and their associated Dirichlet series.

- The function $\delta(1) = 1$ and $\delta(n) = 0$ for all $n \geq 2$ has the associated Dirichlet series 1.

- The function $1(n) = 1$ for all $n \in \mathbb{N}$ has the associated Dirichlet series $\zeta(s)$ which converges absolutely when $\operatorname{Re}(s) > 1$, and whose analytic continuation we discussed in Exercise 1.4.

- For a natural number k , the k -divisor function $d_k(n)$ counts the number of ways of writing n as $a_1 \cdots a_k$. That is, d_k is the k -fold convolution of the function $1(n)$, and its associated Dirichlet series is $\zeta(s)^k$. The function $d_2(n)$ is called the divisor function and denoted simply by $d(n)$. More generally, for any complex number z , the z -th divisor function $d_z(n)$ is defined as the n -th Dirichlet series coefficient of $\zeta(s)^z$.

- The Möbius function $\mu(n)$ is defined to be 0 if n is divisible by the square of some prime, and if n is square-free $\mu(n)$ is 1 or -1 depending on whether n has an even or odd number of prime factors. The associated Dirichlet series $\sum_{n=1}^{\infty} \mu(n)n^{-s} = \zeta(s)^{-1}$ so that μ is the same as d_{-1} .

- The von Mangoldt function $\Lambda(n)$ is not multiplicative, but is of great interest to us. Its associated Dirichlet series is $-\zeta'(s)/\zeta(s)$. The function $\log n$ has associated Dirichlet series $-\zeta'(s)$, and putting these facts together we see that

$$\log n = (1 * \Lambda)(n) = \sum_{d|n} \Lambda(d), \text{ and } \Lambda(n) = (\mu * \log)(n) = \sum_{ab=n} \mu(a) \log b. \quad (1.11) \quad \boxed{\text{Lammu}}$$

Exercise 1.9 If f and g are functions from \mathbb{N} to \mathbb{C} , show that the relation $f = 1 * g$ is equivalent to the relation $g = \mu * f$. This is known as Möbius inversion.

As mentioned earlier, our goal in this chapter is to show that the prime number theorem is equivalent to a statement about the mean value of the multiplicative function μ . We now formulate this equivalence precisely.

PNTM

Theorem 1.10 The prime number theorem, namely $\psi(x) = x + o(x)$, is equivalent to

$$M(x) = \sum_{n \leq x} \mu(n) = o(x). \quad (1.12) \quad \boxed{\text{Mx}}$$

Before we can prove this, we need one more ingredient: namely, we need to understand the average value of the divisor function.

1.4 The average value of the divisor function and Dirichlet's hyperbola method

PrS4

We wish to evaluate asymptotically $\sum_{n \leq x} d(n)$. An immediate idea gives

$$\begin{aligned}
 \sum_{n \leq x} d(n) &= \sum_{n \leq x} \sum_{d|n} 1 = \sum_{d \leq x} \sum_{\substack{n \leq x \\ d|n}} 1 \\
 &= \sum_{d \leq x} \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \leq x} \left(\frac{x}{d} + O(1) \right) \\
 &= x \log x + O(x).
 \end{aligned}$$

Dirichlet realized that one can substantially improve the error term above by pairing each divisor a of an integer n with its complementary divisor $b = n/a$; one minor exception is when $n = m^2$ and the divisor m cannot be so paired. Since a or n/a must be $\leq \sqrt{n}$ we have

$$d(n) = \sum_{d|n} 1 = 2 \sum_{\substack{d|n \\ d < \sqrt{n}}} 1 + \delta_n,$$

where $\delta_n = 1$ if n is a square, and 0 otherwise. Therefore

$$\begin{aligned}
 \sum_{n \leq x} d(n) &= 2 \sum_{n \leq x} \sum_{\substack{d|n \\ d < \sqrt{n}}} 1 + \sum_{\substack{n \leq x \\ n=d^2}} 1 \\
 &= \sum_{d < \sqrt{x}} \left(1 + 2 \sum_{\substack{d^2 < n \leq x \\ d|n}} 1 \right) \\
 &= \sum_{d < \sqrt{x}} (2[x/d] - 2d + 1),
 \end{aligned}$$

and so

$$\sum_{n \leq x} d(n) = 2x \sum_{d < \sqrt{x}} \frac{1}{d} - x + O(\sqrt{x}) = x \log x - x + 2\gamma x + O(\sqrt{x}), \quad (1.13) \quad \boxed{\text{DD}}$$

by Exercise [II.2](#) ^{[ex:harmonic](#)}.

The method described above is called the hyperbola method because we are trying to count the number of lattice points (a, b) with a and b non-negative and lying below the hyperbola $ab = x$. Dirichlet's idea maybe thought of as choosing parameters A, B with $AB = x$, and dividing the points under the hyperbola according to whether $a \leq A$ or $b \leq B$ or both. We remark that an outstanding open problem, known as the Dirichlet divisor problem, is to show that the error term in (1.13) may be improved to $O(x^{\frac{1}{4}+\epsilon})$. ^{[DD](#)}

For our subsequent work, we use Exercise [II.3](#) ^{[ex:stirling](#)} to recast (1.13) as ^{[DD](#)}

$$\sum_{n \leq x} (\log n + 2\gamma - d(n)) = O(\sqrt{x}). \quad (1.14) \quad \boxed{\text{divest}}$$

k-div

Exercise 1.11 Given a natural number k , use the hyperbola method together with induction and partial summation to show that

$$\sum_{n \leq x} d_k(n) = xP_k(\log x) + O(x^{1-1/k+\epsilon})$$

where $P_k(t)$ denotes a polynomial of degree $k-1$ with leading term $t^{k-1}/(k-1)!$.

Primes5

1.5 The prime number theorem and the Möbius function: proof of Theorem 1.10

First we show that the estimate $M(x) = \sum_{n \leq x} \mu(n) = o(x)$ implies the prime number theorem $\psi(x) = x + o(x)$.

Define the arithmetic function $a(n) = \log n - d(n) + 2\gamma$, so that

$$a(n) = (1 * (\Lambda - 1))(n) + 2\gamma.$$

When we convolve a with the Möbius function we therefore obtain

$$(\mu * a)(n) = (\mu * 1 * (\Lambda - 1))(n) + 2\gamma(\mu * 1)(n) = (\Lambda - 1)(n) + 2\gamma\delta(n),$$

where $\delta(1) = 1$, and $\delta(n) = 0$ for $n > 1$. Hence, when we sum $(\mu * a)(n)$ over all $n \leq x$, we obtain

$$\sum_{n \leq x} (\mu * a)(n) = \sum_{n \leq x} (\Lambda(n) - 1) + 2\gamma = \psi(x) - x + O(1).$$

On the other hand, we may write the left hand side above as

$$\sum_{dk \leq x} \mu(d)a(k),$$

and, as in the hyperbola method, split into terms where $k \leq K$ or $k > K$ (in which case $d \leq x/K$). Thus we find that

$$\sum_{dk \leq x} \mu(d)a(k) = \sum_{k \leq K} a(k)M(x/k) + \sum_{d \leq x/K} \mu(d) \sum_{K < k \leq x/d} a(k).$$

Using [\(I.14\)](#) ^{divest} we see that the second term above is

$$= O\left(\sum_{d \leq x/K} \sqrt{x/d}\right) = O(x/\sqrt{K}).$$

Putting everything together, we deduce that

$$\psi(x) - x = \sum_{k \leq K} a(k)M(x/k) + O(x/\sqrt{K}).$$

If we now know that $M(x) = o(x)$, then by letting K tend to infinity very slowly with x , we may conclude that $\psi(x) - x = o(x)$, obtaining the prime number theorem.

Now we turn to the converse. We must show that the prime number theorem implies that $M(x) = o(x)$. Consider the arithmetic function $-\mu(n) \log n$ which is the n -th Dirichlet series coefficient of $(1/\zeta(s))'$. Since

$$\left(\frac{1}{\zeta(s)}\right)' = -\frac{\zeta'(s)}{\zeta(s)^2} = -\frac{\zeta'(s)}{\zeta(s)} \frac{1}{\zeta(s)},$$

we obtain the identity $-\mu(n) \log n = (\mu * \Lambda)(n)$. Since $\mu * 1 = \delta$, we find that

$$\sum_{n \leq x} (\mu * (\Lambda - 1))(n) = -\sum_{n \leq x} \mu(n) \log n - 1. \quad (1.15) \quad \boxed{\text{Pr51}}$$

The right hand side of $\boxed{\text{Pr51}}$ is

$$\begin{aligned} -\log x \sum_{n \leq x} \mu(n) + \sum_{n \leq x} \mu(n) \log(x/n) - 1 &= -(\log x)M(x) + O\left(\sum_{n \leq x} \log(x/n)\right) \\ &= -(\log x)M(x) + O(x), \end{aligned}$$

upon using Exercise $\boxed{\text{ex:stirling}}$. The left hand side of $\boxed{\text{Pr51}}$ is

$$\sum_{ab \leq x} \mu(a)(\Lambda(b) - 1) = \sum_{a \leq x} \mu(a) (\psi(x/a) - x/a).$$

We are assuming the prime number theorem, which means that given $\epsilon > 0$ if $t \geq T$ is large enough then $|\psi(t) - t| \leq \epsilon t$. Using this for $a \leq x/T$ (so that $x/a > T$) and the Chebyshev estimate $|\psi(x/a) - x/a| \ll x/a$ for $x/T \leq a \leq x$ we find that the left hand side of $\boxed{\text{Pr51}}$ is

$$\ll \sum_{a \leq x/T} \epsilon x/a + \sum_{x/T \leq a \leq x} x/a \ll \epsilon x \log x + x \log T.$$

Combining these observations, we find that

$$|M(x)| \ll \epsilon x + x \frac{\log T}{\log x} \ll \epsilon x,$$

if x is sufficiently large. Since ϵ was arbitrary, we have demonstrated that $M(x) = o(x)$.

Exercise 1.12 *Modify the above proof to show that if $M(x) \ll x/(\log x)^A$ then $\psi(x) - x \ll x(\log \log x)^2/(\log x)^A$. And conversely, if $\psi(x) - x \ll x/(\log x)^A$ then $M(x) \ll x/(\log x)^{\min\{1, A\}}$.*

1.6 Selberg's formula

The elementary techniques discussed above were brilliantly used by Selberg to get an asymptotic formula for a suitably weighted sum of primes and products of two primes. Selberg's identity then led Erdős and Selberg to discovering elementary proofs of the prime number theorem. We will not discuss the elementary proof of the prime number theorem here, but let us see how Selberg's identity follows from the ideas developed so far.

Selberg

Theorem 1.13 *We have*

$$\sum_{p \leq x} (\log p)^2 + \sum_{pq \leq x} (\log p)(\log q) = 2x \log x + O(x).$$

Proof We define $\Lambda_2(n) := \Lambda(n) \log n + \sum_{\ell m = n} \Lambda(\ell) \Lambda(m)$. Thus $\Lambda_2(n)$ is the n -th Dirichlet series coefficient of

$$\left(\frac{\zeta'}{\zeta}(s)\right)' + \left(\frac{\zeta'}{\zeta}(s)\right)^2 = \frac{\zeta''(s)}{\zeta(s)},$$

so that $\Lambda_2 = (\mu * (\log)^2)$.

Our previous work exploited that $\Lambda = (\mu * \log)$ and that the function $d(n) - 2\gamma$ had the same average value as $\log n$. Now we search for a divisor type function which has the same average as $(\log n)^2$.

By partial summation we find that

$$\sum_{n \leq x} (\log n)^2 = x(\log x)^2 - 2x \log x + 2x + O((\log x)^2).$$

Using Exercise [17.11](#) ^{|k-div} we may find constants c_2 and c_1 such that

$$\sum_{n \leq x} (2d_3(n) + c_2 d(n) + c_1) = x(\log x)^2 - 2x \log x + 2x + O(x^{2/3+\epsilon}).$$

Set $b(n) = (\log n)^2 - 2d_3(n) - c_2 d(n) - c_1$ so that the above relations give

$$\sum_{n \leq x} b(n) = O(x^{2/3+\epsilon}). \quad (1.16) \quad \text{Pr61}$$

Now consider $(\mu * b)(n) = \Lambda_2(n) - 2d(n) - c_2 - c_1 \delta(n)$, and summing this over all $n \leq x$ we get that

$$\sum_{n \leq x} (\mu * b)(n) = \sum_{n \leq x} \Lambda_2(n) - 2x \log x + O(x).$$

The left hand side above is

$$\sum_{k \leq x} \mu(k) \sum_{l \leq x/k} b(l) \ll \sum_{k \leq x} (x/k)^{2/3+\epsilon} \ll x,$$

and we conclude that

$$\sum_{n \leq x} \Lambda_2(n) = 2x \log x + O(x).$$

The difference between the left hand side above and the left hand side of our desired identity is $\ll \sqrt{x} \log x$, and so our Theorem follows.

□

SelbergIden

Exercise 1.14 *Recast Selberg's identity in the form*

$$(\psi(x) - x) \log x = - \sum_{n \leq x} \Lambda(n) \left(\psi\left(\frac{x}{n}\right) - \frac{x}{n} \right) + O(x)$$

using $\stackrel{\text{Pavg}}{(I.10)}$ is necessary. Deduce that $a + A = 0$ where

$$a = \liminf_{x \rightarrow \infty} \frac{\psi(x) - x}{x}, \quad \text{and} \quad A = \limsup_{x \rightarrow \infty} \frac{\psi(x) - x}{x}.$$

FIRST RESULTS ON MULTIPLICATIVE FUNCTIONS

C2

As we have just seen, understanding the mean value of the Möbius function leads to the prime number theorem. Motivated by this, we now begin our study of mean values of multiplicative functions in general. We begin by giving in this chapter some basic examples and developing some preliminary results in this direction.

S2.1

2.1 A heuristic

In Section ^{|PrS4}1.4 we saw that a profitable way of studying the mean value of the k -divisor function is to write d_k as the convolution $1 * d_{k-1}$. Given a multiplicative function f let us write f as $1 * g$ where g is also multiplicative. Then

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d|n} g(d) = \sum_{d \leq x} g(d) \left[\frac{x}{d} \right].$$

Since $[z] = z + O(1)$ we have

$$\sum_{n \leq x} f(n) = x \sum_{d \leq x} \frac{g(d)}{d} + O\left(\sum_{d \leq x} |g(d)|\right). \quad (2.1) \quad \text{E2.1}$$

In several situations, for example in the case of the k -divisor function treated earlier, the remainder term in ^{|E2.1}(2.1) may be shown to be small. Omitting this term, and thinking of $\sum_{d \leq x} g(d)/d$ as being approximated by $\prod_{p \leq x} (1 + g(p)/p + g(p^2)/p^2 + \dots)$ we arrive at the following heuristic:

$$\sum_{n \leq x} f(n) \approx x \mathcal{P}(f; x) \quad (2.2) \quad \text{E2.2}$$

where

$$\mathcal{P}(f; x) = \prod_{p \leq x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right). \quad (2.3) \quad \text{E2.3}$$

Consider the heuristic ^{|E2.2}(2.2) in the case of the k -divisor function. The heuristic predicts that

$$\sum_{n \leq x} d_k(n) \approx x \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-(k-1)} \sim x (e^\gamma \log x)^{k-1},$$

which is off from the true asymptotic formula $\sim x(\log x)^{k-1}/(k-1)!$ only by a constant factor.

One of our aims will be to obtain results that are uniform over the class of all multiplicative functions. Thus for example we could consider x to be large and consider the multiplicative function f with $f(p^k) = 0$ for $p \leq \sqrt{x}$ and $f(p^k) = 1$ for $p > \sqrt{x}$. In this case, we have $f(n) = 1$ if n is a prime between \sqrt{x} and x and $f(n) = 0$ for other $n \leq x$. Thus, the heuristic suggests that

$$\sum_{n \leq x} f(n) = 1 + \pi(x) - \pi(\sqrt{x}) \approx x \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p}\right) \sim x \frac{e^{-\gamma}}{\log \sqrt{x}} \sim \frac{2e^{-\gamma}x}{\log x}.$$

Again this is comparable to the prime number theorem, but the heuristic is off by the constant $2e^{-\gamma} \approx 1.1\dots$. This discrepancy is significant in prime number theory, and has been exploited beautifully by many authors starting with the pioneering work of Maier.

In the case of the Möbius function, the heuristic suggests comparing

$$M(x) = \sum_{n \leq x} \mu(n) \quad \text{with} \quad x \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^2 \sim \frac{xe^{-2\gamma}}{(\log x)^2},$$

but in fact $\sum_{n \leq x} \mu(n)$ is much smaller. The best bound that we know unconditionally is that $\sum_{n \leq x} \mu(n) \ll x \exp(-c(\log x)^{\frac{3}{5}-\epsilon})$, but we expect that it is as small as $x^{\frac{1}{2}+\epsilon}$ – this is equivalent to the Riemann Hypothesis. In any event, the heuristic certainly suggests the prime number theorem that $M(x) = o(x)$.

S2.2

2.2 Multiplicative functions close to 1

The heuristic ^(E2.2)~~(2.2)~~ is accurate and easy to justify when the function g is small in size, or in other words, when f is close to 1. We give a sample such result which is already quite useful.

pr2.1

Proposition 2.1 *Let $f = 1 * g$ be a multiplicative function, and suppose that $0 \leq \sigma \leq 1$ is such that*

$$\sum_{d=1}^{\infty} \frac{|g(d)|}{d^{\sigma}} = \tilde{G}(\sigma)$$

is convergent. Then, with $\mathcal{P}(f) = \mathcal{P}(f; \infty)$,

$$\left| \sum_{n \leq x} f(n) - x\mathcal{P}(f) \right| \leq x^{\sigma} \tilde{G}(\sigma).$$

Proof The argument giving ^(E2.1)~~(2.1)~~ yields that

$$\left| \sum_{n \leq x} f(n) - x \sum_{d \leq x} \frac{g(d)}{d} \right| \leq \sum_{d \leq x} |g(d)|.$$

Since $\mathcal{P}(f) = \sum_{d \geq 1} g(d)/d$ we have that

$$\left| \sum_{d \leq x} \frac{g(d)}{d} - \mathcal{P}(f) \right| \leq \sum_{d > x} \frac{|g(d)|}{d}.$$

Combining these two inequalities yields that

$$\left| \sum_{n \leq x} f(n) - x \mathcal{P}(f) \right| \leq \sum_{d \leq x} |g(d)| + x \sum_{d > x} \frac{|g(d)|}{d}.$$

The result follows from the following observation, which holds for any sequence of non-negative real numbers: If $a_n \geq 0$ for all $n \geq 1$ then for any σ , $0 \leq \sigma \leq 1$, we have

$$\sum_{n \leq x} a_n + x \sum_{n > x} \frac{a_n}{n} \leq \sum_{n \leq x} a_n \left(\frac{x}{n}\right)^\sigma + x \sum_{n > x} \frac{a_n}{n} \left(\frac{n}{x}\right)^{1-\sigma} = x^\sigma \sum_{n \geq 1} \frac{a_n}{n^\sigma}. \quad (2.4) \quad \boxed{\text{Rankin}}$$

□

ex2.0 **Exercise 2.2** If g is multiplicative, show that the convergence of $\sum_{n=1}^{\infty} |g(n)|/n^\sigma$ is equivalent to the convergence of $\sum_{p^k} |g(p^k)|/p^{k\sigma}$.

ex2.1 **Exercise 2.3** If f is a non-negative arithmetic function, and $\sigma > 0$ is such that $F(\sigma) = \sum_{n=1}^{\infty} f(n)n^{-\sigma}$ is convergent, then $\sum_{n \leq x} f(n) \leq x^\sigma F(\sigma)$. This simple observation is known as Rankin's trick, and is sometimes surprisingly effective.

Remark 2.4 If we are bounding the sum of $f(n)$ for $n \leq x$ then the values of $f(p^k)$ for $p > x$ are not used in determining the sum, yet the $F(\sigma)$ in the upper bound in the previous exercise implicitly uses those values. This suggests that in order to optimize our bound we may select these $f(p)$ to be as helpful as possible, typically taking $f(p^k) = 1$ for all $p > x$, so that $g(p^k) = 0$.

ex2.2 **Exercise 2.5** For any natural number q , prove that for any $0 \leq \sigma \leq 1$

$$\left| \sum_{\substack{n \leq x \\ (n,q)=1}} 1 - \frac{\phi(q)}{q} x \right| \leq x^\sigma \prod_{p|q} \left(1 + \frac{1}{p^\sigma}\right).$$

If one takes $\sigma = 0$, we obtain the sieve of Eratosthenes bound of $2^{\omega(q)}$ (where $\omega(q)$ is the number of distinct primes dividing q) for the right side above. A little calculus shows that, if $\sum_{p|q} (\log p)/(p+1) \leq \log x$, the choice of σ that optimizes our bound, is given by the relation $\sum_{p|q} (\log p)/(p^\sigma + 1) = \log x$.

ex2.3 **Exercise 2.6** Let $\sigma(n) = \sum_{d|n} d$. Prove that

$$\sum_{n \leq x} \frac{\mu(n)^2 \sigma(n)}{\phi(n)} = \frac{15}{\pi^2} x + O(\sqrt{x} \log x).$$

ex2.4 **Exercise 2.7** Let $f = 1 * g$ be a multiplicative function and $\sigma \in [0, 1)$ is such that $\sum_d |g(d)|d^{-\sigma} = \tilde{G}(\sigma) < \infty$. Prove that for $x \geq \exp(1/(1 - \sigma))$

$$\sum_{n \leq x} \frac{f(n)}{n} = \mathcal{P}(f)(\log x + \gamma) - \sum_{d=1}^{\infty} \frac{g(d)}{d} \log d + O(x^{\sigma-1} \log x \tilde{G}(\sigma)).$$

ex2.5 **Exercise 2.8** Let f be multiplicative and write $f = d_k * g$ where $k \in \mathbb{N}$ and d_k denotes the k -divisor function. Assuming that $|g|$ is small, as in Proposition 2.1, develop an asymptotic formula for $\sum_{n \leq x} f(n)$.

Now we refine Proposition 2.1 and establish the heuristic (2.3) under a less restrictive hypothesis.

Prop2.7 **Proposition 2.9** Let $f = 1 * g$ and suppose that

$$\sum_{n=1}^{\infty} \frac{|g(n)|}{n} = \tilde{G}(1)$$

is convergent. Then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \mathcal{P}(f) = \sum_{d=1}^{\infty} \frac{g(d)}{d}.$$

Proof Recall (2.1) which gives $\sum_{n \leq x} f(n) = x \sum_{d \leq x} g(d)/d + O(\sum_{d \leq x} |g(d)|)$. Now

$$\left| \sum_{n \leq x} \frac{g(n)}{n} - \mathcal{P}(f) \right| \leq \sum_{n > x} \frac{|g(n)|}{n} \rightarrow_{x \rightarrow \infty} 0,$$

and

$$\sum_{n \leq x} |g(n)| = \int_0^x \sum_{t < n \leq x} \frac{|g(n)|}{n} dt = o(x),$$

as $\sum_{n=1}^{\infty} |g(n)|/n$ is convergent, and the result follows. \square

2.3 Non-negative multiplicative functions

Let us now consider our heuristic for the special case of non-negative multiplicative functions with suitable growth conditions. Here we shall see that right side of our heuristic (2.2) is at least a good upper bound for $\sum_{n \leq x} f(n)$.

Prop2.1 **Proposition 2.10** Let f be a non-negative multiplicative function, and suppose there are constants A and B such that

$$\sum_{p^k \leq z} f(p^k) \log(p^k) \leq Az + B,$$

for all $z \geq 1$. Then for $x \geq e^{2B}$ we have

$$\sum_{n \leq x} f(n) \leq \frac{(A+1)x}{\log x - B} \prod_{p \leq x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots \right).$$

Proof Consider

$$\sum_{n \leq x} f(n) \log x = \sum_{n \leq x} f(n) \log n + \sum_{n \leq x} f(n) \log(x/n).$$

The first term satisfies

$$\begin{aligned} \sum_{n \leq x} f(n) \log n &= \sum_{n \leq x} \sum_{\substack{n=p^k r \\ (p,r)=1}} f(r) f(p^k) \log(p^k) \leq \sum_{r \leq x} f(r) \sum_{p^k \leq x/r} f(p^k) \log(p^k) \\ &\leq \sum_{r \leq x} f(r) \left(\frac{Ax}{r} + B \right). \end{aligned}$$

Since $\log t \leq t$ the second term is $\leq x \sum_{n \leq x} f(n)/n$. We conclude that

$$\sum_{n \leq x} f(n) \leq \frac{x}{\log x} (A+1) \sum_{n \leq x} \frac{f(n)}{n} + \frac{B}{\log x} \sum_{n \leq x} f(n),$$

and since $\sum_{n \leq x} f(n)/n \leq \prod_{p \leq x} (1 + f(p)/p + f(p^2)/p^2 + \dots)$, the Proposition follows. \square

Note that, by Mertens' Theorem, the upper bound in Proposition [2.10](#) is $\leq (A+1+o(1))x\mathcal{P}(f;x)$.

In Proposition [2.10](#) we have in mind a non-negative multiplicative function dominated by some k -divisor function, and in such a situation we have shown that $\sum_{n \leq x} f(n)$ is bounded above by a constant times the heuristic prediction $x\mathcal{P}(f;x)$. For a non-negative multiplicative function bounded by 1, Propositions [2.9](#) and [2.10](#) establish the heuristic [\(2.3\)](#) in the limit $x \rightarrow \infty$.

cor2.3 **Corollary 2.11** *If $0 \leq f(n) \leq 1$ is a non-negative multiplicative function then*

$$\sum_{n \leq x} f(n) \ll x\mathcal{P}(f;x) \ll x \exp \left(- \sum_{p \leq x} \frac{1-f(p)}{p} \right) \quad (2.5) \quad \text{E2.5}$$

with an absolute implied constant. Moreover we have

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \mathcal{P}(f).$$

Proof The Chebyshev estimates give that

$$\sum_{p^k \leq z} f(p^k) \log(p^k) \leq \sum_{p^k \leq z} \log(p^k) \leq Az + B$$

with any constant $A > \log 4$ being permissible. The estimate [\(2.5\)](#) therefore follows from Proposition [2.10](#). \square

If $\sum_p (1 - f(p))/p$ diverges, then $\stackrel{|\text{E2.5}|}{(2.5)}$ shows that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n) = 0 = \mathcal{P}(f).$$

Suppose now that $\sum_p (1 - f(p))/p$ converges. If we write $f = 1 * g$ then this condition assures us that $\sum_{p^k} |g(p^k)|/p^k$ converges, which in turn is equivalent to the convergence of $\sum_n |g(n)|/n$. $\stackrel{|\text{Prop 2.7}|}{\text{Proposition 2.9}}$ now finishes our proof. \square

We would love to have a uniform result like $\stackrel{|\text{E2.5}|}{(2.5)}$ for real valued multiplicative functions with $-1 \leq f(n) \leq 1$ (and more generally for complex valued multiplicative functions), since that would immediately imply the prime number theorem. Establishing such a result will be one of our goals in the coming chapters. In particular, one may ask if $\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n)$ exists (and equals $\mathcal{P}(f)$) for more general classes of multiplicative functions. Erdős and Wintner conjectured that this is so for real valued multiplicative functions with $-1 \leq f(n) \leq 1$, and this was established by Wirsing whose proof also establishes that $\sum_{n \leq x} \mu(n) = o(x)$. The work of Halasz, which we shall focus on soon, considers the more general case of complex valued multiplicative functions taking values in the unit disc.

INTEGERS WITHOUT LARGE PRIME FACTORS

3.1 “Smooth” or “friable” numbers

Given a real number $y \geq 2$, we let $\mathcal{S}(y)$ denote the set of natural numbers all of whose prime factors are at most y . Such natural numbers are called “smooth” in the English literature, and “friable” (meaning crumbly) in the French literature; the latter usage seems to be spreading, at least partly because the word “smooth” is already overused. Smooth numbers appear all over analytic number theory in connections ranging from computational number theory and factoring algorithms to Waring’s problem. Our interest is in the counting function of smooth numbers:

$$\Psi(x, y) := \sum_{\substack{n \leq x \\ n \in \mathcal{S}(y)}} 1.$$

We can formulate this as a question about multiplicative functions by considering the multiplicative function given by $f(p^k) = 1$ if $p \leq y$, and $f(p^k) = 0$ otherwise.

If $x \leq y$ then clearly $\Psi(x, y) = [x] = x + O(1)$. Next suppose that $y \leq x \leq y^2$. If $n \leq x$ is not y -smooth then it must be divisible by a unique prime $p \in (y, x]$. Thus

$$\begin{aligned} \Psi(x, y) &= [x] - \sum_{y < p \leq x} \sum_{\substack{n \leq x \\ p|n}} 1 = x + O(1) - \sum_{y < p \leq x} \left(\frac{x}{p} + O(1) \right) \\ &= x \left(1 - \log \frac{\log x}{\log y} \right) + O\left(\frac{x}{\log y} \right). \end{aligned}$$

The formula above suggests writing $x = y^u$, and then for $1 \leq u \leq 2$ it gives

$$\Psi(y^u, y) = y^u (1 - \log u) + O\left(\frac{y^u}{\log y} \right).$$

We can continue the process begun above, using the principle of inclusion and exclusion to evaluate $\Psi(y^u, y)$ by subtracting from $[y^u]$ the number of integers which are divisible by a prime larger than y , adding back the contribution from integers divisible by two primes larger than y , and so on. A result of this type for small values of u may be found in Ramanujan’s unpublished manuscripts (collected in “The last notebook”), but the first published uniform results on this problem are due to Dickman and de Bruijn. The answer involves the Dickman-de Bruijn function $\rho(u)$ defined as follows. For $0 \leq u \leq 1$ let $\rho(u) = 1$, and

let $\rho(u) = 1 - \log u$ for $1 \leq u \leq 2$. For $u > 1$ we define ρ by means of the differential-difference equation

$$u\rho'(u) = -\rho(u-1),$$

or, equivalently, the integral equation

$$u\rho(u) = \int_{u-1}^u \rho(t)dt.$$

It is easy to check that the differential-difference equation above has a unique continuous solution, and that $\rho(u)$ is non-negative and decreases rapidly to 0 as u increases. For example, note that $\rho(u) \leq \rho(u-1)/u$ and iterating this we find that $\rho(u) \leq 1/[u]!$.

smooth

Theorem 3.1 *Uniformly for all $u \geq 1$ we have*

$$\Psi(y^u, y) = \rho(u)y^u + O\left(\frac{y^u}{\log y} + 1\right).$$

Proof Let $x = y^u$, and we start with

$$\Psi(x, y) \log x = \sum_{\substack{n \leq x \\ n \in \mathcal{S}(y)}} \log n + O\left(\sum_{n \leq x} \log(x/n)\right) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}(y)}} \log n + O(x).$$

Using $\log n = \sum_{d|n} \Lambda(d)$ we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}(y)}} \log n = \sum_{\substack{d \leq x \\ d \in \mathcal{S}(y)}} \Lambda(d) \Psi(x/d, y) = \sum_{p \leq y} (\log p) \Psi(x/p, y) + O(x),$$

since the contribution of prime powers p^k (with $k \geq 2$) is easily seen to be $O(x)$. Thus

$$\Psi(x, y) \log x = \sum_{p \leq y} \log p \Psi\left(\frac{x}{p}, y\right) + O(x). \quad (3.1) \quad \boxed{\text{E2.10}}$$

Now we show that a similar equation is satisfied by what we think approximates $\Psi(x, y)$, namely $x\rho(u)$. Put $E(t) = \sum_{p \leq t} \frac{\log p}{p} - \log t$ so that $E(t) = O(1)$ by (I.10). Now

$$\sum_{p \leq y} \frac{\log p}{p} \rho\left(\frac{\log(x/p)}{\log y}\right) = \int_1^y \rho\left(u - \frac{\log t}{\log y}\right) d(\log t + E(t)),$$

and making a change of variables $t = y^\nu$ we find that

$$\int_1^y \rho\left(u - \frac{\log t}{\log y}\right) d(\log t) = (\log y) \int_0^1 \rho(u - \nu) d\nu = (\log x) \rho(u).$$

Moreover, since $E(t) \ll 1$ and ρ is monotone decreasing, integration by parts gives

$$\int_1^y \rho\left(u - \frac{\log t}{\log y}\right) d(E(t)) \ll \rho(u-1) + \int_1^y \left| \frac{d}{dt} \rho\left(u - \frac{\log t}{\log y}\right) \right| dt \ll \rho(u-1).$$

Thus we find that

$$(x\rho(u)) \log x = \sum_{p \leq y} \log p \left(\frac{x}{p} \rho\left(\frac{\log(x/p)}{\log y}\right) \right) + O(x). \quad (3.2) \quad \boxed{\text{E2.11}}$$

Subtracting $\boxed{\text{E2.11}}$ from $\boxed{\text{E2.10}}$ we arrive at

$$|\Psi(x, y) - x\rho(u)| \log x \leq \sum_{p \leq y} \log p \left| \Psi\left(\frac{x}{p}, y\right) - \frac{x}{p} \rho\left(\frac{\log x/p}{\log y}\right) \right| + Cx, \quad (3.3) \quad \boxed{\text{E2.12}}$$

for a suitable constant C .

Now suppose that the Theorem has been established for all values until $x/2$, and we now wish to establish it for x . We may suppose that $x \geq y^2$, and our induction hypothesis is that for all $t \leq x/2$ we have

$$\left| \Psi(t, y) - t\rho\left(\frac{\log t}{\log y}\right) \right| \leq C_1 \left(\frac{t}{\log y} + 1 \right),$$

for a suitable constant C_1 . From $\boxed{\text{E2.12}}$ we obtain that

$$|\Psi(x, y) - x\rho(u)| \log x \leq C_1 \sum_{p \leq y} \log p \left(\frac{x}{p \log y} + 1 \right) + Cx \leq C_1 x + O\left(\frac{x}{\log y} + y\right) + Cx.$$

Assuming, as we may, that $C_1 \geq 2C$ and that y is sufficiently large, the right hand side above is $\leq 2C_1 x$, and we conclude that $|\Psi(x, y) - x\rho(u)| \leq C_1 x / \log y$. This completes our proof. \square

ex2.6

Exercise 3.2 Let

$$\zeta(s, y) = \prod_{p \leq y} (1 - 1/p^s)^{-1} = \sum_{n \in \mathcal{S}(y)} n^{-s},$$

be the Dirichlet series associated with the y -smooth numbers. For any real numbers $x \geq 1$ and $y \geq 2$, show that the function $x^\sigma \zeta(\sigma, y)$ for $\sigma \in (0, \infty)$ attains its minimum at $\alpha = \alpha(x, y)$ satisfying

$$\log x = \sum_{p \leq y} \frac{\log p}{p^\alpha - 1}.$$

By Rankin's trick (see Exercise $\boxed{\text{ex2.1}}$) conclude that

$$\Psi(x, y) \leq x^\alpha \zeta(\alpha, y) = \min_{\sigma > 0} x^\sigma \zeta(\sigma, y).$$

ex2.7 **Exercise 3.3** For any given η , $\frac{1}{\log y} \ll \eta < 1$, show that

$$\sum_{p \leq y} \frac{1}{p^{1-\eta}} \leq \log(1/\eta) + O\left(\frac{y^\eta}{\log(y^\eta)}\right).$$

(Hint: Compare the sum for the primes with $p^\eta \ll 1$ to the sum of $1/p$ in the same range. Use upper bounds on $\pi(x)$ for those primes for which $p^\eta \gg 1$.)

ex2.8 **Exercise 3.4** For $x = y^u$ with $y > (\log x)^{2+\epsilon}$ let $\sigma = 1 - \frac{\log(u \log u)}{\log y}$. Deduce from the last two exercises that there exists a constant $C > 0$ such that

$$\Psi(x, y) \ll \left(\frac{C}{u \log u}\right)^u x \log y.$$

ex2.9 **Exercise 3.5** Prove that

$$\rho(u) = \left(\frac{1 + o(1)}{u \log u}\right)^u.$$

(Hint: Select c maximal such that $\rho(u) \gg (c/u \log u)^u$. By using the functional equation for ρ deduce that $c \geq 1$. Take a similar approach for the implicit upper bound.)

3.2 Multiplicative functions which only vary at small prime factors

C2a
GenFundLem

Proposition 3.6 Suppose that $f(p^k) = 1$ for all $p > y$. Let $x = y^u$. Then

$$\frac{1}{x} \sum_{n \leq x} f(n) = \mathcal{P}(f; x) + O(1/u^{u/3}).$$

Can we get an estimate of $\mathcal{P}(f; x)\{1 + O(1/u^u)\}$, and so generalize the Fundamental Lemma of Sieve Theory? We begin with a simple case that follows from the Fundamental Lemma of Sieve Theory:

BabyBuchstab

Lemma 3.7 Suppose that $g(p^k) = 0$ for all $p \leq y$, and $g(p^k) = 1$ for all $p > y$. Let $x = y^u$. Then

$$\sum_{n \leq x} g(n) = x \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \{1 + O(1/u^u)\}.$$

Proof of Proposition 3.6 ^{**GenFundLem**} Define

$$g(p^k) = \begin{cases} 0 & \text{if } p \leq y \\ 1 & \text{if } p > y \end{cases} \quad \text{and} \quad h(p^k) = \begin{cases} f(p^k) & \text{if } p \leq y \\ 0 & \text{if } p > y \end{cases}$$

so that $f = g * h$. Hence if $AB = x$ then

$$\sum_{n \leq x} f(n) = \sum_{a \leq A} h(a) \sum_{b \leq x/a} g(b) + \sum_{b \leq B} g(b) \sum_{A < a < x/b} h(a).$$

Let $A = B = \sqrt{x}$ and use lemma ^{BabyBuchstab}3.7 on the first sum to obtain

$$\sum_{a \leq A} h(a) \kappa_y \frac{x}{a} \{1 + O(1/(u/2)^{u/2})\}.$$

where $\kappa_y := \prod_{p \leq y} \left(1 - \frac{1}{p}\right)$. Hence we have a main term of

$$\kappa_y x \sum_a \frac{h(a)}{a} = x\mathcal{P}(h; y) = x\mathcal{P}(f; x),$$

plus an error term of

$$\kappa_y x \sum_{a > A} \frac{|h(a)|}{a} + (u/2)^{-u/2} \kappa_y x \sum_a \frac{|h(a)|}{a} \ll \frac{x}{\log y} \sum_{a > A} \frac{1}{a} + (u/2)^{-u/2} x \ll (u/2)^{-u/2} x,$$

as $|h(a)| \leq 1$, using our estimate on tail of sums over smooth numbers. We bound the second sum above using our knowledge of smooths to obtain

$$\leq \sum_{b \leq B} g(b) \frac{x}{b} (u/2)^{-u/2} \ll x (u/2)^{1-u/2}$$

□

DISTANCES AND THE THEOREMS OF DELANGE, WIRSING AND HALÁSZ

In Chapter 2 we considered the heuristic that the mean value of a multiplicative function f might be approximated by the Euler product $\mathcal{P}(f; x)$ (see (2.2) and (2.3)). We proved some elementary results towards this heuristic and were most successful when f was “close to 1” (see §2.2) or when f was non-negative (see §2.3). Even for nice non-negative functions the heuristic is not entirely accurate, as revealed by the example of smooth numbers discussed in Chapter 3. We now continue our study of this heuristic, and focus on whether the mean value can be bounded above by something like $|\mathcal{P}(f; x)|$. We begin by making precise the geometric language, already employed in §2.2, of one multiplicative function being “close” to another.

4.1 The distance between two multiplicative functions

The notion of a distance between multiplicative functions makes most sense in the context of functions whose values are restricted to the unit disc $\mathbb{U} = \{|z| \leq 1\}$. In thinking of the distance between two such multiplicative functions f and g , naturally we may focus on the difference between $f(p^k)$ and $g(p^k)$ on prime powers. An obvious candidate for quantifying this distance is

$$\sum_{p^k \leq x} \frac{|f(p^k) - g(p^k)|}{p^k},$$

and implicitly it is this distance which is used in Proposition 2.9 (and a stronger form of such a distance is used in Proposition 2.1). However, it turns out that a better notion of distance involves $1 - \operatorname{Re}(f(p^k)\overline{g(p^k)})$ in place of $|f(p^k) - g(p^k)|$.

lem4.1

Lemma 4.1 *Suppose we have a sequence of functions $\eta_j : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the triangle inequality*

$$\eta_j(z_1, z_3) \leq \eta_j(z_1, z_2) + \eta_j(z_2, z_3),$$

for all $z_1, z_2, z_3 \in \mathbb{U}$. Then we may define a metric $\mathbb{U}^{\mathbb{N}} = \{\mathbf{z} = (z_1, z_2, \dots)\}$ by setting

$$d(\mathbf{z}, \mathbf{w}) = \left(\sum_{j=1}^{\infty} \eta_j(z_j, w_j)^2 \right)^{\frac{1}{2}},$$

assuming that the sum converges. This metric satisfies the triangle inequality

$$d(\mathbf{z}, \mathbf{w}) \leq d(\mathbf{z}, \mathbf{y}) + d(\mathbf{y}, \mathbf{w}).$$

Proof Expanding out we have

$$d(\mathbf{z}, \mathbf{w})^2 = \sum_{j=1}^{\infty} \eta_j(z_j, w_j)^2 \leq \sum_{j=1}^{\infty} (\eta_j(z_j, y_j) + \eta_j(y_j, w_j))^2$$

by the assumed triangle inequality for η_j . Now, using Cauchy-Schwarz, we have

$$\begin{aligned} \sum_{j=1}^{\infty} (\eta_j(z_j, y_j) + \eta_j(y_j, w_j))^2 &= d(\mathbf{z}, \mathbf{y})^2 + d(\mathbf{y}, \mathbf{w})^2 + 2 \sum_{j=1}^{\infty} \eta_j(z_j, y_j) \eta_j(y_j, w_j) \\ &\leq d(\mathbf{z}, \mathbf{y})^2 + d(\mathbf{y}, \mathbf{w})^2 + 2 \left(\sum_{j=1}^{\infty} \eta_j(z_j, y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} \eta_j(y_j, w_j)^2 \right)^{\frac{1}{2}} \\ &= (d(\mathbf{z}, \mathbf{y}) + d(\mathbf{y}, \mathbf{w}))^2, \end{aligned}$$

which proves the triangle inequality. \square

A nice class of examples is provided by taking $\eta_j(z) = a_j(1 - \operatorname{Re}(z_j))$ for non-negative a_j , and we now check that this satisfies the hypothesis of Lemma 4.1.

lem4.0 **Lemma 4.2** Define $\eta : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}_{\geq 0}$ by $\eta(z, w)^2 = 1 - \operatorname{Re}(z\bar{w})$. Then for any z_1, z_2, z_3 in \mathbb{U} we have

$$\eta(z_1, z_3) \leq \eta(z_1, z_2) + \eta(z_2, z_3).$$

Proof Without loss of generality we may suppose that $z_1 = \kappa_1$, $z_2 = \kappa_2 e^{i\theta_2}$ and $z_3 = \kappa_3 e^{i\theta_3}$ with $\kappa_1, \kappa_2, \kappa_3 \in [0, 1]$ and $\theta_2, \theta_3 \in (-\pi, \pi]$. Our claim is that

$$(1 - \kappa_1 \kappa_3 \cos \theta_3)^{\frac{1}{2}} \leq (1 - \kappa_1 \kappa_2 \cos \theta_2)^{\frac{1}{2}} + (1 - \kappa_2 \kappa_3 \cos(\theta_2 - \theta_3))^{\frac{1}{2}}. \quad (4.1) \quad \text{eq:4.2.1}$$

Suppose first that $\cos \theta_2$ and $\cos(\theta_2 - \theta_3)$ have the same sign. If they are both negative then the RHS of (4.1) is clearly ≥ 2 and our claim holds. If they are both positive, then for fixed κ_1 and κ_3 the RHS of (4.1) is minimum for $\kappa_2 = 1$ and our claim is then that

$$(1 - \kappa_1 \kappa_3 \cos \theta_3)^{\frac{1}{2}} \leq (1 - \kappa_1 \cos \theta_2)^{\frac{1}{2}} + (1 - \kappa_2 \cos(\theta_2 - \theta_3))^{\frac{1}{2}}. \quad (4.2) \quad \text{eq:4.2.2}$$

To establish this we square both sides, write $\cos \theta_3 = \cos \theta_2 \cos(\theta_2 - \theta_3) + \sin \theta_2 \sin(\theta_2 - \theta_3)$, and the inequality $(1 - r \cos \theta) \geq \frac{1}{2} r^2 \sin^2 \theta$ (valid for all $0 \leq r \leq 1$).

So we may assume that $\cos \theta_2$ and $\cos(\theta_2 - \theta_3)$ have opposite signs, so that one of the two must have opposite sign from $\cos \theta_3$. Suppose $\cos \theta_3$ and $\cos \theta_2$ have opposite signs. If $\cos \theta_3 \geq 0 \geq \cos \theta_2$ then it suffices to check (4.1) in the case $\kappa_1 = 0$ and clearly this holds. If $\cos \theta_2 \geq 0 \geq \cos \theta_3$ then it suffices to check (4.1) in the case when $\kappa_1 = 1$ and this may be verified in the same manner as (4.2). \square

We can use the above remarks to define distances between multiplicative functions taking values in the unit disc. Taking $a_j = 1/p$ for each prime $p \leq x$ we may define a distance (up to x) of the multiplicative functions f and g by

$$\mathbb{D}(f, g; x)^2 = \sum_{p \leq x} \frac{1 - \operatorname{Re} f(p)\overline{g(p)}}{p}.$$

By Lemma [4.1](#) ^{lem4.1} this satisfies the triangle inequality

$$\mathbb{D}(f, g; x) + \mathbb{D}(g, h; x) \geq \mathbb{D}(f, h; x). \quad (4.3) \quad \text{triangle1}$$

It is natural to multiply multiplicative functions together, and we may wonder: if f_1 and g_1 are close to each other, and f_2 and g_2 are close to each other whether it then follows that $f_1 f_2$ is close to $g_1 g_2$? Indeed this variant of the triangle inequality holds, and we leave its proof as an exercise to the reader:

$$\mathbb{D}(f_1, g_1; x) + \mathbb{D}(f_2, g_2; x) \geq \mathbb{D}(f_1 f_2, g_1 g_2; x). \quad (4.4) \quad \text{triangle2}$$

Alternatively, we can take any $\alpha > 1$ and take the coefficients $a_j = 1/p^\alpha$ and $z_j = f(p)$ as p runs over all primes. In this case we have

$$\mathbb{D}_\alpha(f, g)^2 = \sum_p \frac{1 - \operatorname{Re} f(p)\overline{g(p)}}{p^\alpha},$$

which obeys the analogs of [\(4.3\)](#) and [\(4.4\)](#). ^{triangle1} ^{triangle2}

lem4.3 **Lemma 4.3** For any multiplicative functions f and g taking values in the unit disc we have

$$\mathbb{D}(f, g; x)^2 = \mathbb{D}_\alpha(f, g)^2 + O(1)$$

with $\alpha = 1 + 1/\log x$. Furthermore, if f is completely multiplicative and $F(s) = \sum_{n=1}^{\infty} f(n)/n^s$ is the Dirichlet series associated to f we have

$$|F(1 + 1/\log x)| \asymp \zeta(1 + 1/\log x) \exp\left(-\mathbb{D}(1, f; x)^2\right) \asymp \log x \exp\left(-\mathbb{D}(1, f; x)^2\right).$$

Proof With $\alpha = 1 + 1/\log x$ we have

$$|\mathbb{D}(f, g; x)^2 - \mathbb{D}_\alpha(f, g)^2| \leq 2 \sum_{p \leq x} \left(\frac{1}{p} - \frac{1}{p^{1+\alpha}} \right) + 2 \sum_{p > x} \frac{1}{p^{1+\alpha}} = O(1),$$

proving our first assertion. The second statement follows since $\log |F(\alpha)| = \operatorname{Re} \sum_p f(p)/p^\alpha + O(1)$. \square

Taking $g(n) = n^{it}$ we obtain, for $x \geq 2$

$$\exp\left(\sum_{p \leq x} \frac{f(p)}{p^{1+it}}\right) \asymp \sum_{n \geq 1} \frac{f(n)}{n^{1+\frac{1}{\log x}+it}} = F\left(1 + \frac{1}{\log x} + it\right). \quad (4.5) \quad \text{TruncRight}$$

4.2 Delange's Theorem

Delange

Theorem 4.4 *Let f be a multiplicative function taking values in the unit disc \mathbb{U} . Suppose that*

$$\mathbb{D}(1, f; \infty) = \sum_p \frac{1 - \operatorname{Re} f(p)}{p} < \infty.$$

Then as $x \rightarrow \infty$ we have

$$\sum_{n \leq x} f(n) \sim x \mathcal{P}(f; x).$$

Delange's theorem may be seen as a refinement of Proposition ^{|Prop2.7}2.9. There the hypothesis is essentially that $\sum_p |1 - f(p)|/p < \infty$ which is a stronger requirement than Delange's hypothesis. We warn the reader that the hypothesis of Delange's theorem does not guarantee that $\mathcal{P}(f; x)$ tends to a limiting value $\mathcal{P}(f)$ as $x \rightarrow \infty$ – the reader may have fun coming up with examples. We postpone the proof of Delange's theorem to the next chapter.

4.3 A key example: the multiplicative function $f(n) = n^{i\alpha}$

Delange's theorem gives a satisfactory answer in the case of multiplicative functions at a bounded distance from 1, and we are left to ponder what happens when $\mathbb{D}(1, f; x) \rightarrow \infty$ as $x \rightarrow \infty$. One would be tempted to think that in this case $\frac{1}{x} \sum_{n \leq x} f(n) \rightarrow 0$ as $x \rightarrow \infty$ were it not for the following important counter example. Let $\alpha \neq 0$ be a fixed real number and consider the completely multiplicative function $f(n) = n^{i\alpha}$. By partial summation we find that

$$\sum_{n \leq x} n^{i\alpha} = \int_{1-}^{x+} y^{i\alpha} d[y] \sim \frac{x^{1+i\alpha}}{1+i\alpha}. \quad (4.6) \quad \boxed{\text{eq:4.1}}$$

The mean-value at x then is $\sim x^{i\alpha}/(1+i\alpha)$ which has magnitude $1/|1+i\alpha|$ but whose argument varies with x . In this example it seems plausible enough that $\mathbb{D}(1, p^{i\alpha}; x) \rightarrow \infty$ as $x \rightarrow \infty$ and we now supply a proof of this important fact. We begin with a useful Lemma on the Riemann zeta function.

lem4.3.0

Lemma 4.5 *If $s = \sigma + it$ with $\sigma > 1$ then*

$$\frac{|s|}{|s-1|} - |s| \leq |\zeta(s)| \leq \frac{|s|}{|s-1|} + |s|.$$

If in addition we have $|s-1| \gg 1$ then

$$|\zeta(s)| \ll \log(2 + |s|).$$

Proof The first assertion follows easily from Exercise 1.4. To prove the second assertion, modify the argument of Exercise 1.4 to show that for any integer $N \geq 1$ we have

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{\{y\}}{y^{s+1}} dy.$$

Choose $N = \lfloor s \rfloor + 1$, and bound the sum over n trivially to deduce the stated bound for $|\zeta(s)|$. \square

lem4.3.1

Lemma 4.6 *Let α be any real number. Then for all $x \geq 3$ we have*

$$\mathbb{D}(1, p^{i\alpha}; x)^2 = \log(1 + |\alpha| \log x) + O(1),$$

in the case $|\alpha| \leq 1/10$. When $|\alpha| \geq 1/10$ we have

$$\mathbb{D}(1, p^{i\alpha}; x)^2 \geq \log \log x - \log \log(2 + |\alpha|) + O(1).$$

Proof We have from Lemma 4.3 ^{lem4.3}

$$\mathbb{D}(1, p^{i\alpha}; x)^2 = \log \frac{\log x}{|\zeta(1 + 1/\log x + i\alpha)|}.$$

Now use the bounds of Lemma 4.5 ^{lem4.3.0}. \square

We shall find Lemma 4.6 ^{lem4.3.1} very useful in our work. One important consequence of it and the triangle inequality is that a multiplicative function cannot pretend to be like two different problem examples $n^{i\alpha}$ and $n^{i\beta}$.

cor:repulsive

Corollary 4.7 *Let α and β be two real numbers and let f be a multiplicative function taking values in the unit disc. Then*

$$\left(\mathbb{D}(f, p^{i\alpha}; x) + \mathbb{D}(f, p^{i\beta}; x) \right)^2$$

exceeds

$$\log(1 + |\alpha - \beta| \log x) + O(1)$$

in case $|\alpha - \beta| \leq 1/10$, and in the case $|\alpha - \beta| \geq 1/10$ it exceeds

$$\log \log x - \log \log(2 + |\alpha - \beta|) + O(1).$$

Proof Indeed the triangle inequality gives that $\mathbb{D}(f, p^{i\alpha}; x) + \mathbb{D}(f, p^{i\beta}; x) \geq \mathbb{D}(p^{i\alpha}, p^{i\beta}; x) = \mathbb{D}(1, p^{i(\alpha-\beta)}; x)$ and we may now invoke Lemma 4.6 ^{lem4.3.1}. \square

The problem example $n^{i\alpha}$ discussed above takes on complex values, and one might wonder if there is a real valued multiplicative function f taking values in $[-1, 1]$ for which $\mathbb{D}(1, f; x) \rightarrow \infty$ as $x \rightarrow \infty$ but for which the mean value does not tend to zero. A lovely theorem of Wirsing, a precursor to the important theorem of Halász that we shall next discuss, establishes that this does not happen.

Wirsing

Theorem 4.8 Let f be a real valued multiplicative function with $|f(n)| \leq 1$ and $\mathbb{D}(1, f; x) \rightarrow \infty$ as $x \rightarrow \infty$. Then as $x \rightarrow \infty$

$$\frac{1}{x} \sum_{n \leq x} f(n) \rightarrow 0.$$

Note that Wirsing's theorem applied to $\mu(n)$ immediately yields the prime number theorem. We shall not directly discuss this theorem; instead we shall deduce it as a consequence of Halász's theorem.

4.4 Halász's theorem

We saw in the previous section that the function $f(n) = n^{i\alpha}$ has a large mean value even though $\mathbb{D}(1, f; x) \rightarrow \infty$ as $x \rightarrow \infty$. We may tweak such a function at a small number of primes and expect a similar result to hold. More precisely, one can ask if an analog of Delange's result holds: that is if f is multiplicative with $\mathbb{D}(f(p), p^{i\alpha}; \infty) < \infty$ for some α , can we understand the behavior of $\sum_{n \leq x} f(n)$? This is the content of the first result of Halász.

ex:4.4.1

Exercise 4.9 If f is a multiplicative function with $|f(n)| \leq 1$ show that there is at most one real number α with $\mathbb{D}(f, p^{i\alpha}; \infty) < \infty$.

Hal1

Theorem 4.10 Let f be multiplicative function with $|f(n)| \leq 1$ and suppose there exists $\alpha \in \mathbb{R}$ such that $\mathbb{D}(f, p^{i\alpha}; \infty) < \infty$. Write $f(n) = g(n)n^{i\alpha}$. Then as $x \rightarrow \infty$

$$\sum_{n \leq x} f(n) = \frac{x^{1+i\alpha}}{1+i\alpha} \mathcal{P}(g; x) + o(x).$$

Proof We show how Halász's first theorem may be deduced from Delange's Theorem 4.4. By partial summation we have

$$\sum_{n \leq x} f(n) = \int_1^x t^{i\alpha} d\left(\sum_{n \leq t} g(n)\right) = x^{i\alpha} \sum_{n \leq x} g(n) - i\alpha \int_1^x t^{i\alpha-1} \sum_{n \leq t} g(n) dt.$$

Now $\mathbb{D}(1, g; \infty) = \mathbb{D}(f, p^{i\alpha}; \infty) < \infty$ and so by Delange's theorem, if t is sufficiently large then

$$\sum_{n \leq t} g(n) = t\mathcal{P}(g; t) + o(t).$$

Therefore

$$\sum_{n \leq x} f(n) = x^{i\alpha} \mathcal{P}(g; x) - i\alpha \int_1^x t^{i\alpha} \mathcal{P}(g; t) dt + o(x).$$

Now note that $\mathcal{P}(g; t)$ is slowly varying: $\mathcal{P}(g; t) = \mathcal{P}(g; x) + O(\log(ex/t)/\log x)$ and our result follows. \square

Applying Theorem ^{Hal1}4.10 with f replaced by $f(n)/n^{i\alpha}$ we obtain the following:

Corollary 4.11 *Let f be multiplicative function with $|f(n)| \leq 1$ and suppose there exists $\alpha \in \mathbb{R}$ such that $\mathbb{D}(f, p^{i\alpha}; \infty) < \infty$. Then as $x \rightarrow \infty$*

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{x^{i\alpha}}{1 + i\alpha} \cdot \frac{1}{x} \sum_{n \leq x} \frac{f(n)}{n^{i\alpha}} + o(1).$$

This will be improved considerably in Theorem [AsympT2](#) [12.1](#).

The next result of Halász is central to our book, and it deals with the case when $\mathbb{D}(f, p^{i\alpha}; \infty) = \infty$ for all α . In fact Halász's result is more precise and quantitative.

Hal2 **Theorem 4.12** *Let f be a multiplicative function with $|f(n)| \leq 1$ for all n and let $1 \leq T \leq (\log x)^{10}$ be a parameter. Let*

$$M(x, T) = M_f(x, T) = \min_{|t| \leq T} \mathbb{D}(f, p^{it}; x)^2. \quad (4.7) \quad \text{mindist}$$

Then

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll M(x, T) \exp(-M(x, T)) + \frac{1}{T}.$$

Corollary 4.13 *If f is multiplicative with $|f(n)| \leq 1$ and $\mathbb{D}(f, p^{i\alpha}; \infty) = \infty$ for all real numbers α then as $x \rightarrow \infty$*

$$\frac{1}{x} \sum_{n \leq x} f(n) \rightarrow 0.$$

ex:4.13 **Exercise 4.14** *Show that if $T \geq 1$ then*

$$\frac{1}{2T} \int_{-T}^T \mathbb{D}(f, p^{it}; x)^2 dt \geq \log \log x + O(1).$$

Conclude that $M_f(x, T) \leq \log \log x + O(1)$, and the bound in Halász's theorem is never better than $x \log \log x / \log x$.

Exercise 4.15 *If $x \geq y$ show that*

$$0 \leq M_f(x, T) - M_f(y, T) \leq 2 \sum_{y < p \leq x} \frac{1}{p} = 2 \log \frac{\log x}{\log y} + O(1).$$

PROOF OF DELANGE'S THEOREM

Theorem 4.4 ^{|Delange} Let f be a multiplicative function taking values in the unit disc \mathbb{U} for which $\mathbb{D}(1, f; \infty) < \infty$. Then as $x \rightarrow \infty$ we have

$$\sum_{n \leq x} f(n) \sim x\mathcal{P}(f; x).$$

Let y be large and

$$\epsilon(y) := \sum_{p \geq y} \frac{1 - \operatorname{Re} f(p)}{p} \quad (5.1) \quad \boxed{\text{eq:De121}}$$

so that, by hypothesis, $\epsilon(y) \rightarrow 0$ as $y \rightarrow \infty$. Since $|1 - z|^2 \leq 2(1 - \operatorname{Re} z)$ for $z \in \mathbb{U}$ we have

$$\sum_{p \geq y} \frac{|1 - f(p)|^2}{p} \leq 2\epsilon(y). \quad (5.2) \quad \boxed{\text{eq:De122}}$$

Now we decompose the function f as $f(n) = s(n)\ell(n)$ where $s(n) = s_y(n)$ is the multiplicative function defined by $s(p^k) = f(p^k)$ if $p \leq y$ and $s(p^k) = 1$ otherwise. Correspondingly, $\ell(n) = \ell_y(n)$ is the multiplicative function defined by $\ell(p^k) = f(p^k)$ for $p > y$ and $\ell(p^k) = 1$ otherwise. Fixing y , Proposition 2.9 ^{|Prop2.7} gives that as $x \rightarrow \infty$

$$\sum_{n \leq x} s(n) = x\mathcal{P}(s; \infty) + o(x) = x\mathcal{P}(f; y) + o(x). \quad (5.3) \quad \boxed{\text{eq:De123}}$$

We shall prove Delange's theorem by showing that for large x (henceforth assumed $> y^2$) the function $\ell(n)$ is more or less constant over $n \leq x$.

Exercise 5.1 For any complex numbers w_1, \dots, w_k and z_1, \dots, z_k in the unit disc we have

$$|z_1 \cdots z_k - w_1 \cdots w_k| \leq \sum_{j=1}^k |z_j - w_j|.$$

Define now $g(p) = 0$ if $p \leq y$, $g(p) = f(p) - 1$ for $y < p \leq \sqrt{x}$ and $g(p) = 0$ for $p > \sqrt{x}$. Then consider the additive function

$$g(n) = \sum_{p|n} g(p),$$

where the primes are counted without multiplicity. If $n \leq x$ is not divisible by the square of any prime $> y$, using Exercise 5.1 we have

$$\begin{aligned} |\ell(n) - \exp(g(n))| &\leq \sum_{\substack{p|n \\ \sqrt{x} > p > y}} |f(p) - \exp(f(p) - 1)| + \sum_{\substack{p|n \\ p > \sqrt{x}}} |f(p) - 1| \\ &\ll \sum_{\substack{p|n \\ p > y}} |1 - f(p)|^2 + \sum_{\substack{p|n \\ p > \sqrt{x}}} |f(p) - 1|. \end{aligned}$$

Since the number of integers below x that are divisible by the square of some prime $> y$ is $\leq \sum_{p > y} x/p^2 \leq x/y$, we conclude that

$$\begin{aligned} \sum_{n \leq x} |\ell(n) - \exp(g(n))| &\ll \frac{x}{y} + x \sum_{\sqrt{x} > p > y} \frac{|1 - f(p)|^2}{p} + x \sum_{\sqrt{x} < p \leq x} \frac{|1 - f(p)|}{p} \\ &\ll x(\sqrt{\epsilon(y)} + 1/y), \end{aligned} \tag{5.4}$$

eq:bound1

where the last step follows upon using Cauchy's inequality and (5.2). eq:De122

PropDel1

Proposition 5.2 Suppose that $g(\cdot)$ is additive (as above) with each $|g(p)| \ll 1$. Let

$$\tilde{g} = \sum_{y < p \leq x} \frac{g(p)}{p}.$$

Then, for $x \geq y^2$,

$$\sum_{n \leq x} |g(n) - \tilde{g}|^2 \leq x \sum_{y < p \leq x} \frac{|g(p)|^2}{p} + O\left(\frac{x}{(\log x)^2}\right).$$

Proof Note that since $g(\cdot)$ is additive, and $g(p) = 0$ for $p \leq y$ and $p > \sqrt{x}$ we have

$$\sum_{n \leq x} g(n) = \sum_{\sqrt{x} \geq p > y} g(p) \left(\frac{x}{p} + O(1) \right) = x\tilde{g} + O(\pi(\sqrt{x})).$$

Hence, using $|\tilde{g}| \ll \log \log x$,

$$\sum_{n \leq x} |g(n) - \tilde{g}|^2 = \sum_{n \leq x} |g(n)|^2 - x|\tilde{g}|^2 + O\left(\frac{\sqrt{x} \log \log x}{\log x}\right).$$

Now, if $[p, q]$ is the least common multiple of p and q then

$$\begin{aligned} \sum_{n \leq x} |g(n)|^2 &= \sum_{\sqrt{x} \geq p, q \geq y} g(p) \overline{g(q)} \sum_{\substack{n \leq x \\ p, q | n}} 1 \\ &= \sum_{\sqrt{x} \geq p, q \geq y} g(p) \overline{g(q)} \cdot \frac{x}{[p, q]} + O(\pi(\sqrt{x})^2) \\ &= x|\tilde{g}|^2 + x \sum_{\sqrt{x} \geq p \geq y} |g(p)|^2 \left(\frac{1}{p} - \frac{1}{p^2} \right) + O\left(\frac{x}{(\log x)^2}\right) \end{aligned}$$

and the result follows. □

Now we are ready to prove Delange's theorem. Using [\(eq:bound1\)](#) we have

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} s(n) \exp(g(n)) + O(x(\sqrt{\epsilon(y)} + 1/y)).$$

Now if z and w have negative real parts, $|\exp(z) - \exp(w)| \ll |z - w|$. Therefore

$$\begin{aligned} \sum_{n \leq x} s(n) \exp(g(n)) &= \exp(\tilde{g}) \sum_{n \leq x} s(n) + O\left(\sum_{n \leq x} |g(n) - \tilde{g}|\right) \\ &= \exp(\tilde{g}) \sum_{n \leq x} s(n) + O(x(\sqrt{\epsilon(y)} + 1/\log x)), \end{aligned}$$

upon using [\(eq:Del22\)](#), [Proposition 5.2](#) and Cauchy's inequality. Now using [\(eq:Del23\)](#) we conclude that

$$\sum_{n \leq x} f(n) = \exp(\tilde{g}) x \mathcal{P}(s; x) + o(x) + O(x(\epsilon(y)^{\frac{1}{2}} + y^{-\frac{1}{2}})).$$

Now

$$\mathcal{P}(\ell; x) = \exp\left(\tilde{g} + O\left(\sum_{y < p \leq \sqrt{x}} \frac{1}{p^2} + \sum_{\sqrt{x} < p \leq x} \frac{|f(p) - 1|}{p}\right)\right) = \tilde{g} \left(1 + O\left(\frac{1}{y} + \sqrt{\epsilon(y)}\right)\right).$$

Since $\mathcal{P}(\ell; x) \mathcal{P}(s; x) = \mathcal{P}(f, x)$ we conclude that

$$\sum_{n \leq x} f(n) = x \mathcal{P}(f; x) + o(x) + O(x(\epsilon(y)^{\frac{1}{2}} + y^{-\frac{1}{2}})).$$

Letting $y \rightarrow \infty$ so that $\epsilon(y) \rightarrow 0$, we obtain Delange's theorem.

DEDUCING THE PRIME NUMBER THEOREM FROM HALÁSZ'S THEOREM

6.1 Real valued multiplicative functions: Deducing Wirsing's theorem

Let f be a multiplicative function with $-1 \leq f(n) \leq 1$ for all n . It seems unlikely that f can pretend to be a complex valued multiplicative function $n^{i\alpha}$. The triangle inequality allows us to make this intuition precise:

realdist

Lemma 6.1 *Let f be a multiplicative function with $-1 \leq f(n) \leq 1$ for all n . For any real number $|\alpha| \leq (\log x)^{10}$ we have*

$$\mathbb{D}(f, p^{i\alpha}; x) \geq \min \left(\frac{1}{2} \sqrt{\log \log x} + O(1), \frac{1}{3} \mathbb{D}(1, f; x) + O(1) \right).$$

Proof Since $\mathbb{D}(f, p^{i\alpha}; x) = \mathbb{D}(f, p^{-i\alpha}; x)$ the triangle inequality gives

$$\mathbb{D}(1, p^{2i\alpha}; x) = \mathbb{D}(p^{-i\alpha}, p^{i\alpha}; x) \leq 2\mathbb{D}(f, p^{i\alpha}; x).$$

In the range $1/100 \leq |\alpha| \leq (\log x)^{10}$, we obtain from Lemma 4.6 that $\mathbb{D}(1, p^{2i\alpha}; x)^2 \geq (1 - \epsilon) \log \log x$, and so the lemma follows in this range.

Suppose now that $|\alpha| \leq 1/100$. Then $\mathbb{D}(1, p^{2i\alpha}; x) = \mathbb{D}(1, p^{i\alpha}; x) + O(1)$ by Lemma 4.3.1. Thus, by the triangle inequality and our estimate above

$$\mathbb{D}(f, p^{i\alpha}; x) \geq \mathbb{D}(1, f; x) - \mathbb{D}(1, p^{i\alpha}; x) \geq \mathbb{D}(1, f; x) - 2\mathbb{D}(f, p^{i\alpha}; x) + O(1)$$

so that

$$\mathbb{D}(f, p^{i\alpha}; x) \geq \frac{1}{3} \mathbb{D}(1, f; x) + O(1).$$

□

Using the above Lemma and Halász's theorem with $T = (\log x)^{10}$ we deduce:

Halreal

Corollary 6.2 *If f is a multiplicative function with $-1 \leq f(n) \leq 1$ then*

$$\frac{1}{x} \sum_{n \leq x} f(n) \ll \mathbb{D}(1, f; x)^2 \exp \left(-\frac{1}{9} \mathbb{D}(1, f; x)^2 \right) + \frac{1}{(\log x)^{\frac{1}{4} + o(1)}}.$$

Note that the above Corollary implies a quantitative form of Wirsing's Theorem 4.8. An optimal version of Corollary 6.2 has been obtained by Hall and Tenenbaum.

6.2 Deducing the prime number theorem

Using Corollary ^{Halász}6.2 with $f = \mu$ we get

$$\left| \sum_{n \leq x} \mu(n) \right| \ll \frac{x}{(\log x)^{\frac{2}{9}+o(1)}}$$

and then

$$\psi(x) = x + O\left(\frac{x}{(\log x)^{\frac{2}{9}+o(1)}}\right)$$

by Exercise 1.12 of §1.5.

The classical proof of the Prime Number Theorem yields a much better error term than what we have obtained above; indeed one can obtain

$$\psi(x) = x + O\left(x \exp\left(-(\log x)^{3/5+o(1)}\right)\right).$$

There are also elementary proofs of the prime number theorem that yield an error term of $O\left(x \exp\left(-(\log x)^{1/2+o(1)}\right)\right)$. While we can make some small improvements (see Lemma 6.2 below) to the error term $O(x/(\log x)^{\frac{2}{9}+o(1)})$ obtained by Halász's theorem, the methods from the study of multiplicative functions do not appear capable of giving an error better than $O(x/\log x)$. That is our methods are very far, quantitatively, from what can be obtained by other methods.

SELBERG'S SIEVE AND THE BRUN-TITCHMARSH THEOREM

In order to develop the theory of mean values of multiplicative functions, we shall need an estimate for the distribution of primes in short intervals. We need only an upper estimate for the number of such primes, and this can be achieved by a simple sieve method and does not need results of the strength of the prime number theorem. We describe a beautiful method of Selberg which works well in this and many other applications, but there are many other sieves which would also work. The reader is referred to Friedlander and Iwaniec's *Opera de Cribro* for a thorough treatment of sieves in general and their many applications.

7.1 The Brun-Titchmarsh theorem

Let $a \pmod{q}$ be an arithmetic progression with $(a, q) = 1$ and let $\pi(x; q, a)$ denote the number of primes $p \leq x$ with $p \equiv a \pmod{q}$. The Brun-Titchmarsh theorem gives an estimate for the number of primes in an interval $(x, x+y]$ lying in the arithmetic progression $a \pmod{q}$.

Let $\lambda_1 = 1$ and let λ_d be a sequence of real numbers with $\lambda_d = 0$ if $d > R$ or if d has a common factor with q . Selberg's sieve is based on the simple idea that squares are positive, and so

$$\left(\sum_{d|n} \lambda_d \right)^2 \text{ is } \begin{cases} = 1 & \text{if } n > R \text{ is prime} \\ \geq 0 & \text{always.} \end{cases}$$

Therefore, assuming for simplicity that $R \leq x$,

$$\pi(x+y; q, a) - \pi(x; q, a) \leq \sum_{\substack{x < n \leq x+y \\ n \equiv a \pmod{q}}} \left(\sum_{d|n} \lambda_d \right)^2.$$

Expanding out the inner sum this is

$$\sum_{d_1, d_2} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{x < n \leq x+y \\ n \equiv a \pmod{q} \\ [d_1, d_2] | n}} 1,$$

where $[d_1, d_2]$ denotes the l.c.m. of d_1 and d_2 . Since $\lambda_d = 0$ unless $(d, q) = 1$, the inner sum over n above is over one congruence class $\pmod{q[d_1, d_2]}$, and therefore this inner sum is within 1 of $y/(q[d_1, d_2])$. We conclude that

$$\begin{aligned}
\pi(x+y; q, a) - \pi(x; q, a) &\leq \frac{y}{q} \sum_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} + \sum_{d_1, d_2} |\lambda_{d_1} \lambda_{d_2}| \\
&= \frac{y}{q} \sum_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} + \left(\sum_d |\lambda_d| \right)^2. \tag{7.1} \quad \boxed{\text{E3.1}}
\end{aligned}$$

The ingenious part of Selberg's argument is in determining the optimal choice of λ_d so as to minimize the first term in (7.1). The second term in (7.1) may be viewed as an error term, arising from the error in counting integers in an interval, and this roughly places the restriction that R is at most $\sqrt{y/q}$. In such a range of R , the first term in (7.1) is the more important main term, and observe that it is a quadratic form in the variables λ_d . The problem of minimizing this main term thus takes the shape of minimizing a quadratic form subject to the linear constraint $\lambda_1 = 1$. Selberg's quadratic form admits an elegant diagonalization which allows us to find the optimal choice for λ_d .

Since $[d_1, d_2] = d_1 d_2 / (d_1, d_2)$, and $(d_1, d_2) = \sum_{\ell | (d_1, d_2)} \phi(\ell)$ we have

$$\sum_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} = \sum_{\ell} \phi(\ell) \sum_{\substack{\ell | d_1 \\ \ell | d_2}} \frac{\lambda_{d_1}}{d_1} \frac{\lambda_{d_2}}{d_2} = \sum_{\ell} \frac{\phi(\ell)}{\ell^2} \left(\sum_d \frac{\lambda_{d\ell}}{d} \right)^2.$$

If we set

$$\xi_{\ell} = \sum_d \frac{\lambda_{d\ell}}{d},$$

then we have diagonalized the quadratic form in our main term:

$$\sum_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} = \sum_{\ell} \frac{\phi(\ell)}{\ell^2} \xi_{\ell}^2. \tag{7.2} \quad \boxed{\text{E3.2}}$$

Note that like λ_d , we have that $\xi_{\ell} = 0$ if $\ell > R$ or if $(\ell, q) > 1$.

What does the constraint $\lambda_1 = 1$ mean for the new variables ξ_{ℓ} ? We must invert the linear change of variables that we made in going from the λ 's to the ξ 's, and this is easily done by Möbius inversion. Let $\delta(\ell) = \sum_{r|\ell} \mu(r)$ be 1 if $\ell = 1$ and 0 otherwise. Then

$$\lambda_d = \sum_{\ell} \frac{\lambda_{d\ell}}{\ell} \delta(\ell) = \sum_{\ell} \frac{\lambda_{d\ell}}{\ell} \sum_{r|\ell} \mu(r) = \sum_r \mu(r) \sum_{r|\ell} \frac{\lambda_{d\ell}}{\ell} = \sum_r \frac{\mu(r)}{r} \xi_{dr}.$$

In particular, the linear constraint $\lambda_1 = 1$ becomes

$$1 = \sum_r \frac{\mu(r)}{r} \xi_r. \tag{7.3} \quad \boxed{\text{E3.3}}$$

We have transformed our problem to minimizing the diagonal quadratic form in (7.2) subject to the linear constraint in (7.3). It is clear that the optimal choice

is when ξ_r is proportional to $\mu(r)r/\phi(r)$ for $r \leq R$ and $(r, q) = 1$. The constant of proportionality can be determined from (7.3) and we conclude that the optimal choice is to take (for $r \leq R$ and $(r, q) = 1$)

$$\xi_r = \frac{1}{L_q(R)} \frac{r\mu(r)}{\phi(r)} \quad \text{where} \quad L_q(R) = \sum_{\substack{r \leq R \\ (r, q) = 1}} \frac{\mu(r)^2}{\phi(r)}. \quad (7.4) \quad \boxed{\text{E3.4}}$$

For this choice, the quadratic form in (7.2) attains the minimum value which is $1/L_q(R)$. Note also that for this choice of ξ , we have (for $d \leq R$ and $(d, q) = 1$)

$$\lambda_d = \frac{1}{L_q(R)} \sum_{\substack{r \leq R/d \\ (r, q) = 1}} \frac{d\mu(r)\mu(dr)}{\phi(dr)},$$

and so

$$\sum_{d \leq R} |\lambda_d| \leq \frac{1}{L_q(R)} \sum_{\substack{d, r \\ dr \leq R \\ (dr, q) = 1}} \frac{\mu(dr)^2 d}{\phi(dr)} = \frac{1}{L_q(R)} \sum_{\substack{n \leq R \\ (n, q) = 1}} \frac{\mu(n)^2 \sigma(n)}{\phi(n)}, \quad (7.5) \quad \boxed{\text{E3.5}}$$

where $\sigma(n) = \sum_{d|n} d$.

Putting these estimates into (7.1) we deduce that for any arithmetic progression $a \pmod{q}$ with $(a, q) = 1$, and any $R \leq x$, we have

$$\pi(x+y; q, a) - \pi(x; q, a) \leq \frac{y}{qL_q(R)} + \frac{1}{L_q(R)^2} \left(\sum_{\substack{n \leq R \\ (n, q) = 1}} \frac{\mu(n)^2 \sigma(n)}{\phi(n)} \right)^2, \quad (7.6) \quad \boxed{\text{BTeqn}}$$

This bound looks unwieldy but the techniques developed in Chapter 2 are enough to estimate the sums above. We illustrate this in the case $q = 1$. Note that by Exercise 2.6

$$\sum_{n \leq R} \frac{\mu(n)^2 \sigma(n)}{\phi(n)} = \frac{15}{\pi^2} x + O(\sqrt{x} \log x).$$

Exercise 7.1 Using Exercise 2.7, or otherwise, show that

$$L_1(R) = \log R + \gamma + C + O\left(\frac{(\log R)^2}{\sqrt{R}}\right),$$

where

$$C = \sum_p \frac{\log p}{p(p-1)} = 0.7553 \dots$$

Exercise 7.2 Taking $R = \sqrt{\lambda y}$ and choosing λ optimally as $\pi^4/450$, prove that for any $3 \leq y \leq x$ we have

$$\pi(x+y) - \pi(x) \leq \frac{2y}{\log y} + \frac{2y}{(\log y)^2} \left(1 - 2\gamma - 2C + \log \frac{450}{\pi^4}\right) + O\left(\frac{y}{(\log y)^3}\right),$$

In particular we have:

Theorem 7.1 (Brun-Titchmarsh) *If $y \geq y_0$ is large enough then*

$$\pi(x+y) - \pi(x) \leq \frac{2y}{\log y}.$$

One can go much further than this, using ^{BTeqn}(7.6), to obtain that if $y/q \geq y_0$ then

$$\pi(x+y; q, a) - \pi(x; q, a) \leq \frac{2y}{\phi(q) \log(y/q)}.$$

Exercise 7.2 *Prove this.*

7.2 An alternative lower bound for a key distance

Lemma 7.3 *If $|t| \leq x^{o(1)}$ then*

$$\mathbb{D}^2(\mu(n), n^{it}; x) \geq \left\{1 - \frac{2}{\pi} + o(1)\right\} \log \left(\frac{\log x}{\log(2 + |t|)} \right).$$

Proof Fix $\alpha \in [0, 1)$ and $\epsilon > 0$. Let P be the set of primes for which there exists an integer n such that $p \in I_n := [e^{2\pi(n+\alpha)/|t|}, e^{2\pi(n+\alpha+\epsilon)/|t|}]$, so that $\operatorname{Re}(p^{it})$ lies between $\cos(2\pi\alpha)$ and $\cos(2\pi(\alpha+\epsilon))$. We partition the intervals I_n into subintervals of the form $[y, y+z]$, where $z = o(y)$ and $\log z \sim \log y$, which is possible provided $|t| = o(n/\log n)$ (Exercise). The Brun-Titchmarsh Theorem implies that the number of primes in each such interval is $\leq \{2 + o(1)\}z/\log y$, and so $\sum_{p \in I_n} 1/p \leq \{2 + o(1)\} \log(1 + \frac{\epsilon}{n+\alpha})$, from which we deduce

$$\sum_{\substack{x_0 < p \leq x \\ p \in I_n \text{ for some } n}} \frac{1}{p} \leq \{2\epsilon + o(1)\} \log \left(\frac{\log x}{\log x_0} \right) + O(\epsilon),$$

where $x_0 := (2 + |t|)^{\log u} e^{2\pi/|t|}$ and $2 + |t| = x^{1/u}$, as $u \rightarrow \infty$. Combining this with (1.2.4), we deduce (exercise) that

$$\begin{aligned} \sum_{x_0 < p \leq x} \frac{1 + \cos(t \log p)}{p} &\geq \{2 + o(1)\} \log \left(\frac{\log x}{\log x_0} \right) \int_{1/4}^{3/4} (1 + \cos(2\pi\alpha)) d\alpha + O(1) \\ &\geq \left\{1 - \frac{2}{\pi} + o(1)\right\} \log \left(\frac{\log x}{\log x_0} \right) + O(1). \end{aligned}$$

The result follows if $|t| \geq 1$. If $|t| < 1$ then $\log \left(\frac{\log x}{\log x_0} \right) \sim \log(|t| \log x)$. However, we also have

$$\sum_{p \leq e^{2\pi/3|t|}} \frac{1 + \cos(t \log p)}{p} \geq (1 + \cos(2\pi/3)) \sum_{p \leq e^{2\pi/3|t|}} \frac{1}{p} \geq \frac{1}{2} \log \frac{1}{|t|} + O(1),$$

by (1.2.4), and then adding these lower bounds gives the result. \square

HALÁSZ'S THEOREM

In this chapter we develop the proof of Halász's Theorem (Theorem ^{|Hal2}4.12) that if f is a multiplicative function with $|f(n)| \leq 1$ for all n and let $1 \leq T \leq (\log x)^{10}$ be a parameter with $M(x, T) = M_f(x, T) = \min_{|t| \leq T} \mathbb{D}(f, p^{it}; x)^2$, then

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll M(x, T) \exp(-M(x, T)) + \frac{1}{T}.$$

Throughout the chapter f will be a multiplicative function with $|f(n)| \leq 1$. The sum $\sum_{n \leq x} f(n)$ will be denoted by $S(x)$ and the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ by $F(s)$.

MeanF(n)

8.1 Averages of averages

First we begin with an identity which generalizes the identity ^{|E2,10}(3.1) for smooth numbers.

lemHal1

Lemma 8.1 *For any multiplicative function f with $|f(n)| \leq 1$ we have*

$$S(x) \log x = \sum_{p \leq x} f(p) \log p S(x/p) + O(x). \quad (8.1) \quad \text{HildIdentity}$$

Proof Note that

$$S(x) \log x = \sum_{n \leq x} f(n) \log n + O\left(\sum_{n \leq x} \log(x/n)\right) = \sum_{n \leq x} f(n) \log n + O(x). \quad (8.2) \quad \text{eq:Hal1.1}$$

Next writing $\log n = \sum_{d|n} \Lambda(d)$ we have

$$\sum_{n \leq x} f(n) \log n = \sum_{n \leq x} f(n) \sum_{d|n} \Lambda(d) = \sum_{d \leq x} \Lambda(d) \sum_{\substack{n \leq x \\ d|n}} f(n).$$

The last sum above has size $\leq x/d$, and so the contribution from prime powers $d = p^b$ with $b \geq 2$ is $\ll \sum_{p \leq \sqrt{x}} (\log p)(x/p^2) \ll x$. Further when $d = p$ the final sum over n equals $f(p)S(x/p) + O(x/p^2)$, where the error results from those n that are divisible by p^2 and there are at most x/p^2 such terms. We thus conclude that

$$S(x) \log x = \sum_{p \leq x} \log p \sum_{\substack{n \leq x \\ p|n}} f(n) + O(x) = \sum_{p \leq x} \log p S(x/p) + O(x),$$

proving our Lemma. □

The next step is to bound $S(x)$ by an average involving $S(t)$ for all $t \leq x$.

propHal1 **Proposition 8.2** *With notations as above*

$$\frac{|S(x)|}{x} \ll \frac{1}{\log x} \int_1^x \frac{|S(t)|}{t} \frac{dt}{t} + \frac{1}{\log x}.$$

Note that $|S(t)|/t$ is the average size of $f(n)$ for n up to t , and so the Proposition bounds $|S(x)|$ by an “average of averages”.

Proof Now, for $z = y + y^{1/2}$, using the Brun-Titchmarsh theorem,

$$\begin{aligned} \sum_{y < p \leq z} \log p \left| S\left(\frac{x}{p}\right) \right| &\leq \sum_{y < p \leq z} \log p \max_{y \leq u \leq z} \left| S\left(\frac{x}{u}\right) \right| \ll (z - y) \max_{y \leq u \leq z} \left| S\left(\frac{x}{u}\right) \right| \\ &\leq \int_y^z \left| S\left(\frac{x}{t}\right) \right| dt + (z - y) \max_{y \leq t, u \leq z} \left| S\left(\frac{x}{t}\right) - S\left(\frac{x}{u}\right) \right|, \end{aligned}$$

and

$$\left| S\left(\frac{x}{t}\right) - S\left(\frac{x}{u}\right) \right| \leq \left| \frac{x}{t} - \frac{x}{u} \right| = x \cdot \frac{|u - t|}{tu} \leq x \cdot \frac{z - y}{y^2}.$$

Summing over such intervals between y and $2y$ we obtain

$$\sum_{y < p \leq 2y} \log p \left| S\left(\frac{x}{p}\right) \right| \ll \int_y^{2y} \left| S\left(\frac{x}{t}\right) \right| dt + \frac{x}{y^{1/2}},$$

which implies, by Lemma lemHal1, that

$$\begin{aligned} |S(x)| &\ll \frac{1}{\log x} \int_1^x \left| S\left(\frac{x}{t}\right) \right| dt + \frac{x}{\log x} \\ &= \frac{x}{\log x} \int_1^x |S(t)| \frac{dt}{t^2} + \frac{x}{\log x}. \end{aligned}$$

□

8.2 Applications of the Plancherel formula

Plancherel

Proposition 8.3 *Let a_n be any sequence of complex numbers such that $\mathcal{A}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely in $\operatorname{Re}(s) > 1$. Define also $A(x) = \sum_{n \leq x} a_n$. For any $\alpha > 0$ we have*

$$\int_1^{\infty} \frac{|A(t)|^2}{t^{3+2\alpha}} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\mathcal{A}(1 + \alpha + iy)|^2}{|1 + \alpha + iy|^2} dy.$$

Proof Consider the function $G(y) = A(e^y)/e^{(1+\alpha)y}$. Note that the Fourier transform of G is

$$\hat{G}(\xi) = \int_{-\infty}^{\infty} G(y) e^{-iy\xi} dy = \sum_{n=1}^{\infty} a_n \int_{\log n}^{\infty} e^{-(1+\alpha+iy\xi)y} dy = \frac{\mathcal{A}(1+\alpha+i\xi)}{1+\alpha+i\xi}.$$

Thus Plancherel's formula gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\mathcal{A}(1+\alpha+i\xi)}{1+\alpha+i\xi} \right|^2 d\xi = \int_{-\infty}^{\infty} |G(y)|^2 dy = \int_1^{\infty} \frac{|A(t)|^2}{t^{3+2\alpha}} dt,$$

upon making the substitution $t = e^y$. \square

Plancherel

The Proposition connects weighted averages of $|S(t)|$ with the generating function $F(s)$. It turns out that it is more fruitful to apply the Plancherel formula not directly to F but to F' . The bound that we thus derive is crucial to the proof of Halász's theorem.

keybound

Proposition 8.4 *Let $T \geq 1$ be a parameter. For any $1 \geq \alpha > 0$ we have*

$$\int_1^{\infty} \left| \sum_{n \leq t} f(n) \log n \right|^2 \frac{dt}{t^{3+2\alpha}} \ll \frac{1}{\alpha} \left(\max_{|y| \leq T} |F(1+\alpha+iy)|^2 + 1 + \frac{1}{(\alpha T)^2} \right).$$

Proof We write $F(s) = G(s)H(s)$ where

$$G(s) = \prod_p \left(1 - \frac{f(p)}{p^s} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\tilde{f}(n)}{n^s}.$$

That is, $\tilde{f}(n)$ is the completely multiplicative function which matches f on all primes. Note that $H(s)$ is given then by an Euler product which converges absolutely in $\operatorname{Re}(s) > 1/2$ and that in the region $\operatorname{Re}(s) \geq 1$ we have $|H(s)|$ and $|H'(s)| \ll 1$.

The Plancherel formula gives

$$\begin{aligned} \int_1^{\infty} \left| \sum_{n \leq t} f(n) \log n \right|^2 \frac{dt}{t^{3+2\alpha}} &\ll \int_{-\infty}^{\infty} \left| \frac{F'(1+\alpha+iy)}{1+\alpha+iy} \right|^2 dy \\ &\ll \int_{-\infty}^{\infty} \left(\left| \frac{G'H(1+\alpha+iy)}{1+\alpha+iy} \right|^2 + \left| \frac{GH'(1+\alpha+iy)}{1+\alpha+iy} \right|^2 \right) dy. \end{aligned} \quad (8.3) \quad \text{eq:Hal12}$$

Since $H'(1+\alpha+iy) \ll 1$ the second term above is

$$\ll \int_{-\infty}^{\infty} \left| \frac{G(1+\alpha+iy)}{1+\alpha+iy} \right|^2 dy \ll \int_1^{\infty} \left| \sum_{n \leq t} \tilde{f}(n) \right|^2 \frac{dt}{t^{3+2\alpha}} \ll \frac{1}{\alpha}, \quad (8.4) \quad \text{eq:Hal13}$$

upon using Plancherel again.

Now consider the first term in (8.3)^[eg:Hal12] and split it into the two regions $|y| \leq T$ and $|y| > T$. Consider the contribution of the first region. This is

$$\begin{aligned} &\ll \left(\max_{|y| \leq T} |F(1 + \alpha + iy)|^2 \right) \int_{|y| \leq T} \left| \frac{(G'/G)(1 + \alpha + iy)}{1 + \alpha + iy} \right|^2 dy \\ &\ll \left(\max_{|y| \leq T} |F(1 + \alpha + iy)|^2 \right) \int_{-\infty}^{\infty} \left| \frac{(G'/G)(1 + \alpha + iy)}{1 + \alpha + iy} \right|^2 dy. \end{aligned}$$

Now $G'/G(s) = -\sum_n \tilde{f}(n)\Lambda(n)n^{-s}$, and using Plancherel yet again we have

$$\int_{-\infty}^{\infty} \left| \frac{(G'/G)(1 + \alpha + iy)}{1 + \alpha + iy} \right|^2 dy \ll \int_1^{\infty} \left| \sum_{n \leq t} \tilde{f}(n)\Lambda(n) \right|^2 \frac{dt}{t^{3+2\alpha}} \ll \int_1^{\infty} \frac{dt}{t^{1+2\alpha}} \ll \frac{1}{\alpha},$$

upon using the Chebyshev bound that $\psi(t) \ll t$. This is clearly acceptable.

It remains lastly to consider the contribution of the region $|y| \geq T$. Since $H(1 + \alpha + iy) \ll 1$ we must bound

$$\int_{|y| > T} \left| \frac{G'(1 + \alpha + iy)}{1 + \alpha + iy} \right|^2 dy \ll \int_{-\infty}^{\infty} \left| \frac{G'(1 + \alpha + iy)}{T + 1 + \alpha + iy} \right|^2 dy.$$

Now $G'(1 + \alpha + iy)/(T + 1 + \alpha + iy)$ is the Fourier transform of the function $-e^{-(T+1+\alpha)x} \sum_{n \leq e^x} \tilde{f}(n)n^T \log n$ and so by Plancherel the above quantity is

$$\ll \int_{-\infty}^{\infty} e^{-2x(T+1+\alpha)} \left| \sum_{n \leq e^x} \tilde{f}(n)n^T \log n \right|^2 dx \ll \int_0^{\infty} x^2 e^{-2x(T+1+\alpha)} \left(\sum_{n \leq e^x} n^T \right)^2 dx.$$

Now by splitting into the cases $e^x \leq T$ and $e^x > T$ we can easily establish that $\sum_{n \leq e^x} n^T \ll e^{Tx} + e^{(T+1)x}/(T+1)$. Therefore our integral above is

$$\ll \int_0^{\infty} x^2 \left(e^{-2x(1+\alpha)} + \frac{e^{-2\alpha x}}{(T+1)^2} \right) dx \ll 1 + \frac{1}{\alpha^3 T^2}.$$

This completes our proof. \square

8.3 The key estimate

Combining our work in the preceding two sections we arrive at the following key estimate.

keyProp

Proposition 8.5 *With notations as above, we have for $x \geq 3$,*

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll \frac{1}{\log x} \int_{1/\log x}^1 \max_{|t| \leq T} |F(1 + \alpha + it)| \frac{d\alpha}{\alpha} + \frac{1}{T} + \frac{\log \log x}{\log x}.$$

Proof For any $x \geq y \geq 3$ from eq:Hal1.1 (8.2) we have

$$S(y) = \frac{1}{\log y} \sum_{n \leq y} f(n) \log n + O\left(\frac{y}{\log y}\right) \ll \int_{1/\log x}^1 \left| \sum_{n \leq y} f(n) \log n \right| \frac{d\alpha}{y^{2\alpha}} + \frac{y}{\log y}.$$

Therefore

$$\int_2^x \frac{|S(y)|}{y^2} dy \ll \int_{1/\log x}^1 \left(\int_2^x \left| \sum_{n \leq y} f(n) \log n \right| \frac{dy}{y^{2+2\alpha}} \right) d\alpha + \log \log x. \quad (8.5) \quad \text{eq:Hal14}$$

Applying Cauchy's inequality and Proposition keybound 8.4 we get, for $1 \geq \alpha \geq 1/\log x$,

$$\begin{aligned} \left(\int_2^x \left| \sum_{n \leq y} f(n) \log n \right| \frac{dy}{y^{2+2\alpha}} \right)^2 &\leq \left(\int_1^x \frac{dy}{y^{1+2\alpha}} \right) \left(\int_2^x \left| \sum_{n \leq y} f(n) \log n \right|^2 \frac{dy}{y^{3+2\alpha}} \right) \\ &\ll \frac{1}{\alpha^2} \left(\max_{|y| \leq T} |F(1 + \alpha + iy)|^2 + 1 + \frac{1}{(\alpha T)^2} \right). \end{aligned}$$

Using this in eq:Hal14 (8.5) we conclude that

$$\begin{aligned} \int_1^x \frac{|S(y)|}{y^2} dy &\ll \int_{1/\log x}^1 \frac{1}{\alpha} \left(\max_{|y| \leq T} |F(1 + \alpha + iy)| + 1 + \frac{1}{\alpha T} \right) d\alpha + \log \log x \\ &\ll \int_{1/\log x}^1 \max_{|y| \leq T} |F(1 + \alpha + iy)| \frac{d\alpha}{\alpha} + \frac{\log x}{T} + \log \log x. \end{aligned}$$

Inserting this bound in Proposition propHal1 8.2 we have completed the proof of our Proposition. \square

8.4 Proof of Halász's theorem

We begin with the following general Lemma.

lemHal12

Lemma 8.6 Let a_n be a sequence of complex numbers such that $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$, so that $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent in $\text{Re}(s) \geq 1$. For all real numbers $T \geq 1$, and all $0 \leq \alpha \leq 1$ we have

$$\max_{|t| \leq T} |A(1 + \alpha + it)| \leq \max_{|u| \leq 2T} |A(1 + iu)| + O\left(\frac{\alpha}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n}\right).$$

Exercise 8.7 Prove that, for any integer $n \geq 1$, we have

$$n^{-\alpha} = \frac{1}{\pi} \int_{-T}^T \frac{\alpha}{\alpha^2 + \xi^2} n^{-i\xi} d\xi + O\left(\frac{\alpha}{T}\right).$$

(Hint: Show that $\frac{2\alpha}{\alpha^2 + \xi^2}$ is the Fourier transform of $e^{-\alpha|z|}$.)

Proof Multiplying the result in this exercise through by a_n/n^{1+it} , and summing over all n , we obtain

$$A(1 + \alpha + it) = \frac{1}{\pi} \int_{-T}^T \frac{\alpha}{\alpha^2 + \xi^2} A(1 + it + i\xi) d\xi + O\left(\frac{\alpha}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n}\right)$$

which yields the result when $|t| \leq T$, since then $|u| \leq |t| + |\xi| \leq 2T$ for $u = t + \xi$, and as $\frac{1}{\pi} \int_{-T}^T \frac{\alpha}{\alpha^2 + \xi^2} d\xi \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 + \xi^2} d\xi = 1$ by the exercise with $n = 1$. \square

Now we are ready to complete the proof of Halász's theorem. We will bound the terms in the integral in Proposition 8.5 using Lemma 8.6 above. Applying Lemma 8.6 with $a_n = f(n)n^{-1/\log x}$ we obtain that for any $1/\log x \leq \alpha \leq 1$ we have

$$\begin{aligned} \max_{|y| \leq T} |F(1 + \alpha + iy)| &\leq \max_{|y| \leq 2T} |F(1 + 1/\log x + iy)| + O\left(\frac{\alpha \log x}{T}\right) \\ &\ll (\log x) \exp(-M(x, 2T)) + \frac{\alpha \log x}{T}. \end{aligned}$$

Moreover we have

$$\max_{|y| \leq T} |F(1 + \alpha + iy)| \leq \zeta(1 + \alpha) = \frac{1}{\alpha} + O(1).$$

Using the minimum of the two bounds above in Proposition 8.5 (in other words, using the first bound for $\alpha \leq \exp(-M(x, 2T))/\log x$ and the second for larger α) we conclude that

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll M(x, 2T) \exp(-M(x, 2T)) + \frac{1}{T} + \frac{\log \log x}{\log x}.$$

By Exercise 4.13 we have $M(x, 2T) \exp(-M(x, 2T)) \gg \log \log x / \log x$, so that the $\log \log x / \log x$ term above may be dropped. Now renaming $2T$ as T we obtain Theorem 4.12.

8.5 The logarithmic mean

Since

$$\int_1^x \sum_{n \leq t} f(n) \frac{dt}{t^2} = \sum_{n \leq x} f(n) \int_n^x \frac{dt}{t^2} = \sum_{n \leq x} \frac{f(n)}{n} - \frac{1}{x} \sum_{n \leq x} f(n),$$

and so we deduce that

$$\left| \sum_{n \leq x} \frac{f(n)}{n} \right| \leq \frac{S(x)}{x} + \int_1^x |S(t)| \frac{dt}{t^2}.$$

By Proposition 8.2 and then (8.5) we then deduce

$$\frac{1}{\log x} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| \ll (1 + M(x, T)) e^{-M(x, T)} + \frac{1}{T}. \quad (8.6) \quad \boxed{\text{Halasz4Log}}$$

MULTIPLICATIVE FUNCTIONS

This is where the book gets less organized. This section includes several useful results that will be used, but at the moment are not well tied together in a common theme.

9.1 Upper bounds by averaging further

Suppose that $0 \leq h(p^a) \ll C^a$ for all prime powers p^a , where $C < 2$.

Exercise 9.1 *Use this hypothesis to show that $\sum_{p^a \leq x} h(p^a) \log p^a \ll x$. Give an example to show that this fails for $C = 2$.*

Therefore

$$\sum_{n \leq x} h(n) \log n = \sum_{n \leq x} h(n) \sum_{p^a \parallel n} \log p^a = \sum_{m \leq x} h(m) \sum_{\substack{p^a \leq x/m \\ p \nmid m}} h(p^a) \log p^a \ll x \sum_{m \leq x} \frac{h(m)}{m},$$

by the Brun-Titchmarsh theorem. Moreover, since $\log(x/n) \leq x/n$ whenever $n \leq x$, hence

$$\sum_{n \leq x} h(n) \log(x/n) \leq x \sum_{m \leq x} \frac{h(m)}{m}$$

and adding these together gives

$$\sum_{n \leq x} h(n) \ll \frac{x}{\log x} \sum_{m \leq x} \frac{h(m)}{m}. \quad (9.1) \quad \boxed{(3.2.1)}$$

Using partial summation we deduce from ^(3.2.1)9.1 that for $1 \leq y \leq x^{1/2}$,

$$\sum_{x/y < n \leq x} \frac{h(n)}{n} \ll \frac{\log(2y)}{\log x} \sum_{n \leq x} \frac{h(n)}{n}. \quad (9.2) \quad \boxed{(3.2.2)}$$

If $f = 1 * g$ and we proceed as in the proof of ^{pr2.1}2.1 then

$$\begin{aligned}
\left| \frac{S(x)}{x} - \frac{S(x/y)}{x/y} \right| &\leq \left| \frac{S(x)}{x} - \sum_{d \leq x} \frac{g(d)}{d} \right| + \left| \frac{S(x/y)}{x/y} - \sum_{d \leq x/y} \frac{g(d)}{d} \right| + \left| \sum_{x/y < d \leq x} \frac{g(d)}{d} \right| \\
&\leq \frac{1}{x} \sum_{d \leq x} |g(d)| + \frac{1}{x/y} \sum_{d \leq x/y} |g(d)| + \sum_{x/y < d \leq x} \frac{|g(d)|}{d} \\
&\ll \frac{\log(2y)}{\log x} \sum_{m \leq x} \frac{|g(m)|}{m} \leq \frac{\log(2y)}{\log x} \exp \left(\sum_{p \leq x} \frac{|1 - f(p)|}{p} \right),
\end{aligned} \tag{9.3} \quad \boxed{\text{ConvId}}$$

by (3.2.1) and (3.2.2). Note that this holds trivially for $y > x$. This result may be regarded as a first Lipschitz type estimate, explored in more detail later on in chapter 15.

9.2 Convolutions of Sums

We introduce here an idea that will be of importance later, in which we develop (8.1). If f is totally multiplicative then

$$\begin{aligned}
\int_0^1 S(x^{1-t}) \sum_{r \leq x^t} f(r) \Lambda(r) dt &= \sum_{mr \leq x} f(mr) \Lambda(r) \int_{\frac{\log r}{\log x}}^{\frac{\log x/m}{\log x}} dt \\
&= \sum_{n \leq x} f(n) \log n \frac{\log x/n}{\log x} = \int_1^x S(x/t) \frac{\log(x/t^2)}{\log x} \frac{dt}{t}.
\end{aligned}$$

By (9.3) this equals

$$S(x) + O \left(\frac{S(x)}{\log x} + \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{|1 - f(p)|}{p} \right) \right). \tag{9.4} \quad \boxed{(\text{ConvApprox})}$$

9.3 A first Structure Theorem

Given a multiplicative function f , define $g(p^k) = 1$, $h(p^k) = f(p^k)$ if $p \leq y$ and $g(p^k) = f(p^k)$, $h(p^k) = 1$ if $p > y$. Now $1 * f = g * h$ so that if $h = 1 * H$ then $f = g * H$. Therefore

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{1}{x} \sum_{ab \leq x} H(a) g(b) = \sum_{a \leq x} \frac{H(a)}{a} \frac{1}{x/a} \sum_{b \leq x/a} g(b).$$

By (9.3) this is

$$\sum_{a \leq x} \frac{H(a)}{a} \cdot \frac{1}{x} \sum_{b \leq x} g(b) + O \left(\sum_{a \leq x} \frac{|H(a)|}{a} \frac{\log(2a)}{\log x} \exp \left(\sum_{p \leq x} \frac{|1 - g(p)|}{p} \right) \right).$$

We may extend both sums to be over all integers a since the error term is trivially bigger than the main term when $a > x$. Now

$$\begin{aligned} \sum_{a \geq 1} \frac{|H(a)|}{a} \log a &= \sum_{a \geq 1} \frac{|H(a)|}{a} \sum_{p^k \parallel a} k \log p \\ &\leq 2 \sum_{\substack{p \leq y \\ k \geq 1}} \frac{k \log p}{p^k} \sum_{A \geq 1} \frac{|H(A)|}{A} \ll \log y \cdot \exp \left(\sum_{p \leq y} \frac{|H(p)|}{p} \right), \end{aligned}$$

writing $a = p^k A$ with $(A, p) = 1$ and then extending the sum to all A , since $|H(p^k)| \leq 2$. Now

$$\sum_{p \leq x} \frac{|1 - g(p)| + |H(p)|}{p} = \sum_{p \leq x} \frac{|1 - f(p)|}{p},$$

and so we have proved, applying Proposition [3.6](#), ^{[GenFundLem](#)}

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{1}{x} \sum_{n \leq x} g(n) \cdot \frac{1}{x} \sum_{n \leq x} h(n) + O \left(\frac{\log y}{\log x} \exp \left(\sum_{p \leq x} \frac{|1 - f(p)|}{p} \right) \right). \quad (9.5)$$

1stStructure

This is especially useful for understanding real valued f whose mean-value is large.

9.4 Bounding the tail of a sum

Lemma 9.2 *If f and g are totally multiplicative, with $0 \leq f(p) \leq g(p) \leq p$ for all primes p , then*

$$\prod_{p \leq y} \left(1 - \frac{f(p)}{p} \right) \sum_{\substack{n \leq x \\ P(n) \leq y}} \frac{f(n)}{n} \geq \prod_{p \leq y} \left(1 - \frac{g(p)}{p} \right) \sum_{\substack{n \leq x \\ P(n) \leq y}} \frac{g(n)}{n}$$

Proof We prove this in the case that $f(q) < g(q)$ and $g(p) = f(p)$ otherwise, since then the result follows by induction. Define h so that $g = f * h$, so that $h(q^{b+1}) = (g(q) - f(q))g(q^b)$ for all $b \geq 0$, and $h(p^a) = 0$ otherwise. The left hand side above equals $\prod_{p \leq y} \left(1 - \frac{g(p)}{p} \right)$ times

$$\sum_{m \geq 1} \frac{h(m)}{m} \sum_{\substack{n \leq x \\ P(n) \leq y}} \frac{f(n)}{n} \geq \sum_{\substack{N \leq x \\ P(N) \leq y}} \sum_{mn=N} \frac{h(m)}{m} \cdot \frac{f(n)}{n} = \sum_{\substack{n \leq x \\ P(n) \leq y}} \frac{g(n)}{n},$$

as desired. □

Corollary 9.3 Suppose that f is a totally multiplicative function, with $0 \leq f(p) \leq 1$ for all primes p . Then

$$\prod_{p \leq y} \left(1 - \frac{f(p)}{p}\right) \sum_{\substack{n > x \\ p|n \Rightarrow p \leq y}} \frac{f(n)}{n} \ll \left(\frac{C}{u \log u}\right)^u,$$

where $x = y^u$.

Proof If take $x = \infty$, both sides equal 1 in the Lemma. Hence if we subtract both sides from 1, and let $g = 1$, we obtain

$$\prod_{p \leq y} \left(1 - \frac{f(p)}{p}\right) \sum_{\substack{n > x \\ p|n \Rightarrow p \leq y}} \frac{f(n)}{n} \leq \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{n > x \\ p|n \Rightarrow p \leq y}} \frac{1}{n}.$$

By Mertens' theorem and this is

$$\lesssim \frac{e^{-\gamma}}{\log y} \int_x^\infty \frac{d\Psi(t, y)}{t} \leq \frac{e^{-\gamma}}{\log y} \int_x^\infty \frac{\Psi(t, y)}{t^2} dt,$$

and the result follows from (3.3.3). \square

9.5 Elementary proofs of the prime number theorem

In exercise ^{SelbergIden}1.14, we rewrote Selberg's formula as

$$(\psi(x) - x) \log x = - \sum_{p \leq x} \log p \left(\psi\left(\frac{x}{p}\right) - \frac{x}{p} \right) + O(x).$$

There is an analogous formula for $\mu(n)$, derived from (3.1.1):

$$M(x) \log x = - \sum_{p \leq x} \log p M\left(\frac{x}{p}\right) + O(x).$$

Exercise 9.4 Show that

$$\liminf_{x \rightarrow \infty} \frac{M(x)}{x} + \limsup_{x \rightarrow \infty} \frac{M(x)}{x} = 0.$$

DIRICHLET CHARACTERS

We give a concise introduction to Dirichlet characters. We wish to classify the non-zero homomorphisms $\chi : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{C}$.

Suppose that $q = \prod_{j=1}^k p_j^{e_j}$. We define a homomorphism $\chi_j : \mathbb{Z}/p_j^{e_j}\mathbb{Z} \rightarrow \mathbb{C}$, by taking $\chi_j(a) = \chi(A)$ where $A \equiv a \pmod{p_j^{e_j}}$ and $A \equiv 1 \pmod{q/p_j^{e_j}}$ (as is possible by the Chinese Remainder Theorem). Moreover one can verify that $\chi = \chi_1 \chi_2 \dots \chi_k$, and so the characters mod q can be determined by the characters mod the prime power factors of q .

Now if $\chi_k = 1$ then $\chi = \chi_1 \dots \chi_{k-1}$ is a homomorphism $\mathbb{Z}/(q/p_k^{e_k})\mathbb{Z} \rightarrow \mathbb{C}$. *Dirichlet characters* are those χ that are not (also) a homomorphism $\mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{C}$ for some proper divisor d of q with $(d, q/d) = 1$. Hence we may assume that each $\chi_j \neq 1$.

Now suppose that $q = p^e$. Since $\chi \neq 0$ there exists a such that $\chi(a) \neq 0$. Then $\chi(a) = \chi(a \cdot 1) = \chi(a)\chi(1)$ and so $\chi(1) = 1$. Since $\chi \neq 1$ there exists b such that $\chi(b) \neq 1$. Then $\chi(0) = \chi(b \cdot 0) = \chi(b)\chi(0)$ and so $\chi(0) = 0$. But then $\chi(p)^e = \chi(p^e) = \chi(q) = \chi(0) = 0$ and so $\chi(p) = 0$. Hence $\chi(a) = 0$ if $(a, p) > 1$.

Now let us return to arbitrary q . The last paragraph implies that $\chi(a) = 0$ if $(a, q) > 1$, so we can think of χ as a homomorphism $(\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$. Now suppose that $(\mathbb{Z}/q\mathbb{Z})^*$ is generated by g_1, g_2, \dots, g_ℓ of orders k_1, \dots, k_ℓ , respectively. Any a with $(a, q) = 1$ can be written uniquely as $g_1^{a_1} \dots g_\ell^{a_\ell} \pmod{q}$ where $0 \leq a_i \leq k_i - 1$ for each i , and so $\chi(a) = \chi(g_1)^{a_1} \dots \chi(g_\ell)^{a_\ell}$ and therefore the values of $\chi(g_1), \dots, \chi(g_\ell)$ determine χ . Now $\chi(g_i)^{k_i} = \chi(g_i^{k_i}) = 1$ and so $\chi(g_i)$ is a k_i th root of unity, and in fact we can select any k_i th root of unity. Indeed let ψ_j be that character mod q with $\psi_j(g_j) = e(1/k_j)$, and $\psi_j(g_i) = 1$ for $i \neq j$. Then the set of possible characters mod q is

$$\{\psi_1^{a_1} \dots \psi_\ell^{a_\ell} \text{ where } 0 \leq a_i \leq k_i - 1 \text{ for each } i\}$$

which, we see, can be viewed as a multiplicative group, isomorphic to $(\mathbb{Z}/q\mathbb{Z})^*$.

Exercise 10.1 Prove that if $(a, q) = 1$ but $a \not\equiv 1 \pmod{q}$ then there exists a character $\chi \pmod{q}$ such that $\chi(a) \neq 1$.

We call χ_0 the *principal character* if $\chi_0(a) = 1$ whenever $(a, q) = 1$. If $q = dm$ with $m > 1$ and $\chi = \delta\mu_0$ where δ is a character mod d and μ_0 is the principal character mod m then χ is *induced by* δ . If m is the smallest such integer then m is the *conductor* of χ ; if $m = q$ then χ is *primitive*.

The *orthogonality relations* are of central importance:

$$\frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{otherwise;} \end{cases} \quad (10.1) \quad \boxed{\text{Orthog1}}$$

$$\frac{1}{\phi(q)} \sum_b \chi(b) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases} \quad (10.2) \quad \boxed{\text{Orthog2}}$$

^{|Orthog1}
(10.1) is trivial if $m = 1$. Otherwise select $\psi \pmod{q}$ for which $\psi(m) \neq 1$. As the characters mod q form a group, the set $\{\psi\chi : \chi \pmod{q}\}$ is also the character group, and so $\psi(m) \sum_{\chi} \chi(m) = \sum_{\chi} (\psi\chi)(m) = \sum_{\chi} \chi(m)$, and the result follows.

Exercise 10.2 Prove ^{|Orthog2}
(10.2). (Hint: One proof is analogous to that of ^{|Orthog1}
(10.1).)

For a given character $\chi \pmod{q}$, define the *Gauss sum*

$$g(\chi) := \sum_a \chi(a) e\left(\frac{a}{q}\right).$$

When $(m, q) = 1$ we can change the variable a to bm , as b varies through the residues mod q , coprime to q , so that

$$\bar{\chi}(m)g(\chi) = g(\chi, m), \text{ where } g(\chi, m) := \sum_b \chi(b) e\left(\frac{bm}{q}\right). \quad (10.3) \quad \boxed{\text{GenGSums}}$$

Select b_j to be the inverse of $q/p_j^{e_j} \pmod{p_j^{e_j}}$ so that $1 \equiv \sum_j b_j \cdot q/p_j^{e_j} \pmod{q}$, and therefore

$$g(\chi) = \sum_a (\chi_1 \cdots \chi_k)(a) e\left(\sum_j \frac{ab_j}{p_j^{e_j}}\right) = \prod_j g(\chi_j, b_j) = \prod_j \bar{\chi}_j(b_j) g(\chi_j).$$

This implies that $|g(\chi)| = \prod_j |g(\chi_j)|$, and so we may restrict our attention to prime powers $q = p^e$:

Suppose that χ is a primitive character mod q . We have $g(\chi, 0) = 0$ by ^{|Orthog2}
(10.2). If $e > 1$ then $\chi(1 + q/p) \neq 1$, else χ is a character mod q/p . Now by writing $a \equiv b(1 + q/p) \pmod{q}$, we have

$$\begin{aligned} g(\chi, Mp) &= \sum_a \chi(a) e\left(\frac{aM}{q/p}\right) \\ &= \chi(1 + q/p) \sum_b \chi(b) e\left(\frac{bM}{q/p}\right) = \chi(1 + q/p) g(\chi, Mp), \end{aligned}$$

so that $g(\chi, Mp) = 0$; that is $g(\chi, m) = 0$ whenever $(m, q) \neq 1$. Hence

$$\begin{aligned}
\phi(q)|g(\chi)|^2 &= \sum_{m \pmod{q}} |g(\chi, m)|^2 = \sum_{a, b \pmod{q}} \chi(a)\overline{\chi}(b) \sum_{m \pmod{q}} e\left(\frac{(a-b)m}{q}\right) \\
&= q \sum_{a \pmod{q}} |\chi(a)|^2 = \phi(q)q,
\end{aligned}$$

so that $|g(\chi)| = \sqrt{q}$ for q a prime power and, by the above, this follows for primitive characters modulo composite q as well.

ZETA FUNCTIONS AND DIRICHLET SERIES: A MINIMALIST DISCUSSION

11.1 Dirichlet characters and Dirichlet L -functions

We define the Dirichlet L -function for the character $\chi \pmod{q}$ by

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$$

for $\operatorname{Re}(s) > 1$. One can verify using the fundamental theorem of arithmetic that this has the Euler product expansion

$$L(s, \chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}$$

in the same range.

Exercise 11.1 If $\chi \pmod{q}$ is induced by $\psi \pmod{m}$ then determine $L(s, \chi)/L(s, \psi)$.

Remark 11.2 We will need to add a proof of Dirichlet's class number formula, perhaps a uniform version? (Since this can be used to establish the connection between small class number and small numbers of primes in arithmetic progressions). We also need to discuss the theory of binary quadratic forms, at least enough for the class number formula and to understand prime values of such forms.

Lemma 11.3 For any non-principal Dirichlet character $\chi \pmod{q}$ and any complex number s with real part > 0 , we can define

$$L(s, \chi) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{\chi(n)}{n^s},$$

since this limit exists.

The content of this result is that the right-side of the equation converges. One usually uses the idea of analytic continuation to state that this equals the left-side.

Proof [sketch] We will prove this by suitably bounding

$$\sum_{n=N+1}^{\infty} \frac{\chi(n)}{n^s},$$

for $N \geq q|s|$, where $s = \sigma + it$. If $n = N + j$ we replace the n in the denominator by N , incurring an error of

$$\left| \frac{1}{(N+j)^s} - \frac{1}{N^s} \right| \ll \frac{1}{N^\sigma} \frac{|s|j}{N} \ll \frac{|s|q}{N^{1+\sigma}},$$

for $1 \leq j \leq q$. Summing this over all n in the interval $(N, N+q]$, gives $N^{-s} \sum_n \chi(n) + O(|s|q^2/N^{1+\sigma}) \ll |s|q^2/N^{1+\sigma}$. Summing now over $N, N+q, N+2q, \dots$, we obtain a total error of $\ll |s|q/\sigma N^\sigma$, which implies the result. \square

11.2 Dirichlet series just to the right of the 1-line

Corollary 11.4 Suppose that there exists an integer $k \geq 1$ such that $f(p)^k = 1$ for all primes p . Then $\mathbb{D}(f(n), n^{it}; \infty) = \infty$ for every non-zero real t .

Examples of this include $f = \mu$ the Möbius function, χ a Dirichlet character (though one needs to modify the result to deal with the finitely many primes p for which $\chi(p) = 0$), and even $\mu\chi$.

Proof Suppose that there exists a real number $t \neq 0$ such that $\mathbb{D}(f(n), n^{it}; \infty) < \infty$. Then $\mathbb{D}(1, n^{ikt}; \infty) \leq k\mathbb{D}(f(n), n^{it}; \infty) < \infty$ by the triangle inequality. Let $s = 1 + \frac{1}{\log x} + ikt$. By (4.5), we have

$$\log \zeta(s) = \sum_{p \leq x} \frac{1}{p^{1+ikt}} + O(1),$$

and so

$$\begin{aligned} \log |\zeta(s)| &= \operatorname{Re}(\log \zeta(s)) = \sum_{p \leq x} \frac{\operatorname{Re}(p^{ikt})}{p} + O(1) \\ &= \sum_{p \leq x} \frac{1}{p} - \mathbb{D}(1, n^{ikt}; x) + O(1) = \log \log x + O_t(1), \end{aligned}$$

and therefore $|\zeta(s)| \gg \log x$. However exercise ^{zeta}1.4 yields that

$$\zeta(s) = \frac{1}{s-1} + O(1+|t|) = \frac{1}{it} + O\left(1+|t| + \frac{1}{|t|^2 \log x}\right),$$

a contradiction. \square

Koukouli

Lemma 11.5 If χ is a character mod q and $x \geq y \geq q$ then

$$\sum_{y < p \leq x} \frac{\chi(p)}{p^{1+it}} \ll \log \left(2 + \frac{\log q(1+|t|)}{\log y} \right).$$

Proof (Koukoulopoulos) Taking absolute values we have the upper bound $\log \left(\frac{\log x}{\log y} \right)$. Let m be the product of the primes $\leq y$ that do not divide q . Write $s_X := 1 + \frac{1}{\log X} + it$ for all $X > 0$, and take $s = s_x$ for convenience. Taking absolute values we obtain an acceptable upper bound for the primes in the sum that are $\leq Y := (|s_x|q)^4$. We may therefore now assume that $y \geq Y$. By (4.5) with $f = \chi$ we have that

$$\exp \left(\sum_{y < p \leq x} \frac{\chi(p)}{p^{1+it}} \right) \asymp \sum_{\substack{n \geq 1 \\ (n,m)=1}} \frac{\chi(n)}{n^s}.$$

Take $N \geq y$ with $H = qN^{1/3}$. For $s = s_x = 1 + \frac{1}{\log x} + it$ we have

$$\sum_{\substack{N < n \leq N+qH \\ (n,m)=1}} \frac{\chi(n)}{n^s} = \frac{1}{N^s} \sum_{\substack{N < n \leq N+qH \\ (n,m)=1}} \chi(n) + O \left(\sum_{\substack{N < n \leq N+qH \\ (n,mq)=1}} \left| \frac{1}{n^s} - \frac{1}{N^s} \right| \right).$$

Now $|1/n^s - 1/N^s| \ll (|s|qH/N)/N^{\Re s} \leq |s|qH/N^2$ as $|s|qH \leq N$, which leads to a bound on the second sum; and we bound the first sum by taking $x = N, y = H, z = y$ in Corollary 17.3. Then, partitioning $(N, 2N]$ into intervals of length qH , we obtain

$$\sum_{\substack{N < n \leq 2N \\ (n,m)=1}} \frac{\chi(n)}{n^s} \ll \frac{1}{\log y} \frac{1}{H^{\frac{1}{\log y}}} + \frac{1}{\sqrt{H}} + \frac{|s|qH}{N \log y} \ll \frac{1}{\log y} \frac{1}{N^{\frac{1}{\log y}}} + \frac{1}{N^{\frac{1}{6}}}.$$

Summing over $N = y, 2y, 4y, 8y, \dots$ yields that our sum is bounded, and hence the result. \square

By (4.5) with $f = \chi$ we have that $\log(L(s_x, \chi)/L(s_y, \chi)) = \sum_{y < p \leq x} \frac{\chi(p)}{p^{1+it}} + O(1)$, and so we deduce from the above that if χ is a character mod q and $x \geq y \geq q$ then

$$\left| L \left(1 + \frac{1}{\log x} + it, \chi \right) \right| \ll \left(1 + \frac{\log |t|}{\log y} \right) \left| L \left(1 + \frac{1}{\log y} + it, \chi \right) \right|.$$

There is a proof of this which uses the theory of analytic functions, which is too beautiful to not include:

Proof It is well-known that the completed Dirichlet L -function has a Hadamard factorization; that is if $\delta = (1 - \chi(-1))/2$ then

$$\Lambda(s, \chi) := \left(\frac{\pi}{q} \right)^{-\frac{s+\delta}{2}} \Gamma \left(\frac{s+\delta}{2} \right) L(s, \chi) = e^{A+Bs} \prod_{\rho \in \Lambda(\rho, \chi)=0} \left(1 - \frac{s}{\rho} \right) e^{s/\rho}$$

where $\operatorname{Re}(B + \sum_{\rho} 1/\rho) = 0$ (as in Chapter 12 of Davenport). We deduce that

$$\left| \frac{\Lambda(\alpha + it, \chi)}{\Lambda(\beta + it, \chi)} \right| = \prod_{\rho: \Lambda(\rho, \chi) = 0} \left| \frac{\alpha + it - \rho}{\beta + it - \rho} \right|.$$

Now if $\operatorname{Re} \rho \leq \alpha \leq \beta$ then $|\alpha + it - \rho| \leq |\beta + it - \rho|$ by the (geometric) triangle inequality, and so the above product is ≤ 1 if $1 \leq \alpha \leq \beta$ since we know that $\operatorname{Re}(\rho) \leq 1$. Inserting this inequality into the definition of $\Lambda(s, \chi)$, we deduce the result from the fact that $\Gamma'(s)/\Gamma(s) = \log s + O(1/|s|)$ (as in (6) of Chapter 10 of Davenport), which implies that the ratio of the Gamma factors is $\ll \log |t|/\log y \ll 1$. \square

Exercise 11.6 *The Riemann Hypothesis for $L(s, \chi)$ states that if $\Lambda(\rho, \chi) = 0$ then $\operatorname{Re}(\rho) \leq 1/2$. Prove that this is equivalent to the conjecture that $\Lambda(s, \chi)$ is increasing as one moves in the positive real direction along any horizontal line, from the line $\operatorname{Re}(s) = 1/2$.*

HALÁSZ'S THEOREM: INVERSES AND HYBRIDS

It is evidently useful to evaluate the mean value of $f(n)$ in terms of the mean value of $f(n)/n^{it}$:

AsympT2

Theorem 12.1 Suppose $f(n)$ is a multiplicative function with $|f(n)| \leq 1$ for all n . If $t = t_f(x, \log x)$ then

$$\sum_{n \leq x} f(n) = \frac{x^{it}}{1+it} \sum_{n \leq x} \frac{f(n)}{n^{it}} + O\left(\frac{x \log \log x}{(\log x)^{2-\sqrt{3}}}\right).$$

This also holds if we take $t = t_f(x^A, \log(x^A))$ for some A , $1 \leq A \ll 1$.

This yields a hybrid version of Halász's theorem that takes into account the point $1+it$:

UBdt

Theorem 12.2 Let $t = t(x, \log x)$ and let $L = L(x, \log x)$. Then

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll \frac{L}{1+|t|} \log \frac{2}{L} + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}. \quad (12.1) \quad \text{UBdHyb1}$$

We can obtain a better result when we have no useful information about the size of L :

UBdHyb2

Theorem 12.3 Let f be a multiplicative function with $|f(n)| \leq 1$ for all n . Then

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll \frac{1}{1+|t|} + \frac{\log \log x}{(\log x)^{1-\frac{2}{\pi}}}.$$

Proof of Theorem 12.3 We may suppose that $|t| \geq 10$. Let $y = t_f(x, |t| - 2)$. By Lemma 12.2 and the definition of t , we see that $|F(1+iy)| \ll (\log x)^{\frac{2}{\pi}}$, as $|y| \leq |t| - 2$, and the result follows from (12.1) with $T = |t| - 2$. \square

Exercise 12.1 Prove that if $|t| \ll m$ and $|\delta| \leq 1/2$ then $2m^{it} = (m - \delta)^{it} + (m + \delta)^{it} + O(|t|/m^2)$. Deduce that

$$\sum_{m \leq z} m^{it} = \begin{cases} \frac{z^{1+it}}{1+it} + O(1+t^2) \\ O(z). \end{cases}$$

Generalize this argument to sum other (carefully selected) functions over the integers.

We require the following lemma, which relates the mean value of $f(n)$ to the mean-value of $f(n)n^{it}$.

AsympT1

Lemma 12.4 Suppose $f(n)$ is a multiplicative function with $|f(n)| \leq 1$ for all n . Then for any real number t with $|t| \leq x^{1/3}$ we have

$$\sum_{n \leq x} f(n) = \frac{x^{it}}{1+it} \sum_{n \leq x} \frac{f(n)}{n^{it}} + O\left(\frac{x}{\log x} \log(2+|t|) \exp\left(\mathbb{D}(f(n), n^{it}; x) \sqrt{2 \log \log x}\right)\right).$$

Proof Let g and h denote the multiplicative functions defined by $g(n) = f(n)/n^{it}$, and $h(p^k) = g(p^k) - g(p^{k-1})$, so that $g(n) = \sum_{d|n} h(d)$. Then

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} g(n) n^{it} = \sum_{n \leq x} n^{it} \sum_{d|n} h(d) = \sum_{d \leq x} h(d) d^{it} \sum_{m \leq x/d} m^{it}.$$

We use the first estimate in the exercise when $d \leq x/(1+t^2)$, and the second estimate when $x/(1+t^2) \leq d \leq x$. This gives

$$\sum_{n \leq x} f(n) = \frac{x^{1+it}}{1+it} \sum_{d \leq x} \frac{h(d)}{d} + O\left((1+t^2) \sum_{d \leq x/(1+t^2)} |h(d)| + x \sum_{x/(1+t^2) \leq d \leq x} \frac{|h(d)|}{d}\right).$$

Applying (2.4.5) and (2.4.6) we deduce that

$$\begin{aligned} \sum_{n \leq x} f(n) &= \frac{x^{1+it}}{1+it} \sum_{d \leq x} \frac{h(d)}{d} + O\left(\frac{x}{\log x} \log(2+|t|) \sum_{d \leq x} \frac{|h(d)|}{d}\right) \\ &= \frac{x^{1+it}}{1+it} \sum_{d \leq x} \frac{h(d)}{d} + O\left(\frac{x}{\log x} \log(2+|t|) \exp\left(\sum_{p \leq x} \frac{|1-g(p)|}{p}\right)\right). \end{aligned}$$

We use this estimate twice, once as it is, and then with $f(n)$ replaced by $f(n)/n^{it}$, and t replaced by 0, so that g and h are the same in both cases.

Then, by the Cauchy-Schwarz inequality,

$$\left(\sum_{p \leq x} \frac{|1-g(p)|}{p}\right)^2 \leq 2 \sum_{p \leq x} \frac{1}{p} \cdot \sum_{p \leq x} \frac{1 - \operatorname{Re}(g(p))}{p} \leq 2 \mathbb{D}(g(n), 1; x)^2 (\log \log x + O(1)),$$

and the result follows, since $\mathbb{D}(f(n), n^{it}; x)^2 = \mathbb{D}(g(n), 1; x)^2 \ll \log \log x$. \square

Proof of Theorems 12.1 and 12.2 ^{AsympT2} ^{UBdt} We may assume that $M := M_f(x, \log x) > (2-\sqrt{3}) \log \log x$ else Corollary 12.1 follows immediately from Lemma 12.4. Now, in this case $\sum_{n \leq x} f(n) \ll x \log \log x / (\log x)^{2-\sqrt{3}}$ by Halász's Theorem. Now let $g(n) = f(n)/n^{it}$. If $|t| > \frac{1}{2} \log x$ then $|(x^{it}/(1+it)) \sum_{n \leq x} g(n)| \leq x/(1+|t|) \ll x/\log x$ and Corollary 12.1 ^{AsympT2} follows. But if $|t| \leq \frac{1}{2} \log x$ then $t_g(x, \frac{1}{2} \log x) = 0$, so that $M_g(x, \frac{1}{2} \log x) = M$, and Corollary 12.1 ^{AsympT2} follows from Halász's Theorem applied to g .

Finally Theorem 12.2 ^{UBdt} follows from Corollary 12.1 ^{AsympT2} by the definition of L .

It is left as an exercise for the reader to prove this for $t = t_f(x^A, \log(x^A))$.

\square

12.1 Lower Bounds on mean values

Halász's Theorem states that

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll L(x, T) \log(2/L(x, T)) + T^{-1}.$$

We will see an example which shows that the $L \log(1/L)$ is necessary, but that is for a very special function. Of more interest is whether we really need a function like L in our upper bound for typical f .

Lbdl **Theorem 12.5** *Suppose that $t = t_f(x, T) = 0$ and let $L = L(x, T)$ with $\kappa = 1/\log(1/L)$ and $B = \log(1/\kappa)$. There exists a constant $c > 0$ such that there exists y in the range $x^{L/C} \leq y \leq x^{CB}$ for which*

$$\left| \sum_{n \leq y} f(n) \right| \gg L(x, T)y.$$

If $f(n) \geq 0$ for all n then one can improve this to

$$\sum_{n \leq y} f(n) \gg \kappa L(y, T)y.$$

Proof By $\frac{\text{TruncRight}}{(4.5)}$ we have

$$L \log x \asymp \sum_{n \geq 1} \frac{f(n)}{n^{1+\frac{1}{\log x}}} = \left(1 + \frac{1}{\log x}\right) \int_1^\infty \frac{1}{y^{2+\frac{1}{\log x}}} \sum_{n \leq y} f(n) dy.$$

If $y > x$ then $(1/y) \sum_{n \leq y} f(n) \ll L(y) \log(1/L(y)) \ll L(x)/\kappa$ by Halász's Theorem, and so

$$\int_{x^{CB}}^\infty \frac{1}{y^{2+\frac{1}{\log x}}} \sum_{n \leq y} f(n) dy \ll \frac{L(x)}{\kappa} \int_{x^{CB}}^\infty \frac{dy}{y^{1+\frac{1}{\log x}}} \ll \frac{L(x) \log x}{e^{(C-1)B}}.$$

Also taking $L = L(x)$ we have

$$\int_1^{x^{L/C}} \frac{1}{y^{2+\frac{1}{\log x}}} \sum_{n \leq y} f(n) dy \leq \int_1^{x^{L/C}} \frac{dy}{y} \ll \frac{L \log x}{C}.$$

Now if $(1/y) \sum_{n \leq y} f(n) \ll L(x)/C$ for all $y, x^{L/C} < y < x^{CB}$ we obtain

$$\int_{x^{L/C}}^{x^{CB}} \frac{1}{y^{2+\frac{1}{\log x}}} \sum_{n \leq y} f(n) dy \ll \frac{L}{C} \int_{x^{L/C}}^{x^{CB}} \frac{dy}{y^{1+\frac{1}{\log x}}} \ll \frac{L \log x}{C}.$$

Combining these estimates yields a contradiction if C is sufficiently large and so implies our first result.

Now suppose that $f(n) \geq 0$ for all n . Then $L(x^t) \leq L(x)/t$, and hence if $(1/y) \sum_{n \leq y} f(n) \ll \kappa L(y)/C$ for all y , $x^{L/C} < y < x^{CB}$ then

$$\begin{aligned} \int_{x^{L/C}}^{x^{CB}} \frac{1}{y^{2+\frac{1}{\log x}}} \sum_{n \leq y} f(n) dy &\ll \kappa L/C \int_{x^{L/C}}^x \frac{\log x}{y^{1+\frac{1}{\log x}} \log y} dy + \kappa L/C \int_x^{x^{CB}} \frac{dy}{y^{1+\frac{1}{\log x}}} \\ &\ll \frac{L \log x}{C}, \end{aligned}$$

which implies our second result. \square

Note that this cannot be much improved. The example with $f(p) = 1$ for $p < x^L$ and $f(p) = 0$ thereafter, yields $\left| \sum_{n \leq y} f(n) \right| \asymp y/u^u$ for $y = x^{uL}$, so in our first result we cannot improve the lower bound on the range for y to as much as $y^{L \log(1/L)}$; and in the second result to as much as y^{cL} .

Exercise 12.6 Use Theorem ^{AsympT2}12.1 to obtain an analogous result when $t_f(x, T) \neq 0$.

12.2 Tenenbaum (Selberg)

Developing an idea of Selberg, Tenenbaum showed that if the mean value of $f(p)$ is z , where $z \neq 0, -1$, with very little variance, then

$$\frac{1}{x} \sum_{n \leq x} f(n) \sim \frac{1}{\Gamma(z)} \left(\lim_{s \rightarrow 1^+} (s-1)^z F(s) \right) (\log x)^{z-1}.$$

Our “expected” mean value is the same quantity with $\Gamma(z)$ replaced by $e^{\gamma(1-z)}$. Note that if $z = 0, -1$ then $1/\Gamma(z) = 0$ so we might expect a rather different phenomenon there. Indeed one can show that in both those cases the mean value is $\ll 1/(\log x)^2$. In the case $z = 0$ this “singularity” restricts how much we might believe our heuristic about the mean value of a multiplicative function. In particular when trying to prove a lower bound on the mean value like $\gg L$ we see that it is necessary to include at least a $O(1/\log x)$ term.

This is a very delicate kind of result for real $z \leq 0$. Let $z = -\delta, \delta \geq 0$; the above suggests that $\sum_{n \leq x} f(n) = o(x/(\log x))$. If we now alter the multiplicative function f on the primes $(x/2, x]$ only, then we alter $\sum_{n \leq x} f(n)$ by $\sum_{x/2 < p \leq x} f'(p) - f(p)$ which can be selected to have any size as large as $x/2 \log x$. This implies that to prove the above result we need very precise distribution of the $f(p)$; not something of great general interest.

DISTRIBUTION OF VALUES OF A MULTIPLICATIVE FUNCTION

Suppose that f is a multiplicative function, with $|f(n)| = 1$ for all $n \geq 1$. Define

$$R_f(N, \alpha, \beta) := \frac{1}{N} \# \left\{ n \leq N : \frac{1}{2\pi} \arg(f(n)) \in (\alpha, \beta] \right\}.$$

We say that the $f(n)$ are *uniformly distributed* on the unit circle if $R_f(N, \alpha, \beta) \rightarrow \beta - \alpha$ for all $0 \leq \alpha < \beta < 1$. Jordan Ellenberg asked whether the values $f(n)$ are necessarily equidistributed on the unit circle according to some measure, and if not whether their distribution is entirely predictable. We prove the following response.

Theorem 13.1 *Let f be a completely multiplicative function such that each $f(p)$ is on the unit circle. Either the $f(n)$ are uniformly distributed on the unit circle, or there exists a positive integer k for which $(1/N) \sum_{n \leq N} f(n)^k \not\rightarrow 0$. If k is the smallest such integer then*

$$R_f(N, \alpha, \beta) = \frac{1}{k} R_{f^k}(N, k\alpha, k\beta) + o_{N \rightarrow \infty}(1) \text{ for } 0 \leq \alpha < \beta < 1$$

Exercise 13.2 *Deduce in the final case that $R(N, \alpha + \frac{1}{k}, \beta + \frac{1}{k}) = R(N, \alpha, \beta) + o_{N \rightarrow \infty}(1)$ for all $0 \leq \alpha < \beta < 1$.*

The last parts of the result tell us that if f is not uniformly distributed on the unit circle, then its distribution function is k copies of the distribution function for f^k , a multiplicative function whose mean value does not $\rightarrow 0$. It is easy to construct examples of such functions $f^k = g$ whose distribution function is not uniform: For example, let $g(p) = 1$ for all odd primes p and $g(2) = e(\sqrt{2})$, where g is completely multiplicative.

To prove our distribution theorem we use

Weyl's theorem *Let $\{\xi_n : n \geq 1\}$ be any sequence of points on the unit circle. The set $\{\xi_n : n \geq 1\}$ is uniformly distributed on the unit circle if and only if $(1/N) \sum_{n \leq N} \xi_n^m$ exists and equals 0, for each non-zero integer m .*

We warm up for the proof of the distribution theorem by proving the following result:

WeylCor

Corollary 13.3 *Let f be a completely multiplicative function such that each $f(p)$ is on the unit circle. The following statements are equivalent:*

- (i) *The $f(n)$ are uniformly distributed on the unit circle.*
- (ii) *Fix any $t \in \mathbb{R}$. The $f(n)n^{it}$ are uniformly distributed on the unit circle.*
- (iii) *For each fixed non-zero integer k , we have $\sum_{n \leq N} f(n)^k = o(N)$.*

Proof That (i) is equivalent to (iii) is given by Weyl's equidistribution theorem. By Halász's Theorem we find that (iii) does not hold for some given $k \neq 0$ if and only if $f(n)^k$ is n^{iu} -pretentious for some fixed u . But this holds if and only if $(f(n)n^{it})^k$ is $n^{i(u+kt)}$ -pretentious for some fixed u . But then, by Theorem 12.5, we see that (iii) does not hold with $f(n)$ replaced by $f(n)n^{it}$, and hence the $f(n)n^{it}$ are not uniformly distributed on the unit circle. \square

Proof of the distribution theorem The first part of the result follows from Corollary 13.3. If k is the smallest positive integer for which $\sum_{n \leq N} f(n)^k \gg N$ then, by Halász's Theorem we know that there exists $u_k \ll 1$ such that $\mathbb{D}(f(n)^k, n^{iku_k}, \infty) < \infty$, and that $\mathbb{D}(f^j, n^{iu}, \infty) = \infty$ for $1 \leq j \leq k-1$, whenever $|u| \ll 1$. (And note that $\mathbb{D}(f^{-j}, n^{-iu}, \infty) = \mathbb{D}(f^j, n^{iu}, \infty)$.) Write $f(p) = r(p)p^{iu_k}g(p)$, where $r(p)$ is chosen to be the nearest k th root of unity to $f(p)p^{-iu_k}$, so that $|\arg(g(p))| \leq \pi/k$, and hence $1 - \operatorname{Re}(g(p)) \leq 1 - \operatorname{Re}(g(p)^k)$. Therefore $\mathbb{D}(1, g, \infty) \leq \mathbb{D}(g^k, 1, \infty) = \mathbb{D}(f(n)^k, n^{iku_k}, \infty) < \infty$.

By the triangle inequality, $\mathbb{D}(f^{mk}, n^{ikmu_k}, \infty) \leq m\mathbb{D}(f^k, n^{iku_k}, \infty) < \infty$, and $\mathbb{D}(f^{mk+j}, n^{iu}, \infty) \geq \mathbb{D}(f^j, n^{iv}, \infty) - \mathbb{D}(f^{mk}, n^{ikmu_k}, \infty) = \infty$, where $v = u - kmu_k$ for $1 \leq j \leq k-1$ and any $|u| \ll 1$, and so $\sum_{n \leq N} f(n)^\ell = o_\ell(N)$ if $k \nmid \ell$.

The characteristic function of the interval (α, β) is

$$\sum_{m \in \mathbb{Z}} \frac{e(m\alpha) - e(m\beta)}{2i\pi m} e(mt).$$

We can take this sum in the range $1 \leq |m| \leq M$ with an error $\leq \epsilon$. Hence

$$\begin{aligned} R(N, \alpha, \beta) &= \sum_{1 \leq |m| \leq M} \frac{e(m\alpha) - e(m\beta)}{2i\pi m} \frac{1}{N} \sum_{n \leq N} f(n)^m + O(\epsilon) \\ &= \sum_{1 \leq |r| \leq R} \frac{e(kr\alpha) - e(kr\beta)}{2i\pi kr} \frac{1}{N} \sum_{n \leq N} f(n)^{kr} + O(\epsilon) \end{aligned}$$

writing $m = kr$ (since the other mean values are 0) and $R = [M/k]$. This formula does not change value when we change $\{\alpha, \beta\}$ to $\{\alpha + \frac{1}{k}, \beta + \frac{1}{k}\}$, nor when we change $\{f, \alpha, \beta\}$ to $\frac{1}{k}$ times the formula for $\{f^k, k\alpha, k\beta\}$ and hence the results. \square

It is an interesting problem to prove a uniform version of this result when N is large.

LIPSCHITZ BOUNDS

We wish to determine how mean values of multiplicative functions vary in short intervals. Theorem [12.1](#) shows that this is not straightforward for if the mean values of $f(n)$ at x and x/z are roughly the same and large, and similarly the mean values of $f(n)/n^{it}$ at x and x/w ; then Theorem [12.1](#) implies that $w^{it} \approx 1$ which is not necessarily true. However if we take the t into account then we can prove such a result:

LipschBounds

Corollary 14.1 *For $1 \leq w \leq x^{1-\epsilon}$, we have*

$$\left| \frac{1}{x^{1+it}} \sum_{n \leq x} f(n) - \frac{1}{(x/w)^{1+it}} \sum_{n \leq x/w} f(n) \right| \ll \frac{\left(\frac{\log 2w}{\log x} \right)^\lambda}{1+|t|} \log \left(\frac{\log x}{\log 2w} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}},$$

where $t = t_f(x, \log x)$ if $|t_f(x, \log x)| < \frac{1}{2} \log x$, otherwise $t = 0$, and $\lambda := 1 - \frac{2}{\pi} = 0.36338\dots$

Note that $2 - \sqrt{3} = 0.267949\dots$. When $f(n)$ is non-negative we can improve the $\lambda = 1 - 2/\pi$ in Corollary [14.1](#) to $1 - 1/\pi$, see [\[49\]](#).

As a consequence we can give the same upper bound on the absolute value of the difference of the mean value of f up to x , and the mean value of f up to x/w . However we can do better if f is real-valued:

RealLipsch

Exercise 14.2 *Deduce that if $f(n) \in \mathbb{R}$ for all n then*

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x/w} \sum_{n \leq x/w} f(n) \right| \ll \left(\frac{\log 2w}{\log x} \right)^\lambda \log \left(\frac{\log x}{\log 2w} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}.$$

We deduce Corollary [14.1](#) from the following:

Lipschitz

Theorem 14.3 *For any $x \geq 3$ and all $1 \leq w \leq x/10$, we have, with the same notation as Corollary [14.1](#)*

$$\left| \frac{1}{x} \sum_{n \leq x} \frac{f(n)}{n^{it}} - \frac{w}{x} \sum_{n \leq x/w} \frac{f(n)}{n^{it}} \right| \ll \left(\frac{\log 2w}{\log x} \right)^\lambda \log \left(\frac{\log x}{\log 2w} \right).$$

We would like to increase the exponent λ as much as possible. It must be ≤ 1 since $|\rho(1+\delta) - \rho(1)| = \log(1+\delta) \sim \delta$ for $0 \leq \delta \leq 1$.

Our proof is a modification of the proof of Halasz's Theorem, so that the key is the appropriate modification of Proposition ^{|keyProp}8.5. We again define $S(N) := \sum_{n \leq N} f(n)$.

If we use exercise in section 8.4 to establish that

$$\frac{1}{n^\alpha}(1 - w^{-\alpha-iy}) = \frac{1}{\pi} \int_{-T}^T \frac{\alpha}{\alpha^2 + \xi^2} n^{-i\xi} (1 - w^{-iy-i\xi}) d\xi + O\left(\frac{\alpha}{T}\right),$$

then we obtain a slight variant of Lemma ^{|1emHal12}8.6:

OffLineOn2

Lemma 14.4 *With the same hypothesis as Lemma ^{|OffLineOn}??, for all real numbers $T, w \geq 1$, and all $0 \leq \alpha \leq 1$ we have*

$$\max_{|t| \leq T} |A(1 + \alpha + it)(1 - w^{-\alpha-iy})| \leq \max_{|u| \leq 2T} |A(1 + iu)(1 - w^{-iy})| + O\left(\frac{\alpha}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n}\right).$$

HalModi

Proposition 14.5 *Let f , T , and x be as in Proposition ^{|keyProp}8.5. Then for $1 \leq w \leq x$, we have*

$$\begin{aligned} \left| \frac{S(x)}{x} - \frac{S(x/w)}{x/w} \right| &\ll \frac{1}{\log x} \int_{1/\log x}^1 \frac{1}{\alpha} \left(\max_{|y| \leq T} |(1 - w^{-\alpha-iy})F(1 + \alpha + iy)| \right) d\alpha \\ &\quad + \frac{1}{T} + \frac{\log 2w}{\log x} \log \left(\frac{\log x}{\log 2w} \right). \end{aligned}$$

Proof Since the proof is very similar to that of Proposition ^{|keyProp}8.5, we shall merely sketch it. Arguing as in the proof of Proposition ^{|keyProp}8.5, we get that

$$\begin{aligned} \int_{2w}^x \left| \frac{S(y)}{y} - \frac{S(y/w)}{y/w} \right| \frac{dy}{y} &\ll \int_{1/\log x}^1 \left(\int_{2w}^x \left| \frac{1}{y} \sum_{n \leq y} f(n) \log n - \frac{1}{y/w} \sum_{n \leq y/w} f(n) \log n \right| \frac{dy}{y^{1+2\alpha}} \right) d\alpha \\ &\quad + \log 2w \log \left(\frac{\log x}{\log 2w} \right). \end{aligned}$$

Using Cauchy's inequality, we obtain for $\alpha \geq 1/\log x$,

$$\begin{aligned} &\left(\int_{2w}^x \left| \frac{1}{y} \sum_{n \leq y} f(n) \log n - \frac{1}{y/w} \sum_{n \leq y/w} f(n) \log n \right| \frac{dy}{y^{1+2\alpha}} \right)^2 \\ &\ll \frac{1}{\alpha} \int_{2w}^x \left| \frac{1}{y} \sum_{n \leq y} f(n) \log n - \frac{1}{y/w} \sum_{n \leq y/w} f(n) \log n \right|^2 \frac{dy}{y^{1+2\alpha}}. \end{aligned}$$

As in the proof of Proposition ^{|keyProp}8.5, extending the range of integration for y to \int_1^∞ , substitute $y = e^t$, and use Plancherel's formula. The only difference is that $F'(1 + \alpha + iy)/(1 + \alpha + iy)$ in the right side there must be replaced by the Fourier transform of $e^{-(1+\alpha)t} \sum_{n \leq e^t} \tilde{f}(n) \log n - we^{-(1+\alpha)t} \sum_{n \leq e^t/w} \tilde{f}(n) \log n$ which is $-F'(1 + \alpha + iy)(1 - w^{-\alpha-iy})/(1 + \alpha + iy)$. We make this adjustment, and follow the remainder of the proof of Proposition ^{|keyProp}8.5. \square

Proof of Theorem 14.3 ^{Lipschitz} We may assume that $|t| \leq (\log x)/2$, else the result follows from Theorem 12.3. ^{Uchib2} Let $g(n) = f(n)n^{-it}$, so that $G(s) = F(s + it)$; and therefore

$$|G(1)| = |F(1 + it)| = \max_{|y| \leq (\log x)/2} |G(1 + iy)|.$$

By Proposition 14.5, ^{HalModi} with f there replaced by g , F by G , and $T = (\log x)/2$, we obtain the upper bound

$$\ll \frac{\log 2w}{\log x} \log \left(\frac{\log x}{\log 2w} \right) + \frac{1}{\log x} \int_{1/\log x}^1 \max_{|y| \leq (\log x)/2} |G(1 + \alpha + iy)(1 - w^{-\alpha - iy})| \frac{d\alpha}{\alpha}.$$

Let a_n be the multiplicative function with $a_{p^k} = g(p^k)$ if $n \leq x$ and $a_{p^k} = 0$ so that $\sum_n |a_n|/n \leq \prod_{p \leq x} (1 - 1/p)^{-1} \ll \log x$. By Lemma 14.4 ^{Offline02} with $A(s) = G(s)$, and $T = (\log x)/2$, we have

$$\max_{|y| \leq (\log x)/2} |G(1 + \alpha + iy)(1 - w^{-\alpha - iy})| \leq \max_{|y| \leq \log x} |G(1 + iy)(1 - w^{-iy})| + O(1).$$

Now $|G(1 + iy)| \leq \sum_n |a_n|/n \ll \log x$; and $|G(1 + iy)| \ll (\log x)^{\frac{2}{\pi}} (1 + 1/|y|)^{1 - \frac{2}{\pi}}$ by Lemma 14.7. ^{Repulsion} Moreover, since $|1 - w^{-iy}| \ll \min(1, |y| \log 2w)$, we deduce that

$$\max_{|y| \leq (\log x)/2} |G(1 + \alpha + iy)(1 - w^{-\alpha - iy})| \ll (\log x)^{\frac{2}{\pi}} (\log 2w)^{1 - \frac{2}{\pi}}.$$

In addition, we have the trivial estimate

$$\max_{|y| \leq (\log x)/2} |G(1 + \alpha + iy)(1 - w^{-\alpha - iy})| \ll \zeta(1 + \alpha) \ll \frac{1}{\alpha}.$$

Using the first bound when $\alpha < 1/(\log x)^{\frac{2}{\pi}} (\log 2w)^{1 - \frac{2}{\pi}}$, and the second bound otherwise, in our integral, we obtain our result \square

Proof of Corollary 14.1 ^{LipschBounds} The result follows from Corollary 12.1 ^{AsympT2} followed by Theorem 14.3. ^{Lipschitz} \square

14.1 Consequences

If m is a squarefree integer $\leq x^{1-\epsilon}$ we have, for f totally multiplicative,

$$\begin{aligned} \sum_{\substack{n \leq x \\ (n, m) = 1}} f(n) &= \sum_{n \leq x} f(n) \sum_{d|(m, n)} \mu(d) = \sum_{d|m} \mu(d) f(d) \sum_{r \leq x/d} f(r) \\ &= \sum_{d|m} \frac{\mu(d) f(d)}{d^{1+it}} \sum_{n \leq x} f(n) + O \left(\sum_{d|m} \frac{x}{d} \left(\left(\frac{\log 2m}{\log x} \right)^\lambda \log \left(\frac{\log x}{\log 2m} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}} \right) \right) \\ &= \prod_{p|m} \left(1 - \frac{f(p)}{p^{1+it}} \right) \sum_{n \leq x} f(n) + O \left(\frac{m}{\phi(m)} \cdot x \left(\left(\frac{\log 2m}{\log x} \right)^\lambda \log \left(\frac{\log x}{\log 2m} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}} \right) \right) \end{aligned} \quad (14.1) \quad \boxed{\text{FSieved1}}$$

by Corollary [14.1](#), as $1 + |t| \geq 1$. Combining this further with Corollary [14.1](#) we obtain, for $mw \leq x^{1-\epsilon}$,

$$\sum_{\substack{n \leq x/w \\ (n,m)=1}} f(n) = \frac{1}{w^{1+it}} \prod_{p|m} \left(1 - \frac{f(p)}{p^{1+it}}\right) \sum_{n \leq x} f(n) + O\left(\frac{m}{\phi(m)} \cdot \frac{x}{w} \left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}} + \left(\frac{\log mw}{\log x}\right)^{1-2/\pi} \log\left(\frac{\log x}{\log mw}\right)\right)\right). \quad (14.2) \quad \boxed{\text{FSieved2}}$$

Exercise 14.6 *Verify that*

$$\sum_{y \leq n \leq x} \frac{f(n)}{n} = \frac{S(x)}{x} - \frac{S(y)}{y} + \int_y^x \frac{S(z)}{z^2} dz.$$

Prove that if $\kappa_t(w) = (1 - i/t)(1 - 1/w^{it})/\log w$ if $t \neq 0$, and $\kappa_0(w) = 1$, then

$$\frac{1}{\log w} \sum_{x/w \leq n \leq x} \frac{f(n)}{n} = \kappa_t(w) \cdot \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{\left(\frac{\log 2w}{\log x}\right)^\lambda}{1 + |t|} \log\left(\frac{\log x}{\log 2w}\right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right).$$

Show that we may assume $t = 0$ if f is real-valued.

Up until this point in this book we have developed the theory for all multiplicative functions (which is necessary since we need to work with $\mu(n)$). It is typically easier to develop the theory just for totally multiplicative functions. The point of the next two exercises is to show that this can be done with little loss of generality.

UseTotally

Exercise 14.7 *Given f define g to be that totally multiplicative function with $g(p) = f(p)$ for all primes p . Prove that*

$$\sum_{n \leq x} f(n) = C_t(f) \sum_{n \leq x} g(n) + O\left(x \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right)$$

where $t = t_f(x, \log x) = t_g(x, \log x)$, and the correction factor

$$C_t(f) := \prod_p \left(1 - \frac{f(p)}{p^{1+it}}\right) \left(1 + \frac{f(p)}{p^{1+it}} + \frac{f(p^2)}{p^{2+2it}} + \dots\right).$$

*(Hint: Write $f = g * h$ and bound the size of $h(p^k)$.) Show that we may take $t = 0$ if f is real-valued. Show that $C_t(f) = 0$ if and only if $f(2^k) = -2^{ikt}$ for all $k \geq 1$.*

UseTotally2

Exercise 14.8 Use the last two exercises to show that

$$\sum_{n \leq x} \frac{f(n)}{n} = \mathcal{C}_0(f) \sum_{n \leq x} \frac{g(n)}{n} - \kappa_t(f) \cdot \frac{1}{x} \sum_{n \leq x} g(n) + O\left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right)$$

where $\kappa_t(f) = (1 - i/t)(\mathcal{C}_0(f) - \mathcal{C}_t(f))$ if $t \neq 0$, and

$$\kappa_0(f) = \mathcal{C}_0(f) \left(\sum_{p \text{ prime}} \log p \left(\frac{\sum_{k \geq 0} k f(p^k)/p^k}{\sum_{k \geq 0} f(p^k)/p^k} - \frac{f(p)/p}{1 - f(p)/p} \right) \right).$$

In the special case that $t = 0$ and $f(2^k) = -1$ for all $k \geq 1$ we have

$$\sum_{n \leq x} \frac{f(n)}{n} = \mathcal{C}'_0(f) \log 8 \cdot \frac{1}{x} \sum_{n \leq x} g(n) + O\left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right),$$

where $\mathcal{C}'_0(f) = \prod_{p \geq 3} (1 - f(p)/p)(1 + f(p)/p + f(p^2)/p^2 + \dots)$. Show that we may take $t = 0$ if f is real-valued.

14.2 Truncated Dirichlet series

One can verify the identity (obtained through partial summation) that for every $\sigma > 0$ one has

$$\sum_{n \leq x} \frac{f(n)}{n^\sigma} = \frac{S(x)}{x^\sigma} + \sigma \int_1^x \frac{S(z)}{z^{1+\sigma}} dz.$$

OtherMeans

Exercise 14.9 Use Corollary ^{LipschBounds}14.1 to prove that if $(1 - \sigma) \log x \rightarrow \infty$ then

$$\sum_{n \leq x} \frac{f(n)}{n^\sigma} \Big/ \sum_{n \leq x} \frac{1}{n^\sigma} = \frac{(1 - \sigma)(1 + it)}{1 - \sigma + it} \frac{S(x)}{x} \left(1 + O\left(\frac{1}{x^{1-\sigma}}\right) \right) + O\left(\frac{\log \log x}{(1 + |t|)(\log x)^\lambda} + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right).$$

In particular if $t = 0$ then this equals $S(x)/x + o(1)$.

Exercise 14.10 Show that if $\sigma > 1$ and $(\sigma - 1) \log x \rightarrow \infty$ then

$$\sum_{n \leq x} \frac{f(n)}{n^\sigma} \sim \prod_{p \text{ prime}} \left(1 + \frac{f(p)}{p^\sigma} + \frac{f(p^2)}{p^{2\sigma}} + \dots \right).$$

In analogy to Proposition 2.9, establish that this can be re-written as

$$\sum_{n \leq x} \frac{f(n)}{n^\sigma} \Big/ \sum_{n \leq x} \frac{1}{n^\sigma} \sim \prod_{p \leq x} \left(1 - \frac{1}{p^\sigma} \right) \left(1 + \frac{f(p)}{p^\sigma} + \frac{f(p^2)}{p^{2\sigma}} + \dots \right).$$

In the last two exercise we have seen that the value of the truncated Dirichlet series can be easily understood for all $\sigma \geq 0$ in terms of Euler products and $S(x)$,

except in a small range around $\sigma = 1$. We write $s(t) := S(x^t)/x^t$. Substituting this into the above identity, we obtain for $\sigma = 1 + A/\log x$,

$$\sum_{n \leq x} \frac{f(n)}{n^\sigma} = e^{-A} s(1) + (\log x + A) \int_0^1 e^{-At} s(t) dt.$$

If A is bounded then this implies that

$$\sum_{n \leq x} \frac{f(n)}{n^\sigma} \bigg/ \sum_{n \leq x} \frac{1}{n^\sigma} = \int_0^1 e^{-At} s(t) dt \bigg/ \int_0^1 e^{-At} dt + O\left(\frac{1}{\log x}\right).$$

This seems to be rather more difficult to understand depending, as it does, on the vagaries of the mean value of f .

One can view all of these results as comparison of different weighted mean values.

THE STRUCTURE THEOREM

C20

We have seen two types of mean values of multiplicative functions

- When $f(p) = 0$ if $p|m$ and $f(p) = 1$ otherwise then $\sum_{n \leq x} f(n) \sim x\mathcal{P}(f; x)$.
- When $f(p) = 1$ if $p \leq y$ say, then the mean value of f is obtained from an integral delay equation (as in section 3.1).

One might ask what other possibilities there are. The *Structure Theorem* tells us that all large mean values are the product of the two types, the first for the small prime factors, the latter for the large prime factors:

Given a multiplicative function f , let $t = t_f(x, \log x)$ and define

$$g(p^k) = \begin{cases} 1 & \text{if } p \leq y \\ f(p^k)/(p^k)^{it} & \text{if } y < p \leq x \end{cases} \quad \text{and} \quad h(p^k) = \begin{cases} f(p^k)/(p^k)^{it} & \text{if } p \leq y \\ 1 & \text{if } y < p \leq x, \end{cases}$$

If $t = 0$ then $h * g = 1 * f$.

StructThm

Theorem 15.1 *We have*

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{x^{it}}{1+it} \frac{1}{x} \sum_{n \leq x} g(n) \cdot \frac{1}{x} \sum_{n \leq x} h(n) + O\left(\left(\frac{\log y}{\log x}\right)^\kappa\right)$$

where $\kappa = \lambda/(1+\lambda) < 0.2665288966 \dots$

Proof of Theorem 15.1 ^{StructThm} We begin our proof in the case that $t_f(x, \log x) = 0$. We let $I(x)$ equal

$$\sum_{n \leq x} (g * h)(n) \log(x/n) = \sum_{ab \leq x} g(a)h(b) \int_b^{x/a} \frac{dT}{T} = \int_1^x \sum_{a \leq x/T} g(a) \cdot \sum_{b \leq T} h(b) \frac{dT}{T}.$$

We split this integral into several intervals. First for $T \leq y$ we simply use the trivial bounds to get $\ll x \log y$. For the remaining values of T we simply take $f = h$ in Proposition 3.6 ^{GenFundLem} to obtain a main term, as $\mathcal{P}(h; T) = \mathcal{P}(h; x)$, of

$$\int_y^x \sum_{a \leq x/T} g(a) \mathcal{P}(h; x) dT = \mathcal{P}(h; x) x \int_1^{x/y} \sum_{a \leq A} g(a) \frac{dA}{A^2}$$

plus an error term, again using the trivial bound for $\sum_a g(a)$, and writing $T = y^t$, $x = y^u$, of

$$\ll \log y \int_1^u x t^{-t} dt \ll x \log y,$$

by Proposition [3.6](#)^{|GenFundLem}. Hence if $z = y^v$ where $1 \leq v = u^\kappa \leq u$ then

$$\begin{aligned} \frac{I(x) - zI(x/z)}{x} &= \mathcal{P}(h; x) \int_{x/yz}^{x/y} \frac{1}{A} \sum_{a \leq A} g(a) \frac{dA}{A} + O(\log y) \\ &= \log z \left(\mathcal{P}(h; x) \frac{1}{x} \sum_{n \leq x} g(n) + O((1/v + (v/u)^\lambda)) \right) \\ &= \log z \left(\frac{1}{x} \sum_{n \leq x} g(n) \cdot \frac{1}{x} \sum_{n \leq x} h(n) + O(1/u^\kappa) \right) \end{aligned}$$

by Theorem [14.3](#)^{|Lipschitz}, and then re-applying Proposition [3.6](#)^{|GenFundLem}.

Now since $g * h = f * 1$ we can apply the same observations to the pair f and 1 (though we could easily obtain sharper estimates in this case); comparing the two evaluations of $I(x) - zI(x/z)$ yields the result in the case that $t_f(x, \log x) = 0$.

We now deduce the result when $t_f(x, \log x) \neq 0$ by comparing $f(n)$ to $F(n) := f(n)/n^{it}$ using Corollary [12.1](#)^{|AsympT2}; hence $t_F(x, \frac{1}{2} \log x) = 0$ and we can apply the above. The result follows. □

15.1 Best possible

Let $f(p) = -1$ if $y^{1/2} < p \leq y$ or $x/y^{1/2} < p \leq x$, and $f(p) = 1$ otherwise. Then

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} h(n) &= \frac{1}{2} + O((c/u)^u), \\ \frac{1}{x} \sum_{n \leq x} g(n) &= 1 - 2 \sum_{x/y^{1/2} < p \leq x} \frac{1}{p} = 1 + 2 \log(1 - 1/2u) = 1 - \frac{1}{u} + O\left(\frac{1}{u^2}\right), \\ \frac{1}{x} \sum_{n \leq x} f(n) &= 1/2 + O((c/u)^u) - 2 \sum_{x/y^{1/2} < p \leq x} \frac{1}{p} = \frac{1}{2} - \frac{1}{u} + O\left(\frac{1}{u^2}\right). \end{aligned}$$

Hence

$$\frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x} \sum_{n \leq x} g(n) \cdot \frac{1}{x} \sum_{n \leq x} h(n) = -\frac{1}{2u} + O\left(\frac{1}{u^2}\right)$$

so we see that we must have $\kappa \leq 1$ in Theorem [15.1](#)^{|StructThm}.

One might hope for something like

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{1}{x} \sum_{n \leq x} g(n) \cdot \frac{1}{x} \sum_{n \leq x} h(n) \left(1 + O\left(\frac{1}{u^c}\right)\right)$$

but it is not true in general. Try $f(p) = -1$ for $y^\alpha < p \leq y$ or $x/y^\alpha < p \leq x$, and $f(p) = 1$ otherwise. Then the means for h, g and f are $\alpha^2, 1 - 2\alpha/u$ and $\alpha^2 - 2\alpha/u$,

respectively. Taking $\alpha = 1/u$ gives mean values $1/u^2$, 1 and $-1/u^2$ roughly; ie the above hoped-for estimate is ridiculous. This example does not work if we take 0 instead of -1 since then the mean values are α , $1 - \alpha/u$, $\alpha - \alpha/u$ respectively, so the last displayed equation with $\kappa = 1$ is feasible. This would be a good research project (ie prove the last display for $f(n) \in [0, 1]$)

THE LARGE SIEVE

We are interested in how a given sequence of complex numbers, a_1, a_2, \dots , is distributed in arithmetic progressions mod q . By ^(Orthog1)(10.1), when $(b, q) = 1$, we have

$$\sum_{n \equiv b \pmod{q}} a_n = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(b) \sum_n a_n \chi(n),$$

Therefore, by using ^(Orthog2)(10.2), we deduce that

$$\sum_{(b,q)=1} \left| \sum_{n \equiv b \pmod{q}} a_n \right|^2 = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \left| \sum_n a_n \chi(n) \right|^2. \quad (16.1) \quad \boxed{\text{SumSqs}}$$

Now

$$\begin{aligned} \sum_{(b,q)=1} \left| \sum_{\substack{n \leq N \\ n \equiv b \pmod{q}}} a_n \right|^2 &\leq \sum_{(b,q)=1} \left(\frac{N}{q} + 1 \right) \sum_{\substack{n \leq N \\ n \equiv b \pmod{q}}} |a_n|^2 \\ &= \left(\frac{N}{q} + 1 \right) \sum_n |a_n|^2, \end{aligned}$$

so by ^(SumSqs)(16.1) we deduce that

$$\frac{q}{\phi(q)} \sum_{\chi \pmod{q}} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \leq (q + N) \sum_{n \leq N} |a_n|^2. \quad (16.2) \quad \boxed{\text{1stBound}}$$

Note that if $a_n = \bar{\chi}(n)$ for all n , then the term on the left-side of ^(1stBound)(16.2) corresponding to the character χ has size $\frac{\phi(q)}{q} N^2$, whereas the right-side of ^(1stBound)(16.2) is about $(q + N) \frac{\phi(q)}{q} N$. Hence if $q = o(N)$ and then ^(1stBound)(16.2) is best possible and any of the terms on the left-side could be as large as the right side. It thus makes sense to remove the largest term on the left side (or largest few terms) to determine whether we can get a significantly better upper bound for the remaining terms. This also has arithmetic meaning since the same argument used to prove ^(SumSqs)(16.1) yields, for any choice of χ_1, \dots, χ_k ,

$$\sum_{(b,q)=1} \left| \sum_{n \equiv b \pmod{q}} a_n - \frac{1}{\phi(q)} \sum_{i=1}^k \bar{\chi}_i(b) \sum_n a_n \chi_i(n) \right|^2 = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_1, \dots, \chi_k} \left| \sum_n a_n \chi(n) \right|^2. \quad (16.3) \quad \boxed{\text{SumSqk}}$$

Typically number theorists are interested in sequences where $a_n = 0$ or 1 (which indicates a subset A of the integers up to N), and which are “dense”, that is A contains more than $N/(\log N)^k$ elements, or even a positive proportion of the integers up to N . Given q it is easy enough to find a dense sequence A that is not well distributed mod q (for example let A be the union of about $q/2$ arithmetic progressions mod q), or even one that is not well distributed modulo each q in some finite set. Nonetheless we might expect that A is well-distributed for “almost all” q (say up to \sqrt{N}) though one needs to be cautious, for if A is not well-distributed mod m then it will not be well-distributed mod n whenever m divides n . To see this, suppose that there are $(1 + \delta)|A|/m$ elements of A that are $\equiv b \pmod{m}$. By the pigeonhole principle there exists some residue class $B \pmod{n}$, with $B \equiv b \pmod{m}$, which contains at least $(1 + \delta)|A|/n$ elements of A . Thus we see it makes more sense to compare the number of elements of A that are $\equiv B \pmod{n}$ with the number that are $\equiv B \pmod{m}$ for each proper divisor m of n .

PrimCharsOnly

Exercise 16.1 Show that the “correct” measure of how well the a_n are distributed mod q (with respect to the divisors of q) is

$$\sum_{d|q} \frac{\mu(q/d)\phi(d)}{\phi(q)} \sum_{\substack{n \leq x \\ n \equiv b \pmod{d} \\ (n,q)=1}} a_n = \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \chi(b) \sum_{n \leq x} a_n \bar{\chi}(n).$$

Summing the left-side of (16.2) over $q \leq Q$ is important in applications, which yields a right-side with coefficient $Q^2/2 + QN$. However with the added restriction to primitive characters (which we saw is appropriate in exercise 16.1), we can use some simple linear algebra to improve this to obtain

The large sieve

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \leq (N + Q^2 - 1) \sum_{n=M+1}^{M+N} |a_n|^2. \quad (16.4)$$

(We will prove this initially with $Q^2 - 1$ replaced by $3Q^2 \log Q$.)

Theorem 16.1 (Duality) Let $x_{m,n} \in \mathbb{C}$ for $1 \leq m \leq M$, $1 \leq n \leq N$. For any constant c we have

$$\sum_n \left| \sum_m a_m x_{m,n} \right|^2 \leq c \sum_n |a_n|^2$$

for all $a_m \in \mathbb{C}$, $1 \leq m \leq M$ if and only if

$$\sum_m \left| \sum_n b_n x_{m,n} \right|^2 \leq c \sum_m |b_m|^2$$

for all $b_n \in \mathbb{C}$, $1 \leq n \leq N$.

LargeSieve

Proof This can be rephrased as stating that for any m -by- n matrix X with complex entries (that is $X \in M_{m,n}(\mathbb{C})$), we have

$$\max_{a \in M_{1,m}(\mathbb{C})} \frac{|aX|}{|a|} = \max_{b \in M_{n,1}(\mathbb{C})} \frac{|Xb|}{|b|}.$$

To see this suppose that $|aX| \leq \gamma|a|$ for all $a \in M_{1,m}(\mathbb{C})$. Given any $b \in M_{n,1}(\mathbb{C})$ let $\bar{a} = Xb$ so that

$$|a| |Xb| = |Xb|^2 = a \cdot Xb = aXb = aX \cdot b \leq |aX| |b| \leq \gamma|a| |b|,$$

and therefore either $|Xb| \leq \gamma|b|$, or $a = Xb = 0$ which also yields $|Xb| \leq \gamma|b|$. The reverse implication is analogous. \square

BabyLS

Proposition 16.2 *Let $a_n, M+1 \leq n \leq M+N$ be a set of complex numbers, and $x_r, 1 \leq r \leq R$ be a set of real numbers. Let $\delta := \min_{r \neq s} \|x_r - x_s\| \in [0, 1/2]$, where $\|t\|$ denotes the distance from t to the nearest integer. Then*

$$\sum_r \left| \sum_{n=M+1}^{M+N} a_n e(nx_r) \right|^2 \leq \left(N + \frac{\log(e/\delta)}{\delta} \right) \sum_{n=M+1}^{M+N} |a_n|^2$$

where $e(t) = e^{2i\pi t}$.

Proof For any $b_r \in \mathbb{C}$, $1 \leq r \leq R$, we have

$$\sum_n \left| \sum_r b_r e(nx_r) \right|^2 = \sum_{r,s} b_r \bar{b}_s \sum_{n=M+1}^{M+N} e(n(x_r - x_s)) = N \|b\|^2 + E,$$

since the inner sum is N if $r = s$, where, for $L := M + \frac{1}{2}(N+1)$,

$$E \leq \sum_{r \neq s} b_r \bar{b}_s e(L(x_r - x_s)) \frac{\sin(\pi N(x_r - x_s))}{\sin(\pi(x_r - x_s))}.$$

Taking absolute values we obtain

$$|E| \leq \sum_{r \neq s} \frac{|b_r \bar{b}_s|}{|\sin(\pi(x_r - x_s))|} \leq \sum_{r \neq s} \frac{|b_r \bar{b}_s|}{2\|x_r - x_s\|} \leq \sum_r |b_r|^2 \sum_{s \neq r} \frac{1}{2\|x_r - x_s\|}$$

since $2|b_r \bar{b}_s| \leq |b_r|^2 + |b_s|^2$. Now, for each x_r the nearest two x_s are at distance at least δ away, the next two at distance at least 2δ away, etc, and so

$$|E| \leq \sum_r |b_r|^2 \sum_{j=1}^{\lfloor 1/\delta \rfloor} \frac{2}{2j\delta} \leq \frac{\log(e/\delta)}{\delta} \sum_m |b_m|^2,$$

so that

$$\sum_n \left| \sum_r b_r e(nx_r) \right|^2 \leq \left(N + \frac{\log(e/\delta)}{\delta} \right) \sum_m |b_m|^2.$$

The result follows by the duality principle. \square

We have $|E| \ll \sum_r |b_r|^2 / \min_{s \neq r} \|x_r - x_s\| \ll \sum_r |b_r|^2 / \delta$ by the strong Hilbert inequality (see section *), which leads to the constant $N + O(1/\delta)$ in the result above.

Proof of (16.4). By (10.3) we have

$$\sum_{n=M+1}^{M+N} a_n \chi(n) = \frac{1}{g(\bar{\chi})} \sum_b \bar{\chi}(b) \sum_{n=M+1}^{M+N} a_n e\left(\frac{bn}{q}\right).$$

where $g(\cdot)$ is the Gauss sum. Therefore, using (16.1)

$$\begin{aligned} \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 &\leq \frac{1}{q} \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \left| \sum_b \bar{\chi}(b) \sum_{n=M+1}^{M+N} a_n e\left(\frac{bn}{q}\right) \right|^2 \\ &\leq \frac{\phi(q)}{q} \sum_{\substack{b \pmod{q} \\ (b,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{bn}{q}\right) \right|^2. \end{aligned}$$

We deduce that the left side of (16.4) is

$$\leq \sum_{q \leq Q} \sum_{\substack{b \pmod{q} \\ (b,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{bn}{q}\right) \right|^2$$

We now apply Proposition 16.2 with $\{x_r\} = \{b/q : (b,q) = 1, q \leq Q\}$, so that

$$\delta \geq \min_{q, q' \leq Q} \min_{\substack{b, b' \\ b/q \neq b'/q'}} \left| \frac{b}{q} - \frac{b'}{q'} \right| \geq \min_{q \neq q' \leq Q} \frac{1}{qq'} \geq \frac{1}{Q(Q-1)},$$

and (16.4) follows. \square

16.1 Prime moduli

Primes are the only moduli for which the only imprimitive character is the principal character. Hence an immediate consequence of (16.4) is:

$$\sum_{\substack{p \leq \sqrt{N} \\ p \text{ prime}}} p \sum_{(b,p)=1} \left| \sum_{n \equiv b \pmod{p}} a_n - \frac{1}{p-1} \sum_{(n,p)=1} a_n \right|^2 \ll N \sum_n |a_n|^2, \quad (16.5)$$

which can be re-written as

$$\sum_{\substack{p \leq \sqrt{N} \\ p \text{ prime}}} \frac{1}{p} \sum_{(b,p)=1} \left| p \sum_{n \equiv b \pmod{p}} a_n - \sum_n a_n \right|^2 \ll N \sum_n |a_n|^2. \quad (16.6)$$

(Elliott showed how to also include the $b = 0$ congruence class in the sum.) Typically this corresponds to a massive saving. For example if $a_n = 1$ if n is prime and 0 otherwise, then this gives

$$\sum_{\substack{p \leq \sqrt{x} \\ p \text{ prime}}} p \sum_{(b,p)=1} \left| \pi(x; p, b) - \frac{\pi(x)}{p-1} \right|^2 \ll x\pi(x);$$

and so

$$\sum_{Q < p \leq \sqrt{x}} \sum_{\substack{p \text{ prime} \\ (b,p)=1}} \left| \pi(x; p, b) - \frac{\pi(x)}{p-1} \right|^2 \ll \frac{x^2}{Q \log x}.$$

Schlage-Puchta [AA 2003] proved

$$\sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive}}} \left| \sum_{p \leq N} a_p \chi(p) \right|^2 \leq \frac{N}{\log N} \sum_{p \leq N} |a_p|^2. \quad (16.7) \quad \boxed{\text{LargeSievePrimes}}$$

16.2 Other things to perhaps include on the large sieve

Elliott [MR962733] proved that for $Q < x^{1/2-\epsilon}$, and f multiplicative with $|f(n)| \leq 1$,

$$\sum_{p \leq Q} (p-1) \max_{y \leq x} \max_{(a,p)=1} \left| \sum_{\substack{n \leq y \\ n \equiv a \pmod{p}}} f(n) - \frac{1}{p-1} \sum_{\substack{n \leq y \\ (n,p)=1}} f(n) \right|^2 \ll \frac{x}{\log^A x},$$

where the sum is over all p except one where there might be an exceptional character.

Consequences of the large sieve to be discussed : Least quadratic non-residue.

THE SMALL SIEVE

17.1 List of sieving results used

In this subsection we have collected together many of the simple sieve results that we use. We will need to decide how to present this; whether to prove everything or whether to quote, say, Opera di Cribro. This chapter probably should come a lot earlier.

FLS **Lemma 17.1** (*The Fundamental Lemma of Sieve Theory*) *If $(am, q) = 1$ and all of the prime factors of m are $\leq z$ then*

$$\sum_{\substack{x < n \leq x+qy \\ (n,m)=1 \\ n \equiv a \pmod{q}}} 1 = \{1 + O(u^{-u-2})\} \frac{\phi(m)}{m} y + O(\sqrt{y}),$$

where $y = z^u$.

FLS1 **Corollary 17.2** *If $(am, q) = 1$ and all of the prime factors of m are $\leq x^{1/u}$ then*

$$\sum_{\substack{n \leq x \\ (n,m)=1 \\ n \equiv a \pmod{q}}} \log n = \{1 + O(u^{-u-2})\} \frac{\phi(m)}{m} \frac{x}{q} (\log x - 1) + O(\sqrt{x} \log x).$$

The proof of this and the subsequent corollaries are left as exercises. One approach here is to begin by writing $\log n = \int_1^n \frac{dt}{t}$ and then swap the order of the summation and the integral.

FLS2 **Corollary 17.3** *If χ is a character mod q and all of the prime factors of m are $\leq z = y^{1/u}$ and coprime with q , then*

$$\sum_{\substack{x < n \leq x+qy \\ (n,m)=1}} \chi(n) \ll \frac{1}{u^u} \frac{\phi(mq)}{mq} qy + q\sqrt{y}.$$

Let $p(n), P(n)$ be the smallest and largest prime factors of n , respectively.

FLS3 **Corollary 17.4** *If $(a, q) = 1$ and z is chosen so that $q = z^{O(1)}$ and $z \leq y$ then*

$$\left| \sum_{\substack{x < n \leq x+qy \\ n \equiv a \pmod{q} \\ p(n) > z}} 1 \right| \ll \frac{q}{\phi(q)} \frac{y}{\log z}.$$

17.2 Shiu's Theorem

Suppose that $0 \leq f(n) \leq 1$. Corollary 2.11^{cor2.3} states that the mean value of f up to x is $\ll \mathcal{P}(f; x)$. Shiu's Theorem states that an analogous result is true for the mean value of f in short intervals, in arithmetic progressions, and even in both:

Shiu **Theorem 17.5** *If $(a, q) = 1$ then*

$$\left| \frac{1}{y} \sum_{\substack{x < n \leq x+qy \\ n \equiv a \pmod{q}}} f(n) \right| \ll \prod_{\substack{p \leq y \\ p \nmid q}} \left(1 - \frac{1}{p} \right) \left(1 + \frac{|f(p)|}{p} \right).$$

This is $\asymp \mathcal{P}(|f| \chi_0; y) \asymp \exp \left(- \sum_{p \leq y, p \nmid q} \frac{1-f(p)}{p} \right)$.

Proof Let $g(p) = |f(p)|$ where $p \leq y$, and $g(p^k) = 1$ otherwise. Then $|\sum_n f(n)| \leq \sum_n |f(n)| \leq \sum_n g(n)$, and proving the result for g implies it for f .

Write $n = p_1^{k_1} p_2^{k_2} \dots$ with $p_1 < p_2 < \dots$, and let $d = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ where $d \leq y^{1/2} < dp_{r+1}^{k_{r+1}}$. Therefore $n = dm$ with $p(m) > z_d := \max\{P(d), y^{1/2}/d\}$, $(d, q) = 1$ and $g(n) \leq g(d)$. Now, if we fix d then m is in an interval $(x/d, x/d + qy/d]$ of an arithmetic progression $a/d \pmod{q}$ containing $y/d + O(1)$ integers. Note that $z_d \leq \max\{d, y^{1/2}/d\} \leq y^{1/2} \leq y/d$, and so we may apply Corollary 17.4^{FLS3} to show that there are $\ll qy/d\phi(q) \log(P(d) + y^{1/2}/d)$ such m . This implies that

$$\sum_{\substack{x < n \leq x+qy \\ n \equiv a \pmod{q}}} g(n) \leq \frac{qy}{\phi(q)} \sum_{\substack{d \leq y^{1/2} \\ (d, q) = 1}} \frac{g(d)}{d \log(P(d) + y^{1/2}/d)}.$$

For those terms with $d \leq y^{1/2-\epsilon}$ or $P(d) > y^\epsilon$, we have $\log(P(d) + y^{1/2}/d) \geq \epsilon \log y$, and so they contribute

$$\ll \frac{qy}{\phi(q)} \prod_{p \leq y} \left(1 - \frac{1}{p} \right) \sum_{\substack{d \leq y^{1/2} \\ (d, q) = 1}} \frac{g(d)}{d} \ll y \prod_{\substack{p \leq y \\ p \nmid q}} \left(1 - \frac{1}{p} \right) \left(1 + \frac{g(p)}{p} \right),$$

the upper bound claimed above. We are left with the $d > y^{1/2-\epsilon}$ for which $P(d) \asymp 2^r$ for some r , $1 \leq r \leq k = \lceil \epsilon \log y \rceil$. Hence we obtain an upper bound:

$$\frac{qy}{\phi(q)} \sum_{r=1}^k \frac{1}{r} \sum_{\substack{d > y^{1/2-\epsilon} \\ (d,q)=1 \\ P(d) < 2^r}} \frac{g(d)}{d} \ll \frac{qy}{\phi(q)} \left(\frac{1}{k} \sum_{\substack{d > y^{1/2-\epsilon} \\ (d,q)=1 \\ P(d) \leq 2^k}} \frac{g(d)}{d} + \sum_{r=1}^k \frac{1}{r^2} \sum_{\substack{d > y^{1/2-\epsilon} \\ (d,q)=1 \\ P(d) \leq 2^r}} \frac{g(d)}{d} \right).$$

For the first term we proceed as above. For the remaining terms we use Corollary 3.4.2, with $u_r := (1/2 - \epsilon) \log y / (r \log 2)$, to obtain

$$\ll \frac{qy}{\phi(q)} \sum_{r=1}^k \frac{1}{r^2} \prod_{\substack{p \leq 2^r \\ p \nmid q}} \left(1 + \frac{g(p)}{p} \right) \frac{1}{u_r^{u_r+1}} \ll y \sum_{r=1}^k \frac{1}{r u_r^{u_r}} \prod_{\substack{p \leq y \\ p \nmid q}} \left(1 - \frac{1}{p} \right) \left(1 + \frac{g(p)}{p} \right).$$

Finally note that u_r is decreasing, so that $\sum_{R/2 < r \leq R} 1/(r u_r^{u_r}) \ll 1/u_R^{u_R}$; moreover $u_{2R} = u_R/2$ and so $\sum_{1 \leq r \leq k} 1/(r u_r^{u_r}) \ll 1/u_k^{u_k} \ll 1$, and the result follows. \square

17.3 Consequences

Define

$$\rho_q(f) := \prod_{\substack{p \leq q \\ p \nmid q}} \left(1 - \frac{1}{p} \right) \left(1 + \frac{|f(p)|}{p} \right) \text{ and } \rho'_q(f) = \frac{\phi(q)}{q} \rho_q(f).$$

(Note that $\rho_q(f)$ is an upper bound in Theorem [17.5](#) provided $y \geq q$.) We also define

$$\log_S(n) := \sum_{\substack{d \in S \\ d|n}} \Lambda(d),$$

where S might be an interval $[a, b]$, and we might write “ $\leq Q$ ” in place of “ $[2, Q]$ ”, or “ $\geq R$ ” in place of “ $[R, \infty)$ ”. Note that $\log n = \log_{[2, n]} n$.

Small.1

Lemma 17.6 *Suppose that $x \geq Q^{2+\epsilon}$ and $Q \geq q$. Then, for any character $\chi \pmod{q}$ and any $\alpha \in \mathbb{R}$,*

$$\left| \sum_{\substack{n \in \mathcal{N} \\ n \equiv a \pmod{q}}} f(n) \bar{\chi}(n) e(\alpha n) \mathcal{L}(n) \right| \ll \rho_q(f) \frac{x}{q} = \rho'_q(f) \frac{x}{\phi(q)},$$

where $\mathcal{L}(n) = 1, \log(x/n), \frac{\log_{\leq Q} n}{\log Q}$ or $\frac{\log_{\geq x/Q} n}{\log Q}$, and $\mathcal{N} = \{n : Y < n \leq Y + x\}$ for $Y = 0$ in the second and fourth cases, and for any Y in the other two cases.

Proof The first estimate follows from Shiu's Theorem for $x \geq q^{1+\epsilon}$. One can deduce the second since $\sum_{n \leq x} a_n \log(x/n) = \int_{1 \leq T \leq x} \frac{1}{T} \sum_{n \leq T} a_n dT$ for any a_n .

If d is a power of the prime p then let $f_d(n)$ denote $f(n/p^a)$ where $p^a \parallel n$, so that if $n = dm$ then $|f(n)| \leq |f_d(m)|$. Therefore if $x > Qq^{1+\epsilon}$ then, for the third estimate, times $\log Q$, we have, again using Shiu's Theorem,

$$\begin{aligned} &\leq \sum_{\substack{Y < md \leq Y+x \\ md \equiv a \pmod{q} \\ d \leq Q}} |f(md)| \Lambda(d) \leq \sum_{\substack{d \leq Q \\ (d,q)=1}} \Lambda(d) \sum_{\substack{Y/d < m \leq (Y+x)/d \\ m \equiv a/d \pmod{q}}} |f_d(m)| \\ &\ll \sum_{\substack{d \leq Q \\ (d,q)=1}} \frac{\Lambda(d)}{d} \rho_q(f_d) \frac{x}{q} \ll \rho_q(f) \frac{x}{q} \log Q. \end{aligned}$$

In the final case, writing $n = mp$ where p is a prime $> x/Q$ (and note that $p^2 \nmid n$ as $p > x/Q > \sqrt{x}$), we have

$$\leq \sum_{\substack{m \leq Q \\ (m,q)=1}} |f(m)| \sum_{\substack{x/Q < p \leq x/m \\ p \equiv a/m \pmod{q}}} \log p \ll \sum_{\substack{m \leq Q \\ (m,q)=1}} |f(m)| \frac{x/m}{\phi(q)} \ll \rho_q(f) \frac{x}{q} \log Q.$$

by the Brun-Titchmarsh theorem, and then applying partial summation to Shiu's Theorem. \square

By $\left(\begin{smallmatrix} \text{SumSqs} \\ \text{II6.1} \end{smallmatrix}\right)$ we immediately deduce

Small.2 **Corollary 17.7** *With the hypotheses of Lemma $\left(\begin{smallmatrix} \text{Small.1} \\ \text{II7.6} \end{smallmatrix}\right)$ we have*

$$\sum_{\chi \pmod{q}} \left| \sum_{n \leq x} f(n) \overline{\chi}(n) \mathcal{L}(n) \right|^2 \ll (\rho'_q(f)x)^2.$$

Small.3 **Lemma 17.8** *If $\Delta > q^{1+\epsilon}$ then for any $D \geq 0$ we have*

$$\sum_{\chi \pmod{q}} \left| \sum_{D \leq d \leq D+\Delta} f(d) \overline{\chi}(d) \Lambda(d) \right|^2 \ll \Delta^2.$$

Proof We expand the left side using $\left(\begin{smallmatrix} \text{SumSqs} \\ \text{II6.1} \end{smallmatrix}\right)$ to obtain

$$\phi(q) \sum_{(b,q)=1} \left| \sum_{\substack{d \equiv b \pmod{q} \\ D \leq d \leq D+\Delta}} f(d) \Lambda(d) \right|^2 \leq \phi(q) \sum_{(b,q)=1} \left| \sum_{\substack{d \equiv b \pmod{q} \\ D \leq d \leq D+\Delta}} \Lambda(d) \right|^2 \ll \Delta^2,$$

by the Brun-Titchmarsh theorem. \square

THE PRETENTIOUS LARGE SIEVE

18.1 Mean values of multiplicative functions, on average

Define

$$S_\chi(x) := \sum_{n \leq x} f(n) \bar{\chi}(n),$$

and order the characters $\chi_1, \chi_2, \dots \pmod{q}$ so that the $|S_{\chi_i}(x)|$ are in descending order. Our main result is an averaged version of (1.1) for f twisted by all the characters $\chi \pmod{q}$, but with a better error term:

Corollary 18.1 *Suppose that $x \geq Q^{2+\epsilon}$ and $Q \geq q^{2+\epsilon} \log x$. Then*

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_1, \chi_2, \dots, \chi_{k-1}}} \left| \frac{1}{x} S_\chi(x) \right|^2 \ll \left(e^{O(\sqrt{k})} \rho'_q(f) \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{k}}} \log \left(\frac{\log x}{\log Q} \right) \right)^2,$$

where the implicit constants are independent of f . If $k = 1$, f is real and ψ_1 is not, then we can replace the exponent 0 with $1 - \frac{1}{\sqrt{2}}$.

Let \mathcal{C}_q be any subset of the set of characters \pmod{q} , and define

$$L = L(\mathcal{C}_q) := \frac{1}{\log x} \max_{\chi \in \mathcal{C}_q} \max_{|t| \leq \log^2 x} |F_\chi(1 + it)|,$$

where

$$F_\chi(s) := \prod_{p \leq x} \left(1 + \frac{f(p) \bar{\chi}(p)}{p^s} + \frac{f(p^2) \bar{\chi}(p^2)}{p^{2s}} + \dots \right).$$

Our main result is the following:

Theorem 18.2 *Suppose that $x \geq Q^{2+\epsilon}$ and $Q \geq q^{2+\epsilon} \log x$. Then*

$$\sum_{\chi \in \mathcal{C}_q} \left| \frac{1}{x} S_\chi(x) \right|^2 \ll \left(\left(L(\mathcal{C}_q) + \rho'_q(f) \frac{\log Q}{\log x} \right) \log \left(\frac{\log x}{\log Q} \right) \right)^2.$$

Corollary 18.1 follows immediately from Theorem 18.2 and Proposition 1.2.

To prove Theorem 18.2 we begin with an averaged version of (1.1), which was used in the proof of Halasz's Theorem. Notice that if we simply sum up the square of (1.1) for $S = S_\chi$, for each $\chi \pmod{q}$, then we would get the next lemma but with the much weaker error term $\phi(q)$.

AvLogWt

Lemma 18.3 *Suppose that $x \geq Q^{2+\epsilon}$ and $Q \geq q$. Then*

$$\log^2 x \sum_{\chi \in \mathcal{C}_q} \left| \frac{1}{x} S_\chi(x) \right|^2 \ll \sum_{\chi \in \mathcal{C}_q} \left(\int_Q^{x/Q} \left| \frac{1}{t} S_\chi(t) \right| \frac{dt}{t} \right)^2 + (\rho'_q(f) \log Q)^2.$$

Proof Let $z = x/Q$. We follow the proof in section ^{|MeanF(n)|}8.1 for the main terms, but deal with the error terms differently. By Corollary ^{Small.2}17.7 we have

$$\sum_{\chi \pmod{q}} \left| \sum_{n \leq x} f(n) \bar{\chi}(n) \log(x/n) \right|^2 \ll (\rho'_q(f)x)^2,$$

$$\text{and } \sum_{\chi} \left| \sum_{n \leq x} f(n) \bar{\chi}(n) (\log_{\leq Q} n + \log_{> x/Q} n) \right|^2 \ll (\rho'_q(f)x \log Q)^2,$$

so that, using the identity $\log x = \log(x/n) + \log_{\leq Q} n + \log_{> x/Q} n + \log_{(Q, x/Q)} n$,

$$\sum_{\chi \in \mathcal{C}_q} |S_\chi(x) \log x|^2 \ll \sum_{\chi \in \mathcal{C}_q} \left| \sum_{n \leq x} f(n) \bar{\chi}(n) \log_{(Q, x/Q)} n \right|^2 + (\rho'_q(f)x \log Q)^2.$$

Now for $g = f\bar{\chi}$ we have

$$\begin{aligned} & \sum_{n \leq x} g(n) \log_{(Q, x/Q)} n - \sum_{Q < p < x/Q} g(p) \log p \sum_{m \leq x/p} g(m) \\ &= \sum_{\substack{Q < p^k < x/Q \\ k \geq 2}} \log p \sum_{m \leq x/p^k} g(mp^k) + \sum_{Q < p < x/Q} \log p \sum_{m \leq x/p} (g(mp) - g(p)g(m)). \end{aligned}$$

The last term is 0 unless $p^2 | m$, so this last bound is, in absolute value,

$$\leq x \sum_{\substack{Q < p^k < x/Q \\ k \geq 2}} \frac{\log p}{p^k} + 2x \sum_{Q < p < x/Q} \frac{\log p}{p^2} \ll \frac{x}{Q^{1/2}}.$$

We now bound our main term as in section ^{|MeanF(n)|}8.1; though now we let $z = y + \sqrt{y}$ so we obtain the error term x/\sqrt{y} in the equation before ^{|MeanAveraged|}(7.7). Summing over such dyadic intervals this yields

$$\left| \sum_{Q < p < x/Q} g(p) \log p \sum_{m \leq x/p} g(m) \right| \ll \int_Q^{x/Q} |S_\chi(x/t)| dt + \frac{x}{Q^{1/2}}.$$

The result follows from the change of variable $t \rightarrow x/t$ since $Q \geq q$ and $\rho'_q(f) \log Q \gg 1$.

□

In the next Lemma we create a convolution to work with, as well as removing the small primes.

AvConvol **Lemma 18.4** *Suppose that $x \geq Q^{2+\epsilon}$ and $Q \geq q^{2+\epsilon} \log x$. Then*

$$\sum_{\chi \in \mathcal{C}_q} \left(\int_Q^x \left| \frac{1}{t} S_\chi(t) \right| \frac{dt}{t} \right)^2 \ll \sum_{\chi \in \mathcal{C}_q} \left(\int_Q^x \left| \sum_{n \leq t} f(n) \bar{\chi}(n) \log_{>Q} n \right| \frac{dt}{t^2 \log t} \right)^2 + \left(\rho'_q(f) \log Q \cdot \log \left(\frac{\log x}{\log Q} \right) \right)^2.$$

Proof We expand using the fact that $\log t = \log(t/n) + \log_{<Q} n + \log_{>Q} n$; and the Cauchy-Schwarz inequality so that, for any function $c_\chi(t)$,

$$\sum_\chi \left(\int_Q^x c_\chi(t) \frac{dt}{t^2 \log t} \right)^2 \leq \int_Q^x \frac{dt}{t \log t} \cdot \int_Q^x \sum_\chi c_\chi(t)^2 \frac{dt}{t^3 \log t}$$

By Corollary [Small.2](#) [17.7](#) we then have

$$\int_Q^x \sum_\chi \left| \sum_{m \leq t} f(m) \bar{\chi}(m) \log(t/m) \right|^2 \frac{dt}{t^3 \log t} \ll \rho'_q(f)^2 \int_Q^x \frac{dt}{t \log t} \ll \rho'_q(f)^2 \log \left(\frac{\log x}{\log Q} \right)$$

and

$$\int_Q^x \sum_\chi \left| \sum_{m \leq t} f(m) \bar{\chi}(m) \log_{\leq Q} m \right|^2 \frac{dt}{t^3 \log t} \ll \int_Q^x (\rho'_q(f) t \log Q)^2 \frac{dt}{t^3 \log t},$$

and the result follows. \square

Now we prove the mean square version of Halasz's Theorem, which is at the heart of the pretentious large sieve.

AvParsev **Proposition 18.5** *If $x > Q^{1+\epsilon}$ and $Q \geq q^{1+\epsilon}$ then*

$$\sum_{\chi \in \mathcal{C}_q} \left(\int_Q^x \left| \sum_{Q \leq n \leq t} f(n) \chi(n) \log_{>Q} n \right| \frac{dt}{t^2 \log t} \right)^2 \ll \log \left(\frac{\log x}{\log Q} \right) \left(M^2 \log \left(\frac{\log x}{\log Q} \right) + \frac{\phi(q)}{T} \frac{\log Q}{Q} + \frac{\log^3 x}{T^2} \right)$$

where $M := \max_{\chi \in \mathcal{C}_q} \max_{|u| \leq 2T} |F_\chi(1 + iu)|$.

Proof (Revisiting the proof of Halasz's Theorem (particularly Proposition [8.5](#))).
For a given $g = f\bar{\chi}$ and Q we define

$$h(n) = \sum_{\substack{md=n \\ d>Q}} g(m)g(d)\Lambda(d),$$

so that $G(s)(G'_{>Q}(s)/G_{>Q}(s)) = -\sum_{n\geq 1} h(n)/n^s$ for $\text{Re}(s) > 1$. Now

$$\left| \sum_{n\leq t} g(n) \log_{>Q} n - \sum_{n\leq t} h(n) \right| \leq 2 \sum_{p^b > Q} \log p \sum_{\substack{n\leq t \\ p^{b+1}|n}} 1 \leq 2t \sum_{b\geq 1} \sum_{\substack{p^b > Q \\ p^{b+1}\leq t}} \frac{\log p}{p^{b+1}} \ll \frac{t \log t}{Q},$$

by the prime number theorem. This substitution leads to a total error, in our estimate, of

$$\ll |\mathcal{C}_q| \left(\int_{Q_q}^x \frac{t \log t}{Q} \frac{dt}{t^2 \log t} \right)^2 \ll \frac{q}{Q^2} \log^2 \left(\frac{\log x}{\log Q} \right) \ll \frac{1}{q} \log^2 \left(\frac{\log x}{\log Q} \right),$$

which is smaller than the first term in the given upper bound, since $M \gg 1/\log q$.

Now we use the fact that

$$\frac{1}{\log t} \ll \int_{1/\log x}^{1/\log Q} \frac{d\alpha}{t^{2\alpha}}$$

whenever $x \geq t \geq Q$, as $x > Q^{1+\epsilon}$, so that

$$\int_2^x \left| \sum_{n\leq t} h(n) \right| \frac{dt}{t^2 \log t} \ll \int_{1/\log x}^{1/\log Q} \left(\int_2^x \left| \sum_{n\leq t} h(n) \right| \frac{dt}{t^{2+2\alpha}} \right) d\alpha.$$

Now, Cauchyng, but otherwise proceeding as in the proof of Proposition [8.5](#) (with $f(n) \log n$ there replaced by $h(n)$ here), the square of the left side is

$$\ll \int_{1/\log x}^{1/\log Q} \frac{d\alpha}{\alpha} \cdot \int_{1/\log x}^{1/\log Q} \alpha \cdot \frac{1}{2\pi\alpha} \int_{-\infty}^{\infty} \left| \frac{G(G'_{>Q}/G_{>Q})(1+\alpha+it)}{1+\alpha+it} \right|^2 dt d\alpha.$$

The integral in the region with $|t| \leq T$ is now

$$\leq \max_{|t|\leq T} |G(1+\alpha+it)|^2 \int_1^{\infty} \left| \sum_{Q < n \leq t} g(n)\Lambda(n) \right|^2 \frac{dt}{t^{3+2\alpha}}.$$

If we take $g = f\bar{\chi}$ and sum this over all characters $\chi \in \mathcal{C}_q$ then we obtain an error

$$\begin{aligned} &\leq \max_{\substack{|t|\leq T \\ \chi \in \mathcal{C}_q}} |F_{\chi}(1+\alpha+it)|^2 \int_Q^{\infty} \sum_{\chi \pmod{q}} \left| \sum_{Q < n \leq t} f(n)\bar{\chi}(n)\Lambda(n) \right|^2 \frac{dt}{t^{3+2\alpha}} \\ &\ll \max_{\substack{|t|\leq T \\ \chi \in \mathcal{C}_q}} |F_{\chi}(1+\alpha+it)|^2 \int_Q^{\infty} \frac{dt}{t^{1+2\alpha}} \ll \frac{1}{\alpha} \max_{\substack{|t|\leq T \\ \chi \in \mathcal{C}_q}} |F_{\chi}(1+\alpha+it)|^2, \end{aligned}$$

by Lemma [11.3](#) as $t \geq Q \geq q^{1+\epsilon}$.

For that part of the integral with $|t| > T$, summed over all twists of f by characters $\chi \pmod{q}$, we now proceed as in the proof of Proposition 8.5. We obtain $\phi(q)$ times (7.7), with $f(\ell) \log \ell$ replaced by $h(\ell)$ for $\ell = m$ and n , but now with the sum over $m \equiv n \pmod{q}$ with $m, n \geq Q$. Observing that $|h(\ell)| \leq \log \ell$, we proceed analogously to obtain, in total

$$\ll \frac{\phi(q)}{T} \frac{(\log Q)^2}{Q} + \frac{\phi(q)}{q} \cdot \frac{1}{\alpha^4 T^2}.$$

The result follows by collecting the above. \square

Proof of Theorem 18.2: The result follows by taking $T = \frac{1}{2} \log^2 x$ in Proposition 8.5, and then combining this with Lemmas 18.3 and 18.4, since $\rho'_q(f) \log q \gg 1$. \square

PLSRange

Corollary 18.6 Fix $\epsilon > 0$. There exists an integer $k \ll 1/\epsilon^2$ such that if $x \geq q^{4+5\epsilon}$ then

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_1, \chi_2, \dots, \chi_k}} \left| \frac{1}{y} S_{\chi_j}(y) \right|^2 \ll e^{O(1/\epsilon)} \left(\rho'_q(f) \left(\frac{\log Q}{\log y} \right)^{1-\epsilon} \right)^2,$$

where $Q = (q \log x)^2$, for any y in the range

$$\log x \geq \log y \geq \log x / 2 \left(\frac{\log x}{\log Q} \right)^{\epsilon/2},$$

where the implicit constants are independent of f .

Proof Select k to be the smallest integer for which $1/\sqrt{k} < 3\epsilon$. Let \mathcal{C}_q be the set of all characters mod q except $\chi_1, \chi_2, \dots, \chi_k$. Write $x = Q^B$, so that $y = Q^C$ where $B \geq C \geq \frac{1}{2} B^{1-\epsilon/2}$ and apply Theorem 18.2 with $x = y$. Then, by (7.7) and Proposition 7.7 we have

$$L_y \ll L_x \left(\frac{\log x}{\log y} \right)^2 \ll e^{O(1/\epsilon)} \rho'_q(f) \frac{1}{B^{1-3\epsilon}} B^\epsilon \ll e^{O(1/\epsilon)} \rho'_q(f) \frac{1}{C^{1-4\epsilon}},$$

and the result follows. Note that by bounding L_y in terms of L_x , we can have the same exceptional characters $\chi_1, \chi_2, \dots, \chi_k$ for each y in our range. \square

MULTIPLICATIVE FUNCTIONS IN ARITHMETIC PROGRESSIONS

It is usual to estimate the mean value of a multiplicative function in an arithmetic progression in terms of the mean value of the multiplicative function on all the integers. This approximation is the summand corresponding to the principal character when we decompose our sum in terms of the Dirichlet characters mod q . In what follows we will instead compare our mean value with the summands for the k characters which best correlate with f . So define

$$E_f^{(k)}(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) - \frac{1}{\phi(q)} \sum_{j=1}^k \chi_j(a) \sum_{n \leq x} f(n) \bar{\chi}_j(n).$$

The trivial upper bound $|E_f^{(k-1)}(x; q, a)| \ll k \rho'_q(f) x / \phi(q)$ can be obtained by bounding each sum in the definition using the small sieve. We now improve this:

FnsInAPs

Theorem 19.1 *For any given $k \geq 2$ and sufficiently large x , if $x \geq X \geq \max\{x^{1/2}, q^{6+7\epsilon}\}$ then*

$$|E_f^{(k-1)}(X; q, a)| \ll e^{C\sqrt{k}} \frac{\rho'_q(f)X}{\phi(q)} \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}} \log\left(\frac{\log x}{\log Q}\right),$$

where $Q = (q \log x)^5$ and the implicit constants are independent of f and k . If f is real and χ_1 is not then we can extend this to $k = 1$ with exponent $1 - \frac{1}{\sqrt{2}}$.

To prove this we need the following technical tool, deduced from Corollary [PLSRRange](#) [18.6](#).

LinearPLS

Proposition 19.2 *Fix $\epsilon > 0$. For given $x = q^A$ there exists $K \ll \epsilon^{-3} \log \log A$ such that if $x \geq X \geq x^{1/2}$ and $Q = (q \log x)^5$ then*

$$\frac{1}{\log x} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_j, j=1, \dots, K}} \left| \frac{1}{X} \sum_{n \leq X} f(n) \bar{\chi}(n) \log_{[Q, x/Q]} n \right| \ll e^{O(1/\epsilon)} \rho'_q(f) \left(\frac{\log Q}{\log x}\right)^{1-\epsilon}.$$

Proof Let $\log x_i = 2^{(1+\epsilon/3)^i + 1} \log q$ for $0 \leq i \leq IA$, with I chosen to be the smallest integer for which $x_I > x/Q$, so that $I \ll (1/\epsilon) \log \log A$. In order to apply Corollary [PLSRRange](#) [18.6](#) with $x = x_i$ we must exclude the characters $\chi_{j,i}$, $1 \leq j \leq k$, for

$1 \leq i \leq I$. Let $\chi_1, \chi_2, \dots, \chi_K$ be the union of these sets of characters, so that $K \leq k(I+1) \ll \epsilon^{-3} \log \log A$. Therefore, for all $y \in [Q, x/Q]$, we have

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_1, \chi_2, \dots, \chi_K}} \left| \frac{1}{y} S_{\chi_j}(y) \right|^2 \ll e^{O(1/\epsilon)} \left(\rho'_q(f) \left(\frac{\log Q}{\log y} \right)^{1-\epsilon} \right)^2. \quad (19.1) \quad \boxed{\text{PLSuniform}}$$

We rewrite the sum in the Proposition as

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_j, j=1, \dots, K}} \left| \sum_{\substack{dm \leq X \\ Q \leq d \leq x/Q}} f(m) \overline{\chi}(m) f(d) \overline{\chi}(d) \Lambda(d) \right|,$$

and split this into subsums, depending on the size of d . This is bounded by a sum of sums of the form

$$\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_j, j=1, \dots, K}} \left| \sum_{D \leq d \leq D+\Delta} f(d) \overline{\chi}(d) \Lambda(d) \sum_{m \leq X/d} f(m) \overline{\chi}(m) \right|.$$

where $Q \leq D \leq x/Q$ with $\Delta \approx \frac{D \log(q \log(X/D))}{q \log(X/D)}$. If we approximate the last sum here with the range $m \leq X/D$, then we can Cauchy to obtain

$$\left(\sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_j, j=1, \dots, K}} \left| \sum_{D \leq d \leq D+\Delta} f(d) \overline{\chi}(d) \Lambda(d) \sum_{m \leq X/D} f(m) \overline{\chi}(m) \right| \right)^2 \quad (19.2)$$

$$\leq \sum_{\chi \pmod{q}} \left| \sum_{D \leq d \leq D+\Delta} f(d) \overline{\chi}(d) \Lambda(d) \right|^2 \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_j, j=1, \dots, k-1}} \left| \sum_{m \leq X/D} f(m) \overline{\chi}(m) \right|^2 \quad (19.3)$$

$$\ll e^{O(1/\epsilon)} \left(\Delta \cdot \rho'_q(f) \frac{X}{D} \left(\frac{\log Q}{\log(X/D)} \right)^{1-\epsilon} \right)^2, \quad (19.4) \quad \boxed{\text{OneTermBound}}$$

by Lemma [17.8](#) and [\(19.1\)](#). Summing the square root of this over the D/Δ such intervals for d in $[D, 2D)$ yields an upper bound

$$\ll e^{O(1/\epsilon)} \rho'_q(f) X \left(\frac{\log Q}{\log(X/D)} \right)^{1-\epsilon};$$

and then summing this over $D = X/Q2^j$ for $0 \leq j \leq J \asymp \log X$ we obtain the claimed upper bound.

Finally the error in replacing the range $m \leq X/d$ by $m \leq X/D$ is

$$\leq \sum_{\substack{X/d < m \leq x/D \\ (m,q)=1}} |f(m)| \chi_0(m) \leq \sum_{\substack{X/(D+\Delta) < m \leq x/D \\ (m,q)=1}} |f(m)| \chi_0(m) \ll \rho'_q(f) \frac{X\Delta}{D^2},$$

so an upper bound for the contribution in $[D, 2D)$ is

$$\ll \rho'_q(f) \frac{X\Delta\phi(q)}{D} \sum_{D \leq d < 2D} \frac{\Lambda(d)}{d} \ll \rho'_q(f) X \frac{\log Q}{\log(X/D)},$$

which is smaller than the other error term. \square

Proof of Theorem 19.1 ^{FnsInAPs}: Fix $\epsilon > 0$ sufficiently small with $1/\sqrt{k} > \epsilon$. By applying Lemma 17.6 ^{Small.1}, with $\chi = \chi_0$ we have

$$\log x \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) \log_{[Q, x/Q]} n + O\left(\rho'_q(f) \frac{x}{\phi(q)} \log Q\right).$$

Multiplying this by $\bar{\chi}(a)$, and summing over a we obtain

$$\log x \sum_{n \leq x} f(n) \bar{\chi}(n) = \sum_{n \leq x} f(n) \bar{\chi}(n) \log_{[Q, x/Q]} n + O(\rho'_q(f) x \log Q);$$

so that

$$\begin{aligned} E_f^{(K)}(x; q, a) &= \frac{1}{\phi(q)} \sum_{j=K+1}^{\phi(q)} \chi_j(a) \sum_{n \leq x} f(n) \bar{\chi}_j(n) \frac{\log_{[Q, x/Q]} n}{\log x} + O\left(K \rho'_q(f) \frac{x}{\phi(q)} \frac{\log Q}{\log x}\right) \\ &\ll e^{O(1/\epsilon)} \frac{\rho'_q(f) x}{\phi(q)} \left(\frac{\log Q}{\log x}\right)^{1-\epsilon}, \end{aligned}$$

by Proposition 19.2 ^{LinearPLS}, where $K \ll \epsilon^{-3} \log \log A$. By Cauchy and then Corollary 18.1 ^{PLSk}, we obtain

$$\begin{aligned} |E_f^{(k)}(x; q, a) - E_f^{(K)}(x; q, a)| &\leq \frac{1}{\phi(q)} \sum_{j=k+1}^K |S_{\chi_j}(x)| \\ &\leq \frac{1}{\phi(q)} \left(K \sum_{j=k+1}^K |S_{\chi_j}(x)|^2\right)^{1/2} \ll e^{O(\sqrt{k})} \rho'_q(f) \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}, \end{aligned}$$

since $K \ll \log \log A$, and $1 - \frac{1}{\sqrt{k+1}} > 1 - \frac{1}{\sqrt{k}}$. Applying the same argument again, we also obtain

$$|E_f^{(k-1)}(x; q, a) - E_f^{(k)}(x; q, a)| \ll e^{C\sqrt{k}} \frac{\rho'_q(f) x}{\phi(q)} \left(\frac{\log Q}{\log x}\right)^{1-\frac{1}{\sqrt{k}}} \log\left(\frac{\log x}{\log Q}\right).$$

The result follows from using the triangle inequality and adding the last three inequalities. \square

PRIMES IN ARITHMETIC PROGRESSION

PNTapsk

Theorem 20.1 For any $k \geq 2$ and $x \geq q^2$ there exists an ordering χ_1, \dots of the non-principal characters $\chi \pmod{q}$ such that, for $Q = (q \log x)^2$,

$$\sum_{\substack{n \leq y \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq y} \Lambda(n) - \frac{1}{\phi(q)} \sum_{j=1}^{k-1} \chi_j(a) \sum_{n \leq y} \Lambda(n) \bar{\chi}_j(n) \\ \ll e^{C\sqrt{k}} \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{k}}} \log^3 \left(\frac{\log x}{\log Q} \right).$$

PNTaps1

Corollary 20.2 There exists a character $\chi \pmod{q}$ such that if $x \geq q^2$ then

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq x} \Lambda(n) - \frac{\chi(a)}{\phi(q)} \sum_{n \leq x} \Lambda(n) \bar{\chi}(n) \ll \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{2}} - \varepsilon}.$$

where $Q = (q \log x)^2$. We may remove the χ term unless χ is a real-valued character.

Remark 20.3 Can we obtain the error in terms of $1/|L(1+it, \chi)|/\log x$? And when χ is real, probably $t = 0$.

Proof of Theorem 20.1 ^{PNTapsk} We may assume that $x \geq q^B$ for B sufficiently large, else the result follows from the Brun-Titchmarsh Theorem.

Let $g(\cdot)$ be the totally multiplicative function for which $g(p) = 0$ for $p \leq Q$ and $g(p) = 1$ for $p > Q$, and then $f = \mu g$, so that we have the following variant of von Mangoldt's formula (I.11), ^{Lammu}

$$\Lambda_Q(n) := \sum_{dm=n} f(d)g(m) \log m = \begin{cases} \Lambda(n) & \text{if } p|n \implies p > Q, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\sum_{\substack{n \leq x \\ n \equiv b \pmod{q}}} (\Lambda(n) - \Lambda_Q(n)) \leq \sum_{\substack{n \leq x \\ p|n \implies p \leq Q}} \Lambda(d) \ll \sum_{p \leq Q} \log x \ll Q \frac{\log x}{\log Q}.$$

by the Brun-Titchmarsh theorem. Denote the left side of the equation in the Theorem as $E_{\Lambda, +}^{(k-1)}(x; q, a)$, and note that all of these sums can be expressed as mean-values of $\sum_{n \leq x, n \equiv b \pmod{q}} \Lambda(n)$, as b varies. Hence

$$E_{\Lambda,+}^{(k-1)}(x; q, a) - E_{\Lambda_Q,+}^{(k-1)}(x; q, a) \ll Q \frac{\log x}{\log Q}.$$

Now

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda_Q(n) = \sum_{\substack{d \leq x \\ (d,q)=1}} f(d) \sum_{\substack{m \leq x/d \\ m \equiv a/d \pmod{q}}} g(m) \log m. \quad (20.1) \quad \boxed{\text{LQexpand}}$$

Similar decompositions for the $\sum_n \Lambda_Q(n) \bar{\chi}_j(n)$ imply that $E_{\Lambda_Q,+}^{(k-1)}(x; q, a)$ equals the sum of $f(d)$ over $d \leq x$ with $(d, q) = 1$, times

$$\sum_{\substack{m \leq x/d \\ m \equiv a/d \pmod{q}}} g(m) \log m - \frac{1}{\phi(q)} \sum_{j=0}^{k-1} \chi_j(a/d) \sum_{(b,q)=1} \bar{\chi}_j(b) \sum_{\substack{m \leq x/d \\ m \equiv b \pmod{q}}} g(m) \log m.$$

By Corollary ^{|FLS1|}17.2 (with m the product of the primes $\leq Q$ that do not divide q) this last quantity is

$$\ll \left(\frac{k}{u^{u+2}} \frac{x}{d\phi(q) \log Q} + k \sqrt{\frac{x}{d}} \right) \log x/d$$

where $x/d = Q^u$. Let R be the product of the primes $\leq Q$. We deduce that the sum over d in a range $x/Q^{2u} < d \leq x/Q^u$ with $f(d) \neq 0$, is

$$\ll k \sum_{\substack{x/Q^{2u} < d \leq x/Q^u \\ (d,R)=1}} \left(\frac{1}{u^{u+2}} \frac{x}{d\phi(q) \log Q} + \sqrt{\frac{x}{d}} \right) \log x/d \ll \frac{k}{u^u} \frac{x}{\phi(q)} + \frac{kux}{Q^{u/2}}$$

by Corollary ^{|FLS3|}17.4 (for the sum over d), provided $u \leq \nu := \log \left(\frac{\log x}{\log Q} \right)$. Summing this up over $u = 2, 4, 8, \dots, \nu$, the sum over d in the range $Q^2 < d \leq x/Q^{2\nu}$ is

$$\ll \frac{x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^2.$$

The same argument works to give a much better upper bound for the terms with $d \leq Q^2$, though removing the condition $(d, R) = 1$ in the sum above. Hence we are left to deal with those $d \geq x/Q^\nu$, which implies that $m \leq x/d < Q^\nu$.

The remaining sum in (20.1) is ^{|LQexpand|}

$$\sum_{\substack{m < Q^\nu \\ (m,q)=1}} g(m) \log m \sum_{\substack{x/Q^\nu < d \leq x/m \\ d \equiv a/m \pmod{q}}} f(d).$$

There are analogous sums for the remaining terms in $E_{\Lambda_Q,+}^{(k-1)}(x; q, a)$ and so we need to bound

$$\sum_{\substack{m < Q^\nu \\ (m, q) = 1}} g(m) \log m (E_{f,+}^{(k-1)}(x/m; q, a/m) - E_{f,+}^{(k-1)}(x/Q^\nu; q, a/m)).$$

To do so we need to apply Theorem [18.2](#) with \mathcal{C}_q to be the set of all characters mod q , less $\chi_0, \chi_1, \dots, \chi_{k-1}$. Then we can deduce Corollary [18.1](#) though now with $\chi \neq \chi_0, \dots, \chi_{k-1}$ as the condition on the sum (but otherwise the same). We can then similarly modify Corollary [18.6](#) and finally obtain Theorem [19.1](#) with $E_f^{(k-1)}$ replaced by $E_{f,+}^{(k-1)}$. Therefore we obtain the bound

$$\begin{aligned} \sum_{\substack{m < Q^\nu \\ (m, q) = 1}} g(m) \log m |E_{f,+}^{(k-1)}(x/m; q, a/m) - E_{f,+}^{(k-1)}(x/Q^\nu; q, a/m)| \\ \ll e^{C\sqrt{k}} \frac{\rho'_q(f)x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{k}}} \nu \sum_{\substack{m < Q^\nu \\ (m, q) = 1}} g(m) \frac{\log m}{m} \\ \ll e^{C\sqrt{k}} \frac{\rho'_q(f)x}{\phi(q)} \left(\frac{\log Q}{\log x} \right)^{1 - \frac{1}{\sqrt{k}}} \nu \frac{(\nu \log Q)^2}{\log Q}. \end{aligned}$$

by Corollary [17.4](#), and the result follows since $\rho'_q(f) \ll 1/\log Q$. (*This means we need to change the sieving to go up to Q throughout rather than q .*) \square

Proof of Corollary [20.2](#) We let $k = 2$ in Theorem [11.1](#) to deduce the first part. If χ is not real valued, then we know that

$$\left| \sum_{n \leq x} \Lambda(n) \overline{\chi}(n) \right| = \left| \sum_{n \leq x} \Lambda(n) \chi(n) \right| \leq |E_\Lambda^{(3)}(x; q, a) - E_\Lambda^{(2)}(x; q, a)|$$

and the result follows from Theorem [20.1](#). \square

LINNIK'S THEOREM

In this section we complete the proof of Linnik's famous theorem:

Linnik

Theorem 21.1 *There exist constants $c, L > 0$ such that for any coprime integers a and q there is a prime $\equiv a \pmod{q}$ that is $< cq^L$.*

There are several proofs of this in the literature, none easy. Here we present a new proof as a consequence of the Pretentious Large Sieve, as developed in the previous few sections. Corollary 20.2 implies that if there are no primes $\equiv a \pmod{q}$ up to x , a large power of Q , then the vast majority of primes satisfy $\chi(p) = -\chi(a)$. The difficult part of our current proof is to now show that $\chi(a) = 1$ (which surely should not be difficult!):

LinkNoSieg

Proposition 21.2 *Suppose that $x \geq q^A$ where A is chosen sufficiently large. If*

$$\left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq x} \Lambda(n) \right| \gg \frac{x}{\phi(q)}$$

then there exists a real character $\chi \pmod{q}$ such that $\chi(a) = -1$, and

$$\sum_{\substack{Q < p \leq x \\ \chi(p)=1}} \frac{1}{p} \ll \log \log \left(\frac{\log x}{\log Q} \right).$$

LinkSiegCond

Corollary 21.3 *If there are no primes $p \equiv a \pmod{q}$ with $Q < p \leq x$ then there exists a real character $\chi \pmod{q}$ such that $\chi(a) = -1$, and*

$$\sum_{\substack{Q < p \leq x \\ \chi(p)=1}} \frac{1}{p} \ll 1.$$

HalRevisited

Lemma 21.4 (Halasz's Theorem for sieved functions) *Let f be a multiplicative function with the property that $f(p^k) = 0$ whenever $p \leq Q$. If $x \geq Q$ then*

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) \right| \ll \frac{1}{\log Q} (1 + M) e^{-M} + \frac{1}{T} + \frac{1}{\log x} \left(1 + \frac{1}{\log Q} \log \left(\frac{\log x}{\log Q} \right) \right).$$

where $M := \min_{|t| \leq T} \sum_{Q < p \leq x} \frac{1 - \operatorname{Re}(f(p)p^{-it})}{p}$.

Proof (sketch) We suitably modify the proof of Halasz's Theorem (HalExplic1). We begin by following the proof of Proposition 8.5. First note that $S(N) = 1$ for all $N \leq Q$, so we can reduce the range in the integral for α , throughout the proof of Proposition 8.5, to $\frac{1}{\log x} \leq \alpha \leq \frac{1}{\log Q}$. Moreover in the first displayed equation we can change the error term from $\ll \frac{N}{\log N}$ to $\ll \frac{1}{\log Q} \frac{N}{\log N}$ for $N \geq Q$ by sieving. This allows us to replace the error term in the second displayed equation from $\ll \log \log x$ to $\ll 1 + \frac{1}{\log Q} \log \left(\frac{\log x}{\log Q} \right)$. Hence we can restate Proposition 8.5 with the range for α , and the $\log \log x$ in the error term, changed in this way.

Now we use the bound $|F(1 + \alpha + it)| \leq |F(1 + iu)| + O\left(\frac{\alpha \log x}{T \log Q}\right)$ throughout this range, as in Lemma OffLineOn; and we also note that, in our range for α , $|F(1 + \alpha + it)| \ll 1/(\alpha \log Q)$. We then proceed as in the proof of (HalExplic2), but now splitting the integral at $1/L \log Q \log x$ to obtain the result, since $L \log Q \asymp e^{-M}$. \square

Proof of Proposition 21.2 Write $\nu := \log \left(\frac{\log x}{\log Q} \right)$. We return to the proof of Theorem 20.1, and show, under our hypothesis here, that there exists y in the range $x^{1/2} < y \leq x$ for which

$$\left| \sum_{n \leq y} f(n) \overline{\chi}(n) \right| \gg \frac{y}{\nu^2 \log Q}.$$

For, if not, the proof there implies that

$$\left| \sum_{n \leq x} \Lambda(n) \overline{\chi}(n) \right| = o\left(\frac{x}{\phi(q)}\right),$$

which, by Corollary 20.2, contradicts our hypothesis.

Taking $f = f\overline{\chi}$ in Lemma 21.4, and comparing our upper and lower bounds for $S_\chi(y)$ we deduce that

$$\sum_{Q < p \leq x} \frac{1 + \operatorname{Re}(\chi(p)p^{it})}{p} \ll \log \nu.$$

Let $T := \{z : |z| = 1, \text{ and } \frac{\pi}{3} < \arg(z) < \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} < \arg(z) < \frac{5\pi}{3}\}$. We must have $|t| \ll \nu / \log x$ else $p^{it} \in T$ (and hence $\chi(p)p^{it} \in T$) for enough of the primes in $(x^{c/\nu}, x]$ that the previous estimate cannot hold. Therefore

$$\sum_{\substack{Q < p \leq x \\ \chi(p)=1}} \frac{1}{p} = \frac{1}{2} \sum_{Q < p \leq x} \frac{1 + \operatorname{Re}(\chi(p))}{p} \ll \sum_{Q < p \leq x} \frac{1 + \operatorname{Re}(\chi(p)p^{it}) + |p^{it} - 1|}{p} \ll \log \nu.$$

\square

Proof of Corollary [21.3](#) By [Corollary 20.2](#) we know that for all y in the range $Q \leq y \leq x$ we have

$$\sum_{p \leq y} \Lambda(n)(\chi(p) + \chi(a)) \ll y \left(\frac{\log Q}{\log y} \right)^{1/5}.$$

By partial summation, we deduce that

$$\sum_{Q < p \leq x} \frac{\chi(a) + \chi(p)}{p} \ll 1.$$

Comparing this to the conclusion of [Proposition 21.2](#), we deduce that $\chi(a) = -1$ and we obtain the result. \square

Proposition 21.5 *If the hypotheses of [Corollary 21.3](#) hold for $x = q^A$ where A is sufficiently large, and if $\chi(a) = 1$ then there are primes $\leq x$ that are $\equiv a \pmod{q}$.*

BINARY QUADRATIC FORMS

22.1 The basic theory

Suppose that a, b, c are integers for which $b^2 - 4ac = d$ and define the *binary quadratic form* $F(x, y) := ax^2 + bxy + cy^2$, which has *discriminant* d . We will study the values $am^2 + bmn + cn^2$ when m and n are integers, and in particular the prime values. We say that F *represents* the integer N if there exists integers m, n such that $F(m, n) = N$.

Exercise 22.1 *Prove that if there is an invertible linear transformation (over \mathbb{Z}) between two binary quadratic forms then they represent the same integers; indeed there is a 1-1 correspondence between representations. Show also that the two forms have the same discriminant. These results suggests that we study the equivalence classes of binary quadratic forms of a given discriminant.*

Now $d = b^2 - 4ac \equiv b^2 \equiv 0$ or $1 \pmod{4}$. For such integers d there is always at least one binary quadratic form of discriminant d :

$$\begin{aligned} x^2 - (d/4)y^2 & \quad \text{when } d \equiv 0 \pmod{4} \\ x^2 + xy - ((d-1)/4)y^2 & \quad \text{when } d \equiv 1 \pmod{4}. \end{aligned}$$

The key result is that there are only finitely many equivalence classes of binary quadratic forms of each discriminant d , and we denote this quantity by $h(d)$. We now prove this when $d < 0$: The idea is that every binary quadratic form of negative discriminant is equivalent to a *semi-reduced form*, one for which $|b| \leq a \leq c$. In that case $|d| = 4ac - b^2 \geq 4a^2 - a^2 = 3a^2$ and so $a \leq \sqrt{|d|/3}$, and so for a given d there are only finitely many possibilities since $|b| \leq a \leq \sqrt{|d|/3}$ and once these are chosen $c = (b^2 - d)/4a$. Gauss's proof that every form is equivalent to a semi-reduced form goes as follows: If $c < a$ then the transformation $(x, y) \rightarrow (y, -x)$ swaps a and c ; hence we may assume that $a \leq c$. If $|b| < a$ then let $B \equiv b \pmod{2a}$ with $-a < B \leq a$, so that there exists an integer k with $B = b + ka$. The transformation $(x, y) \rightarrow (x + ky, y)$ changes F to $ax^2 + Bxy + Cy^2$ where $C = (B^2 - d)/4a$. Either this is semi-reduced or $C < a$ in which case we repeat the above process. If we need to then we see that our new pair a, C is smaller than our old pair a, c , so the algorithm must terminate in finitely many steps.

Before we count representations, let's note that given one representation, one can often find a second trivially (the *automorphs*), for example $F(m, n) = F(-m, -n)$.

Exercise 22.2 Show that the only other automorphs when $d < 0$ occur for $d = -3$ and $d = -4$. We denote the number of automorphs by $w(d)$. Deduce that $w(-4) = 4$, $w(-3) = 6$ and $w(d) = 2$ for all other negative discriminants d .

The key result in the theory of binary quadratic forms is to show that there is a 1-1 correspondence between the inequivalent representations of a given integer N by the set of binary quadratic forms of discriminant d , and the number of solutions to $x^2 \equiv d \pmod{4n}$. Once this is established one knows that the total number of representations is

$$R(N) = w(d) \sum_{k|N} \left(\frac{d}{k} \right).$$

Dirichlet had the idea to simply sum $R(N)$ over all $N \leq x$ since the sum equals the total number of values up to x of the inequivalent binary quadratic forms F of discriminant $d < 0$.

Exercise 22.3 Show that the number of pairs m, n of integers for which $am^2 + bmn + cn^2 \leq x$ can be approximated by the area of this shape, with an error term proportional to the perimeter, that is $4\pi x/\sqrt{d} + O(\sqrt{x})$.

Hence

$$\sum_{N \leq x} R(N) = h(d) \left(2\pi \frac{x}{\sqrt{d}} + O(\sqrt{x}) \right).$$

On the other hand

$$\sum_{N \leq x} R(N) = w(d) \sum_{N \leq x} \sum_{k|N} \left(\frac{d}{k} \right) = w(d) \sum_{ab \leq x} \left(\frac{d}{a} \right).$$

The main term comes from summing over $a \leq \sqrt{x}$, since the number of b is $x/a + O(1)$, to obtain

$$\begin{aligned} \sum_{a \leq \sqrt{x}} \left(\frac{d}{a} \right) \frac{x}{a} + O(\sqrt{x}) &= x \sum_{a \geq 1} \left(\frac{d}{a} \right) \frac{1}{a} + O \left(x \sum_{a \geq \sqrt{x}} \left(\frac{d}{a} \right) \frac{1}{a} + \sqrt{x} \right) \\ &= xL(1, (d/.)) + O(d\sqrt{x}), \end{aligned}$$

by partial summation since the sum of (d/a) , over any interval of length $4d$, equals 0. For the same reason

$$\sum_{b \leq \sqrt{x}} \sum_{\sqrt{x} < a < x/b} \left(\frac{d}{a} \right) \leq 4d\sqrt{x}.$$

Dividing through by x , and then letting $x \rightarrow \infty$, we obtain *Dirichlet's class number formula*:

$$h(d) = w(d) \frac{\sqrt{d}}{2\pi} L(1, (d/.)), \quad \text{when } d < 0.$$

When $d < 0$ the binary quadratic forms are positive definite and so can only take each value finitely often. When $d > 0$ there is no obvious limitation on how often a given integer can be represented, and indeed integers can be represented infinitely often. The reason for this is that there are infinitely many automorphs for each d . Fortunately the automorphs can all be generated by two transformations: $F(m, n) = F(-m, -n)$ and $F(m, n) = F(\alpha m + \beta n, \gamma m + \delta n)$ for some linear transformation of infinite order. After taking due consideration this leads to

$$h(d)R_d = \sqrt{d} L(1, (d/.)), \quad \text{when } d > 0,$$

for some constant R_d . In fact $R_d = \log \epsilon_d$ where $\epsilon_d = x + y\sqrt{d}$ corresponds to the smallest solution with $x, y > 0$ to $x^2 - dy^2 = 4$.

22.2 Prime values

Let us suppose that χ is induced from the quadratic character $(./D)$ so that D must be squarefree. We re-write this as $(d/.) = (./D)$ where $d = (-1)^{(D-1)/4}D$, so that $d \equiv 1 \pmod{4}$. To begin with we look at divisibility. For a binary quadratic form $ax^2 + bxy + cy^2$, we know that $(a, b, c)^2 | d$, which is squarefree, and so $(a, b, c) = 1$. Also note that $(m, n)^2$ divides $am^2 + bmn + cn^2$, so we proceed by replacing m by $m/(m, n)$, and n by $n/(m, n)$, and hence we may assume that m and n are coprime.

We now show that if odd prime p divides $am^2 + bmn + cn^2$ then $(d/p) = 0$ or 1. If p divides n then $0 \equiv am^2 + bmn + cn^2 \equiv am^2 \pmod{p}$ and so p divides a as $(m, n) = 1$. Therefore $d = b^2 - 4ac \equiv b^2 \pmod{p}$ and hence $(d/p) = 0$ or 1. If $m \nmid n$ then $4ap$ divides $4a(am^2 + bmn + cn^2) = (2am + bn)^2 - dn^2$, and so

$$\left(\frac{2am + bn}{p}\right)^2 = \left(\frac{(2am + bn)^2}{p}\right) = \left(\frac{dn^2}{p}\right) = \left(\frac{d}{p}\right) \left(\frac{n}{p}\right)^2 = \left(\frac{d}{p}\right),$$

implying that $(d/p) = 0$ or 1.

Exercise 22.1 Show that if p is an odd prime then

$$1 - \frac{1}{p^2} \#\{m, n \pmod{p} : am^2 + bmn + cn^2 \equiv 0 \pmod{p}\} = \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right).$$

We wish to show that $am^2 + bmn + cn^2$ takes on many prime values, that is not many composite values. If $am^2 + bmn + cn^2 \leq x$ is composite then it certainly has a prime factor $\leq \sqrt{x}$ so we will count the number of such values with no small prime factor. To explain our method in an intuitive fashion we will proceed assuming that $d < 0 < a$ (so that $am^2 + bmn + cn^2$ only takes non-negative values); when we give the actual proof we will use sieve weights that are easier to work with but more difficult to understand.

The small sieve shows us that if $x = y^u$ then for $M = \prod_{p \leq y} p$

$$\begin{aligned} & \#\{m, n \in \mathbb{Z} : N := am^2 + bmn + cn^2 \leq x, (N, M) = 1\} = \\ & = \{1 + O(u^{-u})\} \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right) X + O(\sqrt{X}), \end{aligned}$$

where $X := \#\{m, n \in \mathbb{Z} : N := am^2 + bmn + cn^2 \leq x\} = \pi x / \sqrt{d} + O(\sqrt{x})$.

We will use this estimate when y is a small power of x , and then obtain a lower bound by subtracting the number of such integers divisible by a prime in $(y, x^{1/2}]$.

The trick is that if prime ℓ is in this range with $(d/\ell) = 1$ then ℓ can be written as the value of a binary quadratic form of discriminant d in one of two (essentially different) ways, and then N/ℓ similarly. Hence to count the number of such N/ℓ we can use the same estimate, though in this case we use the above simply as an upper bound, particularly as $N/\ell \geq \sqrt{x}$. Hence

$$\begin{aligned} & \#\{m, n \in \mathbb{Z} : N := am^2 + bmn + cn^2 \leq x, (N, M) = 1, \ell | N\} \\ & \ll \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right) \frac{X}{\ell}. \end{aligned}$$

Hence in total, we have

$$\begin{aligned} & \#\{m, n \in \mathbb{Z} : N := am^2 + bmn + cn^2 \leq x, N \text{ is prime}\} \\ & \gg \left\{1 - \sum_{\substack{y < \ell \leq x^{1/2} \\ (d/\ell)=1}} \frac{2}{\ell} - \epsilon\right\} \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{(d/p)}{p}\right) X, \end{aligned}$$

where say $u \gg 1/\epsilon$.

From the first equation in the proof of Corollary [21.3](#) ^{[LinkSiegCond](#)} we deduce that that if there are no primes $\equiv a \pmod{q}$ up to x then

$$\sum_{\substack{y < \ell \leq x^{1/2} \\ (d/\ell)=1}} \frac{1}{\ell} \ll \left(\frac{\log Q}{\log y}\right)^{1/5};$$

hence if $x = q^L$ where L/u is sufficiently large then $\sum_{y \leq p \leq x^{1/2}} (1 + (d/p))/p \leq 1/2$; and so, from the above, we know that there are many prime values of our binary quadratic form.

22.3 Finishing the proof of Linnik's Theorem

To obtain a complete proof without proving all sorts of results about binary quadratic forms (and of positive and negative discriminant), we can proceed working (more-or-less) only with the character χ , though based on what we know about binary quadratic forms. The extra observation to add to the analysis of the previous section is that we should work with the values of all binary quadratic forms of discriminant d , simultaneously, since Gauss showed that the total number of “inequivalent” representations of n is then $\sum_{m|n} \chi(m)$. Hence let $w(n) = \sum_{m|n} \chi(m)$, so that $w(p) = 1 + \chi(p)$. We define

$$A(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} w(n).$$

Exercise 22.2 Show that if f is totally multiplicative and $g = 1 * f$ then

$$g(mn) = \sum_{d|(m,n)} \mu(d) f(d) g(m/d) g(n/d).$$

As usual $A_m(x; q, a) := \sum_n w(n)$ where the sum is over $n \leq x$ with $M|n$ and $n \equiv a \pmod{q}$. Hence, using the exercise with $f = \chi$, if $(m, q) = 1$ then

$$\begin{aligned} A_m(x; q, a) &= \sum_{\substack{N \leq x/m \\ N \equiv a/m \pmod{q}}} w(mN) = \sum_{\substack{N \leq x/m \\ N \equiv a/m \pmod{q}}} \sum_{d|(m,N)} \mu(d) \chi(d) w(m/d) w(N/d) \\ &= \sum_{d|m} \mu(d) \chi(d) w(m/d) \sum_{\substack{N \leq x/m \\ N \equiv a/m \pmod{q} \\ d|N}} w(N/d) \\ &= \sum_{d|m} \mu(d) \chi(d) w(m/d) A(x/md; q, a/md). \end{aligned}$$

Now $w(n) = \sum_{m|n, m \leq \sqrt{n}} \chi(m) + \sum_{m|n, m < \sqrt{n}} \chi(n/m)$. Therefore

$$\begin{aligned} A(x; q, a) &= \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \left(\sum_{m|n, m \leq \sqrt{n}} \chi(m) + \sum_{m|n, m < \sqrt{n}} \chi(n/m) \right) \\ &= \sum_{\substack{m \leq x \\ (m,q)=1}} \chi(m) \sum_{\substack{m^2 \leq n \leq x \\ n \equiv a \pmod{q} \\ m|n}} 1 + \chi(a) \sum_{\substack{m \leq x \\ (m,q)=1}} \bar{\chi}(m) \sum_{\substack{m^2 < n \leq x \\ n \equiv a \pmod{q} \\ m|n}} 1 \\ &= \frac{1}{q} \sum_{\substack{m \leq \sqrt{x} \\ (m,q)=1}} (\chi(m) + \chi(a) \bar{\chi}(m)) \left(\frac{x}{m} - m + O(1) \right). \end{aligned}$$

Now $\sum_{m \pmod{q}} (kq+m)\chi(m) = \sum_{m \pmod{q}} m\chi(m) \ll q^{3/2}$. Moreover $\sum_{m \leq M} \chi(m)/m = L(1, \chi) + O(q/M)$, and so $A(x; q, a) = (1 + \chi(a))L(1, \chi)x/q + O(q\sqrt{x})$ since χ is real. Hence, if m is squarefree and coprime to q , and $\chi(a) = 1$ then

$$\begin{aligned} A_m(x; q, a) &= L(1, \chi) \frac{x}{mq} \sum_{d|m} \frac{\mu(d)\chi(d)}{d} w(m/d)(1 + \chi(a/md)) + O\left(\frac{q}{m} \sum_{d|m} w(m/d)\sqrt{mx/d}\right) \\ &= 2L(1, \chi) \frac{x}{mq} \prod_{p|m} \left(1 + \chi(p) \left(1 - \frac{1}{p}\right)\right) + O\left(\frac{q}{m} \sqrt{x} \prod_{p|m} (1 + (1 + \chi(p))\sqrt{p})\right). \end{aligned}$$

Hence if we write $A_m(x; q, a) = (g(m)/m)A(x; q, a) + r_m(x; q, a)$ then g is a multiplicative function with $g(p) = 1 + \chi(p) \left(1 - \frac{1}{p}\right)$ and

$$\sum_{m \leq M} |r_m(x; q, a)| \ll q\sqrt{Mx} \sum_{m \leq M} \frac{1}{m} \prod_{p|m} (1 + \chi(p) + 1/\sqrt{p}) \ll q\sqrt{Mx} \log^2 M.$$

Sieving Lemma

Lemma 22.4 (Standard sieving lemma) Suppose that a_n are a set of real weights supported on a finite set of integers n . Let $A(x) = \sum_n a_n$ and suppose that there exists a non-negative multiplicative function $g(\cdot)$ such that

$$A_m(x) = \sum_{n: m|n} a_n = \frac{g(m)}{m} A(x) + r_m(x)$$

for all squarefree m , for which there exists $K, \kappa > 0$ such that

$$\prod_{y < p \leq z} \left(1 - \frac{g(p)}{p}\right)^{-1} \leq K \left(\frac{\log z}{\log y}\right)^\kappa,$$

for all $2 \leq y < z \leq x$. Let P be a given set of primes, and $P(z)$ be the product of the elements of P that are $\leq z$. Then

$$\sum_{\substack{n \leq x \\ (n, P(z))=1}} a_n = \{1 + O_{K, \kappa}(e^{-u})\} \prod_{\substack{p \in P \\ p \leq z}} \left(1 - \frac{g(p)}{p}\right) \sum_{n \leq x} a_n + O\left(\sum_{\substack{m|P(z) \\ m \leq z^u}} |r_m(x)|\right)$$

Above we let $x \gg \frac{q^5}{L(1, \chi)^2}$ and $z = x^\epsilon$, with u large and ϵu small, and then apply Lemma 22.4 with $\kappa = 2$ to obtain

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{q} \\ (n, P(z))=1}} w(n) = \{1 + O(e^{-u})\} \prod_{p \leq z} \left(1 - \frac{1}{p}\right) \left(1 - \frac{\chi(p)}{p}\right) A(x; q, a).$$

Now for each primes $p, z < p \leq \sqrt{x}$ we must remove from the left side those n divisible by p . For each prime p write $n = Np$ and so we get an upper bound from

$w(p)$ times the sum of $w(N)$ over $N \leq x/p$, $N \equiv a/p \pmod{q}$ and $(N, P(z)) = 1$. Since $x/p \geq \sqrt{x}$, we can get an upper bound from the same estimate, of the right side with x/p in place of x ; that is divided by p . Hence we deduce that

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod{q} \\ p \text{ prime}}} w(p) = \left\{ 1 + O \left(e^{-u} + \sum_{z < p \leq \sqrt{x}} \frac{1 + \chi(p)}{p} \right) \right\} \prod_{p \leq z} \left(1 - \frac{1}{p} \right) \left(1 - \frac{\chi(p)}{p} \right) A(x; q, a).$$

In the last section we explained that $\sum_{z < p \leq \sqrt{x}} \frac{1 + \chi(p)}{p} \ll \left(\frac{\log Q}{\log z} \right)^{1/5}$, and hence we have proved that

$$\pi(x; q, a) = \{1 + o_{L \rightarrow \infty}(1)\} \prod_{p \leq z} \left(1 - \frac{1}{p} \right) \left(1 - \frac{\chi(p)}{p} \right) L(1, \chi) \frac{x}{q},$$

where $x = q^L$ and $z = q^{\sqrt{L}}$.

EXPONENTIAL SUMS

Given a real number α we consider rational approximations a/q with $(a, q) = 1$ such that

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}. \quad (23.1) \quad \boxed{\text{DiApprox}}$$

23.1 Technical Lemmas

We will work with exponential sums.

Exercise 23.1 Define $e(t) := e^{2i\pi t}$. Let $\|t\|$ be the distance from t to the nearest integer. Prove that for any real β we have

$$\left| \frac{1}{2M} \sum_{A-M \leq m < A+M} e(\beta m) \right| \ll \min \left\{ 1, \frac{1}{M\|\beta\|} \right\};$$

and

$$\left| \frac{1}{2M} \sum_{A-M \leq m < A+M} e(\beta m) \left(1 - \frac{|m-A|}{M} \right) \right| \ll \min \left\{ 1, \frac{1}{M\|\beta\|} \right\}^2.$$

We begin by proving the following:

$$\sum_{n \leq N} \min \left\{ 1, \frac{1}{M\|n\alpha + \gamma\|} \right\} \ll \begin{cases} \frac{q}{M} \log \frac{2MN}{q} & \text{if } M, N \leq q \\ 1 + \frac{q}{M} \log N & \text{if } N \leq q < M \\ \frac{N}{M} \log 2M & \text{if } M \leq q < N \\ \frac{N}{q} + \frac{N}{M} \log M & \text{if } q < M, N \end{cases} \quad (23.2)$$

$$\ll \left(1 + \frac{N}{q} + \frac{q}{M} + \frac{N}{M} \right) \log(2MN/q) \quad (23.3) \quad \boxed{\text{expsum1}}$$

if $q \leq MN$ (and if $q > MN$ the case $\alpha = 1/q$ yields the trivial bound N). In each case there are examples for which these bounds can not be improved. We proceed by writing $\alpha = a/q + \beta$ and $n = mq + r$ with $-q/2 < r \leq q/2$ so that $n\alpha = mq\alpha + ra/q + r\beta$ with $|r\beta| \leq (q/2)(1/q^2) = 1/2q$. Hence, for each m these points are well distributed around the circle (in that for each b , $0 \leq b \leq q-1$ there is at most one such point in the arc of length $1/q$ centered on $mq\alpha + b/q \pmod{1}$). Hence in such an interval our sum is

$$\ll \sum_{\ell=0}^{\min\{N, q/2\}} \min \left\{ 1, \frac{1}{M\|\ell/q\|} \right\} \ll 1 + \frac{q}{M} + \frac{q}{M} \log \left(\frac{\min\{N, q\}}{1 + \frac{q}{M}} \right),$$

and summing up over such intervals we obtain the result. Since

$$\sum_{\ell=0}^{\min\{N, q/2\}} \min \left\{ 1, \frac{1}{M |\ell/q|} \right\}^2 \ll 1 + \frac{q}{M},$$

we also deduce that

$$\sum_{n \leq N} \min \left\{ 1, \frac{1}{M \|n\alpha\|} \right\}^2 \ll \left(1 + \frac{N}{q} \right) \left(1 + \frac{q}{M} \right) = 1 + \frac{N}{q} + \frac{q}{M} + \frac{N}{M}. \quad (23.4) \quad \boxed{\text{expsum2}}$$

This usually wins a $\log(2MN/q)$ over the first moment, which can be important.

Now we wish to do the same for prime differences. That is, instead of summing over $n \leq N$, we sum over $p, p' \leq N$ and let $n = p - p'$. We get $\pi(N) \sim N/\log N$ copies of 0 and the number of prime pairs $p, p+n$ is $\ll (n/\phi(n))N/\log^2 N$. Now $\frac{n}{\phi(n)} \ll \sum_{m|n, m < \sqrt{n}} \mu^2(m)/m$. Hence

$$\begin{aligned} \sum_{p, p' \leq N} \min \left\{ 1, \frac{1}{M \|(p-p')\alpha\|} \right\}^2 &\ll \frac{N}{\log N} + \frac{N}{\log^2 N} \sum_{n \leq N} \min \left\{ 1, \frac{1}{M \|n\alpha\|} \right\}^2 \sum_{\substack{m|n \\ m < \sqrt{n}}} \frac{\mu^2(m)}{m} \\ &\ll \frac{N}{\log N} + \frac{N}{\log^2 N} \sum_{m < \sqrt{N}} \frac{\mu^2(m)}{m} \sum_{m < k \leq N/m} \min \left\{ 1, \frac{1}{M \|km\alpha\|} \right\}^2 \end{aligned}$$

writing $n = mk$. Now if $\alpha = a/q$ then $m\alpha = b/r \pmod{1}$ where $r = q/(q, m)$. Hence by $\boxed{\text{expsum2}}$ this sum is

$$\ll \sum_{m < \sqrt{N}} \frac{\mu^2(m)}{m} \left(1 + \frac{N(m, q)}{mq} + \frac{q}{M(m, q)} + \frac{N}{mM} \right) \ll 1 + \frac{N}{\phi(q)} + \frac{q}{M} \log N + \frac{N}{M};$$

and so we deduce that

$$\sum_{d, d' \leq N} \min \left\{ 1, \frac{1}{M \|(d-d')\alpha\|} \right\}^2 \Lambda(d) \Lambda(d') \ll \frac{N^2}{\phi(q)} + \frac{N^2}{M} + \left(1 + \frac{q}{M} \right) N \log N. \quad (23.5) \quad \boxed{\text{expsum3}}$$

23.2 The bound of Montgomery and Vaughan

We begin by proving Montgomery and Vaughan's celebrated result that if $\boxed{\text{DiApprox}}$ holds then

$$\sum_{n \leq x} f(n) e(\alpha n) \ll \frac{x}{\sqrt{\phi(q)}} + \frac{x}{\log x} + \sqrt{qx \log x}. \quad (23.6) \quad \boxed{\text{MV1}}$$

(The last term can be removed if $q \leq x/(\log x)^3$.)

Montgomery and Vaughan proceeded by multiplying through by $\log x$; converting this to $\log n$ brings in an error of $O(x)$. Then writing $\log n = \sum_{d|n} \Lambda(d)$, we find ourselves with the sum

$$\sum_{dm \leq x} f(dm) e(\alpha dm) \Lambda(d).$$

We break this into intervals (assuming f is totally multiplicative for simplicity) to get sums of the form

$$\leq \sum_m |f(m)| \left| \sum_d f(d) e(\alpha dm) \Lambda(d) \right|$$

and Cauchy, so that the square is

$$\begin{aligned} &\leq \sum_m |f(m)|^2 \cdot \sum_m \left| \sum_d f(d) e(\alpha dm) \Lambda(d) \right|^2 \\ &\ll M \sum_{d_1, d_2} \Lambda(d_1) \Lambda(d_2) f(d_1) \overline{f(d_2)} \sum_m e(\alpha(d_1 - d_2)m) \\ &\ll M \sum_{d_1, d_2} \Lambda(d_1) \Lambda(d_2) \max \left\{ M, \frac{1}{\|\alpha(d_1 - d_2)\|} \right\}. \end{aligned}$$

This can be improved by a minor modification. If the range for m is $A - M/2 < m < A + M/2$ then we bound the top line above by multiplying the m th term by $2(1 - |m - A|/M)$ (which is ≥ 1 in this range), and then extend the sum to all m in the range $A - M < m < A + M$. By the second part of the exercise we then obtain the bound

$$\ll M^2 \sum_{d_1, d_2 \leq D} \Lambda(d_1) \Lambda(d_2) \max \left\{ 1, \frac{1}{M \|\alpha(d_1 - d_2)\|} \right\}^2 \ll \frac{x^2}{\phi(q)} + \frac{x^2}{M} + (M+q)x \log(x/M)$$

by ^(expsum3)(23.5), as $MD \leq x$. We take the square root (since we Cauchy'd) and sum this up over $1 \leq M = 2^i \leq x$ to obtain a total upper bound

$$\ll \frac{x \log x}{\sqrt{\phi(q)}} + x + q^{1/2} x^{1/2} \log^{3/2} x,$$

from which ^(MV1)(23.6) follows.

23.3 How good is this bound?

If we let $f = \chi$, a character mod q with $\alpha = 1/q$, and x a multiple of q , then

$$\sum_{n \leq x} \chi(n) e(n/q) = \frac{x}{q} \sum_{n \leq q} \chi(n) e(n/q) = \frac{g(\chi)}{q} x,$$

where $g(\chi)$ is the Gauss sum. We saw earlier that $|g(\chi)| = \sqrt{q}$, and so if q is prime then this is $\gg x/\sqrt{\phi(q)}$. Hence the first term in ^(MV1)(23.6) needs to be there.

Given the values of $f(p)$ for $p \leq x/2$ let $\sum_n f(n)e(\alpha n) = re(\theta)$ with $r \in \mathbb{R}_{\geq 0}$, where the sum is over all $n \leq x$ other than the primes in $(x/2, x]$. Now consider the multiplicative function f where $f(p) = e(\theta - \alpha p)$ for all primes p , $x/2 < p \leq x$. Then $\sum_{n \leq x} f(n)e(\alpha n) = (r + \pi(x) - \pi(x/2))e(\theta)$; in particular $|\sum_{n \leq x} f(n)e(\alpha n)| \gg x/\log x$. Hence the second term in (23.6) needs to be there.

In both cases we do not need to take f to be exactly the functions described, f should just be pretentious in that way. In the latter case one can most easily avoid such problems by removing all integers that have some large prime factor: As shown by La Bréteche one has, for $q < y + x/e^{2\sqrt{\log x}}$,

$$\sum_{\substack{n \leq x \\ P(n) \leq y}} f(n)e(\alpha n) \ll \left(\sqrt{xy} + \frac{x}{\sqrt{q}} \right) \log^2 x + \frac{x}{e^{\sqrt{\log x}}}. \quad (23.7) \quad \boxed{\text{LaBret1}}$$

In this case we do not multiply through by $\log x$ but rather write each $n = dm$ where $(d, m) = 1$ and d is a power of the largest prime dividing n . Hence

$$\sum_{\substack{n \leq x \\ P(n) \leq y}} f(n)e(\alpha n) = \sum_{\substack{d=p^k \\ p \text{ prime}, \leq y}} f(d) \sum_{\substack{m \leq x/d \\ P(m) < p}} f(m)e(d\alpha m).$$

Taking absolute values, we first deal with the term where d is a prime power. This gives

$$\leq \sum_{\substack{d=p^k, k \geq 2 \\ p \text{ prime}}} \psi(x/p^k, p).$$

Using our estimate (*) for $\psi(x, y)$ it is an exercise to show that this is $\ll x/\exp(\{2 + o(1)\}\sqrt{\log x \log \log x})$ then main contribution coming from p^2 values around $\exp(\{1 + o(1)\}\sqrt{\log x \log \log x})$. We shall similarly approach those terms where $d = p \leq T$ is “small”: they contribute

$$\leq \sum_{\substack{p \leq T \\ p \text{ prime}}} \psi(x/p, p) \ll x/\exp(\{\sqrt{2} + o(1)\}\sqrt{\log x \log \log x})$$

where $T = \exp(\sqrt{\frac{1}{2} \log x \log \log x})$.

To bound the remaining terms we forget that d should only be prime, and arrive at

$$\leq \sum_{T < d \leq y} \left| \sum_{\substack{m \leq x/d \\ P(m) < d}} f(m)e(d\alpha m) \right|.$$

Cauchyng for the terms with $T < d \leq D$ and $m \asymp M$ where $DM \ll x$ we obtain

$$\begin{aligned}
&\ll D \sum_{m, m' \leq 2M} f(m) \overline{f(m')} \sum_{P(m), P(m') < d \leq x / \max\{m, m'\}} e(d\alpha(m - m')) \\
&\ll D \sum_{m, m' \leq 2M} \min \left\{ \frac{x}{\max\{m, m'\}}, \frac{1}{\|\alpha(m - m')\|} \right\}.
\end{aligned}$$

The $m = m'$ terms yield $\ll Dx \log M$. Otherwise let $k = \min\{m, m'\}$ and $k + j = \max\{m, m'\}$, so our sum becomes

$$\ll D \sum_{j, k \leq 2M} \min \left\{ \frac{x}{k + j}, \frac{1}{\|\alpha j\|} \right\}$$

For each k we partition the j -values into intervals $[1, k]$ and $[2^i k, 2^{i+1} k)$ for $i = 0, 1, 2, \dots, I$ where I is minimal such that $2^I k > 2M$, and then apply (23.3) assuming $x \geq q$. We obtain

$$\begin{aligned}
&\ll D \sum_{k \leq 2M} \left(\frac{x}{k} + \left(\frac{x}{q} + q \right) \log(M/k) + M \right) \log(2x/q) \\
&\ll \left(xD \log 2M + \left(\frac{x}{q} + q \right) DM + DM^2 \right) \log(2x/q).
\end{aligned}$$

Now we take the square root and sum this over all $M = 2^j \leq X/T$ with $D = \min\{y, x/M\}$ to obtain (23.7).

23.4 When f is pretentious

We have seen that Montgomery and Vaughan's bound can be considerably improved if one removes the effect of the large prime factors, unless f is χ -pretentious for some character χ of modulus q . Here we will be interested in obtaining better estimates in this special case.

We deduce

$$\log x \sum_{n \leq x} \sum_{(\text{mod } q)} f(n) e(\alpha n) = \sum_{n \leq x} f(n) e(\alpha n) \log_{[Q, x/Q]} n + O(\rho(f)x \log Q).$$

(23.8)

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from Lemma 17.6. If $(b, d) = 1$ then

$$e\left(\frac{b}{d}\right) = \sum_{j=0}^{d-1} e\left(\frac{j}{d}\right) \cdot \frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \bar{\chi}(b) \chi(j) = \frac{1}{\phi(d)} \sum_{\chi \pmod{d}} \bar{\chi}(b) g(\chi);$$

ExpSums2Chars

therefore if $(a, q) = 1$ then, writing $n = mq/d$ when $(n, q) = q/d$ (so that $(m, d) = 1$),

$$\begin{aligned}
\sum_{n \leq x} f(n) e\left(\frac{an}{q}\right) &= \sum_{d|q} \sum_{\substack{m \leq x/(q/d) \\ (m,d)=1}} f(mq/d) e\left(\frac{am}{d}\right) \\
&= \sum_{d|q} \frac{f(q/d)}{\phi(d)} \sum_{\substack{\psi \pmod{d} \\ \psi \text{ induces some } \chi_j, 1 \leq j \leq k}} \bar{\psi}(a) g(\psi) \sum_{m \leq x/(q/d)} f(m) \bar{\psi}(m)
\end{aligned}$$

Proceeding as in the proof of Theorem [19.1](#) ^{|FnsInAPs} we have

$$\begin{aligned}
&\left| \sum_{n \leq x} f(n) e\left(\frac{an}{q}\right) - \sum_{d|q} \frac{f(q/d)}{\phi(d)} \sum_{\substack{\psi \pmod{d} \\ \psi \text{ induces some } \chi_j, 1 \leq j \leq k}} \bar{\psi}(a) g(\psi) \sum_{m \leq x/(q/d)} f(m) \bar{\psi}(m) \right| \\
&\leq \sum_{d|q} \frac{\sqrt{d}}{\phi(d)} \sum_{\substack{\psi \pmod{d} \\ \psi \text{ induces some } \chi_j, k+1 \leq j \leq K}} \left| \sum_{m \leq x/(q/d)} f(m) \bar{\psi}(m) \right| \\
&\quad + \sum_{d|q} \frac{\sqrt{d}}{\phi(d)} \sum_{\substack{\psi \pmod{d} \\ \psi \text{ induces } \chi_j, j > K}} \left| \sum_{m \leq x/(q/d)} f(m) \bar{\psi}(m) \frac{\log_{[Q, x/Q]} m}{\log x} \right|,
\end{aligned}$$

since all the prime factors of q/d are $< Q$. By Proposition [19.2](#) ^{|LinearPLS} $x \equiv Q^A$, the second error term is

$$\ll \sum_{d|q} \frac{\sqrt{d}}{\phi(d)} \frac{\rho'(f)}{A^{1-\epsilon}} \frac{x}{q/d} \ll \rho'(f) \frac{x}{\sqrt{q}} \prod_{p|q} \left(1 + \frac{1}{\sqrt{p}} + \frac{2}{p}\right) \cdot \frac{1}{A^{1-\epsilon}}.$$

For the first error term we Cauchy it, in two parts, and proceed as in the proof of Theorem [19.1](#) ^{|FnsInAPs} to obtain

$$\ll \rho'(f) \frac{x}{\sqrt{q}} \prod_{p|q} \left(1 + \frac{1}{\sqrt{p}} + \frac{2}{p}\right) \cdot \frac{\log A}{A^{1-\frac{1}{\sqrt{k+1}}}}$$

which dominates.

We now deal with the “main terms”: Suppose that the primitive character $\psi \pmod{r}$ induces some $\chi_j \pmod{q}$. If $\chi \pmod{kr}$ is induced from $\psi \pmod{r}$ then $g(\chi) = \mu(k) \psi(k) g(\psi)$, so we may assume $(k, r) = 1$ else $g(\chi) = 0$. Therefore the total contribution is

$$= \bar{\psi}(a) g(\psi) \sum_{\substack{k|q/r \\ (k,r)=1}} \frac{f(q/kr)}{\phi(kr)} \mu(k) \psi(k) \sum_{\substack{m \leq kr x/q \\ (m,k)=1}} f(m) \bar{\psi}(m).$$

By [\(14.2\)](#) ^{|FSieved2} the error terms add up to

$$\begin{aligned}
&\ll \frac{\sqrt{r}}{\phi(r)} \sum_{\substack{k|q/r \\ (k,r)=1}} \frac{\mu(k)^2}{\phi(k)} \frac{k}{\phi(k)} \cdot \frac{krx}{q} \left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}} + \left(\frac{\log q/r}{\log x} \right)^{1-2/\pi} \log \left(\frac{\log x}{\log q/r} \right) \right) \\
&\ll \frac{2^{\omega(q/r)} \sqrt{r}}{\phi(q)} x \left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}} + \left(\frac{\log q/r}{\log x} \right)^{1-2/\pi} \log \left(\frac{\log x}{\log q/r} \right) \right) \ll \frac{\sqrt{q}}{\phi(q)} x \frac{\log \log x}{(\log x)^{2-\sqrt{3}}},
\end{aligned}$$

as $1 - 2/\pi > 2 - \sqrt{3}$, and since the maximum is attained when $r \asymp q$. The main terms add up to

$$\bar{\psi}(a)g(\psi) \sum_{\substack{k|q/r \\ (k,r)=1}} \frac{f(q/kr)}{\phi(kr)} \mu(k)\psi(k) \frac{1}{(q/kr)^{1+it}} \prod_{p|k} \left(1 - \frac{F(p)}{p^{1+it}} \right) \sum_{n \leq x} F(n)$$

with $F(n) = f(n)\bar{\psi}(n)$; and this equals

$$\Theta(f, \psi, t; q) \frac{\bar{\psi}(a)g(\psi)}{\phi(q)} \sum_{n \leq x} F(n)$$

where

$$\Theta(f, \psi, t; q) := \prod_{\substack{p^e \| q/r \\ p \nmid r}} \psi(p)^e ((F(p)p^{-it})^e - (F(p)p^{-it})^{e-1}) \prod_{\substack{p^e \| q/r \\ p \nmid r}} (f(p)/p^{it})^e.$$

Hence in total we have

$$\begin{aligned}
\sum_{n \leq x} f(n) e\left(\frac{an}{q}\right) &= \sum_{j=1}^k \Theta(f, \psi_j, t; q) \frac{\bar{\psi}_j(a)g(\psi_j)}{\phi(q)} \sum_{n \leq x} f(n) \bar{\psi}_j(n) \\
&\quad + O\left(\frac{\sqrt{q}}{\phi(q)} x \left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}} + \rho'(f) \prod_{p|q} \left(1 + \frac{1}{\sqrt{p}} + \frac{1}{p} \right) \cdot \frac{\log A}{A^{1-\frac{1}{\sqrt{k}+1}}} \right) \right).
\end{aligned}$$

In particular if we have $\log q = (\log x)^{o(1)}$ then since $1 - 2/\pi, 1 - 1/\sqrt{2} > 2 - \sqrt{3}$ we deduce that

$$\begin{aligned}
\sum_{n \leq x} f(n) e\left(\frac{an}{q}\right) &- \sum_{j=1}^k \frac{\bar{\psi}_j(a)g(\psi_j)}{\phi(q)} \left\{ \Theta(f, \psi_j, t; q) \sum_{n \leq x} f(n) \bar{\psi}_j(n) + O\left(2^{\omega(q/r)} \frac{x}{(\log x)^{2-\sqrt{3}+o(1)}} \right) \right\} \\
&\ll \frac{\sqrt{q}}{\phi(q)} \frac{x}{(\log x)^{1-\frac{1}{\sqrt{2}}+o(1)}}
\end{aligned}$$

if $\log q \ll (\log \log x)^2$.

In the special case that q is prime we have that $\frac{\phi(q)}{x} \sum_{n \leq x} f(n) e\left(\frac{an}{q}\right)$ equals

$$\left(1 - \frac{f(q)}{q^{it}}\right) \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}} + \frac{\log A}{A^{1-\frac{2}{\pi}}}\right)$$

if some $\psi_j = 1$; plus $\bar{\psi}(a)g(\psi)$ times

$$\frac{f(q)}{q^{it}} \frac{1}{x} \sum_{n \leq x} f(n) \bar{\psi}(n) + O\left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right)$$

for each $\psi = \psi_j$ of conductor q , plus $O(\sqrt{q}(\log A)/A^{1-\frac{1}{\sqrt{k+1}}})$.

THE EXPONENTS η_K

We wish to find the largest exponents $\eta_1 \geq \eta_2 \geq \dots$ that can be used in Proposition [1.1](#); that is if $\chi_1, \chi_2, \dots, \chi_k$ are distinct characters mod q , with $q < Q < x$ then

$$\max_{1 \leq j \leq k} \sum_{Q \leq p \leq x} \frac{1 - \operatorname{Re}((f\bar{\chi}_j)(p)/p^{it_j})}{p} \geq \{1 - \eta_k + o(1)\} \log \left(\frac{\log x}{\log Q} \right) + O_k(1),$$

where the implicit constants are independent of f . Proposition [1.1](#) shows that $\eta_k \leq \frac{1}{\sqrt{k}}$.

It is evident that $\eta_1 = 1$ taking the example $f(n) = 1$ along with $\chi = \chi_0$.

eta2 **Proposition 24.1** *We have $\eta_2 \leq 1/3$. In fact $\eta_2 = 1/3$ assuming that*

$$\pi(x; q, a) = \frac{\pi(x)}{\phi(q)} \left(1 + O \left(\left(\frac{\log q}{\log x} \right)^\epsilon \right) \right)$$

for $x \geq q^2$.

Proof To prove the lower bound, suppose that $\chi \pmod{q}$ has order 3. Define

$$f(p) = \begin{cases} 1 & \text{if } \chi(p) = 1 \\ -1 & \text{if } \chi(p) = \omega \text{ or } \omega^2 \end{cases},$$

so that

$$1 - \operatorname{Re} f(p) \bar{\chi}(p) = 1 - \operatorname{Re} f(p) \chi(p) = \frac{1 - \operatorname{Re} \chi(p)}{3}.$$

Therefore our two sums are

$$\frac{1}{3} \sum_{Q \leq p \leq x} \frac{1 - \operatorname{Re} \chi(p)}{p} = \frac{1}{3} \log \left(\frac{\log x}{\log Q} \right) + O(1),$$

as in the proof of Proposition [1.1](#), and so $\eta_2 \leq 1/3$.

To prove our lower bound it suffices, as in the proof of Proposition [1.1](#), to suitably bound

$$\frac{1}{2} \sum_{Q < p \leq x} \frac{1}{p} \left| \sum_{j=1}^2 \frac{\bar{\chi}_j(p)}{p^{it_j}} \right| = \frac{1}{2} \sum_{Q < p \leq x} \frac{1}{p} |1 + \chi(p)p^{2it}|$$

where $\chi = \chi_1 \bar{\chi}_2$ and $t = (t_2 - t_1)/2$. We will suppose $t > 0$ (if $t < 0$ we simply replace $\chi(p)p^{2it}$ by $\bar{\chi}(p)p^{-2it}$.) By our assumption on $\pi(x; q, a)$, this equals

$$\frac{1}{2\phi(q)} \sum_{\substack{a \pmod{q} \\ (a, q)=1}} \int_{\log Q}^{\log x} \left| 1 + \chi(a)e^{2itv} \right| \frac{dv}{v} + O(1).$$

If χ has order $m > 1$ then there are exactly $\phi(q)/m$ values of $j \pmod{m}$ for which $\chi(a) = e^{2i\pi j/m}$, and so our integral equals

$$\frac{1}{m} \sum_{j=0}^{m-1} \int_{\log Q}^{\log x} \left| \cos(tv + \pi j/m) \right| \frac{dv}{v} = \frac{1}{m} \sum_{j=0}^{m-1} \int_{t \log Q}^{t \log x} \left| \cos(\pi j/m + \theta) \right| \frac{d\theta}{\theta}$$

We handle that part of the integral with $\theta \geq 1$ using the first part of exercise [ex7.2](#). When $\theta < 1$ we substitute $\cos(\pi j/m + \theta) = \cos(\pi j/m) + O(\theta)$. Hence our integral equals

$$\frac{2}{\pi} \int_{\max\{1, t \log Q\}}^{\max\{1, t \log x\}} \frac{d\theta}{\theta} + c_m \int_{\min\{1, t \log Q\}}^{\min\{1, t \log x\}} \frac{d\theta}{\theta} + O(1)$$

where $c_m := \frac{1}{m} \sum_{j=0}^{m-1} |\cos(\pi j/m)|$ equals

$$\frac{1}{m} \sum_{-m/2 < j \leq m/2} \cos(\pi j/m) = \begin{cases} \frac{1}{m \sin(\pi/2m)} & \text{if } m \text{ is odd,} \\ \frac{1}{m \tan(\pi/2m)} & \text{if } m \text{ is even.} \end{cases}$$

The maximum of c_m thus occurs for $m = 3$, and equals $2/3$. Therefore, since $\frac{2}{\pi} < \frac{2}{3}$ our integral is

$$\leq \frac{2}{3} \int_{t \log Q}^{t \log x} \frac{d\theta}{\theta} = \frac{2}{3} \log \left(\frac{\log x}{\log Q} \right) + O(1)$$

as claimed.

When $t = 0$ we can simplify the above proof to obtain $c_m \log \left(\frac{\log x}{\log Q} \right) + O(1)$. □

etak **Proposition 24.2** We have $\frac{1}{2\sqrt{m}} < \eta_m \leq \frac{1}{\sqrt{m}}$ for all $m \geq 1$.

Proof The upper bound was obtained in Proposition [kRepulsion](#). For the lower bound, let ℓ be the smallest prime in $(2m, 4m]$, say $= 2k + 1$.

Suppose that $\chi \pmod{q}$ has prime order ℓ . Define $f(p) = 1$ if $\chi(p) = 1$, and

$$f(p) = \left(\frac{a}{\ell} \right) \text{ when } \chi(p) = e \left(\frac{a}{\ell} \right)$$

whenever $a \not\equiv 0 \pmod{\ell}$. In this case we note that

$$f(p)g_\ell = \sum_{n \pmod{\ell}} \left(\frac{a^{-1}n}{\ell} \right) e\left(\frac{n}{\ell}\right) = \sum_{m \pmod{\ell}} \left(\frac{m}{\ell} \right) e\left(\frac{am}{\ell}\right) = \sum_{m \pmod{\ell}} \left(\frac{m}{\ell} \right) \chi^m(p),$$

where $g_\ell = g\left(\left(\frac{\cdot}{\ell}\right)\right)$, so that

$$f(p) = \frac{1}{g_\ell} \sum_{m=0}^{\ell-1} \left(\frac{m}{\ell} \right) \chi^m(p) + \frac{1}{\ell} \sum_{m=0}^{\ell-1} \chi^m(p),$$

for all p . As in the proof of Proposition [17.7](#) ^{[kRepulsion](#)} we deduce that

$$\sum_{q \leq p \leq x} \frac{\operatorname{Re}(f(p)\bar{\chi}^j(p))}{p} = \left(\operatorname{Re}\left(\frac{1}{g_\ell}\right) \left(\frac{j}{\ell}\right) + \frac{1}{\ell} \right) \log\left(\frac{\log x}{\log q}\right) + O_q(1).$$

Now $g_\ell \in \mathbb{R}$ if and only if $\ell \equiv 1 \pmod{4}$. Moreover there are exactly $\frac{\ell-1}{2} = k$ values of $j \pmod{\ell}$ for which $\left(\frac{j}{\ell}\right)$ has the same sign as g_ℓ , and for these the above implies that

$$\eta_m \geq \eta_k \geq \frac{1}{g_\ell} + \frac{1}{\ell} = \frac{1}{\sqrt{\ell}} + \frac{1}{\ell} = \frac{1}{\sqrt{2k+1}} + \frac{1}{2k+1} > \frac{1}{2\sqrt{m}}.$$

□

24.1 How to determine a better upper bound on η_k in general

We may proceed much as in the proofs above. Given the χ_j and t_j we select $f(p)$ to have size 1 in the same direction as $\sum_{j=1}^k \chi_j(p)p^{it_j}$ so that

$$\sum_{j=1}^k \operatorname{Re}((f\bar{\chi}_j)(p)/p^{it_j}) = \operatorname{Re}\left(f(p) \sum_{j=1}^k \bar{\chi}_j(p)/p^{it_j}\right) = \left| \sum_{j=1}^k \chi_j(p)p^{it_j} \right|.$$

Hence in this case

$$\sum_{j=1}^k \sum_{Q \leq p \leq x} \frac{\operatorname{Re}((f\bar{\chi}_j)(p)/p^{it_j})}{p} = \sum_{Q \leq p \leq x} \frac{\left| \sum_{j=1}^k \chi_j(p)p^{it_j} \right|}{p}.$$

Using the hypothesis of Proposition [24.1](#) ^{[eta2](#)}, this equals

$$\frac{1}{\phi(q)} \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \int_{\log Q}^{\log x} \left| \sum_{j=1}^k \chi_j(a)e^{it_j v} \right| \frac{dv}{v} + O(1).$$

It is not so easy to proceed as before since the quantity inside $|\cdot|$ is no longer periodic. Certainly one can do something similar but not the exact same thing.

One important special case is where each $t_j = 0$, since this was the worst case when $k = 2$. In this case suppose that each $\chi_j^m = 1$, that q is prime and if g has order $m \bmod q$ then $\chi_j(g) = e(b_j/m)$. (Here the b_j must be distinct mod m , as the χ_j are distinct.) Hence the above becomes

$$\frac{1}{m} \sum_{n=0}^{m-1} \left| \sum_{j=1}^k e(nb_j/m) \right| \cdot \log \left(\frac{\log x}{\log Q} \right) + O(1).$$

We therefore wish to find the maximum of this as we vary over all possible b_j .

By computer we found optimal examples for $2 \leq m \leq 6$ by an exhaustive search. Writing the example as $[b_1, \dots, b_k; m]$, we have $[0, 1; 3]$, $[0, 1, 3; 7]$, $[0, 1, 3, 9; 13]$, $[0, 1, 4, 14, 16; 21]$, $[0, 1, 3, 8, 12, 18; 31]$. One observes that $m = k^2 - k + 1$ and that these are all perfect difference sets; that is the numbers $\{b_i - b_j \pmod{m} : 1 \leq i \neq j \leq k\} = \{\ell \pmod{m} : 1 \leq \ell \leq m-1\}$. This case is easy to analyze because then we have

$$\left| \sum_{j=1}^k e(nb_j/m) \right|^2 = \sum_{1 \leq i, j \leq k} e(n(b_i - b_j)/m) = k + \sum_{1 \leq \ell \leq m-1} e(n\ell/m) = k - 1,$$

if $n \neq 0$. Therefore

$$\frac{1}{m} \sum_{n=0}^{m-1} \left| \sum_{j=1}^k e(nb_j/m) \right| = \frac{(m-1)\sqrt{k-1} + k}{m}.$$

Exercise 24.3 Use the Cauchy-Schwarz inequality to show that this is indeed maximal. (Hint: Under what circumstances do we get equality in the Cauchy-Schwarz inequality?)

Although there are perfect difference sets for $k = 2, 3, 4, 5, 6$ and 7 , there is none for $k = 8$. The existence of a perfect difference set is equivalent to the existence of a “cyclic projective plane” mod $m = k^2 - k + 1$.¹ There are always perfect difference sets for k a prime power.

The next question is to understand the size of the individual sums, if we want a lower bound. What we get is that

$$\sum_{Q \leq p \leq x} \frac{\operatorname{Re}((f\bar{\chi}_i)(p))}{p} = c_i \log \left(\frac{\log x}{\log Q} \right) + O(1),$$

where

¹This is Theorem 2.1 in *Cyclic projective planes* by Marshall Hall Jr, Duke 14 (1947) 1079–1090.

$$\begin{aligned}
c_i &= \frac{1}{m} \sum_{n=0}^{m-1} \sum_{j=1}^k e\left(\frac{nb_j}{m}\right) e\left(\frac{-nb_i}{m}\right) / \left| \sum_{j=1}^k e\left(\frac{nb_j}{m}\right) \right| \\
&= \frac{1}{m} \left(1 + \frac{1}{\sqrt{k-1}} \sum_{j=1}^k \sum_{n=1}^{m-1} e\left(\frac{n(b_j - b_i)}{m}\right) \right) \\
&= \frac{1}{m} \left(1 + \frac{m-k}{\sqrt{k-1}} \right) > \frac{1}{\sqrt{k+1}},
\end{aligned}$$

for $k > 1$. Evidently, because of equalities throughout, this best possible (when the $t_j = 0$). It also supplies us with a lower bound in general, at least if k is prime.

We can use short gaps between primes to extend this to all k . For example, the prime number theorem implies that there is always a prime in $[m, m + o(m))$, and so

$$\eta_k \sim \frac{1}{\sqrt{k}}.$$

LOWER BOUNDS ON $L(1, \chi)$, AND ZEROS; THE WORK OF
PINTZ

Exercise 25.1 For $\eta, 0 < \eta < 1$ show that

$$\lim_{x \rightarrow \infty} \sum_{m \leq x} \frac{1}{m^{1-\eta}} - \frac{x^\eta - 1}{\eta}$$

exists, and call it γ_η . Prove that

$$\left| \sum_{m \leq x} \frac{1}{m^{1-\eta}} - \frac{x^\eta - 1}{\eta} - \gamma_\eta \right| \leq \frac{1}{x^{1-\eta}}.$$

Proposition 25.2 Suppose that $L(s, \chi) \neq 0$ for real s , $1 - \frac{1}{\log q} \leq s \leq 1$ where χ is a real non-principal character mod q . Then $L(1, \chi) \gg 1/\log q$.

Proof Let $\eta = c/\log q$. For any real character χ define $g = 1 * \chi$ so that $g(n) \geq 0$ for all n and $g(m^2) \geq 1$. Hence

$$\begin{aligned} \sum_{m^2 \leq x} \frac{1}{m^{2-2\eta}} &\leq \sum_{n \leq x} \frac{g(n)}{n^{1-\eta}} = \sum_{d \leq x} \frac{\chi(d)}{d^{1-\eta}} \sum_{m \leq x/d} \frac{1}{m^{1-\eta}} \\ &\leq \sum_{d \leq x} \frac{\chi(d)}{d^{1-\eta}} \left(\frac{(x/d)^\eta - 1}{\eta} + \gamma_\eta + \frac{d^{1-\eta}}{x^{1-\eta}} \right) \\ &= \frac{x^\eta}{\eta} \sum_{d \leq x} \frac{\chi(d)}{d} + \left(\gamma_\eta - \frac{1}{\eta} \right) \sum_{d \leq x} \frac{\chi(d)}{d^{1-\eta}} + \frac{1}{x^{1-\eta}} \sum_{d \leq x} \chi(d) \\ &\leq \frac{x^\eta}{\eta} L(1, \chi) + \left(\gamma_\eta - \frac{1}{\eta} \right) L(1 - \eta, \chi) + O(q/\eta x^{1-\eta}) \end{aligned}$$

since $\sum_{d > x} \chi(d)/d^\rho \ll q/x^\rho$ for all $\rho > 0$. Now as there are no zeros in $[1 - \eta, 1]$ hence $L(1 - \eta, \chi) > 0$ (like $L(1, \chi)$) and $\gamma_\eta < 1/\eta$ so that term is < 0 . Taking $x = q^2$ we obtain the result. \square

25.1 Siegel's Theorem

If $L(s, \chi) \neq 0$ for real s , $1 - \frac{1}{\log q} \leq s \leq 1$, for all real quadratic characters $\chi \pmod{q}$, for all q then we can use the above Proposition. Otherwise we suppose

that there exists a character $\psi \pmod{k}$ and a real number ρ such that $L(\rho, \psi) = 0$. Now

$$\begin{aligned} \sum_{n \leq x} \frac{(1 * \chi)(n) \psi(n)}{n^\rho} &= \sum_{ab \leq x} \frac{\chi(a) \psi(a)}{a^\rho} \cdot \frac{\psi(b)}{b^\rho} \\ &= \sum_{a \leq \sqrt{x}} \frac{\chi(a) \psi(a)}{a^\rho} \sum_{b \leq x/a} \frac{\psi(b)}{b^\rho} + \sum_{b \leq \sqrt{x}} \frac{\psi(b)}{b^\rho} \sum_{\sqrt{x} < a \leq x/b} \frac{\chi(a) \psi(a)}{a^\rho} \\ &= \sum_{a \leq \sqrt{x}} \frac{\chi(a) \psi(a)}{a^\rho} L(\rho, \psi) + O \left(k \sum_{a \leq \sqrt{x}} \frac{1}{x^\rho} + \frac{qk}{x^{\rho/2}} \sum_{b \leq \sqrt{x}} \frac{1}{b^\rho} \right) \ll \frac{qk}{x^{\rho-1/2}}. \end{aligned}$$

Now $(1 * \lambda)(n) = 1$ if $n = m^2$, and 0 otherwise, where λ is Liouville's function. We write $\lambda = \chi * h$ (so that $h(p^k) = \lambda(p^k)(1 + \chi(p))$). Now

$$1 \ll \sum_{\substack{m^2 \leq x \\ (m, k)=1}} \frac{1}{m^{2-2\eta}} = \sum_{n \leq x} \frac{\psi(n)(1 * \lambda)(n)}{n^\rho} = \sum_{n \leq x} \frac{\psi(n)(1 * \chi * h)(n)}{n^\rho} = \sum_{ab \leq x} \frac{\psi(a)(1 * \chi)(a)}{a^\rho} \cdot \frac{\psi(b)h(b)}{b^\rho}.$$

Now the terms with $b \leq \sqrt{x}$ are, since $|h(b)| \leq 2^{\nu(b)}$, and using the bound above,

$$\leq \sum_{b \leq \sqrt{x}} \frac{|h(b)|}{b^\rho} \left| \sum_{a \leq x/b} \frac{(1 * \chi)(a) \psi(a)}{a^\rho} \right| \ll \sum_{b \leq \sqrt{x}} \frac{2^{\nu(b)}}{b^\rho} \frac{qk}{(x/b)^{\rho-1/2}} \ll \frac{qk}{x^{\rho-1/2}} \sum_{b \leq \sqrt{x}} \frac{2^{\nu(b)}}{b^{1/2}} \ll \frac{qk \log x}{x^{\rho-3/4}}.$$

The remaining terms, since $|(1 * \chi)(a)| \leq d(a)$, $0 \leq |h(b)| \leq (1 * \chi)(b)$ and $1/(ab)^\rho \leq x^{1-\rho}/ab$, are

$$\leq x^{1-\rho} \sum_{a \leq \sqrt{x}} \frac{d(a)}{a} \sum_{\sqrt{x} < b \leq x} \frac{(1 * \chi)(b)}{b} = x^{1-\rho} \sum_{a \leq \sqrt{x}} \frac{d(a)}{a} \sum_{\sqrt{x} < mn \leq x} \frac{\chi(m)}{mn}.$$

The first sum here is $\ll x^{(1-\rho)/2} \log x$. For the second we have

$$\begin{aligned} &\sum_{m \leq x^{1/3}} \frac{\chi(m)}{m} \sum_{\sqrt{x}/m < n \leq x/m} \frac{1}{n} + \sum_{n \leq x^{2/3}} \frac{1}{n} \sum_{x^{1/3} < m \leq x/n} \frac{\chi(m)}{m} \\ &= \sum_{m \leq x^{1/3}} \frac{\chi(m)}{m} \frac{1}{2} \log x + O \left(\sum_{m \leq x^{1/3}} \frac{1}{\sqrt{x}} \right) + O \left(\sum_{n \leq x^{2/3}} \frac{1}{n} \frac{q}{x^{1/3}} \right) \\ &= \frac{1}{2} L(1, \chi) \log x + O \left(\frac{1}{x^{1/6}} \right) \end{aligned}$$

if $x > q^7$. Combining the above we obtain, provided $\rho \geq 9/10$ and taking $x = q^7$, that

$$L(1, \chi) \gg 1/q^{11(1-\rho)}.$$

THE SIEGEL-WALFISZ THEOREM

We saw in our discussion of Selberg-Tenenbaum that if the mean value of $f(p)$ is about δ , with $\delta \neq 0, -1$ then the mean value of $f(n)$ for $n \leq x$ is about $c_f/(\log x)^{1-\delta}$. In both the two cases $\delta = 0$ and -1 one can show that the mean value of $f(n)$ is $\ll_f 1/(\log x)^2$. In our first subsection we shall sketch an argument to show that if the mean value of $f(p)$ is about 0 then the mean value of $f(n)$ is correspondingly small. The case when the mean value of $f(p)$ is -1 is rather more difficult but fortunately featured in [7]. This is relevant to a strong version of the prime number theorem, since their argument can be used to bound the mean value of $\mu(n)$. In a future version we shall give a stronger version of their argument.

The main point of this section is to prove a strong converse theorem when the mean value of $f(n)$ is around 0 and the mean value of $f(p)$ cannot be close to -1 . Since this is what we know about Dirichlet characters this will lead us to a pretentious proof of the Siegel-Walfisz Theorem. This proof is due to Dimitris Koukoulopoulos. In this version of the book we include a preliminary version of his paper; he will present a more complete version at the Snowbird meeting.

26.1 Primes well distributed implies...

Let $S(x) = \sum_{n \leq x} f(n)$ and $P(x) = \sum_{d \leq x} \Lambda(d)f(d)$. Assume $|P(x)| \leq cx/(\log x)^A$ with $A > 2$.

Select B in the range $2 < B < A$ and then $c_B > 0$ minimal for which there exists a constant x_B such that if $x \geq x_B$ then $|S(x)| \leq c_B x/(\log x)^B$. Let $D = x^\beta$ with $\beta > 0$ so that $(B-1)(1-\beta)^{B-1} > 1$.

Suppose f is totally multiplicative

$$\sum_{n \leq x} f(n) \log n = \sum_{dm \leq x} f(md) \Lambda(d) = \sum_{d \leq D} \Lambda(d) S(x/d) + \sum_{m \leq x/D} f(m) (P(x/m) - P(D)).$$

The second term is, in absolute value,

$$\leq 2c \sum_{m \leq x/D} \frac{x}{m \log(x/m)^A} \sim \frac{2c}{A-1} \frac{x}{(\log D)^{A-1}} \ll \frac{x}{(\log x)^{A-1}}.$$

If our bound is proved up to $x/2$ then we can insert into the first term to obtain

$$\leq c_B \sum_{d \leq D} \Lambda(d) \frac{x}{d \log(x/d)^B} \sim \frac{c_B}{B-1} \frac{x}{(\log x/D)^{B-1}} < (1-2\epsilon) c_B \frac{x}{(\log x)^{B-1}}.$$

Hence the total bound that we get is $|S(x)| \leq (1-\epsilon) c_B x/(\log x)^B$.

We use this argument several times

1) To show that $|S(x)| \ll x/(\log x)^B$.

2) Letting $c_B = \liminf |S(x)|/x/(\log x)^B$ to show that $c_B = 0$.

Hence we have proved that for any $B < A$ we have $S(x) = o(x/(\log x)^B)$. I suspect that the argument can be used to show that $S(x) \ll x\psi(x)/(\log x)^A$ where $\psi(x)$ is any function going monotonically to infinity, no matter how slowly.

Note that we need to have $A > 2$ for this argument to work, which seems to fit the sort of things we know from Selberg-Delange-Tenenbaum.

The above argument is written in a uniform manner. I am interested in what happens if, say, $P(x) \ll x/e^{\sqrt{\log x}}$. The key remark is that we can take $\beta = \log(B/2)/B$ roughly. To make the argument then work we need

$$A(\log D)^{A-1} \gg B(\log x)^{B-1}$$

If say A is roughly $(\log x)^\alpha$ and B is roughly this size we get something like $(1-\alpha)(A-1) \gg B$ from the powers of $\log x$; that is B is roughly $(1-\alpha)A$. One can be precise, I think, and show that one can obtain

$$S(x) \ll xA^A/(\log x)^A$$

provided $A \rightarrow \infty$. Hence if $P(x) \ll x/e^{2\sqrt{\log x}}$ and $S(x) \ll x/e^{\sqrt{\log x}}$.

26.2 Main results

For an arithmetic function $f : \mathbb{N} \rightarrow \mathbb{C}$ we set

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad L_y(s, f) = \sum_{P^-(n) > y} \frac{f(n)}{n^s},$$

provided the series converge. We will use pretentious methods to prove:

pnt **Corollary 26.1** *Let $x \geq 1$ and $(a, q) = 1$ such that*

$$\frac{\log q}{L_q(1, \chi)} \leq c\sqrt{\log x}$$

for all real characters $\chi \pmod{q}$, for some sufficiently small $c > 0$. Then

$$\pi(x; q, a) = \frac{\pi(x)}{\phi(q)} + O\left(\frac{x}{e^{c\sqrt{\log x}}}\right).$$

Using Siegel's Theorem this allows us to recover the Siegel-Walfisz Theorem. That is

The Siegel-Walfisz Theorem. Fix $A > 0$. Uniformly we have

$$\pi(x; q, a) = \frac{\pi(x)}{\phi(q)} + O\left(\frac{x}{(\log x)^A}\right).$$

If $L_y(1+it, f)$ converges for all $t \in \mathbb{R}$ and all $y \geq 1$, we set

$$\mathcal{L}_y^{(1)}(f) = \min_{|t| \leq y} |L_{y+|t|}(1+it, f)| \quad \text{and} \quad \mathcal{L}_y^{(2)}(f) = \min_{|t| > y} |L_{y+|t|}(1+it, f)|.$$

In the special case that $y = 1$ we omit the subscript y .

bounded**Theorem 26.2** *Let $f : \mathbb{N} \rightarrow \mathbb{D}$ be a completely multiplicative function such that*

$$\left| \sum_{n \leq x} f(n) \right| \leq x^{1-\epsilon} \quad (x \geq Q) \quad (26.1) \quad \text{e0}$$

for some $\epsilon > 0$ and some $Q \geq 2$. Then we have $\mathcal{L}_Q^{(j)}(f) \ll_\epsilon 1$ for $j = 1, 2$. Furthermore, if we assume that $\mathcal{L}_Q^{(2)}(f) \geq \eta$, then there are constants c_1 and c_2 , depending at most on ϵ and η , such that

$$\sum_{p \leq x} f(p) \ll \frac{x}{e^{c_1 \sqrt{\log x}}} \quad \text{whenever} \quad \log x \geq c_2 \left(\frac{\log Q}{\mathcal{L}_Q^{(1)}(f)} \right)^2.$$

char**Theorem 26.3** *Let $f : \mathbb{N} \rightarrow \mathbb{D}$ be a completely multiplicative function which satisfies (26.1) for some $\epsilon > 0$ and some $Q \geq 2$.*

1. *If (26.1) holds for f^2 as well, then $\mathcal{L}_Q^{(j)}(f) \asymp_\epsilon 1$ for $j = 1, 2$.*
2. *If $f(p) \in \{-1, +1\}$ for $p > Q$, then $\mathcal{L}_Q^{(1)}(f) \asymp_\epsilon L_Q(1, f)$ and $\mathcal{L}_Q^{(2)}(f) \asymp_\epsilon 1$. Furthermore, $L_Q(1, f) \neq 0$ if the sum $\sum_{n=1}^\infty f(n)/\sqrt{n}$ converges. Lastly, if $\log Q \geq 2/\epsilon$ and $L(\sigma, f) \neq 0$ for $1 - 1/\log Q \leq \sigma \leq 1$, then $\mathcal{L}_Q(f) \asymp_\epsilon 1$.*

26.3 Technical results

Let $Q \geq 3$, $k \in \mathbb{N} \cup \{0\}$, $A \geq 2$ and $M : [(\log Q)/3, +\infty) \rightarrow (0, +\infty)$ a differentiable function such that

$$\frac{1}{A} \leq M(u) \leq e^{2u/3} \quad (u \geq Q)$$

and for $j \in \{0, 1, \dots, k\}$ the function $M(u)/u^j$ increases for $u \geq Aj$. We call (Q, k, A, M) an *admissible quadruple*. Given such a quadruple and $t \in \mathbb{R}$ we define Q_t by

$$Q_t = \min\{z \geq Q : M((\log z)/3) \geq |t|\}. \quad (26.2) \quad \text{qt}$$

Also, we let $\mathcal{F}(Q, k, A, M)$ be the family of completely multiplicative functions $f : \mathbb{N} \rightarrow \mathbb{D} := \{z \in \mathbb{N} : |z| \leq 1\}$ such that

$$\left| \sum_{n \leq x} f(n) \right| \leq \frac{x}{(\log x)^2 M(\log x)} \quad (x \geq Q).$$

For such an f we define

$$\mathcal{L}_+(f; Q, k, A, M) = \min_{|t| \geq M((\log Q)/3)} |L_{Q_t}(1 + it, f)|$$

and

$$\mathcal{L}(f; Q, k, A, M) = \min_{t \in \mathbb{R}} |L_{Q_t}(1 + it, f)|.$$

The notation

$$g(x) \ll_{a,b,\dots} h(x) \quad (x \geq x_0)$$

means that $|g(x)| \leq Ch(x)$ for $x \geq x_0$, where C is a constant which depends at most on a, b, \dots . Lastly, the letters c and C denote generic constants, possibly

different at each case and possibly depending on certain parameters which will always be specified, e.g. by $c = c(a, b, \dots)$.

main

Theorem 26.4 *Let (Q, k, A, M) be an admissible quadruple and consider $f \in \mathcal{F}(Q, k, A, M)$. For $t \in \mathbb{R}$ define Q_t by (26.2).*

1. *We have*

$$\frac{1}{x} \sum_{p \leq x} f(p) \ll_A \left(\frac{ck \log Q}{\mathcal{L}(f; Q, k, A, M) \log x} \right)^{\frac{k-1}{2}} + \left(\frac{ck^2}{\mathcal{L}_+(f; Q, k, A, M) \log x} \right)^{\frac{k-1}{2}} \quad (x \geq Q)$$

for some constant $c = c(A)$.

2. *We have the estimate $\mathcal{L}(f; Q, k, A, M) \ll_A 1$. Moreover, if for some $t \in \mathbb{R}$ we have*

$$\sum_{Q_t < p \leq z} \frac{1 + \Re(f(p)p^{-it})}{p} \geq \delta \log \left(\frac{\log z}{\log Q_t} \right) - C \quad (z \geq Q_t, t \in \mathbb{R}), \quad (26.3) \quad \text{notmobius}$$

where $\delta > 0$ and $C \geq 0$ are some constants, then $|L_{Q_t}(1 + it, f)| \asymp_{A, \delta, C} 1$.

3. *If $f^2 \in \mathcal{F}(Q, k, A, M)$ as well, then $\mathcal{L}(f; Q, k, A, M) \asymp_A 1$.*

4. *If $f(p) \in \{-1, +1\}$ for all primes $p > Q$, then*

$$\mathcal{L}_+(f; Q, k, A, M) \asymp_A 1 \quad \text{and} \quad \mathcal{L}(f; Q, k, A, M) \asymp_A L_Q(1, f).$$

The key estimate in proving Theorem ^{main}26.4 is the following theorem.

derivative

Theorem 26.5 *Let (Q, k, A, M) be an admissible quadruple and consider $f \in \mathcal{F}(Q, k, A, M)$. For $x \geq y \geq Q$ we have*

$$\sum_p \frac{f(p) \log^m p}{p^{1+1/\log x}} \ll_A \left(\frac{cm \log y}{|L_y(1, f)|} \right)^m \quad (1 \leq m \leq k)$$

for some constant $c = c(A)$. Moreover, $|L_y(1, f)| \ll_A 1$.

26.3.1 Preliminaries

series

Lemma 26.6 *Let $\{a_n\}_{n=1}^\infty$ be a sequence of elements of \mathbb{D} . If $\sum_{n=1}^\infty a_n/n$ converges, then*

$$\lim_{\epsilon \rightarrow 0+} \sum_{n=1}^\infty \frac{a_n}{n^{1+\epsilon}} = \sum_{n=1}^\infty \frac{a_n}{n}.$$

Proof The lemma follows by an easy partial summation argument. \square

The following result is Lemma 5 in ^{fi}[?].

fund

Lemma 26.7 *Let $y \geq 2$ and $D = y^s$ with $s \geq 2$. Let $\mathbf{1}[P^-(n) > y]$ denote the indicator function of integers n all of whose prime factors are greater than y .*

Then there exist two sequences $\{\lambda^\pm(d)\}_{d \leq D}$ whose elements lie in $[-1, 1]$ and such that

$$(\lambda^- * 1)(n) \leq \mathbf{1}[P^-(n) > y] \leq (\lambda^+ * 1)(n).$$

Moreover, if $f : \mathbb{N} \rightarrow [0, 1]$ is a multiplicative function then

$$\sum_{d \leq D} \frac{\lambda^\pm(d)f(d)}{d} = (1 + O(e^{-s})) \prod_{p \leq y} \left(1 - \frac{f(p)}{p}\right).$$

quadruple

Lemma 26.8 *Let (Q, k, A, M) be an admissible quadruple.*

1. For $0 \leq j \leq k$ and $\lambda \geq \max\{j, \log Q\}$ we have

$$\int_1^\infty \frac{u^{j-2}}{M(\lambda u)} du \leq \frac{A^j}{M(\lambda)}.$$

2. For $\rho > 0$ and $\lambda \geq \max\{k, \log Q\}$ we have

$$\int_1^\infty \frac{u^{k-1}}{M(\lambda u)e^{u/\rho}} du \leq \frac{A^k(2 + \log \max\{1, \rho\})}{M(\lambda)}.$$

Proof (a) If $j = 0$, then the result follows immediately because M is increasing. Fix $1 \leq j \leq k$ and $\lambda \geq \max\{j, \log Q\}$. Then

$$\int_1^\infty \frac{u^{j-2}}{M(\lambda u)} du \leq \frac{1}{M(\lambda)} \int_1^A u^{j-2} du + \frac{A^k}{M(\lambda A)} \int_A^\infty \frac{du}{u^2} \leq \frac{A^k}{M(\lambda)}.$$

(b) It suffices to consider the case $\rho \geq 1$. So fix such a ρ and some $\lambda \geq \max\{\log Q, k\}$. If $k = 0$, the result follows immediately by the fact that M is increasing and by the estimate

$$\int_1^\infty \frac{du}{e^{u/\rho} u} \leq \log \rho + \int_\rho^\infty \frac{du}{e^{u/\rho} u} \leq \log \rho + 1.$$

So assume that $k \geq 1$. Then

$$\int_1^\infty \frac{u^{k-1}}{M(\lambda u)e^{u/\rho}} du \leq \frac{A^k}{k \cdot M(\lambda)} + \frac{A^k}{M(\lambda A)} \int_A^\infty \frac{du}{e^{u/\rho} u} \leq \frac{A^k}{M(\lambda)} \leq \frac{A^k}{M(\lambda)} \left(\frac{1}{k} + \log \rho + 1\right),$$

which completes the proof. \square

sieve

Lemma 26.9 *Let (Q, k, A, M) be an admissible quadruple and consider $f \in \mathcal{F}(Q, k, A, M)$. For $x \geq y \geq Q$ and $0 \leq m \leq k$ we have*

$$\left| \sum_{P^-(n) > y} \frac{f(n) \log^m n}{n^{1+1/\log x}} \right| \ll A(2A(m+1) \log y)^m.$$

Also, we have

$$\left| \sum_{P^-(n) > y} \frac{f(n) \log^{k+1} n}{n^{1+1/\log x}} \right| \ll (2A(k+1) \log y)^{k+1} \left(1 + A \log \left(\frac{\log x}{\log y}\right)\right).$$

Proof Let $0 \leq m \leq k+1$. Lemma ^{|fund}26.7 with $D = \sqrt{x}$ and $x \geq y^{2m+2}$ implies that

$$\begin{aligned} \sum_{\substack{n \leq x \\ P^-(n) > y}} f(n) &= \sum_{n \leq x} f(n)(\lambda^+ * 1)(n) + O\left(\sum_{n \leq x} (\lambda^+ * 1 - \lambda^- * 1)(n)\right) \\ &\ll \frac{x \log y}{(\log x)^2 M(\log x/2)} + \frac{x e^{-\log x/2 \log y}}{\log y}. \end{aligned}$$

By partial summation then we find that

$$\begin{aligned} \sum_{P^-(n) > y} \frac{f(n) \log^m n}{n^{1+1/\log x}} &\ll (2(m+1) \log y)^m + \int_{y^{2m+2}}^{\infty} \frac{m \log^{m-1} u + \log^m u}{u^{1+1/\log x}} \frac{\log y}{(\log u)^2 M(\log u/2)} du \\ &\quad + \int_{y^{2m+2}}^{\infty} \frac{m \log^{m-1} u + \log^m u}{u^{1+1/\log x}} \frac{e^{-\log u/2 \log y}}{\log y} du \\ &\ll (2(m+1) \log y)^m + (2(m+1) \log y)^m \int_1^{\infty} \frac{t^{m-2}}{e^{\frac{(2m+2) \log y}{\log x} t} M((m+1)(\log y)t)} dt. \end{aligned}$$

Lemma ^{|quadruple}26.8 and our assumption that $M(\log Q) \geq 1/A$ then complete the proof of the lemma. \square

distance

Lemma 26.10 Let (Q, k, A, M) be an admissible quadruple and consider $f \in \mathcal{F}(Q, k, A, M)$. Let $y_2 \geq y_1 \geq y_0 \geq Q$. Assume that

$$\sum_{y_0 < p \leq z} \frac{1 + \Re(f(p))}{p} \geq \delta \log \frac{\log z}{\log y} - C \quad (z \geq y_1)$$

for some $\delta > 0$ and $C \geq 0$. Then

$$\left| \sum_{y_1 < p \leq y_2} \frac{f(p)}{p} \right| \ll_{A, \delta, C} 1.$$

Proof By our assumption and Lemma ^{|sieve}26.9 we have

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{f(n) \Lambda(n)}{n^{1+1/\log x}} \right| &\leq \left| \sum_{P^-(n) \leq y_0} \frac{f(n) \Lambda(n)}{n^{1+1/\log x}} \right| + \left| \sum_{P^-(n) > y_0} \frac{\mu(n) f(n)}{n^{1+1/\log x}} \right| \cdot \left| \sum_{P^-(n) > y_0} \frac{f(n) \log n}{n^{1+1/\log x}} \right| \\ &\ll_C \log y_0 + \left(\frac{\log x}{\log y_0} \right)^{1-\delta} \left| \sum_{P^-(n) > y_0} \frac{f(n) \log n}{n^{1+1/\log x}} \right| \\ &\ll_A \log y_0 + \left(\frac{\log x}{\log y_0} \right)^{1-\delta} (\log y_0) \left(1 + \log \left(\frac{\log x}{\log y_0} \right) \right) \\ &\ll_{\delta} (\log y_0) \left(\frac{\log x}{\log y_0} \right)^{1-\delta/2}. \end{aligned}$$

So we deduce that

$$\begin{aligned} \sum_{y_1 < p \leq y_2} \frac{f(p)}{p} &= O(1) + \log \left\{ F \left(1 + \frac{1}{\log y_2} \right) \right\} - \log \left\{ F \left(1 + \frac{1}{\log y_1} \right) \right\} \\ &= O(1) + \int_{y_1}^{y_2} \frac{-F'}{F} \left(1 + \frac{1}{\log u} \right) \frac{du}{u \log^2 u} \\ &\ll_{A, \delta, C} 1 + (\log y_0)^{\delta/2} \int_{y_1}^{y_2} \frac{du}{u (\log u)^{1+\delta/2}} \ll_{\delta} 1. \end{aligned}$$

This completes the proof of the lemma. \square

26.3.2 Proofs

Proof [Proof of Theorem 26.5] ^{|derivative|}(a) We have that

$$\left| \sum_p \frac{f(p) \log^m p}{p^{1+1/\log x}} \right| \ll (cm \log y)^m + \left| \sum_{P^-(n) > y} \frac{f(n) \Lambda(n) \log^{m-1} n}{n^{1+1/\log x}} \right|.$$

Set $F(s) = L_y(s, f)$ and note that

$$\sum_{P^-(n) > y} \frac{f(n) \Lambda(n) \log^{m-1} n}{n^{1+1/\log x}} = \left(\frac{-F'}{F} \right)^{(m-1)} \left(1 + \frac{1}{\log x} \right).$$

Moreover, we have

$$\left(\frac{-F'}{F} \right)^{(m-1)}(s) = m! \sum_{a_1 + 2a_2 + \dots = m} \frac{(-1 + a_1 + a_2 + \dots)!}{a_1! a_2! \dots} \left(\frac{-F'(s)}{1! F(s)} \right)^{a_1} \left(\frac{-F''(s)}{2! F(s)} \right)^{a_2} \dots \quad (26.4) \quad \boxed{\text{identity}}$$

Lemma 26.9 ^{|sieve|} implies that

$$\left| F^{(j)} \left(1 + \frac{1}{\log x} \right) \right| \ll (j \log y)^j \quad (1 \leq j \leq m).$$

In addition, for every $x' \geq x$ we have that

$$\sum_{x < p \leq x'} \frac{\Re(f(p))}{p} = O(1) + \log \left| L_x \left(1 + \frac{1}{\log x'}, f \right) \right| \leq C_1$$

for some constant $C_1 = C_1(A)$, by Lemma 26.9 ^{|sieve|}. Therefore

$$\begin{aligned} \left| F \left(1 + \frac{1}{\log x} \right) \right| &\asymp \exp \left\{ \sum_{y < p \leq x} \frac{\Re(f(p))}{p} \right\} \gg_A \exp \left\{ \sum_{y < p \leq x'} \frac{\Re(f(p))}{p} \right\} \\ &\asymp \left| \sum_{P^-(n) > y} \frac{f(n)}{n^{1+1/\log x'}} \right| \rightarrow |L_y(1, f)| \end{aligned}$$

as $x' \rightarrow \infty$, by Lemma [26.6](#). Inserting the above estimates into [26.4](#) with $s = 1 + 1/\log x$ and observing that

$$|L_y(1, f)| = \lim_{\epsilon \rightarrow 0^+} |L_y(1 + \epsilon, f)| \ll_{A, \delta, C} 1,$$

by Lemma [26.9](#), yields

$$\sum_p \frac{f(p) \log^m p}{p^{1+1/\log x}} \ll_A \left(\frac{C_2 m \log y}{|L_y(1, f)|} \right)^m \sum_{a_1+2a_2+\dots=m} 1$$

for some $C_2 = C_2(A)$. To complete the proof of part (a), note that

$$\sum_{a_1+2a_2+\dots=m} 1 = \sum_{I \subset \{1, \dots, m\}} \sum_{\substack{i \in I \\ a_i \geq 1}} \sum_{\substack{a_i = m \\ (i \in I)}} 1 \leq \sum_{I \subset \{1, \dots, m\}} \prod_{i \in I} \frac{m}{i} \leq 2^m \frac{m^m}{m!} \asymp \frac{(2e)^m}{\sqrt{m}},$$

by Stirling's formula. \square

Proof [Proof of Theorem [26.4](#)] [\(main\)](#) (a) We may assume that $x \geq Q_1$. For every $T \geq 1$ we have

$$\begin{aligned} \frac{1}{x} \sum_{p \leq x} f(p) (\log p)^{k-1} \log(x/p) &= \frac{1}{2\pi i} \int_{|t| \leq T} \sum_p \frac{f(p) (\log p)^{k-1}}{p^{1+1/\log x+it}} \frac{x^{1/\log x+it}}{(1+1/\log x+it)^2} \\ &\quad + O\left(\frac{(k \log x)^{k-1}}{T}\right). \end{aligned} \tag{26.5} \quad \boxed{\text{e3}}$$

Call $I(T)$ the integral above. By partial summation, we have that

$$\sum_{n \leq x} f(n) n^{-it} \ll x^{1/3} + (1+|t|) \int_{x^{1/3}}^x \left(\sum_{n \leq u} f(n) \right) \frac{du}{u^2} \ll \frac{(1+|t|)x}{M((\log x)/3)} \quad (x \geq Q),$$

that is

$$f(n) n^{-it} \in \mathcal{F}\left(Q_t, k, c_1 A, \frac{M(\cdot/3)}{c_1(1+|t|)}\right)$$

for some absolute constant $c_1 \geq 1$. For $|t| \leq T := M(\frac{\log x}{3})$ we have

$$\sum_p \frac{f(p) (\log p)^{k-1}}{p^{1+1/\log x+it}} \ll_A \left(\frac{c_2 k \log Q_t}{|L_{Q_t}(1+it, f)|} \right)^{k-1}$$

for some $c_2 = c_2(A)$. So

$$I(T) \ll \left(\frac{c_2 k \log Q}{\mathcal{L}(f; Q, A, M)} \right)^{k-1} + \int_{M(\frac{\log Q}{3})}^{M(\frac{\log x}{3})} \left(\frac{c_2 k \log Q_t}{\mathcal{L}_+(f; Q, A, M)} \right)^{k-1} \frac{dt}{t^2}.$$

Making the change of variable $t = M(u)$, we find that $\log Q_t = 3u$ and thus

$$\begin{aligned} \int_{M(\log Q/3)}^{M(\log x/3)} (\log Q_t)^{k-1} \frac{dt}{t^2} &= 3^{k-1} \int_{\frac{\log Q}{3}}^{\frac{\log x}{3}} \frac{u^{k-1} M'(u)}{(M(u))^2} du \\ &= \frac{((\log Q)/3)^{k-1}}{M((\log Q)/3)} - \frac{((\log x)/3)^{k-1}}{M((\log x)/3)} \\ &\quad + (k-1) \int_{\frac{\log Q}{3}}^{\frac{\log x}{3}} \frac{u^{k-2}}{M(u)} du. \end{aligned} \quad (26.6) \quad \boxed{\text{e5}}$$

We have

$$\int_{\max\{k, \frac{\log Q}{3}\}}^{\frac{\log x}{3}} \frac{u^{k-2}}{M(u)} du \leq (A \max\{\log Q, k\})^k,$$

by Lemma 26.8. Also, if $k \geq \log Q/3$, then

$$\int_{\frac{\log Q}{3}}^k \frac{u^{k-2}}{M(u)} du \leq k^{k-1}.$$

Combining the above inequalities with (26.5) and (26.6) yields

$$\frac{1}{x} \sum_{p \leq x} f(p) (\log p)^{k-1} \log(x/p) \ll_A \left(\frac{c_3 k \log Q}{\mathcal{L}(f; Q, A, M)} \right)^{k-1} + \left(\frac{c_3 k^2}{\mathcal{L}_+(f; Q, A, M)} \right)^{k-1}$$

for some $c_3 = c_3(A)$. By a standard differentiation argument, this implies

$$\frac{1}{x} \sum_{p \leq x} f(p) (\log p)^{k-1} \ll_A \left(\frac{c_3 k \log Q}{\mathcal{L}(f; Q, A, M)} \right)^{\frac{k-1}{2}} + \left(\frac{c_3 k^2}{\mathcal{L}_+(f; Q, A, M)} \right)^{\frac{k-1}{2}}.$$

Finally, summing by parts completes the proof of part (a).

(b) For $y_2 \geq y_1 \geq Q_t$ Lemma 26.10 implies that

$$\left| L_{y_1} \left(1 + \frac{1}{\log y_2}, f \right) \right| \asymp \exp \left\{ \sum_{y_1 < p \leq y_2} \frac{\Re(f(p))}{p} \right\} \asymp_{A, \delta, C} 1.$$

Setting $y_1 = Q_t$, letting $y_2 \rightarrow \infty$ and applying Lemma 26.6 completes the proof.

(c) By Lemma 26.9, for $t \in \mathbb{R}$ and $z \geq y \geq Q_t$ we have

$$\sum_{y < p \leq z} \frac{\Re(f^2(p) p^{-2it})}{p} = O(1) + \log \left| L_y \left(1 + \frac{1}{\log z} + it \right) \right| \leq C_1$$

for some absolute constant C_1 . So we find that

$$\begin{aligned} \sum_{y < p \leq z} \frac{\Re(f(p) p^{-it})}{p} &\geq -\frac{1}{2} \left(\sum_{y < p \leq z} \frac{1}{p} \right)^{1/2} \left(\sum_{y < p \leq z} \frac{\Re^2(f(p) p^{-it})}{p} \right)^{1/2} \\ &\geq -\frac{\sqrt{2}}{2} \log \frac{\log z}{\log y} - O(1), \end{aligned}$$

since $\cos^2 x = (1 + \cos(2x))/2$. Lemma 26.10 then completes the proof.

(d) For $|t| \geq 1/\log Q$ we have

$$\sum_{y < p \leq z} \frac{\cos(2t \log p)}{p} = O(1) + \log \left| \sum_{P^-(n) > y} \frac{1}{n^{1+2it+1/\log z}} \right| \leq C_2$$

for some absolute constant C_2 , by partial summation and Lemma [26.7](#)^{|fund|}. So an argument as the one in part (c) shows that $|L_{Q_t}(1+it, f)| \asymp 1$ for $|t| \geq 1/\log Q$. Fix now $t \in \mathbb{R}$ with $1/\log x \leq |t| \leq 1/\log Q$. We claim that

$$\sum_{e^{1/|t|} < p \leq x} \frac{1 + \cos(t \log p)}{p} \geq c \log(|t| \log x) - c' \quad (26.7) \quad \boxed{\text{e4}}$$

for some appropriate constants c and c' . In order to show this we use the argument in [\[7, Lemma 4.2.1\]](#). Fix some $\epsilon \in (1/10 \log Q, 1/3)$ to be chosen later and let \mathcal{P} be the set of primes for which there exists an integer n with

$$p \in I_n := [e^{\frac{\pi(n-\epsilon)}{|t|}}, e^{\frac{\pi(n+\epsilon)}{|t|}}] \quad (n \in \mathbb{N} \cup \{0\}).$$

Since $\epsilon/|t| \geq 1/10$, Mertens' theorem yields

$$\sum_{e^{1/|t|} < p \leq x, p \in \mathcal{P}} \frac{1}{p} \ll \epsilon \log(|t| \log x).$$

Thus

$$\begin{aligned} \sum_{e^{1/|t|} < p \leq x} \frac{1 + \Re(\chi(p)p^{-it})}{p} &\geq \sum_{\substack{e^{1/|t|} < p \leq x \\ p \notin \mathcal{P}}} \frac{1 + \Re(f(p)p^{-it})}{p} \gg \epsilon \sum_{\substack{e^{1/|t|} < p \leq x \\ p \notin \mathcal{P}}} \frac{1}{p} \\ &\geq \epsilon(1 - O(\epsilon)) \log(|t| \log x). \end{aligned}$$

Choosing a small enough ϵ proves [\(26.7\)](#)^{|e4|}. Next, notice that Lemma [26.10](#)^{|distance|} and [\(26.7\)](#)^{|e4|} yield that

$$\sum_{e^{1/|t|} < p \leq x} \frac{f(p)}{p^{1+it}} \ll 1.$$

Therefore for $x \geq e^{1/|t|} \geq Q$ we have

$$\sum_{Q < p \leq x} \frac{\Re(f(p)p^{-it})}{p} = \sum_{Q < p \leq e^{1/|t|}} \frac{f(p)}{p} + O(1) \geq \sum_{Q < p \leq x} \frac{f(p)}{p} + O(1),$$

by Lemma [26.9](#)^{|sieve|}. This completes the proof of the theorem. \square

Proof [Proof of Theorem [26.2](#)]^{|bounded|} For the function $M(u) = e^{\epsilon u}$ the quadruple $(Q, k, 1/\epsilon, M)$ is admissible for all $k \in \mathbb{N}$. Applying Theorem [26.4](#)^{|main|} with $k \asymp \sqrt{\log x}$ proves the desired result. \square

PRIMES IN PROGRESSIONS, ON AVERAGE

Suppose that the character $\chi \pmod{q}$ is induced from the primitive character $\psi \pmod{r}$. Then we write $\text{cond } \chi = q$ and $\text{cond}^* \chi = r$.

We shall use the *Siegel-Walfisz Theorem* which states that for any fixed $A, B > 0$ one has

$$\psi(N; q, a) - \frac{\psi(N)}{\phi(q)} \ll \frac{N}{\phi(q) \log^B N},$$

uniformly for $q \ll \log^A N$ and $(a, q) = 1$. This may also be phrased as

$$\sum_{n \leq N} \chi(n) \Lambda(n) \ll \frac{N}{\log^B N},$$

for all primitive characters $\chi \pmod{q}$, uniformly for $q \ll \log^A N$. We also make use of a strong form of the prime number theorem: For any fixed $A > 0$ we have

$$\psi(N) - N \ll \frac{N}{(\log N)^A}.$$

All of these estimates were proved in the previous section.

27.1 The Barban-Davenport-Halberstam-Montgomery-Hooley Theorem

The first result shows that the mean square of the error term in the prime number theorem for arithmetic progressions can be well understood.

BDHLMH **Theorem 27.1** *If $N/(\log N)^C \leq Q \leq N$ then*

$$\sum_{q \leq Q} \sum_{(a, q)=1} \left| \psi(N; q, a) - \frac{\psi(N)}{\phi(q)} \right|^2 = NQ \log N + O(NQ \log(N/Q)).$$

We begin with a technical lemma; most of the proof is left as an exercise.

LemLS1 **Lemma 27.2** *Let $c := \prod_p \left(1 + \frac{1}{p(p-1)}\right)$ and $\gamma' := \gamma - \sum_p \frac{\log p}{p^2 - p + 1}$. Then*

$$\begin{aligned}\sum_{r \leq R} \frac{1}{\phi(r)} &= c \log R + c\gamma' + O\left(\frac{\log R}{R}\right), \\ \sum_{r \leq R} \frac{r}{\phi(r)} &= cR + O(\log R), \\ \sum_{r \leq R} \frac{r^2}{\phi(r)} &= \frac{c}{2}R^2 + O(R \log R).\end{aligned}$$

Also

$$\sum_{\substack{r \leq R \\ m|r}} \frac{1}{\phi(r)} = \frac{1}{\phi(m)} \prod_{p \nmid m} \left(1 + \frac{1}{p(p-1)}\right) \left(\log \frac{R}{m} + \gamma - \sum_{p \nmid m} \frac{\log p}{p^2 - p + 1}\right) + O\left(\frac{\log R/m}{R}\right).$$

Proof We can write $r/\phi(r) = \sum_{d|r} \mu^2(d)/\phi(d)$ to obtain in the first case

$$\begin{aligned}\sum_{r \leq R} \frac{1}{\phi(r)} &= \sum_{r \leq R} \frac{1}{r} \sum_{d|r} \frac{\mu^2(d)}{\phi(d)} = \sum_{d \leq R} \frac{\mu^2(d)}{\phi(d)} \sum_{\substack{r \leq R \\ d|r}} \frac{1}{r} = \sum_{d \leq R} \frac{\mu^2(d)}{d\phi(d)} \left(\log \frac{R}{d} + \gamma + O\left(\frac{d}{R}\right)\right) \\ &= c(\log R + \gamma') + O\left(\frac{\log R}{R}\right),\end{aligned}$$

by (1.2.1). The next two estimates follow analogously but more easily. The last estimate is an easy generalization of the first.

□

PropLS2

Proposition 27.3

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}^* \chi \geq R}} \left| \sum_{n=X+1}^{X+N} a_n \chi(n) \right|^2 \ll \left(\frac{N}{R} \log Q + Q \right) \log \log Q \sum_{n=X+1}^{X+N} |a_n|^2.$$

Proof Suppose that the character $\chi \pmod{q}$ is induced from the primitive character $\psi \pmod{r}$. Let m be the product of the the primes that divide q but not r and write $q = rm\ell$ so that $(r, m) = 1$, and $p|\ell \implies p|rm$. Hence $\phi(q) = \phi(r)\phi(m)\ell$ and

$$\sum_n a_n \chi(n) = \sum_{n: (n, m)=1} a_n \psi(n);$$

and therefore the left side of the above equation equals

$$\sum_{m \leq Q} \frac{\mu^2(m)}{\phi(m)} \sum_{\substack{R \leq r \leq Q/m \\ (r, m)=1}} \frac{1}{\phi(r)} \sum_{\psi \pmod{r}}^* \left| \sum_{\substack{X < n \leq X+N \\ (n, m)=1}} a_n \psi(n) \right|^2 \sum_{\substack{\ell \leq Q/rm \\ p|\ell \implies p|rm}} \frac{1}{\ell}.$$

The last sum is $\leq \frac{r}{\phi(r)} \cdot \frac{m}{\phi(m)}$. We partition the sum over r into dyadic intervals $y < r \leq 2y$; in such an interval we have $\frac{r}{\phi(r)^2} \ll \frac{r}{\phi(r)} \cdot \frac{\log \log y}{y}$, and so by [\(I6.4\)](#) [LargeSieve](#) the above becomes

$$\begin{aligned} &\ll \log \log Q \sum_{m \leq Q} \frac{\mu^2(m)m}{\phi(m)^2} \sum_{\substack{y=2^i R, \ i=0, \dots, I \\ 2^I R := Q/m}} \frac{1}{y} (N + y^2) \sum_{n=X+1}^{X+N} |a_n|^2 \\ &\ll \log \log Q \sum_{m \leq Q} \frac{\mu^2(m)m}{\phi(m)^2} \left(\frac{N}{R} + \frac{Q}{m} \right) \sum_{n=X+1}^{X+N} |a_n|^2, \end{aligned}$$

which implies the result. \square

Let

$$\psi^{(R)}(x; q, a) = \psi(x; q, a) - \frac{1}{\phi(q)} \sum_{\substack{r \leq R \\ r|q}} \sum_{\substack{\chi \pmod{q} \\ \text{cond}^* \chi = r}} \bar{\chi}(a) \sum_{n \leq x} \chi(n) \Lambda(n),$$

so that $\psi^{(1)}(N; q, a) = \psi(N; q, a) - \frac{\psi(N)}{\phi(q)}$.

CorLS3 **Corollary 27.4** *For $\log N \leq R \leq \sqrt{Q}$ with $Q \leq N$ we have*

$$\begin{aligned} \sum_{q \leq Q} \sum_{(a, q)=1} \left| \psi(N; q, a) - \frac{\psi(N)}{\phi(q)} \right|^2 &\ll \log Q \sum_{r \leq R} \frac{1}{\phi(r)} \sum_{\psi \pmod{r}}^* \left| \sum_{n \leq N} \psi(n) \Lambda(n) \right|^2 \\ &\quad + O \left(\frac{N^2 \log^{2+o(1)} N}{R} + QN \log N \log \log N \right). \end{aligned}$$

Proof By [\(I6.3\)](#) [SumSqk](#), and taking $a_n = \Lambda(n)$, $X = 0$ in Proposition [27.3](#) [PropLS2](#), we deduce that

$$\sum_{q \leq Q} \sum_{(a, q)=1} \left| \psi^{(R)}(N; q, a) \right|^2 \ll \left(\frac{N}{R} \log N + Q \right) N \log N \log \log N.$$

by using the prime number theorem. Now, if $\chi \pmod{q}$ is induced from $\psi \pmod{r}$ then

$$\sum_{n \leq N} \chi(n) \Lambda(n) = \sum_{n \leq N} \psi(n) \Lambda(n) - \sum_{\substack{p^a \leq N \\ p|q, \ p \nmid r}} \psi(p^a) \log p,$$

hence the error term in replacing χ by ψ here is $\ll (\omega(q) - \omega(r)) \log N$, and in the square is $\ll (\omega(q) - \omega(r)) N \log N$, Therefore the total such error is

$$\ll \sum_{r \leq R} \sum_{\substack{q \leq Q \\ r|q}} \frac{\omega(q) - \omega(r)}{\phi(q)} N \log N \ll N \log N \log Q \log R (\log \log Q)^2 \ll N (\log N)^{2+\epsilon},$$

which is smaller than the above. What remains is, by ^(SumSqs)(16.1),

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\substack{r \leq R \\ r|q}} \sum_{\psi}^* \left| \sum_{n \leq N} \psi(n) \Lambda(n) \right|^2 = \sum_{r \leq R} \frac{1}{\phi(r)} \sum_{\psi}^* \left| \sum_{n \leq N} \psi(n) \Lambda(n) \right|^2 \sum_{\substack{q \leq Q \\ r|q}} \frac{\phi(r)}{\phi(q)},$$

and the result follows. \square

Using this we can now prove the Barban-Davenport-Halberstam-Lavrik-Montgomery-Hooley theorem.

Proof of Theorem 27.1 ^(BDHLMH) Let $Q' = Q/\log^2 N$ and $R = (N \log^3 N)/Q$, and use the Siegel-Walfisz Theorem with $A = 2C + 6$ and $B = C + 2$ so that Corollary ^(CorLS3)27.4 yields

$$\sum_{q \leq Q'} \sum_{(a,q)=1} \left| \psi(N; q, a) - \frac{\psi(N)}{\phi(q)} \right|^2 \ll QN.$$

We are left with the sum for $Q' < q \leq Q$, which we will treat as the sum for $Q' < q \leq N$, minus the sum for $Q < q \leq N$. We describe only how we manipulate the second sum, as the first is entirely analogous.

Now the q th term in our sum equals

$$\sum_{p \leq N} \log^2 p + 2 \sum_{\substack{p_1 < p_2 \leq N \\ p_2 \equiv p_1 \pmod{q}}} \log p_1 \log p_2 - \frac{\psi(N)^2}{\phi(q)},$$

plus a small, irrelevant error term made up of contributions from prime powers that divide q . We sum the middle term over all q in the range $Q < q \leq N$. Writing $p_2 = p_1 + qr$ we have $r \leq N/q < N/Q$, so that $p_2 \equiv p_1 \pmod{r}$ with $N \geq p_2 \geq p_1 + Qr$, and therefore the sum equals

$$\begin{aligned} & 2 \sum_{r \leq N/Q} \sum_{p \leq N-Qr} \{ \psi(N; r, p) - \psi(p + Qr; r, p) \} \log p \\ &= \sum_{r \leq N/Q} \frac{2}{\phi(r)} \sum_{p \leq N-Qr} (N - p - Qr) \log p + O \left(\sum_{r \leq N/Q} \frac{N^2}{\phi(r) \log^B N} \right) \\ &= \sum_{r \leq N/Q} \frac{(N - Qr)^2}{\phi(r)} + O(NQ) \\ &= cN^2 (\log N/Q + \gamma' - 3/2) + O(QN \log(N/Q)), \end{aligned}$$

by the Siegel-Walfisz theorem and Lemma ^(LemLS1)27.2. We deduce that the sum of the middle terms over all q in the range $Q' < q \leq Q$ is therefore $cN^2 \log Q/Q' +$

$O(QN \log(N/Q))$. On the other hand the sum of the final term over all $Q' < q \leq Q$ is, by Lemma 27.2, $c\psi(N)^2 \log Q/Q' + O(N^2/Q' \cdot \log N)$. Using the strong version of the prime number theorem these two terms sum to $O(QN \log(N/Q))$. By the prime number theorem with error term $O(N/\log N)$, the first term sums to $QN \log N + O(QN \log(N/Q))$, yielding the result. \square

27.2 The Bombieri-Vinogradov Theorem

This is an extremely useful tool in analytic number theory, showing that the primes up to x are well distributed in arithmetic progressions mod q , “on average” over $q \leq x^{1/2+o(1)}$.

The Bombieri-Vinogradov Theorem. For any fixed $A > 0$ there exists $B = B(A) > 0$ such that

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \right| \ll \frac{x}{(\log x)^A}$$

where $Q = \sqrt{x}/(\log x)^B$. In fact one can take any fixed $B > A + 3$.

Select χ_1 to be that primitive character with conductor in $(1, R]$ for which $|\sum_{n \leq x} \chi(n) \Lambda(n)|$ is maximized. The strong form of the Siegel-Walfisz Theorem (which needs to be given in the previous section) states that if primitive $\chi \neq 1$ or χ_1 then $|\sum_{n \leq x} \chi(n) \Lambda(n)| \ll x/e^{4c\sqrt{\log x}}$.

StrongBV

Corollary 27.5 If $x^{1/2}/e^{c\sqrt{\log x}} \leq Q \leq x^{1/2}$ then

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \psi(x; q, a) - \frac{\psi(x)}{\phi(q)} - \bar{\chi}_1(a) \frac{\psi(x, \chi_1)}{\phi(q)} \right| \ll Q\sqrt{x} \log^{3+o(1)} x,$$

where the χ_1 term is only included if $\text{cond}(\chi_1) | q$.

With $Q = x^{1/2}/e^{c\sqrt{\log x}}$ we see that we get a much stronger bound than in the Bombieri-Vinogradov Theorem at the cost of including χ in our terms.

In order to prove these results we continue to develop the large sieve.

PropLS6

Proposition 27.6 We have

$$\begin{aligned} & \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}^* \chi \geq R}} \left| \sum_{m=X+1}^{X+M} a_m \chi(m) \right| \left| \sum_{n=Y+1}^{Y+N} b_n \chi(n) \right| \\ & \ll \left(\frac{\sqrt{MN}}{R} \log Q + Q + (\sqrt{M+N}) \log^2 Q \right) \log \log Q \sqrt{\sum_{m=X+1}^{X+M} |a_m|^2 \cdot \sum_{n=Y+1}^{Y+N} |b_n|^2}. \end{aligned}$$

Exercise 27.7 *Prove this result. Remarks: If one Cauchy's the result in Proposition 27.3 one obtains a weaker result, with $\log^2 Q$ replaced by $\sqrt{(Q/R)} \log Q$ as the coefficient of $\sqrt{M+N}$ in the bound given. To prove the above one proceeds analogously to the proof of Proposition 13.2. One can Cauchy in this exercise with m fixed, to obtain the result given here.*

PropLS8

Proposition 27.8 *Suppose that a_n, b_n are given sequences with $a_n, b_n = 0$ for $n \leq R^2$, and $|a_n| \leq a_0$, $|b_n| \leq b_0$ for all $n \leq x$. If $c_N := \sum_{mn=N} a_m b_n$ then*

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}^* \chi \geq R}} \left| \sum_{N \leq x} c_N \chi(N) \right| \ll a_0 b_0 \left(\frac{x}{R} + Q\sqrt{x} \right) \log^2 x \log \log x.$$

Proof We begin by noting that

$$\sum_{N \leq x} c_N \chi(N) = \sum_{mn \leq x} a_m \chi(m) \cdot b_n \chi(n).$$

We will partition the pairs m, n with $mn \leq x$ in order to apply Proposition 27.6. For the intervals $X < m \leq X + M$, $Y < n \leq Y + N$, Proposition 27.6 yields the upper bound

$$a_0 b_0 \sqrt{MN} \left(\frac{\sqrt{MN}}{R} \log Q + Q + (\sqrt{M+N}) \log^2 Q \right) \log \log Q$$

We now describe the partition for m in the range $X < m \leq 2X$. Let $Y = x/X$. We begin with all $X < m \leq 2X$, $n \leq Y/2$. Then in step k , with $k = 1, 2, \dots, K$, we take

$$\left(1 + \frac{2^j}{2^k}\right) X < m \leq \left(1 + \frac{2^{j+1}}{2^k}\right) X, \quad Y / \left(1 + \frac{2^{j+2}}{2^k}\right) < n \leq Y / \left(1 + \frac{2^{j+1}}{2^k}\right),$$

for $0 \leq j \leq 2^{k-1} - 1$. The total upper bound from all these terms is

$$\ll a_0 b_0 \sqrt{XY} \left(\frac{\sqrt{XY}}{R} \log Q + KQ + (\sqrt{X+Y}) \log^2 Q \right) \log \log Q.$$

Let K be such that $2^K \asymp Y$. Then, for each m , $X < m < 2X$ there are $\ll 1$ values of $n \leq x/m$ not yet accounted. Hence these missing pairs contribute $\ll a_0 b_0 QX$, and so in our construction we interchange X and Y to guarantee that $X \leq Y$. Hence the total error from these unaccounted-for points is $\ll Q\sqrt{x}$ in total.

We now sum up the upper bound over $X = 2^j R^2$ for $j = 0, 1, 2, \dots, J$ where $2^J = x/R^4$ (since if $m < R^2$ then $b_m = 0$, and if $m > x/R^2$ then $n < R^2$ and so $c_n = 0$), to obtain the claimed upper bound. \square

We now prove a version of the Bombieri-Vinogradov Theorem:

CorLS9 **Corollary 27.9** *If $R \leq e^{\sqrt{\log x}}$ and $Q \leq x^{1/2}$ then*

$$\sum_{q \leq Q} \max_{(a,q)=1} |\psi^{(R)}(x; q, a)| \ll \left(\frac{x}{R} + Q\sqrt{x} \right) \log^3 x \log \log x.$$

Proof The left side is evidently

$$\leq \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \text{cond}^* \chi \geq R}} |\psi(x, \chi)|.$$

Our goal is to bound this using Proposition 27.8 so, as in the proof of Theorem 20.1, we let g be totally multiplicative with $g(p) = 0$ if $p \leq R^2$ and $g(p) = 1$ otherwise. Then we let $a_n = g(n)\mu(n)$ for $n > 1$ and $b_m = g(m) \log m$. To be able to apply Proposition 27.8 we are forced to take $a_1 = 0$ (rather than 1 as in the proof of Theorem 20.1) and so $(a * b)(n) = \Lambda_{R^2}(n) - g(n) \log n$. We substitute this into Proposition 27.8, and bound the contribution of the powers of the primes $\leq R^2$ by $\leq Q \sum_{p \leq R^2} \log x \ll QR^2 \log x / \log R \ll Q\sqrt{x}$, which yields the above upper bound for the sum of $|\psi(x, \chi) - G(x, \chi)|$, and hence for the sum of $\max_{(a,q)=1} |\psi^{(R)}(x; q, a) - G^{(R)}(x; q, a)|$ where $G(x, \chi) := \sum_{n \leq x} \chi(n)g(n) \log n$ and $G(x; q, a) := \sum_{n \leq x, n \equiv a \pmod{q}} g(n) \log n$.

Now, by the small sieve, we know that

$$G^{(1)}(x; q, a) \ll \frac{x \log x}{\phi(q) \log R} \cdot \frac{1}{u^u} + \left(\frac{x}{q} \right)^{1/2}$$

where $x = R^{2u}$, so that this is $\ll x/\phi(q)e^{4\sqrt{\log x}}$. We immediately deduce that

$$\sum_{q \leq Q} \max_{(a,q)=1} |G^{(R)}(x; q, a)| \ll \sum_{q \leq Q} \left(\sum_{\substack{r \leq R \\ r|q}} \phi(r) \right) \cdot \frac{x}{\phi(q)e^{4\sqrt{\log x}}} \ll \frac{x}{e^{\sqrt{\log x}}},$$

and the result follows. \square

Corollary 27.10 *Fix $A > 0$. If $x^{1/2}/\log^A x < Q \leq x^{1/2}$ then*

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \right| \ll Q\sqrt{x} \log^3 x \log \log x$$

Proof Let $R = \log^{A+1} x$ in Corollary 27.9, and bound $|\psi^{(R)}(x; q, a) - \psi^{(1)}(x; q, a)|$ by the Siegel-Walfisz Theorem. \square

The Bombieri-Vinogradov Theorem is an immediate consequence of this result.

Proof of Theorem 27.5 Let $R = e^{c\sqrt{\log x}}$ in Corollary 27.9. There are at most R^2 characters $\neq 1$ or χ_1 in the sum $\psi^{(R)}(x; q, a)$, and hence their contribution is $\ll (R^2/\phi(q))x/e^{4c\sqrt{\log x}}$. Summing over all $q \leq Q$, their total contribution is $\ll x/R$. The result follows. \square

INTEGRAL DELAY EQUATIONS

We have seen two basic examples of multiplicative functions:

- Those for which $f(p) = 1$ for all primes $p > y$, and typically the mean value of $f(n)$ tends to $\mathcal{P}(f, x)$, an Euler product (see Proposition 3.6). GenFundLem
- Those for which $f(p) = 1$ for all primes $p \leq y$. We saw the example of the smooth numbers, for which the mean value up to y^u is given by $\rho(u)$, a function which we defined in terms of an integral delay equation. We will now show that this is typical.

FnToDelEqn

Proposition 28.1 *Suppose that f is a totally multiplicative function with $f(p) = 1$ for all $p \leq y$. Define*

$$\chi(t) := \frac{1}{\psi(y^t)} \sum_{d \leq y^t} f(d) \Lambda(d),$$

(where $\psi(x) = \sum_{m \leq x} \Lambda(m)$ as usual) so that $|\chi(t)| \leq 1$ for all t , and $\chi(t) = 1$ if $t \leq 1$. Let $\sigma(t) = 1$ if $t \leq 1$, and

$$\sigma(u) = \frac{1}{u} \int_0^u \sigma(u-t) \chi(t) dt \quad \text{for all } u \geq 1. \quad (28.1) \quad \text{IntDelEqn}$$

(Typically one writes $(g * h)(u) := \int_0^u g(u-t) h(t) dt$ for the (integral) convolution of the two functions g and h .) Then, for $x = y^u$, we have

$$\frac{1}{x} \sum_{m \leq x} f(m) = \sigma(u) + O\left(\frac{u}{\log y}\right). \quad (28.2) \quad \text{IntDel1}$$

Exercise 28.2 *Convince yourself that the functional equation for estimating smooth numbers, that we gave earlier, is a special case of this result.*

Proof Define $s(t) := S(y^t)/y^t = y^{-t} \sum_{m \leq y^t} f(m)$ so that $s(t) = 1 + O(y^{-t})$ if $t \leq 1$. We note that $\sum_{p \leq x} |1 - f(p)|/p \leq 2 \sum_{y < p \leq x} 1/p \leq 2 \log u + O(1/\log y)$. Then, by (7.7) and the prime number theorem in the form $\psi(D) = D + O(D/(\log D)^{1+\epsilon})$, we obtain

$$s(u) = \frac{1}{u} \int_0^u s(u-t) \chi(t) dt + O\left(\frac{u}{\log y}\right).$$

Now if $\Delta(u) = |s(u) - \sigma(u)|$ then we deduce that $\Delta(t) \leq y^{-t}$ if $t \leq 1$, and

$$\Delta(u) \leq \frac{1}{u} \int_0^u \Delta(u-t) dt + \frac{Cu}{\log y},$$

for some constant $C > 0$. We claim that $\Delta(v) < 2Cv/\log y$ for all $v > 0$ for if not, let u be minimal for which $\Delta(u) \geq 2Cu/\log y$, so that

$$\frac{Cu}{\log y} \leq \Delta(u) - \frac{Cu}{\log y} < \frac{1}{u} \int_0^u \frac{2C(u-t)}{\log y} dt = \frac{Cu}{\log y},$$

a contradiction. Thus $\frac{\text{IntDel1}}{(28.2)}$ follows. \square

This result shows that the mean value of every such multiplicative function can be determined in terms of an integral delay equation.

28.1 Remarks on $\frac{\text{IntDelEqn}}{(28.1)}$

We shall suppose that χ is a measurable function $\chi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{U}$ with $\chi(t) = 1$ for $0 \leq t \leq 1$, and then define $\sigma(t)$ as in Proposition $\frac{\text{FnToDelEqn}}{28.1}$. We make a few straightforward observations:

- Since each $|\chi(t)| \leq 1$ hence $|\sigma(u)| \leq M_\sigma(u) := \frac{1}{u} \int_0^u |\sigma(t)| dt$.
- $|\sigma(u)| \leq 1$ for all $u \geq 0$ for, if not, there exists $u > 1$ for which $|\sigma(u)| \geq |\sigma(t)|$ for all $0 \leq t \leq u$ and hence $|\sigma(u)| \geq M_\sigma(u)$. But this would imply $|\sigma(u)| = M_\sigma(u) = |\sigma(t)|$ for all $0 \leq t \leq u$, and in particular $|\sigma(u)| = 1$.
- $M_\sigma(u)$ is a non-increasing function since $M'_\sigma(u) = (|\sigma(u)| - M_\sigma(u))/u \leq 0$.

We will now show that there is a unique solution $\sigma(u)$ to $\frac{\text{IntDelEqn}}{(28.1)}$ which can be given as follows: Define $I_0(u) = I_0(u; \chi) = 1$, and for $k \geq 1$,

$$I_k(u) = I_k(u; \chi) = \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \frac{1 - \chi(t_1)}{t_1} \dots \frac{1 - \chi(t_k)}{t_k} dt_1 \dots dt_k.$$

Define for all $k \geq 0$,

$$\sigma_k(u) = \sum_{j=0}^k \frac{(-1)^j}{j!} I_j(u; \chi), \quad \text{and} \quad \sigma_\infty(u) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} I_j(u; \chi).$$

Our goal is to show that $\sigma = \sigma_\infty$. We will see how this representation of σ is a manifestation of the inclusion-exclusion principle.

Exercise 28.3 Show that for all $j \geq 1$,

$$uI_j(u) = (1 * I_j)(u) + j((1 - \chi) * I_{j-1})(u).$$

Deduce that $u\sigma_k(u) = (1 * \sigma_k)(u) - ((1 - \chi) * \sigma_{k-1})(u)$. Then show that $\sigma_\infty(u) = 1$ for $0 \leq u \leq 1$, and that $u\sigma_\infty(u) = (\sigma_\infty * \chi)(u)$ for $u > 0$.

To show that σ_∞ is the unique such function, suppose that we have another solution σ . Note that $|\sigma(u) - \sigma_\infty(u)| = 0$ for $0 \leq u \leq 1$ and

$$u|\sigma(u) - \sigma_\infty(u)| = \left| \int_0^u (\sigma(t) - \sigma_\infty(t))\chi(u-t)dt \right| \leq \int_0^u |\sigma(t) - \sigma_\infty(t)|dt.$$

Exercise 28.4 Modify the proof given above to show that $|\sigma(u)| \leq 1$ to now prove that $|\sigma(u) - \sigma_\infty(u)| = 0$ for all $u \geq 0$.

CompareTwoChi

Exercise 28.5 Suppose that χ and χ' are two such functions, and let σ and σ' be the corresponding solutions to (28.1). ^{IntDelEqn} Prove that $\sigma(u) - \sigma'(u)$ equals

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{\substack{t_1, \dots, t_j \geq 1 \\ t_1 + \dots + t_j \leq u}} \frac{\chi'(t_1) - \chi(t_1)}{t_1} \dots \frac{\chi'(t_j) - \chi(t_j)}{t_j} \sigma'(u - t_1 - \dots - t_j) dt_1 \dots dt_j.$$

Deduce that if $|\chi(t) - \chi'(t)| \leq \epsilon$ for all t then $|\sigma(u) - \sigma'(u)| \leq u^\epsilon - 1$, for all $u \geq 1$.

Exercise 28.6 Suppose that χ and χ' are two such functions with $\chi(t) = \chi'(t)$ for $0 \leq t \leq u/2$. Deduce that

$$\sigma(u) - \sigma'(u) = \int_{u/2 \leq t \leq u} \frac{\chi(t) - \chi'(t)}{t} \sigma'(u - t) dt.$$

28.2 Inclusion-Exclusion inequalities

Our formula for $\sigma_\infty (= \sigma)$ looks like an inclusion-exclusion type identity. For a real-valued function χ , we now show how to obtain inclusion-exclusion inequalities for σ .

Proposition 28.7 Suppose that $\chi(t) \in \mathbb{R}$ for each t . Then, for all integers $k \geq 0$, and all $u \geq 0$, we have

$$\sigma_{2k+1}(u) \leq \sigma(u) \leq \sigma_{2k}(u).$$

Proof In (28.1) ^{IntDelEqn} we had $u\sigma = 1 * \sigma - (1 - \chi) * \sigma$. Subtracting $u\sigma_k = 1 * \sigma_k - (1 - \chi) * \sigma_{k-1}$ (which we proved in exercise 3.1) we obtain

$$u\alpha_k = 1 * \alpha_k + (1 - \chi) * \alpha_{k-1}.$$

where $\alpha_k(u) = (-1)^{k+1}(\sigma(u) - \sigma_k(u))$. We wish to prove that $\alpha_k(u) \geq 0$ for all $u \geq 0$, for each $k \geq 0$. For $k = 0$ we have $\sigma_0 = 1$ and so $\alpha_0(u) = 1 - \sigma(u) \geq 0$ by the above. Then, by induction, we have that $u\alpha_k(u) \geq (1 * \alpha_k)(u)$ as $1 - \chi(t), \alpha_{k-1}(u - t) \geq 0$, and then we deduce our result as in the proof given to show that $|\sigma(u)| \leq 1$. \square

Remark 28.8 It would be good to improve Proposition ^{GenFundLem} 3.6 to an estimate like $(1 + o_{u \rightarrow \infty}(1))\mathcal{P}(f; x) + o(1/(\log x)^A)$, as in the Fundamental Lemma of Sieve Theory. The proof there works for f with $0 \leq f(n) \leq 1$. As a first goal we could aim for all real-valued f , that is where $-1 \leq f(n) \leq 1$, for all n . This Proposition perhaps can help us use the technology of sieve theory to do this?

28.3 A converse Theorem

We now show that for every (appropriate) such integral delay equation there is an appropriate multiplicative function whose mean value can be determined in terms of that integral delay equation.

Converse

Proposition 28.9 *Let S be a closed subset of \mathbb{U} and suppose that χ is a measurable function whose values lie in, $K(S)$, the convex hull of S , with $\chi(t) = 1$ for all $t \leq 1$. Given $\epsilon > 0$ and $u \geq 1$ there exist arbitrarily large y and $f \in \mathcal{F}(S)$ with $f(n) = 1$ for $n \leq y$ and*

$$\left| \chi(t) - \frac{1}{\psi(y^t)} \sum_{m \leq y^t} f(m) \Lambda(m) \right| \leq \epsilon \text{ for almost all } 0 \leq t \leq u.$$

Consequently, if $\sigma(u)$ is the solution to $\frac{\text{IntDelEqn}}{(28.1)}$ for this χ then

$$\frac{1}{y^u} \sum_{n \leq y^u} f(n) = \sigma(u) + O(u^\epsilon - 1) + O\left(\frac{u}{\log y}\right).$$

In particular if $\epsilon = \sigma(u)/u \log u$ then

$$\frac{1}{y^u} \sum_{n \leq y^u} f(n) = \sigma(u) + O\left(\frac{\sigma(u)}{u} + \frac{u}{\log y}\right).$$

Proof Since χ is measurable and $\chi(t)$ belongs to the convex hull of S , we can find a step function χ_1 within the convex hull of S such that $\chi_1(t) = 1$ for $t \leq 1$, and $|\chi(t) - \chi_1(t)| \leq \epsilon/2$ for almost all $t \in [0, u]$.²

Now $\chi_1(t)$ belongs to the convex hull of S and so can be arbitrarily well-approximated by (integral) linear combinations of elements of S . Hence if $\chi_1(t)$ has a fixed value in (t_1, t_2) then we can select the set of values of $f(p) \in S$ when $y^{t_1} < p < y^{t_2}$ to reflect such a linear combination, and therefore if

$$\chi'(t) := \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} f(p) \log p,$$

then $|\chi'(t) - \chi_1(t)| \leq \epsilon/2$ for almost all $t \in [0, u]$. Hence $|\chi(t) - \chi'(t)| \leq \epsilon$ for almost all $t \in [0, u]$. The proof in exercise 3.3 then implies that $|\sigma(u) - \sigma'(u)| \leq u^\epsilon - 1$, for all $u \geq 1$, and the result then follows from $\frac{\text{IntDef1}}{(28.2)}$. \square

28.4 An example for Halasz's Theorem

Now suppose that $\chi'(t) = 1$ if $t \leq 1$, $\chi'(t) = i$ if $1 < t \leq u/2$ and $\chi'(t) = 0$ if $t > u/2$. We let $\chi(t) = \chi'(t)$ for $t \leq u/2$. Suppose that $\sigma'(u) = e^{i\theta} |\sigma'(u)|$. For

²By *almost all*, we mean that the inequality is only violated on a set of measure 0.

$u/2 < t \leq u$ we let $\chi(t) = e^{i(\theta-\psi)}$ where $\sigma'(u-t) = e^{i\psi}|\sigma'(u-t)|$. Hence, by the previous exercise

$$\sigma(u) = \sigma'(u) + \int_{u/2 \leq t \leq u} \frac{\chi(t)}{t} \sigma'(u-t) dt = e^{i\theta} \left(|\sigma'(u)| + \int_{u/2 \leq t \leq u} \frac{|\sigma'(u-t)|}{t} dt \right).$$

Let α be a complex number with $\operatorname{Re}(\alpha) < 1$, and let ρ_α denote the unique continuous solution to $u\rho'_\alpha(u) = -(1-\alpha)\rho_\alpha(u-1)$, for $u \geq 1$, with the initial condition $\rho_\alpha(u) = 1$ for $u \leq 1$ (The Dickman-De Bruijn function is the case $\alpha = 0$).³ For $\alpha \in [0, 1]$, Goldston and McCurley [5] gave an asymptotic expansion of ρ_α ,⁴ and showed that when α is not an integer

$$\rho_\alpha(u) \sim \frac{e^{\gamma(1-\alpha)}}{\Gamma(\alpha)u^{1-\alpha}},$$

as $u \rightarrow \infty$.⁵ In our example $\sigma'(v) = \rho_i(v)$ for $v \leq u/2$, and so, for $c = e^\gamma/|\Gamma(i)| = 3.414868086\dots$ we have $|\sigma'(v)| \sim c/v$. Hence taking $v = u - t$ above

$$|\sigma(u)| \gtrsim \int_{1 \leq v \leq u/2} \frac{c}{v(u-v)} dv = \frac{c \log(u-1)}{u} \gg \frac{\log u}{u} \gg Me^{-M},$$

since, in this example we have

$$\begin{aligned} M(x, T) &\geq \min_{y \in \mathbb{R}} \left(\int_0^1 \frac{1 - \cos(vy)}{v} dv + \int_1^{u/2} \frac{1 - \sin(vy)}{v} dv \right), \\ &\geq \log u/2 + \min_{y \in \mathbb{R}} \int_0^y \frac{1 - \cos t + \sin t}{t} dt - \max_{Y \in \mathbb{R}} \int_0^Y \frac{\sin t}{t} dt \geq \log u - O(1). \end{aligned}$$

³We will discuss this example in more detail a little later. Perhaps we should combine the two discussions.

⁴Their proof is in fact valid for all complex α with $\operatorname{Re}(\alpha) < 1$

⁵Just as we saw in the Selberg-Tenenbaum Theorem, when α is an integer the behaviour of ρ_α is very different; in fact $\rho_\alpha(u) = 1/u^{u+o(u)}$. *Exercise:* Use the Structure theorem to compare these results.

LAPLACE TRANSFORMS

For a measurable function $g : [0, \infty) \rightarrow \mathbb{C}$ we will denote the Laplace transform of g by $\mathcal{L}(g, s) := \int_0^\infty g(t)e^{-st} dt$. If g is integrable and grows sub-exponentially (that is, for every $\epsilon > 0$, $|g(t)| \ll_\epsilon e^{\epsilon t}$ almost everywhere) then the Laplace transform is well defined for all complex numbers s with $\operatorname{Re}(s) > 0$. We begin with (28.1). Multiplying through by ue^{-su} and integrating over all $u \geq 0$ yields

$$\begin{aligned} -\mathcal{L}'(\sigma, s) &= \int_0^\infty u\sigma(u)e^{-su} du = \int_0^\infty \int_{t=0}^u \chi(t)e^{-st}\sigma(u-t)e^{-s(u-t)} du \\ &= \mathcal{L}(\sigma, s)\mathcal{L}(\chi, s). \end{aligned}$$

Dividing through by $\mathcal{L}(\sigma, s)$ and integrating yields

$$\mathcal{L}(\sigma, w) = \mathcal{L}(\sigma, 0) \exp\left(-\int_0^w \mathcal{L}(\chi, s) ds\right).$$

Exercise 29.1 Show that

$$\mathcal{L}(I_k(u, \chi), w) = \frac{1}{s} \mathcal{L}\left(\frac{1 - \chi(v)}{v}, s\right)^k.$$

Since $\mathcal{L}(\sigma, s) = \mathcal{L}(\sigma_\infty, s) = \sum_{k \geq 0} (-1)^k \mathcal{L}(I_k(u, \chi), w)/k!$ deduce that

$$\mathcal{L}(\sigma, s) = \frac{1}{s} \exp\left(-\mathcal{L}\left(\frac{1 - \chi(v)}{v}, s\right)\right).$$

Deduce, or use exercise [28.5](#) to show that, more generally

$$\mathcal{L}(\sigma_1, s) = \mathcal{L}(\sigma_2, s) \exp\left(\mathcal{L}\left(\frac{\chi_1(v) - \chi_2(v)}{v}, s\right)\right).$$

We define

$$E(u) = E_\chi(u) := \exp\left(\int_0^u \frac{1 - \chi(t)}{t} dt\right).$$

sigmaUB

Lemma 29.2 Suppose that $\chi(t) = 1$ for $t \leq 1$ and $0 \leq \chi(t) \leq 1$ for all t . Given u define $\hat{\chi}(t) = \chi(t)$ for $t \leq u$ and $\hat{\chi}(t) = 0$ for $t > u$. We have

$$\sigma(u) \leq \frac{e^\gamma}{E(u)} - \frac{1}{u} \int_u^\infty \hat{\sigma}(t) dt.$$

Proof By definition $\sigma(v) = \hat{\sigma}(v)$ and $E_\chi(v) = E_{\hat{\chi}}(v)$ for $v \leq u$. Now

$$\sigma(u) = \hat{\sigma}(u) = \frac{1}{u} \int_0^u \hat{\sigma}(t) \hat{\chi}(u-t) dt \leq \frac{1}{u} \int_0^u \hat{\sigma}(t) dt = \frac{1}{u} \int_0^\infty \hat{\sigma}(t) dt - \frac{1}{u} \int_u^\infty \hat{\sigma}(t) dt.$$

For s a small positive real $< 1/u$, we have

$$\begin{aligned} \mathcal{L}\left(\frac{1-\hat{\chi}(t)}{t}, s\right) - \log E(u) &= \int_0^\infty \left(\frac{1-\hat{\chi}(t)}{t}\right) e^{-st} dt - \int_0^u \frac{1-\chi(t)}{t} dt \\ &= \int_0^u \left(\frac{1-\chi(t)}{t}\right) (e^{-st} - 1) dt + \int_u^\infty \frac{e^{-st}}{t} dt \\ &= -\gamma - \log(us) + O(us), \end{aligned}$$

since $\gamma = \int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt$. Hence

$$\frac{1}{u} \int_0^\infty \hat{\sigma}(t) dt = \frac{1}{u} \lim_{s \rightarrow 0} \mathcal{L}(\hat{\sigma}, s) = \lim_{s \rightarrow 0} \frac{1}{us} \exp\left(-\mathcal{L}\left(\frac{1-\hat{\chi}(t)}{t}, s\right)\right) = \frac{e^\gamma}{E(u)},$$

and so we have the result. \square

THE SPECTRUM

30.1 The Mean Value Spectrum

We are interested in what are the possible mean values of multiplicative functions in certain classes; for example, characters of order m . To this end we let S be a given subset of the unit disc \mathbb{U} , and let \mathbb{T} be the unit circle. Let $\mathcal{F}(S)$ denote the class of completely multiplicative functions f such that $f(p^k) \in S$ for all prime powers p^k .⁶ Our main concern is:

What numbers arise as mean-values of functions in $\mathcal{F}(S)$?

That is, we define $\Gamma_N(S) = \{ \frac{1}{N} \sum_{n \leq N} f(n) : f \in \mathcal{F}(S) \}$ and then seek to understand the *(mean value) spectrum*

$$\Gamma(S) = \lim_{N \rightarrow \infty} \Gamma_N(S).^7$$

The case of most interest to us is S_m , defined as 0 together with the m th roots of unity, because $\mathcal{F}(S_m)$ yields the possible character sums of characters of order m . We begin by making some simple observations.

We see that $\Gamma(\{1\}) = \{1\}$, and if $S_1 \subset S_2$ then $\Gamma(S_1) \subset \Gamma(S_2)$. Moreover $\Gamma(S)$ is a closed subset of the unit disc \mathbb{U} , and $\Gamma(S) = \Gamma(\overline{S})$ where \overline{S} denotes the closure of S , and so, henceforth, we shall assume that S is closed.

The hypothesis implies that the set S is closed under taking integer powers, for if $\alpha \in S$ then let $f(p) = \alpha$ and so $\alpha^k = f(p^k) \in S$ for all k .

RealElt

Exercise 30.1 *Deduce that if there exists $\alpha \in S$ with $\alpha \neq 1$ then there exists a real number $\beta \in K(S)$, the convex hull of S .*

1SinS_m

Lemma 30.2 $\Gamma(S) = \mathbb{U}$ or $S \cap \mathbb{T}$ is finite and only contains roots of unity.

Proof If $\alpha \in \mathbb{T}$ but is not a root of unity then the set $\{\alpha^k : k \geq 1\}$ is (well-known to be) dense on the unit circle, \mathbb{T} . Hence if $\alpha \in S$ then the closure of $\{\alpha^k : k \geq 1\} \subset S$ is \mathbb{T} , and so $\mathbb{T} \subset S$, since S is closed.

But then the multiplicative function $f(n) = n^{it} \in S$ has mean value $\sim N^{it}/(1+it)$ up to N . As we let $N \rightarrow \infty$ we deduce that $\Gamma(S)$ contains the circle $\{z : |z| = 1/|1+it|\}$. By letting t range in $(0, \infty)$ we deduce that $\Gamma(S) = \mathbb{U}$.

⁶One can develop this theory under the less stringent conditions that (i) f is multiplicative but not necessarily completely multiplicative; (ii) $f(p) \in S$ for all primes p , but not necessarily for prime powers. Change (i) requires only minor adjustments, whereas change (ii) makes the theory somewhat more complicated.

⁷Here and henceforth, if we have a sequence of subsets J_N of the unit disc $\mathbb{U} := \{|z| \leq 1\}$, then by writing $\lim_{N \rightarrow \infty} J_N = J$ we mean that $z \in J$ if and only if there is a sequence of points $z_N \in J_N$ with $z_N \rightarrow z$ as $N \rightarrow \infty$.

Hence if $\Gamma(S) \neq \mathbb{U}$ then all elements of $S \cap \mathbb{T}$ are roots of unity. Moreover there are only finitely many, else they have an accumulation point which is not a root of unity, and since S (and hence $S \cap \mathbb{T}$) is closed, this point belongs to S . \square

Henceforth we assume that $S \cap \mathbb{T}$ is finite and only contains roots of unity.⁸

Exercise 30.3 Show that if $1 \in S$ then $1 \in \Gamma(S)$.

NoPtsNrT

Exercise 30.4 Suppose that $S \cap \mathbb{T}$ is finite. Fix $\epsilon > 0$. Show that there exists $\delta > 0$ such that if $z \in \mathbb{T}$ and $|z - s| \geq \epsilon$ for all $s \in S \cap \mathbb{T}$, then $|z - s| \geq \delta$ for all $s \in S$.

Exercise 30.5 Show that if there exists $s \in S$ such that $s \neq 1$ then $0 \in \Gamma(S)$. Show that if $1 \notin S$ then $\Gamma(S) = \{0\}$. (Hint: Use Proposition 3.6). (GenFundLem)

Henceforth we may assume that there exists $1, \alpha \in S$ with $\alpha \neq 1$ and therefore there exists a real number $\beta \in K(S)$ by exercise 30.1. (RealElt)

If $z \in \mathbb{U} \setminus \{1\}$ then define $\text{Ang}(z) = \arg(1 - z)$, so that $-\pi/2 < \text{Ang}(z) < \pi/2$. For any $V \subseteq \mathbb{U}$, define $\text{Ang}(V)$ to be the supremum of $|\arg(1 - v)|$ as we range over all $v \in V$ with $v \neq 1$. We will obtain the following improvement of the last lemma:

SinS_m

Proposition 30.6 $\Gamma(S) = \mathbb{U}$ if and only if $\text{Ang}(S) = \pi/2$. If $\Gamma(S) \neq \mathbb{U}$ then there exists an integer m such that S lies within the convex hull of the m th roots of unity; that is $S \subset K(S) \subset K(S_m)$.

AngMaxRegion

Exercise 30.7 Suppose that $\text{Ang}(S) = \frac{\pi}{2} - \delta$. Prove that S is contained in the convex hull of $\{1\} \cup \{e^{i\theta} : 2\delta \leq |\theta| \leq \pi\}$.

30.2 Factoring mean values

Our first step in understanding the spectrum is to prove that when $S \cap \mathbb{T}$ is finite a version of Theorem 15.1 holds with $t = 0$: (StructThm)

MeanValueStructure

Theorem 30.8 Suppose that S is a closed, proper subset of \mathbb{U} , and that $f \in \mathbb{F}(S)$. Let $g(p^k) = 1$, $h(p^k) = f(p^k)$ if $p \leq y$, and $g(p^k) = f(p^k)$, $h(p^k) = 1$ if $p > y$, for all $k \geq 1$. If $x = y^u$ then

$$\frac{1}{x} \sum_{n \leq x} f(n) = \frac{1}{x} \sum_{n \leq x} g(n) \cdot \frac{1}{x} \sum_{n \leq x} h(n) + o_{u, y \rightarrow \infty}(1).$$

Throughout this section we will suppose that the mean value of f , up to N , is $\geq \delta$ in absolute value. Then $|t_f(x, \log x)| \ll 1/\delta$ by Theorem 15.1. We can also obtain upper bounds on the mean value directly from Halasz's Theorem: (StructThm)

⁸Had we required all $f(n) \in S$ then S would be closed under multiplication, and so $S \cap \mathbb{T}$ would be the set of m th roots of unity, for some integer $m \geq 1$.

Proposition 30.9 Suppose that S is a proper, closed subset of \mathbb{U} . Define

$$C_S := \int_0^1 \min_{s \in S} (1 - \operatorname{Re}(se^{-2i\pi\theta})) d\theta.$$

If $t = t_f(x, \log x)$ then

$$\frac{1}{x} \sum_{n \leq x} f(n) \ll C_S \frac{|\log(|t| \log x)|}{(|t| \log x)^{C_S}} + \frac{\log \log x}{\log x}.$$

Proof To obtain this from ^[HallExplicit1](7.7) we need to bound $\mathbb{D}(f, n^{it}, x)$ from below. We may assume that $t \geq 1/\log x$ else the result is trivial. Now

$$\begin{aligned} \mathbb{D}(f, n^{it}, x)^2 &= \sum_{p \leq x} \frac{1 - \operatorname{Re}(f(p)p^{-it})}{p} \geq \sum_{e^{1/t} < p \leq x} \min_{s \in S} \frac{1 - \operatorname{Re}(sp^{-it})}{p} \\ &= \int_1^{t \log x} \min_{s \in S} \frac{1 - \operatorname{Re}(se^{-iv})}{v} dv + O(1) = C_S \log(t \log x) + O(1). \end{aligned}$$

□

Exercise 30.10 Show that

$$C_S = \int_0^1 \min_{z \in K(S)} (1 - \operatorname{Re}(ze^{-2i\pi\theta})) d\theta.$$

Suppose that $|\sum_{n \leq x} f(n)| > \delta x$. Our proposition yields that $|t| \ll_S 1/\log x$. Now for small t ,

$$\mathbb{D}(f, n^{it}, x)^2 - \mathbb{D}(f, 1, x)^2 = \sum_{p \leq x} \frac{\operatorname{Re}(f(p)(1 - p^{-it}))}{p} \geq - \sum_{p \leq x} \frac{|1 - p^{-it}|}{p}$$

which is $\ll |t| \log x$ if $|t| \leq 1/\log x$, and otherwise $\geq -2 \log(|t| \log x) + O(1)$. Therefore we deduce that $\mathbb{D}(f, 1, x)^2 \ll_S \log(1/\delta)$. This implies that $1 \in S$.

Remark 30.11 In a similar vein to the Proposition, Hall [1] asked for the largest constant κ such that $\mathbb{D}(f, n^{it}, x)^2 \geq \kappa \mathbb{D}(f, 1, x)^2$ whenever $f(p) \in S$. This can be re-expressed as

$$\sum_{p \leq x} \frac{\operatorname{Re}((1 - f(p))(\kappa - p^{-it}))}{p} \leq \sum_{p \leq x} \frac{1 - \operatorname{Re}(p^{-it})}{p},$$

and then, by the prime number theorem, as

$$\int_0^1 \min_{s \in S} \operatorname{Re}((1 - s)(\kappa - e^{-2i\pi\theta})) d\theta \leq 1 + O(1/\log(t \log x)).$$

To approximate this we define $\kappa(S)$ to be the maximum κ for which

$$\int_0^1 \min_{s \in S} \operatorname{Re}((1 - s)(\kappa - e^{-2i\pi\theta})) d\theta \leq 1;$$

then $\mathbb{D}(f, n^{it}, x)^2 \geq \kappa(S) \mathbb{D}(f, 1, x)^2 + O(1)$.

Exercise 30.12 Prove that $\kappa(S) \geq \frac{1}{2}(1 - \frac{\lambda(S)}{2\pi})$ where $\lambda(S)$ is the length of the perimeter of S .

Proof of Theorem 30.8. ^{|MeanValueStructure|}

Suppose that $|\sum_{n \leq x} f(n)| > \delta x$ and let $t = t_f(x, \log x)$. We have just proved that $|t| \ll_S 1/\log x$ and $\mathbb{D}(f, 1, x)^2 \ll_S \log(1/\delta)$. Taking $y = x^\epsilon$ where $\epsilon \rightarrow 0$ very slowly, Theorem 15.1 ^{|StructureThm|} then implies that

$$\frac{1}{x} \sum_{n \leq x} f(n) \sim x^{it} \frac{1}{x} \sum_{n \leq x} g_t(n) \cdot \frac{1}{x} \sum_{n \leq x} h_t(n),$$

where $g_t(p^k) = 1$, $h_t(p^k) = f(p^k)/p^{ikt}$ if $p \leq y$, and $g_t(p^k) = f(p^k)/p^{ikt}$, $h_t(p^k) = 1$ if $p > y$. Hence the mean values of g_t and of h_t are both $\geq \delta$ in absolute value. ^{|GenFundLem|}

We focus first on the mean value of $h_t(n)$. By Proposition 3.6 we see that

$$\frac{1}{x} \sum_{n \leq x} h_t(n) \sim \mathcal{P}(h_t; x) = \mathcal{P}(h_t; y) \sim \mathcal{P}(f, y) = \mathcal{P}(h, y) \sim \frac{1}{x} \sum_{n \leq x} h(n),$$

since

$$\left| \left(1 + \frac{h_t(p)}{p} + \frac{h_t(p^2)}{p^2} + \dots \right) - \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots \right) \right| \ll \frac{|t| \log p}{p},$$

and hence

$$\frac{\mathcal{P}(h_t; y)}{\mathcal{P}(f, y)} = \prod_{p \leq y} \left(1 + O\left(\frac{|t| \log p}{p} \right) \right) = 1 + O(|t| \log y) \sim 1.$$

Now, by Lemma 12.4 ^{|AsympT1|} we see that

$$x^{it} \frac{1}{x} \sum_{n \leq x} g_t(n) \sim \frac{1}{x} \sum_{n \leq x} g_t(n) n^{it} = \frac{1}{x} \sum_{n \leq x} g(n) + O\left(\frac{1}{x} \sum_{n \leq x} |g(n) - g_t(n) n^{it}| \right).$$

Now if $|y| = |z| \leq 1$ then $|y - z| \leq |1 - z/y| = |1 - e^{\log(z/y)}| \ll |\log(z/y)|$, and so

$$\begin{aligned} \sum_{n \leq x} |g(n) - g_t(n) n^{it}| &\ll \sum_{n \leq x} |\log(g(n)/g_t(n) n^{it})| \leq \sum_{n \leq x} \sum_{\substack{p^k \parallel n \\ p \leq y}} k |t| \log p \\ &\ll |t| \sum_{\substack{p \leq y \\ k \geq 1}} \log p \cdot \frac{x}{p^k} \ll x |t| \log y = o(x), \end{aligned}$$

since $g(p^k)/g_t(p^k)(p^k)^{it} = (p^{-it})^k$ if $p \leq y$ and $= 1$ if $p > y$. The result follows. \square

30.3 The Structure of the Mean Value Spectrum

Theorem ^{|MeanValueStructure}30.8 allows us to factor the spectrum $\Gamma(S)$ into two parts:

- The first corresponds to mean values for multiplicative functions that only vary from 1 on the small primes. These mean values can be realized in terms of Euler products. We denote this *Euler product spectrum* by $\Gamma_{\mathcal{P}}(S)$.

- The second corresponds to mean values for multiplicative functions that only vary from 1 on the large primes. These mean values can be realized in terms of solutions to integral delay equations. We denote this *delay equation spectrum* by $\Lambda(S)$.

Hence Theorem ^{|MeanValueStructure}30.8 implies that⁹

$$\Gamma(S) = \Gamma_{\mathcal{P}}(S)\Lambda(S).$$

We will now be more precise in analyzing the sets $\Gamma_{\mathcal{P}}(S)$ and $\Lambda(S)$.

30.4 The Euler product spectrum

$\Gamma_{\mathcal{P}}(S)$ is the set of mean values $\frac{1}{x} \sum_{y < p \leq x} f(n)$ where $f \in \mathcal{F}(S)$ with $f(p^k) = 1$ if $p > y$, and $\frac{\log x}{\log y} \rightarrow \infty$. Proposition ^{|GenFundLem}3.6 implies that this is the same as the set of (finite) Euler products $\mathcal{P}(f; x)$ where $f \in \mathcal{F}(S)$.

EulSpec

Proposition 30.13

$$\Gamma_{\mathcal{P}}(S) = \{e^{-(1-\alpha)t} : t \geq 0, \alpha \in K(S)\}.$$

Proof Since f is totally multiplicative then

$$\mathcal{P}(f; x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{f(p)}{p}\right)^{-1}$$

and so

$$\log \mathcal{P}(f; x) = \sum_{\substack{p \leq x \\ m \geq 1}} \frac{f(p^m) - 1}{mp^m} = (1 - \alpha) \sum_{p \leq x} \log \left(1 - \frac{1}{p}\right)$$

for some $\alpha \in K(S)$ since each $f(p^m) \in S$. Hence $\Gamma_{\mathcal{P}}^*(S) \subset \{e^{-(1-\alpha)t} : t \geq 0, \alpha \in K(S)\}$. In the other direction, for a given α and t , select $y < x$ very large so that $\sum_{y < p \leq x} 1/p = t + O(1/z)$, and then the $f(p) \in S$ so that $\sum_{y < p \leq x} f(p)/p = \alpha t + O(1/y)$, which is certainly possible if y is sufficiently large. But then $\log \mathcal{P}(f; x) = -(1 - \alpha)t + O(1/y)$. Letting $y \rightarrow \infty$ gives the result. \square

If $\alpha \notin \mathbb{R}$ then $\{e^{-(1-\alpha)t}, t \geq 0\}$ is a *spiral* which begins at 1 and ends at 0.

Exercise 30.14 Deduce that $\Gamma_{\mathcal{P}}(S) = \Gamma_{\mathcal{P}}(K(S))$.

Exercise 30.15 Prove that $\Gamma_{\mathcal{P}}(S) \Gamma_{\mathcal{P}}(S) = \Gamma_{\mathcal{P}}(S)$.

⁹We define the product of two sets $A, B \in \mathbb{C}$ to be $AB = \{ab : a \in A, b \in B\}$.

Exercise 30.16 We showed above that we can assume that there exists real $\alpha \neq 1$ with $\alpha \in S \cap \mathbb{R}$. Deduce that $\Gamma_{\mathcal{P}}(S)$ contains the straight-line connecting 0 to 1. Deduce further that $\Gamma_{\mathcal{P}}(S)$ is starlike; that is the straight-line connecting any point of $\Gamma_{\mathcal{P}}(S)$ to the origin, lies entirely inside $\Gamma_{\mathcal{P}}(S)$.

size

Exercise 30.17 Prove that if $z \in \Gamma_{\mathcal{P}}(S)$ then $|z| \leq \exp(-|\arg(z)| \cot(\text{Ang}(S)))$.

Corollary 30.18 $\text{Ang}(\Gamma_{\mathcal{P}}(S)) \geq \text{Ang}(\Gamma(S)) \geq \text{Ang}(S)$.

Proof Suppose that $\alpha \in S$ with $|\arg(1 - \alpha)| \geq \text{Ang}(S) - \epsilon$. Now if $v_t = e^{-(1-\alpha)t} \in \Gamma_{\mathcal{P}}(S)$ then $1 - v_t = (1 - \alpha)t - (1 - \alpha)^2 t^2 / 2 + \dots$. If t is sufficiently small then $|\arg(1 - v_t)| \geq |\arg(1 - \alpha)| - \epsilon \geq \text{Ang}(S) - 2\epsilon$, and the result follows. \square

The interior of the unit disk \mathbb{U} is $\{e^{-re^{i\theta}} : -\pi/2 \leq \theta \leq \pi/2, r > 0\}$

Corollary 30.19 Select $s_+, s_- \in S \setminus \{1\}$ so that $\phi_+ = -\arg(1 - s_+)$ is maximal and $\phi_- = -\arg(1 - s_-)$ is minimal, with $-\pi/2 < \phi_- \leq 0 \leq \phi_+ < \pi/2$. Then

$$\Gamma_{\mathcal{P}}(S) = 1 \cup \{e^{-re^{i\theta}} : \phi_- \leq \theta \leq \phi_+, r > 0\}.$$

This can be written as the boundary and interior of the curves given by

$$e^{-t \cos \phi_- (\cos \phi_+ + i \sin \phi_+)} \quad \text{and} \quad e^{-t \cos \phi_+ (\cos \phi_- + i \sin \phi_-)} \quad \text{for} \quad 0 \leq t \leq \frac{2\pi}{\sin(\phi_+ - \phi_-)}.$$

In particular, the circle of radius e^{-N} , where $N = 2\pi/|\tan \phi_+ - \tan \phi_-|$, centered at 0, is the largest such circle which lies inside $\Gamma_{\mathcal{P}}(S)$.

Proof By definition for every point s of S we may write $1 - s = re^{i\theta}$ with $\phi_- \leq \theta \leq \phi_+$ for some $r > 0$. This is therefore true for every $\alpha \in K(S)$.

Now, if $\phi = -\arg(1 - s)$ where $s = x + iy$ then $(1 - s) = (1 - x)(1 + i \tan \phi)$. Therefore $\frac{n}{1-x}(1 - s) = n + in \tan \phi$. Hence every point on the line between $n + in \tan \phi_+$ and $n + in \tan \phi_-$ takes the form $t(1 - \alpha)$ with $t \geq 0$ and $\alpha \in K(S)$ for each $n \geq 0$. This completes the proof of the first part of our result.

Now if $n \geq N$ then all numbers of the form $n + in \tan \phi_- i\theta$, with $0 \leq \theta \leq 2\pi$ are of this form and so Proposition [30.13](#) yields that $\Gamma_{\mathcal{P}}^*(S)$ contains the circle of radius e^{-n} . \square

Exercise 30.20 Prove that $\text{Ang}(S) = \pi/2$ if and only if there is an infinite sequence of points $r_n e^{i\theta_n} \in S$ such that $\theta_n \rightarrow 0$ as $n \rightarrow \infty$ with $r_n \leq 1$ and $r_n = 1 + o(\theta_n)$.

Proof of Proposition [30.6](#) Suppose that $\text{Ang}(S) = \pi/2$. Take the points $r_n e^{i\theta_n} \in S$ from the last exercise. By Proposition [30.13](#), the points on the spiral $e^{-(1-r_n e^{i\theta_n})t} \in \Gamma(S)$ for $t \geq 0$. For each $\theta \in (-\pi, \pi]$ the consecutive points on the spiral with argument θ differ by a multiplicative distance $e^{-2\pi(1-r_n \cos \theta_n)/|\sin \theta_n|} = 1 + o_{n \rightarrow \infty}(1)$, and hence as $n \rightarrow \infty$ we see that every point on this ray is a limit point of $\Gamma(S)$. Hence $\Gamma(S) = \mathbb{U}$.

Now suppose that $\text{Ang}(S) < \pi/2$. Note that if there exists $\alpha \in S \cap \mathbb{T}$ which is not a root of unity then $\mathbb{T} \subset S$ (as in the proof of Lemma 30.2) and so $\text{Ang}(S) = \pi/2$. Hence we may assume that $S \cap \mathbb{T}$ is finite and consists only of roots of unity.

Now suppose that $\zeta \in S \cap \mathbb{T}$ which is an m th root of unity. We now show that $\text{Ang}(\zeta S) < \pi/2$. If not then we have a sequence of points $r_n e^{i\theta_n} \zeta \in S$ with $\theta_n \rightarrow 0$ as $n \rightarrow \infty$ with $r_n \leq 1$ and $r_n = 1 + o(\theta_n)$. Then $r_n^m e^{im\theta_n} = (r_n e^{i\theta_n} \zeta)^m \in S$ but here $m\theta_n \rightarrow 0$ as $n \rightarrow \infty$ with $r_n^m \leq 1$ and $r_n^m = 1 + o(m\theta_n)$, so that $\text{Ang}(S) = \pi/2$, by the previous exercise, a contradiction.

Let us suppose that every element of $S \cap \mathbb{T}$ is an m th root of unity, and select M divisible by m , so that $\text{Ang}(S_M) > \max_{\zeta \in S \cap \mathbb{T}} \text{Ang}(\zeta S)$ and sufficiently large that the largest distance from $S \cap \mathbb{T}$ to the perimeter of $K(S_M)$ is $< \delta$ then $K(S) \subset K(S_M)$ by exercise 30.4. \square

To simplify our treatment of $\Gamma_{\mathcal{P}}(S)$ we shall now restrict attention to *totally* multiplicative functions. Hence we define $\Gamma^*(S)$ to be the spectrum of mean-values of totally multiplicative functions, and similarly $\Gamma_{\mathcal{P}}^*(S)$ and $\Lambda^*(S)$. All of the above proofs are still valid, and so we deduce that $\Gamma^*(S) = \Gamma_{\mathcal{P}}^*(S) \Lambda^*(S)$ and $\Lambda(S) = \Lambda^*(S)$.

Exercise 30.21 (Open problem) Define $\Gamma^*(S)$ to be the spectrum of mean-values of all multiplicative functions with $f(p^k) \in S$ (but not necessarily totally multiplicative). Similarly define $\Gamma_{\mathcal{P}}^*(S)$. It is evident that $\Gamma_{\mathcal{P}}(S) \subset \Gamma_{\mathcal{P}}^*(S)$. Can you find elements of $\Gamma_{\mathcal{P}}^*(S)$ that do not belong to $\Gamma_{\mathcal{P}}(S)$? Can you determine $\Gamma_{\mathcal{P}}^*(S)$?¹⁰

30.5 The Delay Equation Spectrum

Let $\Lambda(S)$ denote the values $\sigma(u) = \sigma_{\chi}(u)$ obtained from (28.1) when χ is a measurable function with $\chi(t) \in K(S)$ for all $t \geq 0$, with $\chi(t) = 1$ for $0 \leq t \leq 1$. Proposition 28.1 implies that any mean value of a multiplicative function that only varies from 1 on the large primes, belongs to $\Lambda(S)$. On the other hand if $\sigma(u) \in \Lambda(S)$, then there is an $f \in \mathcal{F}(S)$ whose mean value up to x is $\sim \sigma(u)$, by Proposition 28.9.

Exercise 30.22 Explain why $\Lambda(S) = \Lambda(K(S))$. Deduce that $\Gamma(S) = \Gamma(K(S))$, so we can assume throughout that S is a convex, closed, proper subset of \mathbb{U} .

Lemma 30.23

$$\Gamma_{\mathcal{P}}(S) \Lambda(S) \subset \Lambda(S)$$

Exercise 30.24 Deduce that $\Lambda(S)$ and $\Gamma(S)$ are all also starlike. Then deduce that $\Gamma(S) = \Lambda(S)$.

¹⁰The easiest examples arise by simply taking the $p = 2$ term. If $S = S_4$ let $f(2^k) = i$ so that $(1 - \frac{1}{2})(1 + \frac{i}{2} + \frac{i^2}{4} + \dots) = \frac{1+i}{2}$. However the spirals in S_4 look like $e^{-t \pm it}$, so with angle $\pi/4$ the maximum in size is $e^{-\pi/4}(1+i)$ and $e^{-\pi/4} = 0.4559381277 < 1/2$.

Proof Suppose that we are given $e^{-(1-\alpha)t} \in \Gamma_{\mathcal{P}}^*(S)$ and $\sigma(u) \in \Lambda(S)$. Let x be sufficiently large that we can choose $f(p)$ with $z < p \leq y = x^{1/u}$, as in the proof of Proposition 30.13, for which $\mathcal{P}(f, y) = e^{-(1-\alpha)t} + O(1/z)$. We select $f(p)$ for $y < p \leq x$, as in Proposition 28.9. We let $f(p) = 1$ for all $p \leq z$. Applying Theorem 30.8 and then Proposition 3.6 we deduce that the mean value of f is $\sim e^{-(1-\alpha)t}\sigma(u)$. Now applying Theorem 30.8 again, but this time with y there equal to z here, we find that $h = 1, g = f$ and so the mean value of f belongs to $\Lambda(S)$. \square

This result implies that $\Gamma_{\mathcal{P}}(S) \subset \Lambda(S)$. Are there elements of $\Lambda(S)$ that do not belong to $\Gamma_{\mathcal{P}}(S)$? In general, the spectrum contains more elements than simply the Euler products. For example, the spectrum of Euler products for $S = [-1, 1]$ is simply the interval $[0, 1]$, whereas negative numbers are part of $\Lambda(S)$. We have seen that $\Gamma_{\mathcal{P}}(S)$ is straightforward to fully understand, whereas $\Lambda(S)$ remains somewhat mysterious. We will discuss this in more detail in the next chapter.

RESULTS ON SPECTRA

31.1 The spectrum for real-valued multiplicative functions

The spectrum has been fully determined in only one interesting case, where $S = [-1, 1]$; that is real-valued multiplicative functions. In that case, in [GS], we proved that $\Gamma([-1, 1]) = [\delta_1, 1]$ where

$$\delta_1 = 1 - 2\log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log t}{t+1} dt = -0.656999 \dots$$

In other words, for any real-valued completely multiplicative function f with $-1 \leq f(n) \leq 1$, we have

$$\sum_{n \leq x} f(n) \geq (\delta_1 + o(1))x;$$

with equality if and only if $\mathbb{D}(f, f_1, x) = o(1)$ where

$$f_1(p) = \begin{cases} 1 & \text{for primes } p \leq x^{1/(1+\sqrt{e})} \\ -1 & \text{for primes } x^{1/(1+\sqrt{e})} \leq p \leq x. \end{cases}$$

Applying this to the totally multiplicative function $f(n) = \left(\frac{n}{p}\right)$, for some prime p , we deduce that the number of integers below x that are quadratic residues $(\bmod p)$ is

$$\frac{1}{2} \sum_{n \leq x} \left(1 + \left(\frac{n}{p}\right)\right) \geq \frac{1 + \delta_1}{2} x + o(x) = (\delta_0 + o(1))x,$$

where $\delta_0 = 0.171500 \dots$ ¹¹ More colloquially we have:

If x is sufficiently large then, for all primes p , more than 17.15% of the integers up to x are quadratic residues $(\bmod p)$.

Exercise 31.1 *Prove that the constant δ_0 here is best possible.*

¹¹One can derive the following curious expression for δ_0 (from the definition of δ_1):

$$\delta_0 = 1 - \frac{\pi^2}{6} - \log(1 + \sqrt{e}) \log \frac{e}{1 + \sqrt{e}} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{(1 + \sqrt{e})^n}.$$

31.2 The number of m th power residues up to x

We now establish that similar results hold for m -th power residues. For each integer $m \geq 2$, define the minimal density and minimal logarithmic density of m th power residues modulo primes, to be

$$\gamma_m = \liminf_{x \rightarrow \infty} \inf_{\ell \text{ prime}} \frac{1}{x} \sum_{\substack{n \leq x \\ n \equiv a^m \pmod{\ell}}} 1,$$

$$\text{and } \gamma'_m = \liminf_{x \rightarrow \infty} \inf_{\ell \text{ prime}} \frac{1}{\log x} \sum_{\substack{n \leq x \\ n \equiv a^m \pmod{\ell}}} \frac{1}{n}.$$

We already know that $\gamma_2 = \delta_0$ and we will see that $\gamma'_2 = 1/2$. For $m \geq 3$ we show that

$$0 < \gamma_m \leq \rho(m) \left(\approx \frac{1}{m^m} \right) < \frac{1}{2^{m-1}} \leq \gamma'_m \leq \min_{\beta \geq 0} \frac{1}{e^\beta} \sum_{k=0}^{\infty} \frac{\beta^{km}}{(km)!} \left(\sim \frac{1}{e^{m/e}} \right).$$

We do not know the exact values of γ_m and γ'_m for any $m \geq 3$. Calculating the minimum over β , we found that $\gamma'_3 \leq 0.3245$, $\gamma'_4 \leq 0.2187$, $\gamma'_5 \leq 0.14792$, and $\gamma'_6 \leq 0.1003$. However we do obtain the following consequence:

For any given integer $m \geq 2$, there exists a constant $\pi_m > 0$ such that if x is sufficiently large then, for all primes p , more than $\pi_m\%$ of the integers up to x are m th power residues \pmod{p} .

ImportantEx

31.3 An important example

Consider the multiplicative function f with $f(p) = 1$ for $p \leq y = x^{1/u}$ and $f(p) = \alpha \in S$ for $y < p \leq x$. Write $\sigma(u) = \rho_\alpha(u)$ which satisfies the integral delay equation

$$u\rho_\alpha(u) = \int_{u-1}^u \rho_\alpha(t)dt + \alpha \int_1^{u-1} \rho_\alpha(t)dt,$$

and therefore $\rho'_\alpha(u) = -(1-\alpha)\rho_\alpha(u-1)/u$. The case $\alpha = 0$ has already been discussed in detail. In general we can compute the mean value for small u , using our results that $\sigma = \sigma_\infty(u)$, and $I_j(u, \chi) = 0$ if $j \geq u$. Hence:

If $1 \leq u \leq 2$ then the mean value is $1 - (1-\alpha)\log u$. Therefore if $\alpha = re^{i\theta} \in K(S)$ (with $\alpha \neq 1$) then $z = 1 - (1-\alpha)v \in \Lambda(S)$ for $0 \leq v \leq \log 2$. If $z = me^{i\nu}$ then one can show that $m = \sin \theta / \sin(\theta + \nu)$. On the other hand, if $z \in \Gamma_{\mathcal{P}}(S)$ then by exercise 30.17

$$|z| \leq \exp(-\nu \cot \theta) \leq \frac{1}{1 + \nu \cot \theta} < \frac{1}{\cos \nu + \sin \nu \cot \theta} = \frac{\sin \theta}{\sin(\theta + \nu)},$$

which is a contradiction. Hence z is in $\Lambda(S)$ but not in $\Gamma_{\mathcal{P}}(S)$.

If $2 \leq u \leq 3$ then the mean value is $1 - (1 - \alpha) \log u + \frac{(1-\alpha)^2}{2} \int_{\substack{t_1, t_2 \geq 1 \\ t_1 + t_2 \leq u}} \frac{dt_1}{t_1} \frac{dt_2}{t_2}$.

For $\alpha = -1$ we see that $\rho_{-1}(\sqrt{e}) = 0$ and hence $\rho'_{-1}(1 + \sqrt{e}) = 0$. In fact $\rho_{-1}(1 + \sqrt{e}) = \delta_1$ and one can show that this is the absolute minimum value ρ_{-1} takes. Moreover, by continuity, $\rho_{-1}(u)$ takes on all values in the interval $[\delta_1, 1]$ showing that $\Lambda(S) \supset [\delta_1, 1]$. This leads us to the multiplicative function f_1 in the first section of this chapter.

AverageIntEqn

Exercise 31.2 Show that for any χ, σ satisfying (28.1) (that is $u\sigma(u) = (\chi * \sigma)(u)$ for all $u \geq 0$) we have $u(1 * \sigma)(u) = ((1 * \sigma) * (1 + \chi))(u)$. Go on to show that if χ_j, σ_j satisfy (28.1) for $j = 1, 2$ then $u\sigma(u) = (\chi * \sigma)(u)$ for all $u \geq 0$ where $\chi = \chi_1 + \chi_2$ and $\sigma = \sigma_1 * \sigma_2$.

Define $M(u) = M_\chi(u) := \int_0^u \sigma_\chi(t) dt$; that is $M = (1 * \sigma)$. If $\chi(t) = -1$ for all $t > 1$ then $M_{-1}(u) = u$ for $0 \leq u \leq 1$ and, by exercise 31.2,

$$uM_{-1}(u) = 2 \int_{u-1}^u M_{-1}(t) dt \quad \text{for } u > 1.$$

This is much like the functional equation for $\rho(u)$ and can be analyzed in much the same way:

Exercise 31.3 Prove that $M_{-1}(u) = ((2e + o(1))/u \log u)^u$. Use the fact that this is decreasing so fast to deduce that for all sufficiently large v there exists u with $v < u < v + v/\log v$ such that $-\rho_{-1}(u) \gg ((2e + o(1))/u \log u)^u$.

31.4 Open questions of interest

What is $\Gamma([- \alpha, 1])$? That is the spectrum for real-valued f with the each $f(p) \in [- \alpha, 1]$. An easy Corollary of Corollary 27.18 is that if S contains a non-real point that $\Gamma(S)$ contains a negative real-number. We want to know here if this is true when S is real but contains negative real numbers. Evidently $[0, 1] \in \text{Gamma}(S)$ is this case.

What is the spectrum for the mean-value of real-valued multiplicative functions up to x , when $f(p) = 0$ for all $p \leq y$? We will see that this is useful in understanding the distribution of quadratic residues.

THE NUMBER OF UNSIEVED INTEGERS UP TO X

This is the article original, more-or-less unedited

One expects around $x \prod_{p \notin P, p \leq x} (1 - 1/p)$ integers up to x , all of whose prime factors come from the set P . Of course for some choices of P one may get rather more integers, and for some choices of P one may get rather less. Hall [4] showed that one never gets more than $e^\gamma + o(1)$ times the expected amount (where γ is the Euler-Mascheroni constant), which was improved slightly by Hildebrand [5]. Hildebrand [6] also showed that for a given value of $\prod_{p \notin P, p \leq x} (1 - 1/p)$, the smallest count that you get (asymptotically) is when P consists of all the primes up to a given point. In this paper we shall improve Hildebrand's upper bound, obtaining a result close to optimal, and also give a substantially shorter proof of Hildebrand's lower bound. As part of the proof we give an improved Lipschitz-type bound for such counts.

Define

$$g(w) := \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n), \quad \text{and} \quad G(w) := \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n),$$

where both limits are taken over the class of multiplicative functions f with $\mathcal{P}(f, x) = 1/w + o(1)$.

If f is completely multiplicative with $f(p) = 1$ for $p \leq x^{1/u}$ and $f(p) = 0$ for $x^{1/w} \leq p \leq x$ then $\mathcal{P}(f; x) = 1/w + o(1)$ and $\sum_{n \leq x} f(n) = \psi(x, x^{1/w}) \sim x\rho(w)$. Hence $g(w) \leq \rho(w)$ and A. Hildebrand [6] established that in fact $g(w) = \rho(w)$. Since $\rho(w) = w^{-w+o(w)}$ note that $g(w)$ decays very rapidly as w increases.

Regarding $G(w)$, R. Hall [4] established that $G(w) \leq e^\gamma/w$ and Hildebrand [5] improved this slightly by showing that $G(w) \leq \frac{1}{w} \int_0^w \rho(t) dt$. Since $\int_0^\infty \rho(t) dt = e^\gamma$ this does mark an improvement over Hall's result, but the difference from e^γ/w is $\frac{1}{w} \int_w^\infty \rho(t) dt = w^{-w+o(w)}$ which is very small. In this paper we shall prove that $G(w) = e^\gamma/w - 1/w^{2+o(1)}$, but it remains to determine $G(w)$ more precisely. We shall also give a shorter proof of Hildebrand's result that $g(w) = \rho(w)$.

[1] Theorem 32.1 *For all $w \geq 1$ we have that*

$$G(w) \geq \max_{w \geq \Delta \geq 0} \left(\rho(w + \Delta) + \int_0^\Delta \frac{\rho(t)}{w + \Delta - t} dt \right).$$

When w is large, the maximum is attained for $\Delta \sim \log w / \log \log w$, and yields

$$G(w) \geq \frac{e^\gamma}{w} - \frac{(e^\gamma + o(1)) \log w}{w^2 \log \log w}.$$

[2] **Theorem 32.2** For all large w we have

$$G(w) \leq \frac{e^\gamma}{w} - \frac{1}{w^2 \exp(c(\log w)^{2/3}(\log \log w)^{1/3})}$$

for a positive constant c .

We also give an explicit upper bound for $G(w)$ valid for all w .

[3] **Theorem 32.3** For $1 \leq w$ we have that $G(w) \leq 1 - \log w + (\log w)^2/2$ and equality holds here for $1 \leq w \leq 3/2$. For $w \geq 1$ put $\Lambda(w) := \frac{1}{2}(w + 1/w) + \frac{\log w}{2}(w - 1/w)$. Then $G(w) \leq \Lambda(w) \log(1 + e^\gamma/(w\Lambda(w)))$.

The first bound in Theorem 3 is better than the second for $w \leq 3.21 \dots$, when the second bound takes over. Note that the second bound in Theorem 3 equals $e^\gamma/w - (e^{2\gamma} + o(1))/w^3 \log w$, only a little weaker than the bound in Theorem 2, while being totally explicit.

In the range $1 \leq w \leq 3/2$ we may check that the right side of (1.3) equals $1 - \log w + (\log w)^2/2 = G(w)$. Perhaps it is true that $G(w)$ is given by the right side of (1.3) for all w .

We end this section by giving a simple construction that proves Theorem 1.

Proof of Theorem 1 Let y be large and consider the completely multiplicative function f defined by $f(p) = 0$ for $p \in [y, y^w]$ and $f(p) = 1$ for all other primes p . Put $x = y^{w+\Delta}$ where $0 \leq \Delta \leq w$ and note that $\mathcal{P}(f, x) = \prod_{y \leq p \leq y^w} (1 - 1/p) \sim 1/w$. An integer $n \leq x$ with $f(n) = 1$ has at most one prime factor between y^w and x , and all its other prime factors are below y . Hence

$$\sum_{n \leq x} f(n) = \psi(x, y) + \sum_{y^w \leq p \leq x} \psi(x/p, y),$$

and using (1.2) and the prime number theorem this is

$$\sim x\rho(w+\Delta) + x \sum_{y^w \leq p \leq x} \frac{1}{p} \rho\left(w+\Delta - \frac{\log p}{\log y}\right) \sim x\left(\rho(w+\Delta) + \int_0^\Delta \frac{\rho(t)}{w+\Delta-t} dt\right),$$

which gives the lower bound (1.3) for $G(w)$. For large w we see that

$$\rho(w+\Delta) + \int_0^\Delta \frac{\rho(t)}{w+\Delta-t} dt = \frac{1}{w+\Delta} \int_0^\Delta \rho(t) dt + \int_0^\Delta \frac{t\rho(t)}{(w+\Delta)(w+\Delta-t)} dt + \rho(w+\Delta)$$

and since $\int_0^\infty t\rho(t) dt < \infty$ and $\int_0^\Delta \rho(t) dt = e^\gamma - \Delta^{-(1+o(1))\Delta}$ the above is

$$\frac{1}{w+\Delta} (e^\gamma - \Delta^{-(1+o(1))\Delta}) + O\left(\frac{1}{w^2}\right).$$

The quantity above attains a maximum for $\Delta = (1 + o(1)) \log w / \log \log w$, completing the proof of Theorem 1. \square

We noted above that $G(w) = 1 - \log w + (\log w)^2/2$ for $1 \leq w \leq 1.5$ (with the maximum attained in (1.3) at $\Delta = w$). Next we record the bounds obtained for $1.5 \leq w \leq 2$ (though here the maximum is attained with Δ a little smaller than w).

w	1.5	1.6	1.7	1.8	1.9	2.0
$G(w) \geq$.676735	.640255	.608806	.581685	.557392	.535905
$G(w) \leq$.676736	.640449	.610155	.584960	.564135	.547080

The upper and lower bounds for $G(w)$ given by Theorems 1 and 3.

32.1 Reformulation in terms of integral equations

Note that $\mathcal{P}(f, y^u) \sim 1/E(u)$. Analogously to $g(w)$ and $G(w)$ we may define

$$\tilde{g}(w) = \liminf_{\substack{u, \chi \\ E_\chi(u)=w}} \sigma(u), \quad \text{and} \quad \tilde{G}(w) = \limsup_{\substack{u, \chi \\ E_\chi(u)=w}} \sigma(u),$$

where the limits are taken over all pairs u, χ with $u \geq 1$, where χ is a measurable function for which $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) \in [0, 1]$ for all t , and with $E_\chi(u) = w$. We shall show that these quantities are in fact equal to $g(w)$ and $G(w)$ respectively. Something similar was stated (but not very precisely) by Hildebrand in his discussion paper [7].

2.2 Theorem 32.4 *We have $g(w) = \tilde{g}(w)$ and $G(w) = \tilde{G}(w)$.*

To prove Theorem 2.2 we need to know how small primes affect the mean-values of multiplicative functions

Prove that

$$\tilde{g}(w) \geq g(w) \geq \min_{w \geq v \geq 1} \frac{1}{v} \tilde{g}\left(\frac{w}{v}\right), \quad \text{and} \quad \tilde{G}(w) \leq G(w) \leq \max_{w \geq v \geq 1} \frac{1}{v} \tilde{G}\left(\frac{w}{v}\right).$$

32.2 An open problem or two

Fix Θ , $0 < \Theta < 1$. Let f be a multiplicative function such that $0 \leq f(n) \leq 1$, and

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = (\Theta + o(1)) \log x.$$

Prove that

$$\sum_{n \leq x} \frac{f(n)}{n} \leq (e^{-\gamma} + o(1)) \int_0^{1/\Theta} \rho(t) dt \prod_{p \leq x} \left(1 - \frac{f(p)}{p}\right)^{-1},$$

where ρ is the Dickman-de Bruijn function. (Note that $\int_0^{+\infty} \rho(t) dt = e^\gamma$). This inequality is sharp. To see that take f such that $f(p) = 1$ for all primes $p \leq x^\Theta$ and $f(p) = 0$ otherwise.

We can reformulate this in terms of integral equations. Define $\Theta_\chi(u) := \int_0^u \chi(t) dt$, then Hall's conjecture is the following

Conjecture 32.5

$$\int_0^u \sigma(t) dt \leq \left(\int_0^{\frac{u}{\Theta_\chi(u)}} \rho(t) dt \right) \exp \left(\int_1^u \frac{\chi(t)}{t} dt \right).$$

A stronger conjecture asserts that

$$\sigma(u) \geq \rho \left(\frac{u}{\Theta_\chi(u)} \right).$$

If true, this implies the result of Hildebrand that $\liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n)$ exists and is equal to $\rho(\omega)$, where the limit is taken over the class of multiplicative functions f with

$$\prod_{p \leq x} \left(1 - \frac{1}{p} \right) \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots \right) = \frac{1}{\omega} + o(1).$$

32.3 Upper bounds for $G(w)$ and Lipschitz estimates

We are able to improve “ $1 - 2/\pi$ ” to “ $1 - 1/\pi$ ” in the special case that $\chi(t) \in [0, 1]$ for all t .

4 **Theorem 32.6** *Let χ be a measurable function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) \in [0, 1]$ for $t > 1$, and let σ denote the corresponding solution to (2.1). Then*

$$|\sigma(u) - \sigma(v)| \ll \left(\frac{u-v}{u} \right)^{1-\frac{1}{\pi}} \left(1 + \log \frac{u}{u-v} \right) \text{ whenever } 1 \leq v \leq u.$$

Theorem 4 follows immediately from the stronger but more complicated Proposition 4.2 below, and the fact that $|\sigma(u) - \sigma(v)| \leq \frac{3(u-v)}{u}$ whenever $v \leq u(1 - 1/E(u))$. This is trivial for $v \leq 2u/3$, whereas for larger v in the range, we obtain

$$|\sigma(u) - \sigma(v)| \leq \frac{e^\gamma}{E(v)} \leq \frac{ue^\gamma}{vE(u)} \leq \frac{3(u-v)}{u},$$

using Hall’s result that $\sigma(u) \leq e^\gamma/E(u)$.

Using (3.3) in (3.2) leads to the bound $\tilde{G}(w) \leq e^\gamma/w - C_\kappa/(w^{1+1/\kappa} \log w)$ for some positive constant C_κ . Thus if (3.3) holds with $\kappa = 1$ then we would be able to deduce that $G(w) = e^\gamma/w - (\log w)^{O(1)}/w^2$ by Theorem 1.

In order to prove Theorem 3 we give the following explicit Lipschitz estimate (see also Proposition 4.1 of [2]).

3.1 **Proposition 32.7** *Let χ be a measurable function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) \in [0, 1]$ for all t , and let $\sigma(u)$ denote the corresponding solution to (2.1). Then for all $u \geq 1$ and $1 \geq \delta > 0$ we have*

$$\log(1 + \delta) \left(\frac{E(u) - 1/E(u)}{2} + \log E(u) \frac{E(u) + 1/E(u)}{2} \right) \geq \sigma(u(1 + \delta)) - \sigma(u),$$

and

$$\sigma(u(1 + \delta)) - \sigma(u) \geq -\log(1 + \delta) \left(\frac{E(u) + 1/E(u)}{2} + \log E(u) \frac{E(u) - 1/E(u)}{2} \right).$$

Proof We shall only prove the lower bound, the proof of the upper bound is similar. From (2.2a,b) we see that

$$\sigma(u(1+\delta)) - \sigma(u) \geq - \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{1}{j!} (I_j(u(1+\delta); \chi) - I_j(u; \chi)).$$

By symmetry we see that $I_j(u(1+\delta); \chi) - I_j(u; \chi)$ equals

$$j \int_{t_1, \dots, t_{j-1} \geq 1} \frac{1 - \chi(t_1)}{t_1} \dots \frac{1 - \chi(t_{j-1})}{t_{j-1}} \int_{\substack{\max(t_1, \dots, t_{j-1}, u-t_1-\dots-t_{j-1}) \leq t_j \\ t_j \leq u(1+\delta)-t_1-\dots-t_{j-1}}} \frac{1 - \chi(t_j)}{t_j} dt_1 \dots dt_j.$$

The integral over t_j is

$$\leq \log \frac{u/j + u\delta}{u/j} = \log(1 + j\delta) \leq j \log(1 + \delta),$$

since $\max(t_1, \dots, t_{j-1}, u-t_1-\dots-t_{j-1}) \geq u/j$. Further since $\delta < 1$ we have $t_1, \dots, t_{j-1} \leq u$ and so these integrals contribute $\leq (\log E(u))^{j-1}$. Thus we have

$$\sigma(u(1+\delta)) - \sigma(u) \geq - \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{1}{j!} j^2 \log(1 + \delta) (\log E(u))^{j-1},$$

and the result follows easily. \square

Proof of Theorem 2.2 Fix $w \geq v \geq 1$. Suppose $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) \in [0, 1]$ for all t and let $\sigma(u)$ denote the corresponding solution to (2.1) (we will think of χ as giving the optimal function for either $\tilde{g}(w/v)$ or $\tilde{G}(w/v)$). Let $U \geq 1$ be a parameter which we will let tend to infinity. Put $\chi_1(t) = \chi(t/U)$ and note that the corresponding solution to (2.1) is $\sigma_1(u) = \sigma(u/U)$. Define $\chi_2(t) = 0$ for $1 \leq t \leq v$ and $\chi_2(t) = \chi_1(t)$ for all other t , and let $\sigma_2(u)$ denote the corresponding solution to (2.1). By Lemma 2.5 we see that for $U \geq v$

$$\sigma_2(uU) = \sigma_1(uU) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{\substack{v \geq t_1, \dots, t_j \geq 1 \\ t_1 + \dots + t_j \leq uU}} \frac{1}{t_1} \frac{1}{t_2} \dots \frac{1}{t_j} \sigma_1(uU - t_1 - \dots - t_j) dt_1 \dots dt_j.$$

By Proposition 3.1 we know that

$$\sigma_1(uU - t_1 - \dots - t_j) = \sigma_1(uU) + O\left(\min\left(1, E_{\chi}(u) \log E_{\chi}(u) \frac{ju}{uU}\right)\right).$$

Using this above we see easily that for large U with u, v, w fixed we have $\sigma_2(uU) \sim \sigma_1(uU)/v = \sigma(u)/v$ and note further that $E_{\chi_2}(uU) = vE_{\chi_1}(uU) = vE_{\chi}(u)$.

This scaling argument shows that for $1 \leq v \leq w$ we have $\tilde{g}(w/v) \geq v\tilde{g}(w)$ and that $\tilde{G}(w/v) \leq v\tilde{G}(w)$. Using these inequalities in (2.4a) we deduce that $g(w) \geq \tilde{g}(w)$ and that $G(w) \leq \tilde{G}(w)$ and combining this with (2.4b) we obtain Theorem 2.2.

□

Now that Theorem 2.2 has been established, to prove Theorem 3 it suffices to establish the analogous bounds for $\tilde{G}(w)$ and we establish these next.

Proof of Theorem 3 Using the inclusion-exclusion upper bound (2.5) with $n = 2$ we see that $\sigma(u) \leq 1 - \log E(u) + (\log E(u))^2/2$. It follows that $G(w) = \tilde{G}(w) \leq 1 - \log w + (\log w)^2/2$. If $w \leq 3/2$ then consider $\chi(t) = 0$ for $1 \leq t \leq w$ and $\chi(t) = 1$ for all other t . Then we see that the corresponding solution $\sigma(u)$ satisfies $\sigma(u) = 1 - \log w + (\log w)^2/2$ for $3 \geq u \geq 2w$. Thus $\tilde{G}(w) = 1 - \log w + (\log w)^2/2$ for $1 \leq w \leq 3/2$.

We now establish the second bound of the Theorem. As noted in the introduction the second bound is worse than the first for $w \leq 3.21$ and so we may suppose that $w \geq 2$. With $\hat{\chi}, \hat{\sigma}$ as above, note that $\hat{\sigma}(t) \geq 0$ for all t , and

$$\hat{\sigma}(u(1+\delta)) \geq \hat{\sigma}(u) - \Lambda(E(u)) \log(1+\delta) \text{ for } 0 \leq \delta \leq 1$$

by Proposition 3.1. If $E(u) \geq 2$ then $\Lambda(E(u)) \geq 7/4 > 1/\log 2$ so that $\exp(\sigma(u)/\Lambda(E(u))) - 1 < 1$. Hence we obtain that

$$\frac{1}{u} \int_u^\infty \hat{\sigma}(t) dt \geq \int_0^{\exp(\sigma(u)/\Lambda(E(u))) - 1} (\sigma(u) - \Lambda(E(u)) \log(1+\delta)) d\delta \quad (32.1)$$

$$= -\sigma(u) + \Lambda(E(u)) \left(\exp\left(\frac{\sigma(u)}{\Lambda(E(u))}\right) - 1 \right), \quad (32.2)$$

$$(32.3)$$

and inserting this into (3.2) we get the Theorem. □

32.4 An improved upper bound: Proof of Theorem 2

Our proof of Theorem 2 is also based on (3.2) and obtaining lower bounds for $\frac{1}{u} \int_u^\infty \hat{\sigma}(t) dt$. However Theorem 4 is not quite strong enough to obtain this conclusion and so, in this section, we develop a hybrid Lipschitz estimate which for our problem is almost as good as (3.3) with $\kappa = 1$. We begin with the following Proposition (compare Lemma 2.2 and Proposition 3.3 of [3]).

4.1 Proposition 32.8 *Let χ be a measurable function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t)$ in the unit disc for all t . Let σ be the corresponding solution to (2.1). Let $1 \leq v \leq u$ be given real numbers, and put $\delta = u - v$. Define*

$$F := \max_{y \in \mathbb{R}} \exp \left(\gamma - \int_0^u \operatorname{Re} \left(\frac{1 - \chi(t)e^{-ity}}{t} \right) dt \right) |1 - e^{-iy\delta}|.$$

Then

$$|\sigma(u) - \sigma(v)| \leq \frac{\delta}{u} \log \frac{eu}{\delta} + F + F \int_0^{2/(uF)} \frac{1 - e^{-2xu}}{x} dx \quad (32.4)$$

$$\leq \frac{\delta}{u} \log \frac{eu}{\delta} + F \log \frac{e^3}{F}. \quad (32.5)$$

$$(32.6)$$

Proof As in the proof of Theorem 3 take $\hat{\chi}(t) = \chi(t)$ for $t \leq u$ and $\hat{\chi}(t) = 0$ for $t > u$, and let $\hat{\sigma}$ be the corresponding solution to (2.1). Set $\sigma(t) = \hat{\sigma}(t) = 0$ for $t < 0$. Note that

$$|u\sigma(u) - v\sigma(v)| = |u\hat{\sigma}(u) - v\hat{\sigma}(v)| = \left| \int_0^u \chi(t)(\hat{\sigma}(u-t) - \hat{\sigma}(v-t))dt \right| \quad (32.7)$$

$$\leq \int_0^u |\hat{\sigma}(t) - \hat{\sigma}(t-\delta)|dt = \int_0^u 2t|\hat{\sigma}(t) - \hat{\sigma}(t-\delta)| \left(\int_0^\infty e^{-2xt}dx \right) dt \quad (32.8)$$

$$\leq 2 \int_0^\infty \int_0^u \{ |t\hat{\sigma}(t) - (t-\delta)\hat{\sigma}(t-\delta)| + \delta|\hat{\sigma}(t-\delta)| \} e^{-2tx} dt dx \quad (32.9)$$

$$\leq \int_0^\infty I(x)dx + \int_0^\infty \int_\delta^u 2\delta e^{-2tx} dt dx = \delta \log \frac{u}{\delta} + \int_0^\infty I(x)dx, \quad (32.10)$$

$$(32.11)$$

where

$$I(x) = \int_0^u 2|t\hat{\sigma}(t) - (t-\delta)\hat{\sigma}(t-\delta)|e^{-2tx}dt.$$

As $|\sigma(u) - \sigma(v)| \leq \frac{1}{u}(|u\sigma(u) - v\sigma(v)| + \delta|\sigma(v)|) \leq \frac{\delta}{u} + \frac{1}{u}|u\sigma(u) - v\sigma(v)|$, it follows that

$$|\sigma(u) - \sigma(v)| \leq \frac{\delta}{u} \log \frac{eu}{\delta} + \frac{1}{u} \int_0^\infty I(x)dx.$$

By Cauchy's inequality

$$I(x)^2 \leq \left(4 \int_0^u e^{-2tx} dt \right) \left(\int_0^u |t\hat{\sigma}(t) - (t-\delta)\hat{\sigma}(t-\delta)|^2 e^{-2tx} dt \right) \quad (32.12)$$

$$\leq 2 \left(\frac{1 - e^{-2xu}}{x} \right) \left(\int_0^\infty |t\hat{\sigma}(t) - (t-\delta)\hat{\sigma}(t-\delta)|^2 e^{-2tx} dt \right). \quad (32.13)$$

$$(32.14)$$

By Plancherel's formula the second term above is

$$= \frac{1}{2\pi} \int_{-\infty}^\infty |\mathcal{L}(t\hat{\sigma}(t) - (t-\delta)\hat{\sigma}(t-\delta), x+iy)|^2 dy = \frac{1}{2\pi} \int_{-\infty}^\infty |\mathcal{L}(t\hat{\sigma}(t), x+iy)|^2 |1 - e^{-(x+iy)\delta}|^2 dy.$$

From (2.1) we see that $\mathcal{L}(t\hat{\sigma}(t), x + iy) = \mathcal{L}(\hat{\sigma}, x + iy)\mathcal{L}(\hat{\chi}, x + iy)$ and so the above equals

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{L}(\hat{\sigma}, x + iy)\mathcal{L}(\hat{\chi}, x + iy)|^2 |1 - e^{-(x+iy)\delta}|^2 dy \leq F(x)^2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{L}(\hat{\chi}, x + iy)|^2 dy$$

where

$$F(x) := \max_{y \in \mathbb{R}} |1 - e^{-(x+iy)\delta}| |\mathcal{L}(\hat{\sigma}, x + iy)|.$$

Now, using Plancherel's formula again,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{L}(\hat{\chi}, x + iy)|^2 dy = \int_0^{\infty} |\hat{\chi}(t)|^2 e^{-2tx} dt \leq \int_0^x e^{-2tx} dt = \frac{1 - e^{-2xu}}{2x},$$

and so

$$I(x) \leq \frac{1 - e^{-2xu}}{x} F(x).$$

We now demonstrate that $F(x)$ is a decreasing function of x . Suppose that $\beta > 0$ is real, and recall that the Fourier transform of $k(z) := e^{-\beta|z|}$ is $\hat{k}(\xi) = \int_{-\infty}^{\infty} e^{-\beta|z| - i\xi z} dz = \frac{2\beta}{\beta^2 + \xi^2}$. Hence $e^{-\beta z} = k(z) = k(-z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta}{\beta^2 + \xi^2} e^{-i\xi z} d\xi$ by Fourier inversion for $z > 0$. It follows that for $\delta + t > 0$ we have

$$(1 - e^{-\delta(x+\beta+iy)})e^{-t(x+\beta+iy)} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta}{\beta^2 + \xi^2} e^{-t(x+iy+i\xi)} (1 - e^{-\delta(x+iy+i\xi)}) d\xi.$$

Multiplying both sides by $\hat{\sigma}(t)$, and integrating t from 0 to ∞ , we deduce that

$$(1 - e^{-\delta(x+\beta+iy)})\mathcal{L}(\hat{\sigma}, x + \beta + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta}{\beta^2 + \xi^2} \mathcal{L}(\hat{\sigma}, x + iy + i\xi) (1 - e^{-\delta(x+iy+i\xi)}) d\xi \quad (32.15)$$

$$\leq \left(\max_{y \in \mathbb{R}} |(1 - e^{-\delta(x+iy)})\mathcal{L}(\hat{\sigma}, x + iy)| \right) \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta}{\beta^2 + \xi^2} d\xi, \quad (32.16)$$

$$(32.17)$$

and so $F(x + \beta) \leq F(x)$ as claimed. Therefore $F(x) \leq \lim_{x \rightarrow 0^+} F(x)$.

Now if $s = x + iy$ with $x > 0$ then

$$\mathcal{L}\left(\frac{1 - \chi(v)}{v}, s\right) = \int_0^{\infty} \left(\frac{1 - \chi(v)e^{-ivy}}{v}\right) e^{-vx} dv + \int_0^{\infty} \frac{e^{-vs} - e^{-vx}}{v} dv \quad (32.18)$$

$$= \int_0^{\infty} \left(\frac{1 - \chi(v)e^{-ivy}}{v}\right) e^{-vx} dv + \log(x/s), \quad (32.19)$$

$$(32.20)$$

so that

$$\mathcal{L}(\sigma, s) = \frac{1}{x} \exp\left(-\int_0^\infty \left(\frac{1 - \chi(v)e^{-ivy}}{v}\right) e^{-vx} dv\right).$$

Using this for $\hat{\sigma}$ we have

$$|\mathcal{L}(\hat{\sigma}, x + iy)| = \frac{1}{x} \exp\left(-\int_u^\infty \frac{e^{-tx}}{t} dt - \int_0^u \operatorname{Re}\left(\frac{1 - \chi(t)e^{-ity}}{t}\right) e^{-tx} dt\right).$$

For $x \ll 1/u$ we get

$$\int_u^\infty \frac{e^{-tx}}{t} dt = \int_{ux}^\infty \frac{e^{-t}}{t} dt = \int_1^\infty \frac{e^{-t}}{t} dt + \int_{ux}^1 \frac{e^{-t} - 1}{t} dt + \log \frac{1}{ux} = -\gamma + \log \frac{1}{ux} + O(ux),$$

since $\gamma = \int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt$, so that

$$|\mathcal{L}(\hat{\sigma}, x + iy)| = e^\gamma u \exp\left(-\int_0^u \operatorname{Re}\left(\frac{1 - \chi(t)e^{-ity}}{t}\right) dt + O(ux)\right).$$

Note that this is $\ll_u 1$, so that the maximum of $|1 - e^{-(x+iy)\delta}| |\mathcal{L}(\hat{\sigma}, x + iy)|$ cannot occur with $\|y\delta/2\pi\| \rightarrow 0$ as $x \rightarrow 0^+$ (here $\|t\|$ denotes the distance from the nearest integer to t), else $F(x) \ll_u x + \|y\delta/2\pi\| \rightarrow 0$ as $x \rightarrow 0^+$, implying that $F(x) = 0$ which is ridiculous. Thus the maximum occurs with $\|y\delta/2\pi\| \gg 1$ as $x \rightarrow 0^+$ so that $1 - e^{-(x+iy)\delta} = 1 - e^{-iy\delta} + O(x\delta) = (1 - e^{-iy\delta})\{1 + O(x\delta)\}$, so that

$$|1 - e^{-(x+iy)\delta}| |\mathcal{L}(\hat{\sigma}, x + iy)| = u |1 - e^{-iy\delta}| \exp\left(\gamma - \int_0^u \operatorname{Re}\left(\frac{1 - \chi(t)e^{-ity}}{t}\right) dt + O(ux)\right).$$

Therefore $F(x) \leq uF\{1 + O(ux)\}$ for sufficiently small x ; and so $F(x) \leq uF$. Also $F(x) \leq 2 \max_{y \in \mathbb{R}} |\mathcal{L}(\hat{\sigma}, x + iy)| \leq 2/x$. Therefore, by (4.2), we get that

$$I(x) \leq \begin{cases} \frac{1 - e^{-2xu}}{x} uF & \text{if } x \leq 2/uF \\ \frac{2}{x^2} & \text{if } x > 2/uF, \end{cases}$$

which when inserted in (4.1) yields the first estimate in the Proposition.

Now if $F \leq 1$ then

$$\int_0^{2/(uF)} \frac{1 - e^{-2xu}}{x} dx \leq \int_0^{2/u} \frac{1 - e^{-2xu}}{x} dx + \int_{2/u}^{2/(uF)} \frac{1}{x} dx \leq 2 + \log(1/F),$$

and so we deduce the second estimate of Proposition 4.1. If $F > 1$ this holds trivially since $|\sigma(u) - \sigma(v)| \leq 2$.

□

As an application of this Proposition, we establish the following strange-looking Lipschitz estimate in the case that $\chi(t) \in [0, 1]$ for all $t \geq 1$.

4.2 Proposition 32.9 *Let χ be a measurable function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) \in [0, 1]$ for $t > 1$, and let σ denote the corresponding solution to (2.1). Let $1 \leq v \leq u$ be given and write $E(u) = (u/(u-v))^{\mathcal{P}}$ for $\mathcal{P} > 0$. Then*

$$|\sigma(u) - \sigma(v)| \ll \left(\frac{u-v}{u} \right)^{\min\{1, 1 - \frac{1}{\pi} \sin(\pi \mathcal{P})\}} \left(1 + \log \frac{u}{u-v} \right).$$

Proof Let $\delta = u - v$ and $A = \int_0^u \frac{1-\chi(t)}{t} dt = \log E(u)$. We will show that

$$\exp \left(- \int_0^u \frac{1-\chi(t) \cos(ty)}{t} dt \right) \min(1, \delta y) \ll \left(\frac{\delta}{u} \right)^{\min\{1, 1 - \frac{1}{\pi} \sin(\frac{\pi A}{\log(u/\delta)})\}},$$

for all positive y . The result then follows from Proposition 4.1 since $F \ll$ Left side of (4.3).

If $y \leq e/u$ then the left side of (4.3) is $\leq e\delta/u$ and the result follows. Henceforth we may suppose that $y > e/u$. Since $\cos(x) = 1 + O(x^2)$, we get that $\int_0^{1/y} \frac{1-\chi(t) \cos(ty)}{t} dt = \int_0^{1/y} \frac{1-\chi(t)}{t} dt + O(1)$. Thus if we let $z := \int_{1/y}^u \frac{1-\chi(t)}{t} dt$ then

$$\int_0^u \frac{1-\chi(t) \cos(ty)}{t} dt = A - z + O(1) + \int_{1/y}^u \frac{1-\chi(t) \cos(ty)}{t} dt \quad (32.21)$$

$$= A - z + O(1) + \int_{1/y}^u \frac{1-\cos(ty)}{t} dt + \int_{1/y}^u \frac{1-\chi(t)}{t} \cos(ty) dt \quad (32.22)$$

$$= A - z + \log(uy) + O(1) + \int_1^{uy} \frac{1-\chi(t/y)}{t} \cos(t) dt, \quad (32.23)$$

$$(32.24)$$

by making a change of variables, and since (integrating by parts)

$$\int_{1/y}^u \frac{\cos(ty)}{t} dt = \frac{\sin(ty)}{yt} \Big|_{1/y}^u + \int_{1/y}^u \frac{\sin(ty)}{yt^2} dt = O(1).$$

By periodicity

$$\int_1^{uy} \frac{1-\chi(t/y)}{t} \cos(t) dt = \int_0^\pi G(\mathcal{P}) \cos \mathcal{P} d\mathcal{P}, \quad \text{where } G(\mathcal{P}) := \sum_{\substack{t \pm \mathcal{P} \in 2\pi\mathbb{Z} \\ 1 \leq t \leq uy}} \frac{1-\chi(t/y)}{t}$$

and the sum over t above is over real values of t in the range $[1, uy]$ such that $t \pm \mathcal{P}$ is an integer multiple of 2π . Note that

$$0 \leq G(\mathcal{P}) \leq \frac{1}{\pi} \log(uy) + O(1) \quad \text{for all } \mathcal{P}, \quad (32.25)$$

$$\text{and } \int_0^\pi G(\mathcal{P}) d\mathcal{P} = \int_{1/y}^u \frac{1-\chi(t)}{t} dt = z. \quad (32.26)$$

$$(32.27)$$

Consider the problem of minimizing $\int_0^\pi G(\mathcal{P}) \cos \mathcal{P} d\mathcal{P}$ over all functions G satisfying these two constraints. Since $\cos \mathcal{P}$ decreases from 1 to -1 in the range $[0, \pi]$, we see that this is achieved by taking $G(\mathcal{P}) = 0$ for $\mathcal{P} \in [0, \pi - \mathcal{P}_0]$, and $G(\mathcal{P}) = \frac{1}{\pi} \log(uy) + O(1)$ for $\mathcal{P} \in [\pi - \mathcal{P}_0, \pi]$, where \mathcal{P}_0 satisfies $\mathcal{P}_0(\frac{1}{\pi} \log(uy) + O(1)) = z$. We conclude that

$$\int_0^\pi G(\mathcal{P}) \cos \mathcal{P} d\mathcal{P} \geq \int_{\pi - \mathcal{P}_0}^\pi \cos \mathcal{P} \left(\frac{1}{\pi} \log(uy) + O(1) \right) d\mathcal{P} = -\frac{1}{\pi} \log(uy) \sin \mathcal{P}_0 + O(1) \quad (32.28)$$

$$= -\frac{1}{\pi} \log(uy) \sin \left(\frac{\pi z}{\log(uy) + O(1)} \right) + O(1) \quad (32.29)$$

$$= -\frac{1}{\pi} \log(uy) \sin \left(\frac{\pi z}{\log(uy)} \right) + O(1), \quad (32.30)$$

$$(32.31)$$

since $0 \leq z \leq \log(uy)$. Therefore

$$\int_0^u \frac{1 - \chi(t) \cos(ty)}{t} dt \geq A - z + \log(uy) \left(1 - \frac{1}{\pi} \sin \left(\frac{\pi z}{\log(uy)} \right) \right) + O(1).$$

In the domain $0 \leq z \leq \log(uy)$, the right side of (4.4) is a non-increasing function of z , so that it is greater than the value with z replaced by $\log(uy)$, that is, it is $> A + O(1)$. Therefore the left side of (4.3) is $\ll e^{-A} \min(1, \delta y)$, which is $\leq \delta/u$ if $A \geq \log(uy)$, as required. If $A < \log(uy)$ then the right side of (4.4) is greater than the value with z replaced by A , which is $\log(uy) - \frac{\log(uy)}{\pi} \sin(\pi A / \log(uy)) + O(1)$, so that the left side of (4.3) is

$$\ll \frac{\min(1, \delta y)}{uy} (uy)^{\frac{1}{\pi} \sin(\frac{\pi A}{\log(uy)})}.$$

This function is maximized when $y = 1/\delta$ in the range $\log(uy) \geq A$, at which point it yields the right side of (4.3), completing the proof. \square

Proof of Theorem 2 Let $\alpha = E(u) = e^A$. We may assume that α is large, and that $\sigma(u) \geq 1/\alpha$, else our result follows trivially. Let $v = (1 + e^{-\Lambda})u$ for some parameter $\Lambda > A$, and select $\hat{\chi}(t) = \chi(t)$ for $t \leq u$ and $\hat{\chi}(t) = 0$ for $t > u$, as earlier. Using Proposition 4.2 we deduce that there is a constant C such that

$$|\hat{\sigma}(u) - \hat{\sigma}(v)| \leq C(1 + \Lambda) \exp \left(-\Lambda + \frac{\Lambda}{\pi} \sin \left(\frac{\pi A}{\Lambda} \right) \right).$$

If $\Lambda \geq 2A$, then this is $\leq C(1 + \Lambda) \exp(-\Lambda(1 - 1/\pi))$ which is easily verified to be $\leq 1/(2\alpha)$ if α is sufficiently large. If $A < \Lambda \leq 2A$, then the right side of (4.5) is $\leq 2C(1 + A) \exp(-\Lambda + \frac{\Lambda}{\pi} \sin(\frac{\pi A}{\Lambda}))$, which is a decreasing function of Λ in our range. For $\Lambda = A + \xi$ where $\xi := cA^{2/3}(\log A)^{1/3}$, with $c > (6/\pi^2)^{1/3}$, this equals

$$2C(1+A) \exp\left(-A-\xi+\frac{A+\xi}{\pi} \sin\left(\frac{\pi A}{A+\xi}\right)\right) = 2C(1+A) \exp\left(-A-\frac{\pi^2}{6} \frac{\xi^3}{A^2}+O\left(\frac{\xi^4}{A^3}\right)\right) \leq \frac{1}{2\alpha}.$$

Thus we have proved that $|\hat{\sigma}(u)-\hat{\sigma}(v)| \leq 1/(2\alpha)$ for all $\Lambda \geq A+\xi$, which implies that $\hat{\sigma}(v) \geq 1/(2\alpha)$ for $u \leq v \leq u(1+e^{-A-\xi})$. Therefore

$$\frac{1}{u} \int_u^\infty \hat{\sigma}(t) dt \geq \frac{1}{u} \int_u^{u(1+e^{-A-\xi})} \hat{\sigma}(v) dv \geq \frac{1}{u} \cdot ue^{-A-\xi} \cdot \frac{1}{2\alpha} > \frac{1}{2\alpha^2 \exp(\xi)},$$

which implies the theorem, by (3.2).

□

THE LOGARITHMIC SPECTRUM

We saw in section ~~??~~ ^{|TruncDirSeries|} that for any fixed $\sigma > 0$, the spectrum of

$$\lim_{x \rightarrow \infty} \left\{ \sum_{n \leq x} \frac{f(n)}{n^\sigma} \bigg/ \sum_{n \leq x} \frac{1}{n^\sigma} : f \in \mathcal{F}(S) \right\}$$

is easily understood in terms of Euler products and $\Gamma(S)$, except when $\sigma = 1$, in which case we have the *logarithmic spectrum*, $\Gamma_0(S)$, which is easier to study than $\Gamma(S)$: The fact that

$$\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{u} \int_0^u \frac{1}{y^t} \sum_{n \leq y^t} f(n) dt + \frac{1}{x \log x} \sum_{n \leq x} f(n),$$

implies that $\Gamma_0(S) \subset K(\Gamma(S))$, the convex linear combinations of elements of $\Gamma(S)$. Similarly we deduce that

$$\Lambda_0(S) = \left\{ \frac{1}{u} \int_0^t \sigma(t) dt : \chi, \sigma \text{ as in } \left(\frac{|IntDelEqn|}{28.1}, u \geq 1 \right) \right\},$$

so that $\Lambda_0(S) \subset K(\Lambda(S))$. We need to see how much of the theory for $\Gamma(S)$ carries over to $\Gamma_0(S)$:

33.1 Results for logarithmic means

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Proposition 33.1 *Let f be a multiplicative function with $|f(n)| \leq 1$ for all n , and put $g(n) = \sum_{d|n} f(d)$. Then*

$$\frac{1}{\log x} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| \ll \exp \left(-\frac{1}{2} \sum_{p \leq x} \frac{1 - \operatorname{Re}(f(p))}{p} \right).$$

Proof Let $g = 1 * f$. Since

$$\sum_{n \leq x} g(n) = \sum_{n \leq x} \sum_{d|n} f(d) = \sum_{d \leq x} f(d) \left(\frac{x}{d} + O(1) \right) = x \sum_{d \leq x} \frac{f(d)}{d} + O(x),$$

we see that

$$\begin{aligned}
\frac{1}{\log x} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| &\leq \frac{1}{x \log x} \sum_{n \leq x} |g(n)| + O\left(\frac{1}{\log x}\right) \\
&\ll \frac{1}{\log^2 x} \sum_{n \leq x} \frac{|g(n)|}{n} + \frac{1}{\log x} \\
&\ll \exp\left(\sum_{p \leq x} \frac{|1 + f(p)| - 2}{p}\right) + \frac{1}{\log x}
\end{aligned}$$

by [\(3.2.1\)](#) and then Mertens' theorem. Now $\frac{1}{2}(1 - \operatorname{Re}(z)) \leq 2 - |1 + z| \leq 1 - \operatorname{Re}(z)$ whenever $|z| \leq 1$, and so the result follows. \square

Now [\(8.6\)](#) together with Proposition [33.1](#) implies that $t = t_f(x, \log x)$ is small if the mean value is large. Indeed if $|\sum_{n \leq x} f(n)/n| \geq (\log x)^{1-\epsilon}$ then $\mathbb{D}^2(f, n^{it}, x) \leq (\epsilon + o(1)) \log \log x$ and $\mathbb{D}^2(f, 1, x) \leq (2\epsilon + o(1)) \log \log x$, so that

$$\begin{aligned}
\log(1 + |t| \log x) + O(1) &= \mathbb{D}^2(1, n^{it}, x) \leq (\mathbb{D}(f, n^{it}, x) + \mathbb{D}(f, 1, x))^2 \\
&\leq ((1 + \sqrt{2})^2 \epsilon + o(1)) \log \log x.
\end{aligned}$$

Hence

$$|t_f(x, \log x)| \ll \frac{1}{(\log x)^{1-6\epsilon}}.$$

It is not entirely surprising that t must be small since if $f(n) = n^{it}$ with $|t| \geq 1/\log x$ then $\sum_{n \leq x} f(n)/n = \log(1/|t|) + O(1)$.

LogLipsch

Exercise 33.2 Prove the Lipschitz-type estimate

$$\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} - \frac{1}{\log(x/y)} \sum_{n \leq x/y} \frac{f(n)}{n} \ll \frac{\log 2y}{\log x},$$

for all functions f with $|f(n)| \leq 1$. (Hint: Do this from first principles.)

Exercise 33.3 By using partial summation in Proposition [3.6](#) or [GenFundLem](#), show that if $f(p^k) = 1$ for all primes $p > y$ then for $x = y^u$ we have

$$\sum_{n \leq x} \frac{f(n)}{n} = \mathcal{P}(f; y) \log x + O(\log y).$$

This implies that the Euler product spectrum here is the same as before (see, e.g., Proposition [30.13](#)).

LogStructThm

Proposition 33.4 Let f be any multiplicative function with $|f(n)| \leq 1$. Let g be the completely multiplicative function defined by $g(p) = 1$ for $p \leq y$ and $g(p) = f(p)$ for $p > y$. If $x = y^u$ then

$$\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} = \mathcal{P}(f, y) \frac{1}{\log x} \sum_{n \leq x} \frac{g(n)}{n} + O\left(\frac{1}{u^{1/2}}\right).$$

Proof Let $f * 1 = g * h$ so that $h(p) = 1$ for all $p > y$. Using the last exercise we obtain, for $v = \sqrt{u}$ and $x = y^u$,

$$\begin{aligned} \sum_{n \leq x} \frac{(g * h)(n)}{n} &= \sum_{a \leq x/y^v} \frac{g(a)}{a} \sum_{b \leq x/a} \frac{h(b)}{b} + \sum_{b \leq y^v} \frac{h(b)}{b} \sum_{x/y^v < a \leq x/b} \frac{g(a)}{a} \\ &= \sum_{a \leq x/y^v} \frac{g(a)}{a} (\mathcal{P}(h; y) \log x/a + O(\log y)) + O\left(\sum_{b \leq y^v} \frac{\log(y^v/b)}{b}\right) \\ &= \mathcal{P}(h; y) \int_1^x \sum_{a \leq t} \frac{g(a)}{a} \frac{dt}{t} + O(u \log^2 y), \end{aligned}$$

extending the sum over a to all $a \leq x$. Since $g * h = f * 1$ and $\mathcal{P}(1, y) = 1$ we deduce that

$$\int_1^x \sum_{a \leq t} \frac{f(a)}{a} \frac{dt}{t} = \mathcal{P}(f; y) \int_1^x \sum_{a \leq t} \frac{g(a)}{a} \frac{dt}{t} + O(u \log^2 y).$$

We subtract the expression for x from that for xy^w , with $w = \sqrt{u}$, and then use exercise 33.2 to note that

$$\begin{aligned} \int_x^{xy^w} \sum_{a \leq t} \frac{f(a)}{a} \frac{dt}{t} &= \int_x^{xy^w} \left(\frac{\log t}{\log x} \sum_{n \leq x} \frac{f(n)}{n} + O(\log(t/x)) \right) \frac{dt}{t} \\ &= uw \log^2 y \cdot \frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} + O(u \log^2 y). \end{aligned}$$

Combining all this information yields the Proposition. \square

This is our structure theorem, and allows us to assert that $\Gamma_0(S) = \Gamma_{\mathcal{P}}(S) \Lambda_0(S)$, and hence $\Gamma_0(S)$ is starlike.

Exercise 33.5 Prove that $\Gamma_0(S) = \Lambda_0(S)$.

One can easily show that there are elements of $\Lambda_0(S)$ that do not belong to $\Gamma_{\mathcal{P}}(S)$: Let $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = \alpha \in S$ for $t > 1$, so that $\sigma(t) = 1 - (1 - \alpha) \log t$ for $1 \leq t \leq 2$, and hence

$$\frac{1}{u} \int_0^u \sigma(t) dt = 1 - (1 - \alpha) \frac{1}{u} \int_1^u \log t dt = 1 - (1 - \alpha) \left(\log u - 1 + \frac{1}{u} \right)$$

for $1 \leq u \leq 2$. This implies that $\{1 - (1 - \alpha)t : 0 \leq t \leq \log 2 - 1/2\} \subset \Lambda_0(S)$ and if z belongs to this set then $\arg(1 - z) = \arg(1 - \alpha)$. In particular this implies that $\text{Ang}(\Gamma_0(S)) \geq \text{Ang}(S)$. Moreover if $0 < \text{Ang}(S) < \frac{\pi}{2}$ then one can show, as in section 31.3 then $z \notin \Gamma_{\mathcal{P}}(S)$, and hence we have proved that there are elements of $\Lambda_0(S)$ are not in $\Gamma_{\mathcal{P}}(S)$.

The elements of $\Lambda_0(S)$ are of the form $(1 * \sigma)(u)/u$, which arise naturally, as we saw in exercise 31.2. Let us suppose that $S = [-1, 1]$ and $\chi \in \mathcal{F}(S)$. Then $1 + \chi(t) = 2$ for all $t \leq 1$ and $1 + \chi(t) \geq 0$ for all $t \geq 1$. Now we know that $M(u) = M_-(u) = u$ for $0 \leq u \leq 1$. If $M(t) \geq 0$ for all $t < u$ then, by exercise 31.2,

$$uM(u) = \int_0^u M(u-t)(1+\chi)(t)dt \geq 2 \int_0^1 M(u-t)dt.$$

Gamma0-11

Exercise 33.6 Use this functional equation to prove that $M(u) > 0$ for all $u > 0$, or even $M(u) \gg ((2 + o(1))/u \log u)^u$. Deduce that $\Gamma_0([-1, 1]) = [0, 1]$.

33.2 Bounding $\Gamma_0(S)$

We are able to say much more about the structure of $\Gamma_0(S)$ thanks to the following result:

Hull0fG0(S)

Proposition 33.7 Suppose S is a closed subset of \mathbb{U} with $1 \in S$. Then $\Gamma_0(S) \subset \mathcal{R}$, the closure of the convex hull of the points $\prod_{i=1}^n \frac{1+s_i}{2}$, for all $n \geq 1$, and all choices of points s_1, \dots, s_n lying in the convex hull of S .

By exercise 30.7 we know that the elements of $K(S)$ are all convex linear combinations of the points $e^{i\theta}$ with $\theta = 0$ or $2\delta \leq |\theta| \leq \pi$. Hence $\Gamma_0(S)$ is a subset of the convex hull of the points $\prod_{j=1}^n \frac{1+e^{i\theta_j}}{2}$ where $2\delta \leq |\theta_j| \leq \pi$, with $n \geq 0$, by Theorem 33.7. Such a product has magnitude $\leq (\cos \delta)^n \leq \cos \delta$ if $n \geq 1$, and so $\Gamma_0(S)$ is a subset of the convex hull of $\{1\} \cup \{|z| \leq \cos \delta\}$. Now, if $|z| \leq \cos \delta$ then one can show that $\text{Ang}(z) \leq \arcsin(|z|) \leq \frac{\pi}{2} - \delta$, and so it follows that $\text{Ang}(\Gamma_0(S)) \leq \frac{\pi}{2} - \delta = \text{Ang}(S)$. In the previous section we showed that $\text{Ang}(\Gamma_0(S)) \geq \text{Ang}(S)$, and so we can now deduce that

$$\text{Ang}(\Gamma_0(S)) = \text{Ang}(S).$$

Proof of Proposition 33.7 By exercise 28.5 with $\chi'(t) = -1$ for all $t \geq 1$ we have that $\sigma(u)$ equals

$$\sigma_{-1}(u) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \prod_{i=1}^k \frac{1 + \chi(t_i)}{t_i} \sigma_{-1}(u - t_1 - \dots - t_k) dt_1 \dots dt_k.$$

Integrating yields that $M(u)$ equals

$$M_{-1}(u) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \prod_{i=1}^k \frac{1 + \chi(t_i)}{t_i} M_{-1}(u - t_1 - \dots - t_k) dt_1 \dots dt_k. \quad (33.1)$$

LinearCombo

We have shown that $M_{-1}(v) > 0$ for all $v > 0$, so this is a linear combination of elements of \mathcal{R} , with non-negative coefficients. The sum of those coefficients is

$$M_{-1}(u) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\substack{t_1, \dots, t_k \geq 1 \\ t_1 + \dots + t_k \leq u}} \prod_{i=1}^k \frac{2}{t_i} M_{-1}(u - t_1 - \dots - t_k) dt_1 \dots dt_k,$$

which equals $M_1(u)$, that is the case that $\chi(t) = 1$ for all t ; that is $\sigma(t) = 1$ for all t , and so $M_1(u) = u$. Hence we have proved that $M(u)/u$, which equals the quantity in (33.1) divided by u , lies in the convex hull of \mathcal{R} , as desired. \square

33.3 Negative truncations

In exercise 33.6 we saw that $\Gamma_0([-1, 1]) = [0, 1]$, which might mistake one into surmising that $\sum_{n \leq N} f(n)/n \geq 0$ whenever $f \in \mathcal{F}([-1, 1])$; however all one can deduce is that $\sum_{n \leq N} f(n)/n \geq -o_{N \rightarrow \infty}(\log N)$. In 1958 Haselgrove showed that $\sum_{n \leq N} \lambda(n)/n$ gets negative, where $\lambda(p^k) = (-1)^k$, and recently it was shown [8] that the first such value is $N = 72185376951205$. Moreover the sum equals $-2.075 \dots \times 10^{29}$ when $N = 72204113780255$. This leads to several questions: What is the minimum possible value of $\sum_{n \leq N} f(n)/n$ for each large N ? For any N ? To begin with we show that this is easily bounded below: If $g(n) = \sum_{d|n} f(d)$ then each $g(n) \geq 0$ and so

$$0 \leq \sum_{n \leq x} g(n) = \sum_{d \leq x} f(d) \left[\frac{x}{d} \right] \leq \sum_{d \leq x} \left(x \frac{f(d)}{d} + 1 \right),$$

and hence for any $f \in \mathcal{F}([-1, 1])$ and any $N \geq 1$ we have

$$\sum_{n \leq N} \frac{f(n)}{n} \geq -1.$$

This can be somewhat improved:

NegTruncPrecise

Proposition 33.8 *If $f \in \mathcal{F}([-1, 1])$ and $g(n) = \sum_{d|n} f(d)$ then*

$$\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{x} \sum_{n \leq x} g(n) + (1 - \gamma) \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{(\log \log x)^2}{(\log x)^{2-\sqrt{3}}}\right).$$

Proof Proceeding as above we have

$$\sum_{n \leq x} g(n) = \sum_{d \leq x} f(d) \left[\frac{x}{d} \right] = x \sum_{d \leq x} \frac{f(d)}{d} - \sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\},$$

and so for $K = \lfloor \log x \rfloor$, we have

$$x \sum_{n \leq x} \frac{f(n)}{n} - \sum_{n \leq x} g(n) = \sum_{k=1}^K \sum_{x/(k+1) < m \leq x/k} f(m) \int_m^{x/k} \frac{x}{t^2} dt + O(x/K).$$

We can rewrite each such sum as

$$\begin{aligned}
& \int_{x/(k+1)}^{x/k} \sum_{x/(k+1) < m \leq t} f(m) \frac{x}{t^2} dt \\
&= \frac{1}{x} \sum_{n \leq x} f(n) \int_{x/(k+1)}^{x/k} \left(t - \frac{x}{k+1} \right) \frac{x}{t^2} dt + O \left(\frac{x \log \log x}{k(\log x)^{2-\sqrt{3}}} \int_{x/(k+1)}^{x/k} \frac{x}{t^2} dt \right) \\
&= \sum_{n \leq x} f(n) \left(\log \left(\frac{k+1}{k} \right) - \frac{1}{k+1} \right) + O \left(\frac{\log \log x}{k(\log x)^{2-\sqrt{3}}} \right)
\end{aligned}$$

by exercise [14.2](#). Summing up over all k , $1 \leq k \leq K$ yields the result since $\sum_{k=1}^K 1/(k+1) = \log(K+1) + \gamma - 1 + O(1/K)$. \square

GenTruncPrecise

Exercise 33.9 Modify the above proof, using Corollary [14.1](#), to show that for any totally multiplicative f with $|f(n)| \leq 1$ we have the same estimate but with $1 - \gamma$ replaced by

$$c_t := (1 + it) \int_1^\infty \frac{\{z\}}{z^{2+it}} dz, \quad \text{where } t = t_f(x, \log x).$$

NegVals

Proposition 33.10 There exists a constant $c > 0$ such that if x is sufficiently large then there exists $f = f_x \in \mathcal{F}([-1, 1])$ for which

$$\sum_{n \leq x} \frac{f(n)}{n} \leq -\frac{c}{\log x}$$

Proof We discussed above that there exists an integer N such that $\sum_{n \leq N} \frac{\lambda(n)}{n} = -\delta$ for some $\delta > 0$. Now let $x > N^2$ be large and define $f(p) = 1$ if $x/(N+1) < p \leq x/N$ and $f(p) = -1$ for all other p . If $n \leq x$ then we see that $f(n) = \lambda(n)$ unless $n = p\ell$ for a (unique) prime $p \in (x/(N+1), x/N]$ in which case $f(n) = \lambda(\ell) = \lambda(n) + 2\lambda(\ell)$. Therefore

$$\begin{aligned}
\sum_{n \leq x} \frac{f(n)}{n} &= \sum_{n \leq x} \frac{\lambda(n)}{n} + 2 \sum_{x/(N+1) < p \leq x/N} \frac{1}{p} \sum_{\ell \leq x/p} \frac{\lambda(\ell)}{\ell} \\
&= \sum_{n \leq x} \frac{\lambda(n)}{n} - 2\delta \sum_{x/(N+1) < p \leq x/N} \frac{1}{p} \sim -\frac{2\delta}{N \log x},
\end{aligned}$$

by the prime number theorem. \square

This next part needs editing:

Set $u = \sum_{p \leq x} (1 - f(p))/p$. By Theorem 2 of A. Hildebrand [\[5\]](#) (with f there being our function g , $K = 2$, $K_2 = 1.1$, and $z = 2$) we obtain that

$$\begin{aligned}
\frac{1}{x} \sum_{n \leq x} g(n) &\gg \prod_{p \leq x} \left(1 - \frac{1}{p} \right) \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots \right) \sigma_- \left(\exp \left(\sum_{p \leq x} \frac{\max(0, 1 - g(p))}{p} \right) \right) \\
&\quad + O(\exp(-(\log x)^\beta)),
\end{aligned}$$

where β is some positive constant and $\sigma_-(\xi) = \xi\rho(\xi)$ with ρ being the Dickman function¹². Since $\max(0, 1 - g(p)) \leq (1 - f(p))/2$ we deduce that

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} g(n) &\gg (e^{-u} \log x)(e^{u/2} \rho(e^{u/2})) + O(\exp(-(\log x)^\beta)) \\ &\gg e^{-ue^{u/2}} (\log x) + O(\exp(-(\log x)^\beta)), \end{aligned} \quad (33.2) \quad \boxed{3.5}$$

since $\rho(\xi) = \xi^{-\xi+o(\xi)}$.

On the other hand, a special case of the main result in [3] implies that

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll e^{-\kappa u}, \quad (33.3) \quad \boxed{3.6}$$

where $\kappa = 0.32867 \dots$. Combining Proposition 3.1 with (3.5) and (3.6) we immediately get that $\delta(x) \geq -c/(\log \log x)^\xi$ for any $\xi < 2\kappa$. This completes the proof of Theorem 32.1.

Remark 33.11 *The bound (3.5) is attained only in certain very special cases, that is when there are very few primes $p > x^{e^{-u}}$ for which $f(p) = 1 + o(1)$. In this case one can get a far stronger bound than (3.6). Since the first part of Theorem 32.1 depends on an interaction between these two bounds, this suggests that one might be able to improve Theorem 1 significantly by determining how (3.5) and (3.6) depend upon one another.*

Now what about the class of all multiplicative functions, not necessarily totally multiplicative, with values in $[-1, 1]$? We will sketch a proof that we have the same lower bound $\gg -1/(\log \log x)^{3/5}$ unless $\sum_{k \geq 1} (1 + f(2^k))/2^k \ll 1/(\log x)^{1/20}$. Now $\sum_{n \leq x} f(n)/n \geq -\delta_1 \log 2 + o(1)$, with equality if and only if $\mathbb{D}^*(f, f_2; x) = o(1)$ where $\mathbb{D}^*(f, g; x) := \sum_{p \leq x} \sum_{k \geq 1} (1 - (fg)(p^k))/p^k$, and $f_2(2^k) = -1$ for all $k \geq 1$, otherwise $f_2(\cdot)$ is totally multiplicative with $f_2(p) = f_1(p)$ for all $p \geq 3$.

Exercise 33.12 *Use the special case of exercise 14.8 and (14.1) to prove that*

$$\sum_{n \leq x} \frac{f_2(n)}{n} = \log 2 \cdot \frac{1}{x} \sum_{n \leq x} f_1(n) + O\left(\frac{\log \log x}{(\log x)^{2-\sqrt{3}}}\right).$$

Combining exercises 14.8 and 33.9: Given $|f(n)| \leq 1$, let g be totally multiplicative with $g(p) = f(p)$ for all primes p , and $G(n) = \sum_{d|n} g(d)$ then

$$\sum_{n \leq x} \frac{f(n)}{n} = \mathcal{C}_0(f) \frac{1}{x} \sum_{n \leq x} G(n) + (c_t \mathcal{C}_0(f) - \kappa_t(f)) \frac{1}{x} \sum_{n \leq x} g(n) + O\left(\frac{x(\log \log x)^2}{(\log x)^{2-\sqrt{3}}}\right).$$

¹²The Dickman function is defined as $\rho(u) = 1$ for $u \leq 1$, and $\rho(u) = (1/u) \int_{u-1}^u \rho(t) dt$ for $u \geq 1$.

If $-1 \leq f(n) \leq 1$ then $t = 0$ and so we have

$$\sum_{n \leq x} \frac{f(n)}{n} = \mathcal{C}_0(f) \frac{1}{x} \sum_{n \leq x} G(n) + ((1-\gamma)\mathcal{C}_0(f) - \kappa_0(f)) \frac{1}{x} \sum_{n \leq x} g(n) + O\left(\frac{x(\log \log x)^2}{(\log x)^{2-\sqrt{3}}}\right).$$

For $\sum_{n \leq x} \frac{f(n)}{n}$ to have a not-too-small negative value, $\mathcal{C}_0(f)$ must be small else one can argue as above. If $\mathcal{C}_0(f)$ is small then $\sum_{k \geq 1} (1 + f(2^k))/2^k$ must be very small and the main term comes from $\kappa_0(f)$ times the mean value of $g(n)$. We can easily then show that the largest negative value comes from when g is close to f_1 .

33.4 Convergence

We observe that if $\sum_n f(n)/n$ converges then $\sum_{p \leq x} f(p)/p$ is bounded.

To see this we begin by observing that if $t \geq t_0$ then $|E(t)| \leq \epsilon$ where $E(t) := \sum_{n > t} f(n)/n$. This implies, by partial summation, that if $N \geq t_\epsilon$ then $\sum_{n \geq N} f(n)/n^{1+\frac{1}{\log x}} = \int_{t \geq N} dE(t)/t^{\frac{1}{\log x}} \ll \epsilon$. Hence if $\log^2 N < \epsilon \log x$ then

$$\sum_{n \leq N} \frac{f(n)}{n} = \sum_{n \geq 1} \frac{f(n)}{n^{1+\frac{1}{\log x}}} + O(\epsilon),$$

and so the value of $\sum_n f(n)/n$ is simply the limit of $\sum_{n \geq 1} f(n)/n^{1+\alpha}$ as $\alpha \rightarrow 0$. Taking logarithms and limits this means that $\sum_p f(p)/p^{1+\alpha}$ exists as $\alpha \rightarrow 0$. Now if $\alpha = 1/\log x$ then we have seen that this equals $\sum_{p \leq x} f(p)/p + O(1)$. The result follows.

33.5 Upper bounds revisited

Let us suppose that $t = t_f(x, T)$, and write $f(n) = \sum_{ab=n} a^{it} g(b)$.

Exercise 33.13 Show that $1/a^{1+it} = \int_{a-1/2}^{a+1/2} du/u^{1+it} + O((1+|t|)^2/a^3)$, and deduce that if $x \geq A \geq 1+|t|$ then

$$\sum_{x > a \geq A} \frac{1}{a^{1-it}} = \frac{x^{it} - A^{it}}{it} + O\left(\frac{(1+|t|)^2}{A^2}\right)$$

We therefore deduce, for $A \geq (1+|t|)^2$,

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{n} &= \sum_{ab \leq x} \frac{1}{a^{1-it}} \cdot \frac{g(b)}{b} \\ &= \sum_{b \leq x/A} \frac{g(b)}{b} \sum_{a \leq x/b} \frac{1}{a^{1-it}} + \sum_{a \leq A} \frac{1}{a^{1-it}} \sum_{x/A < b \leq x/a} \frac{g(b)}{b} \\ &= \frac{x^{it}}{it} \sum_{b \leq x/A} \left(\frac{g(b)}{b} - \frac{g(b)}{b^{1+it}} \right) + O\left(\frac{(1+|t|)^2}{x} \sum_{b \leq x/A} \frac{|g(b)|}{(x/b)} + \sum_{a \leq A} \frac{1}{a} \sum_{x/A < b \leq x} \frac{|g(b)|}{b} \right). \end{aligned}$$

The second-to-last term is $\ll 1$. The last term is $\ll \frac{(\log A)^2}{\log x} \sum_{b \leq x} \frac{|g(b)|}{b}$. by $\frac{(3.2.2)}{(9.2)}$.
 Just taking absolute values above, with $A = (1 + |t|)^2$, we deduce when $T \leq \log x$

$$\sum_{n \leq x} \frac{f(n)}{n} \ll \left(\frac{1}{|t|} + \frac{(\log \log x)^2}{\log x} \right) \sum_{b \leq x} \frac{|g(b)|}{b}.$$

THE POLYA-VINOGRADOV INEQUALITY

By $\stackrel{\text{GenGSums}}{(\text{I0.3})}$ we have for a primitive character $\chi \pmod{q}$,

$$\begin{aligned} \sum_{n=M+1}^{M+N} \chi(n) &= \frac{1}{g(\bar{\chi})} \sum_{n=M+1}^{M+N} \sum_{a \pmod{q}} \bar{\chi}(a) e\left(\frac{an}{q}\right) \\ &= \frac{1}{g(\bar{\chi})} \sum_{a \pmod{q}} \bar{\chi}(a) e\left(\frac{a(2M+N+1)}{2q}\right) \frac{\sin \pi Na/q}{\sin \pi a/q}. \end{aligned}$$

Taking absolute values, we obtain

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| \leq \frac{1}{\sqrt{q}} \sum_{(a,q)=1} \frac{1}{|\sin \pi a/q|} \leq \sqrt{q} \log q.$$

Exercise 34.1 *Justify this last step. Indicate how one might improve this to $\leq (2/\pi + o(1))\sqrt{q} \log q$.*

There are various ways one can develop the series above. The most useful is due to Polya:

Exercise 34.2 *Prove that if $0 < \alpha \leq 1$ and χ is a character mod q then*

$$\sum_{n \leq \alpha q} \chi(n) = \frac{g(\chi)}{2i\pi} \sum_{1 \leq |n| \leq N} \frac{\bar{\chi}(n)}{n} (1 - e(-n\alpha)) + O\left(1 + \frac{q \log q}{N}\right) \quad (34.1) \quad \boxed{\text{Polya}}$$

for any $N \geq 1$. (Hint: Think: “Fourier analysis.”)

Exercise 34.3 *Deduce that*

$$\sum_{n < q/2} \chi(n) = \frac{g(\chi)}{2i\pi} (2 - \bar{\chi}(2))(1 - \bar{\chi}(-1))L(1, \bar{\chi}) + O(1).$$

$\boxed{\text{ExptoChi}}$

Exercise 34.4 *Using $\stackrel{\text{ExpSums2Chars}}{(\text{23.9})}$ deduce that if $(b, r) = 1$ then*

$$\sum_{n \leq N} \frac{f(n)e(bn/r)}{n} = \sum_{d|r} \frac{f(d)}{d\phi(r/d)} \sum_{\psi \pmod{r/d}} \bar{\psi}(b)g(\psi) \sum_{n \leq N/d} \frac{f(n)\bar{\psi}(n)}{n}.$$

Exercise 34.5 *Deduce that if $(b, r) = 1$ then*

$$\sum_{n \leq bq/r} \chi(n) = \frac{g(\chi)}{2i\pi} \left((1 - \bar{\chi}(-1))L(1, \bar{\chi}) - \sum_{d|r} \frac{2\bar{\chi}(d)}{d\phi(r/d)} \sum_{\substack{\psi \pmod{r/d} \\ (\bar{\chi}\bar{\psi})(-1)=-1}} \bar{\psi}(-b)g(\psi)L(1, \bar{\chi}\bar{\psi}) \right) + O(1)$$

Let $X = x/\log^A x$. If $0 \leq \alpha < 1$ then Dirichlet's approximation theorem tells us that there exists a rational number b/r with $1 \leq r \leq X$ such that $|\alpha - b/r| \leq 1/rX$. Therefore if $n \leq R := 1/|\alpha - b/r|$ then $|e(\alpha n) - e(bn/r)| \ll n|\alpha - b/r|$; and otherwise $|e(\alpha n) - e(bn/r)| \ll 1$. Hence

$$\left| \sum_{n \leq x} \frac{f(n)(e(\alpha n) - e(bn/r))}{n} \right| \ll \sum_{n \leq \min\{R, x\}} |\alpha - b/r| + \sum_{\min\{R, x\} < n \leq x} \frac{1}{n} \\ \ll \log(1 + |\alpha - b/r|x) \ll \log \log x.$$

Select $\psi_1 \pmod{r/d_1}$ as that character with conductor dividing r for which $M = M_{f\bar{\psi}_1}(x, \log x)$ is minimal.¹³ We now bound the contribution of the other terms in exercise 34.4. For the other characters ψ_j we know that $M_{f\bar{\psi}_j}(x, \log x) \geq (2/3 - o(1)) \log(\frac{\log x}{\log r}) + O(1)$ by Proposition 24.1^{eta2}; and that if k is sufficiently large then $M_{f\bar{\psi}_j}(x, \log x) \geq (1 - \epsilon) \log(\frac{\log x}{\log r}) + O(1)$ by Proposition 24.2^{etak}. Substituting these bounds into (8.6)^{Halasz4Log}, and then the bounds from there into exercise 34.4, we obtain¹⁴

$$\sum_{n \leq x} \frac{f(n)e(bn/r)}{n} - f(d_1)\bar{\psi}_1(b) \frac{g(\psi_1)}{d_1\phi(r/d_1)} \sum_{n \leq x/d} \frac{f(n)\bar{\psi}_1(n)}{n} \\ \ll \left(r^{-1/2}(\log x)^{1/3} + r^{1/2} \right) (r \log x)^{o(1)}.$$

34.1 A lower bound on distances

When χ has given order $g > 1$, we wish to bound

$$\mathbb{D}(\chi(n), \psi(n)n^{it}, x)^2 = \sum_{p \leq x} \frac{1 - \operatorname{Re}((\chi\bar{\psi})(p)/p^{it})}{p}$$

from below, where $|t| < (\log x)^2$. The smallest the p th term can be, for given $\psi(p)$ and p^{it} , is when $\chi(p)$ is that g th root of unity nearest to $\psi(p)p^{it}$. If ψ is a character mod r the Siegel-Walfisz Theorem tells us that there are roughly equal number of primes $p \equiv h \pmod{r}$ for each $(h, r) = 1$ in the interval $[z, z + z/(\log z)^{3A}]$ provided $\log \log z > (1/A) \log \log x$. If ψ has order k we may write each $\psi(p) = e(-\ell/k)$, the ℓ depending on the arithmetic progression that p belongs to mod k . Also $p^{it} = z^{it} + o(1) = e^{2i\pi\theta} + o(1)$ where $\theta := (t \log z)/2\pi$. Hence

$$\sum_p \frac{1 - \operatorname{Re}((\chi\bar{\psi})(p)/p^{it})}{p} \geq \sum_p \frac{1}{p} \left\{ \frac{1}{k} \sum_{\ell=0}^{k-1} \left(1 - \min_{0 \leq a \leq g-1} \cos \left(2\pi \left(\frac{a}{g} + \frac{\ell}{k} - \theta \right) \right) \right) + o(1) \right\}$$

where the sum is over the primes in $[z, z + z/(\log z)^{3A}]$.

¹³This is not quite correct. We need to work ex 31.4 by writing it in terms of primitive characters and then use those. Nonetheless the calculations done here are the correct ones.

¹⁴Can we improve the last term using the Pretentious large sieve?

Exercise 34.6 Show that if $L = [g, k]$ and L/g is even then,

$$1 - \frac{\frac{g}{\pi} \sin \frac{\pi}{g}}{\frac{L}{\pi} \sin \frac{\pi}{L}} \cos \left(\frac{2\pi}{L} \left(\{L\theta\} - \frac{1}{2} \right) \right).$$

Show that if L/g is odd then we replace $\{L\theta\} - 1/2$ by $\{L\theta\}$.

Exercise 34.7 Deduce that if L/g is even then the mean of this last function, for $\theta \in [0, u]$ is $\geq 1 - \frac{g}{\pi} \sin \frac{\pi}{g}$ for all u . Deduce that if $r < (\log x)^2$ with $A > 1/\epsilon$ and ψ has even order k then

$$\frac{\mathbb{D}(\chi(n), \psi(n)n^{it}, x)^2}{\log \log x} \geq 1 - \frac{g}{\pi} \sin \frac{\pi}{g} - O(\epsilon).$$

(Also deduce that if L/g is odd and $\mathbb{D}(\chi(n), \psi(n)n^{it}, x)^2 = o(\log \log x)$ then $\mathbb{D}(\chi(n), \psi(n), x)^2 = o(\log \log x)$ and $g = L$ (i.e. k divides g).)

We deduce from the above that if ($g \geq 3$ is odd and)

$$\left| \sum_{n \leq x} \frac{\chi(n)e(bn/r)}{n} \right| \ll \frac{1}{\sqrt{rd_1}} \frac{r}{\phi(r)} (\log x)^{\frac{g}{\pi} \sin \frac{\pi}{g} + o(1)} + r^{1/2} (r \log x)^{o(1)}.$$

We apply this bound when $r \leq \log x$. By partial summation on $\stackrel{\text{MV1}}{(23.6)}$ for the sum between $r^{1+\epsilon}$ and x , for $x \geq r^2$, we obtain

$$\left| \sum_{n \leq x} \frac{f(n)e(bn/r)}{n} \right| \ll \log r + \frac{\log x}{\sqrt{\phi(r)}} + \log \log x.$$

We use this bound for $r > \log x$.

Combining the above (and this needs tidying up) we obtain that if χ is a primitive character of order g then, by $\stackrel{\text{Polya}}{(34.1)}$, for any $N \geq 1$ we have

$$\sum_{n \leq N} \chi(n) \ll \sqrt{q} (\log q)^{\frac{g}{\pi} \sin \frac{\pi}{g} + o(1)}. \quad (34.2) \quad \boxed{\text{PVimprove}}$$

We believe that this exponent is “best possible” with this method (this needs some explanation!).

34.2 Using the Pretentious Generalized Riemann Hypothesis

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