Chapter 4

Angle Modulation and Multiplexing

Contents

- Continuing from Chapter 3, we now focus on the $\phi(t)$ term (angle) in the general modulated carrier waveform

$$x_c(t) = A(t)\cos\left[2\pi f_c t + \phi(t)\right]$$

4.1 Angle Modulation

- A general angle modulated signal is of the form

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

- <u>Definition</u>: *Instantaneous phase* of $x_c(t)$ is

$$\theta_i(t) = \omega_c t + \phi(t)$$

where $\phi(t)$ is the *phase deviation*

- <u>Define</u>: *Instantaneous frequency* of $x_c(t)$ is

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + \frac{d\phi(t)}{dt}$$

where $d\phi(t)/dt$ is the *frequency deviation*

- There are two basic types of angle modulation

1. *Phase modulation* (PM)

$$\phi(t) = \underbrace{k_p}_{\text{phase dev. const.}} m(t)$$

which implies

$$x_c(t) = A_c \cos[\omega_c t + k_p m(t)]$$

- Note: the units of k_p is radians per unit of $m(t)$
- If $m(t)$ is a voltage, k_p has units of radians/volt

2. *Frequency modulation* (FM)

$$\frac{d\phi(t)}{dt} = \underbrace{k_f}_{\text{freq. dev. const.}} m(t)$$

or

$$\phi(t) = k_f \int_{t_0}^{t} m(\alpha)\, d\alpha + \phi_0$$

- Note: the units of k_f is radians/sec per unit of $m(t)$
- If $m(t)$ is a voltage, k_f has units of radians/sec/volt
- An alternative expression for k_f is

$$k_f = 2\pi f_d$$

where f_d is the *frequency-deviation constant* in Hz/unit of $m(t)$

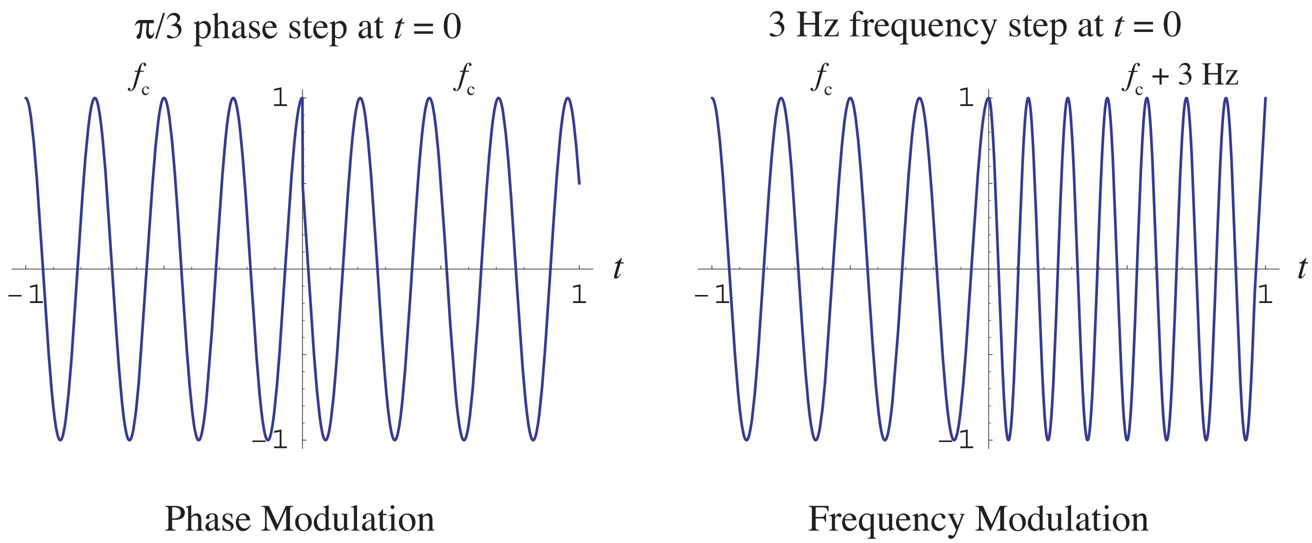
Example 4.1: Phase and Frequency Step Modulation

- Consider $m(t) = u(t)$ v
- We form the PM signal

$$x_{\text{PM}}(t) = A_c \cos\left[\omega_c t + k_p u(t)\right], \quad k_p = \pi/3 \text{ rad/v}$$

- We form the FM signal

$$x_{\text{FM}}(t) = A_c \cos\left[\omega_c t + 2\pi f_d \int^{t} m(\alpha)\, d\alpha\right], \quad f_d = 3 \text{ Hz/v}$$



Phase and frequency step modulation

4.1.1 Narrowband Angle Modulation

- Begin by writing an angle modulated signal in complex form

$$x_c(t) = \text{Re}\Big(A_c e^{j\omega_c t} e^{j\phi(t)}\Big)$$

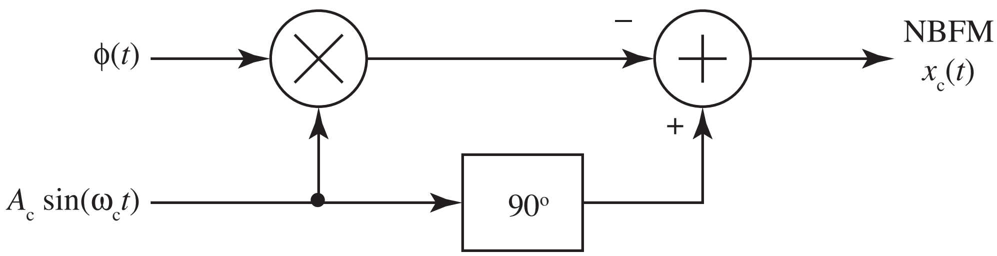
- Expand $e^{j\phi(t)}$ in a power series

$$x_c(t) = \text{Re}\left(A_c e^{j\omega_c t}\left[1 + j\phi(t) - \frac{\phi^2(t)}{2!} - \cdots\right]\right)$$

- The *narrowband approximation* is $|\phi(t)| \ll 1$, then

$$\begin{aligned} x_c(t) &\simeq \text{Re}\Big(A_c e^{j\omega_c t} + jA_c\phi(t)e^{j\omega_c t}\Big) \\ &= A_c\cos(\omega_c t) - A_c\phi(t)\sin(\omega_c t) \end{aligned}$$

- Under the narrowband approximation we see that the signal is similar to AM except it is carrier plus modulated quadrature carrier



NBFM modulator block diagram

Example 4.2: **Single tone narrowband FM**

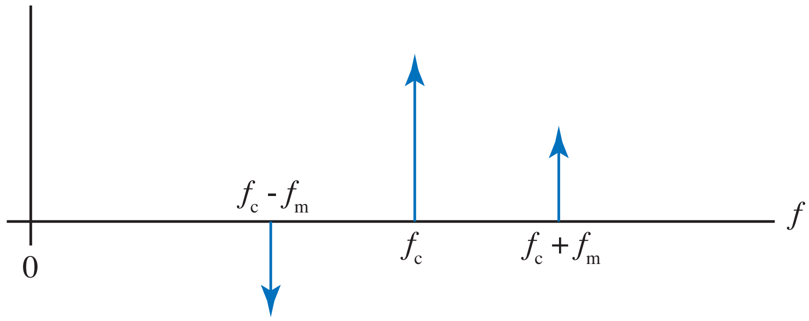
- Consider NBFM with $m(t) = A_m \cos\omega_m t$

$$\begin{aligned}\phi(t) &= 2\pi f_d \int^t A_m \cos\omega_m\alpha \, d\alpha \\ &= A_m \frac{2\pi f_d}{2\pi f_m} \sin\omega_m t = A_m \frac{f_d}{f_m}\sin\omega_m t\end{aligned}$$

- Now,

$$\begin{aligned}x_c(t) &= A_c \cos\left(\omega_c t + A_m \frac{f_d}{f_m}\sin\omega_m t\right) \\ &\simeq A_c\left(\cos\omega_c t - A_m \frac{f_d}{f_m}\sin\omega_m t \sin\omega_c t\right) \\ &= A_c\cos\omega_c t + \frac{A_m f_d}{2f_m}\sin(f_c + f_m)t \\ &\quad - \frac{A_m f_d}{2f_m}\sin(f_c - f_m)t\end{aligned}$$

- This looks very much like AM



Single tone NBFM spectra

4.1.2 Spectrum of an Angle-Modulated Signal

- The development in this obtains the exact spectrum of an angle modulated carrier for the case of

$$\phi(t) = \beta \sin \omega_m t$$

where β is the *modulation index* for sinusoidal angle modulation

- The transmitted signal is of the form

$$\begin{aligned} x_c(t) &= A_c \cos\left(\omega_c t + \beta \sin \omega_m t\right) \\ &= A_c \mathrm{Re}\left\{e^{j\omega_c t} \cdot e^{j\beta \sin \omega_m t}\right\} \end{aligned}$$

- Note that $e^{j\beta \sin \omega_m t}$ is periodic with period $T = 2\pi/\omega_m$, thus we can obtain a Fourier series expansion of this signal, i.e.,

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_m t}$$

- The coefficients are

$$\begin{aligned} Y_n &= \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta\sin\omega_m t}e^{-jn\omega_m t}\,dt \\ &= \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} e^{-j(n\omega_m t-\beta\sin\omega_m t)}\,dt \end{aligned}$$

- Change variables in the integral by letting $x = \omega_m t$, then $dx = \omega_m dt$, $t = \pi/\omega_m \rightarrow x = \pi$, and $t = -\pi/\omega_m \rightarrow x = -\pi$

- With the above substitutions, we have

$$\begin{aligned} Y_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(nx-\beta\sin x)}\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(nx-\beta\sin x)\,dx = J_n(\beta) \end{aligned}$$

which is a *Bessel function* of the first kind order n with argument β

$J_n(\beta)$ Properties

- Recurrence equation:

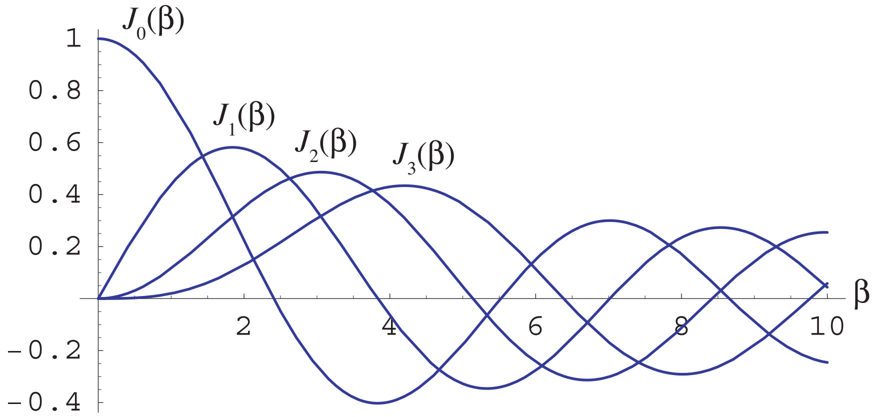
$$J_{n+1}(\beta) = \frac{2n}{\beta}J_n(\beta) - J_{n-1}(\beta)$$

- n – even:

$$J_{-n}(\beta) = J_n(\beta)$$

- n – odd:

$$J_{-n}(\beta) = -J_n(\beta)$$



Bessel function of order 0–3 plotted

- The zeros of the Bessel functions are important in spectral analysis

First five Bessel function zeros for order 0 – 5

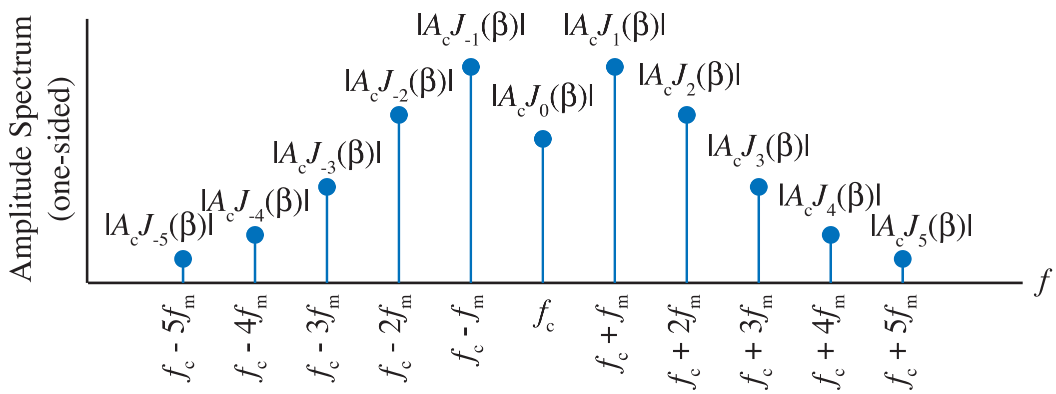
$J_0(\beta) = 0$	2.40483, 5.52008, 8.65373, 11.7915, 14.9309
$J_1(\beta) = 0$	3.83171, 7.01559, 10.1735, 13.3237, 16.4706
$J_2(\beta) = 0$	5.13562, 8.41724, 11.6198, 14.796, 17.9598
$J_3(\beta) = 0$	6.38016, 9.76102, 13.0152, 16.2235, 19.4094
$J_4(\beta) = 0$	7.58834, 11.0647, 14.3725, 17.616, 20.8269
$J_5(\beta) = 0$	8.77148, 12.3386, 15.7002, 18.9801, 22.2178

Spectrum cont.

- We obtain the spectrum of $x_c(t)$ by inserting the series representation for $e^{j\beta \sin \omega_m t}$

$$x_c(t) = A_c \text{Re}\left[e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t}\right]$$
$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[(\omega_c + n\omega_m)t\right]$$

- We see that the amplitude spectrum is symmetrical about f_c due to the symmetry properties of the Bessel functions



- For PM

$$\beta \sin \omega_m t = k_p \underbrace{(A \sin \omega_m t)}_{m(t)}$$

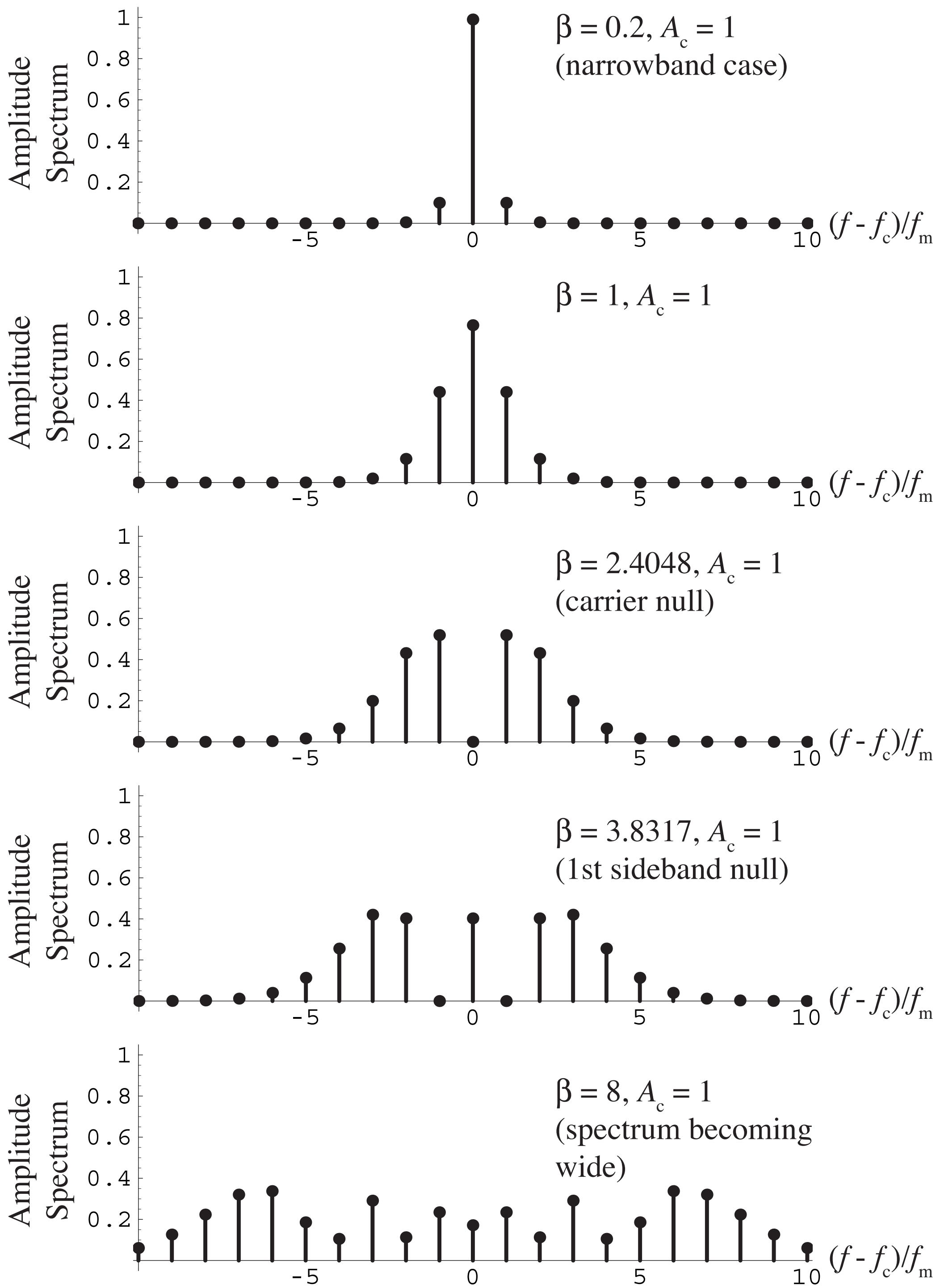
$$\Rightarrow \beta = k_p A$$

- For FM

$$\beta \sin \omega_m t = k_f \int^t A \cos \omega_m \alpha \, d\alpha = \frac{f_d}{f_m} A \sin \omega_m t$$

$$\Rightarrow \beta = (f_d/f_m)A$$

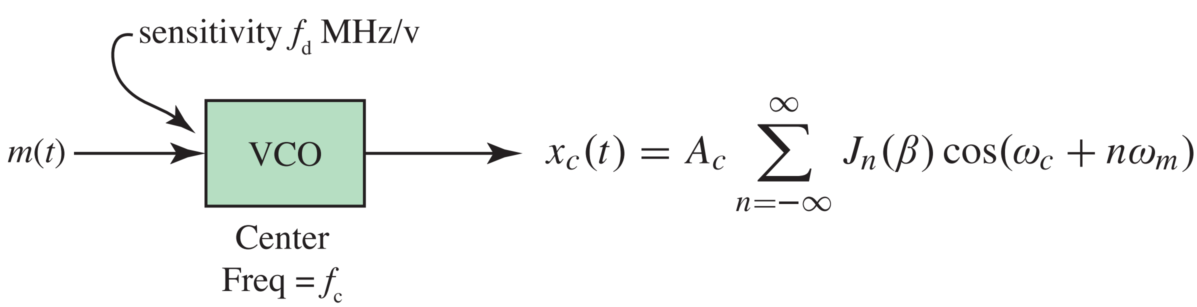
- When β is small we have the narrowband case and as β gets larger the spectrum spreads over wider bandwidth



The amplitude spectrum relative to f_c as β increases

Example 4.3: VCO FM Modulator

- Consider again single-tone FM, that is $m(t) = A\cos(2\pi f_m t)$
- We assume that we know f_m and the modulator deviation constant f_d
- Find A such that the spectrum of $x_c(t)$ contains no carrier component
- An FM modulator can be implemented using a *voltage controlled oscillator* (VCO)



A VCO used as an FM modulator

- The carrier term is $A_c J_0(\beta)\cos\omega_c t$
- We know that $J_0(\beta) = 0$ for $\beta = 2.4048, 5.5201, \ldots$
- The smallest β that will make the carrier component zero is

$$\beta = 2.4048 = \frac{f_d}{f_m}A$$

which implies that we need to set

$$A = 2.4048 \cdot \frac{f_m}{f_d}$$

- Suppose that $f_m = 1$ kHz and $f_d = 2.5$ MHz/v, then we would need to set

$$A = 2.4048 \cdot \frac{1 \times 10^3}{2.5 \times 10^6} = 9.6192 \times 10^{-4}$$

4.1.3 Power in an Angle-Modulated Signal

- The average power in an angle modulated signal is

$$\begin{aligned} \langle x_c^2(t) \rangle &= A_c^2 \langle \cos^2 \left[\omega_c t + \phi(t) \right] \rangle \\ &= \frac{1}{2} A_c^2 + \frac{1}{2} A_c^2 \langle \cos \left\{ 2 \left[\omega_c t + \phi(t) \right] \right\} \rangle \end{aligned}$$

- For large f_c the second term is approximately zero (why?), thus

$$P_{\text{angle mod}} = \langle x_c^2(t) \rangle = \frac{1}{2} A_c^2$$

which makes the power independent of the modulation $m(t)$ (the assumptions must remain valid however)

4.1.4 Bandwidth of Angle-Modulated Signals

- With sinusoidal angle modulation we know that the occupied bandwidth gets larger as β increases

- There are an infinite number of sidebands, but

$$\lim_{n \to \infty} J_n(\beta) \approx \lim_{n \to \infty} \frac{\beta^n}{2^n n!} = 0,$$

so consider the fractional power bandwidth

- Define the power ratio

$$P_r = \frac{P_{\text{carrier}} + P_{\pm k \text{ sidebands}}}{P_{\text{total}}} = \frac{\frac{1}{2}A_c^2 \sum_{n=-k}^{k} J_n^2(\beta)}{\frac{1}{2}A_c^2}$$
$$= J_0^2(\beta) + 2\sum_{n=1}^{k} J_n^2(\beta)$$

- Given an acceptable P_r implies a fractional bandwidth of

$$B = 2kf_m \ \text{(Hz)}$$

- In the text values of $P_r \geq 0.7$ and $P_r \geq 0.98$ are single and double underlined respectively

- It turns out that for $P_r \geq 0.98$ the value of k is IP$[1 + \beta]$, thus

$$B = B_{98} \simeq 2(\beta + 1)f_m \text{ for sinusoidal mod. only}$$

- For arbitrary modulation $m(t)$, define the deviation ratio

$$D = \frac{\text{peak freq. deviation}}{\text{bandwidth of } m(t)} = \frac{f_d}{W}\Big[\max|m(t)|\Big]$$

- In the sinusoidal modulation bandwidth definition let $\beta \rightarrow D$ and $f_m \rightarrow W$, then we obtain what is known as *Carson's rule*

$$B = 2(D + 1)W$$

  - Another view of Carson's rule is to consider the maximum frequency deviation $\Delta f = \max|m(t)|f_d$, then $B = 2(W + \Delta f)$

- Two extremes in angle modulation are
  1. Narrowband: $D \ll 1 \;\Rightarrow\; B = 2W$
  2. Wideband: $D \gg 1 \;\Rightarrow\; B = 2DW = 2\Delta f$

Example 4.4: **Single Tone FM**

- Consider an FM modulator for broadcasting with

$$x_c(t) = 100\cos\left[2\pi(101.1 \times 10^6)t + \phi(t)\right]$$

where $f_d = 75$ kHz/v and

$$m(t) = \cos\left[2\pi(1000)t\right] \text{ v}$$

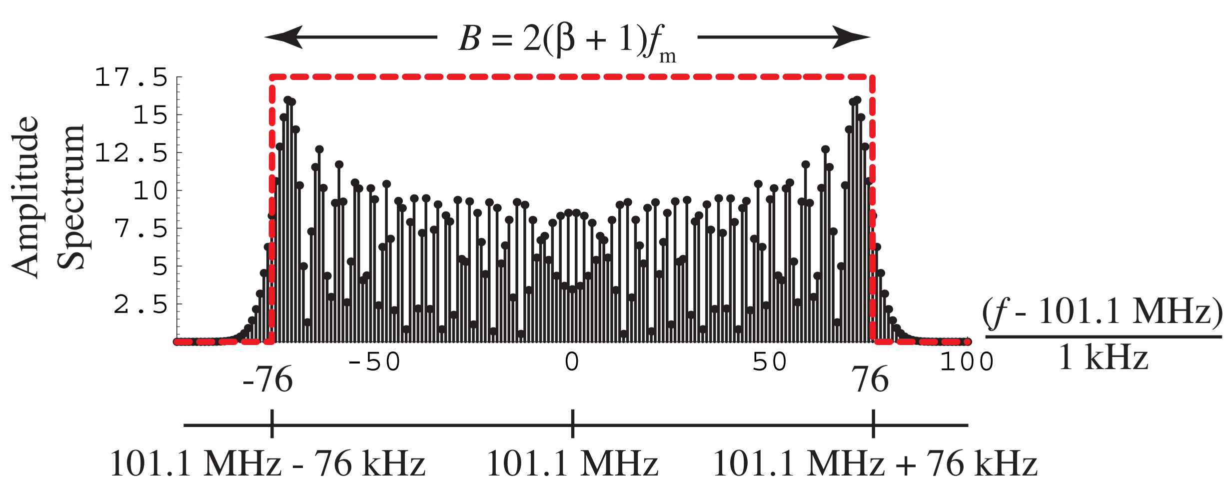
- The β value for the transmitter is

$$\beta = \frac{f_d}{f_m}A = \frac{75 \times 10^3}{10^3} = 75$$

- Note that the carrier frquency is 101.1 MHz and the peak deviation is $\Delta f = 75$ kHz

- The bandwidth of the signal is thus

$$B \simeq 2(1 + 75)1000 = 152 \text{ kHz}$$



- Suppose that this signal is passed through an ideal bandpass filter of bandwidth 11 kHz centered on $f_c = 101.1$ MHz, i.e.,

$$H(f) = \Pi\left(\frac{f - f_c}{11000}\right) + \Pi\left(\frac{f + f_c}{11000}\right)$$

- The carrier term and five sidebands either side of the carrier pass through this filter, resulting an output power of

$$P_{\text{out}} = \frac{A_c^2}{2}\left[J_0^2(75) + 2\sum_{n=1}^{5} J_n^2(75)\right] = 241.93 \text{ W}$$

- Note the input power is $A_c^2/2 = 5000$ W

Example 4.5: **Two Tone FM**

- Finding the exact spectrum of an angle modulated carrier is not always possible
- The single-tone sinusoid case can be extended to multiple tone with increasing complexity
- Suppose that

$$m(t) = A\cos\omega_1 t + B\cos\omega_2 t$$

- The phase deviation is given by $2\pi f_d$ times the integral of the frequency modulation, i.e.,

$$\phi(t) = \beta_1 \sin\omega_1 t + \beta_2 \sin\omega_2 t$$

where $\beta_1 = Af_d/f_1$ and $\beta_2 = Af_d/f_2$

- The transmitted signal is of the form

$$\begin{aligned} x_c(t) &= A_c \cos\left[\omega_c t + \beta_1 \sin \omega_1 t + \beta_2 \sin \omega_2 t\right] \\ &= A_c \text{Re}\left[e^{j\omega_c t} e^{j\beta_1 \sin \omega_1 t} e^{j\beta_2 \sin \beta_2 t}\right] \end{aligned}$$

- We have previously seen that via Fourier series expansion

$$\begin{aligned} e^{j\beta_1 \sin \omega_1 t} &= \sum_{n=-\infty}^{\infty} J_n(\beta_1) e^{jn\omega_1 t} \\ e^{j\beta_2 \sin \omega_1 t} &= \sum_{n=-\infty}^{\infty} J_n(\beta_2) e^{jn\omega_2 t} \end{aligned}$$

- Inserting the above Fourier series expansions into $x_c(t)$, we have

$$\begin{aligned} x_c(t) &= A_c \text{Re}\left\{ e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta_1) e^{jn\omega_1 t} \cdot \sum_{m=-\infty}^{\infty} J_m(\beta_2) e^{jm\omega_2 t} \right\} \\ &= A_c \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(\beta_1) J_m(\beta_2) \cos\left[(\omega_c + n\omega_1 + m\omega_2)t\right] \end{aligned}$$

- The nonlinear nature of angle modulation is clear, since we see not only components at $\omega_c + n\omega_1$ and $\omega_c + m\omega_2$, but also at all combinations of $\omega_c + n\omega_1 + m\omega_2$

- To find the bandwidth of this signal we can use Carson's rule (the sinusoidal formula only works for one tone)

- Recall that $B = 2(\Delta f + W)$, where Δf is the peak frequency deviation

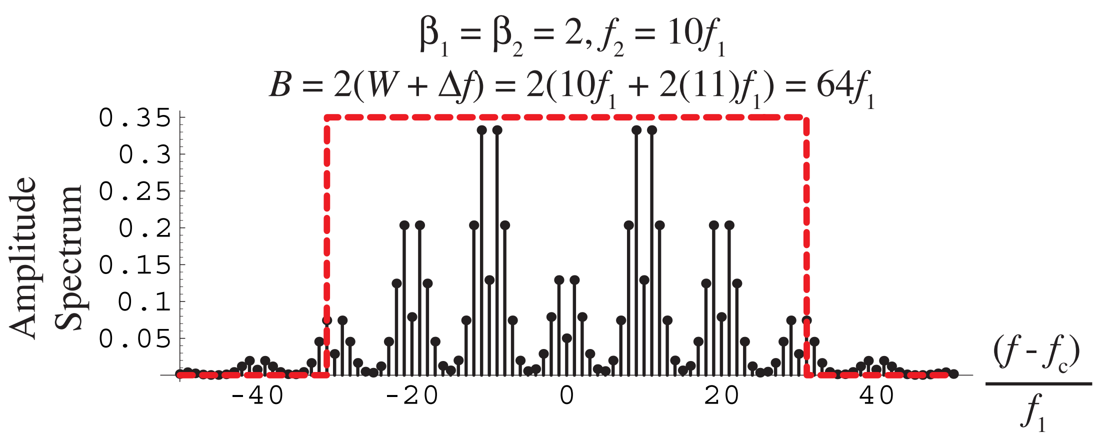
- The frequency deviation is

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\left[\beta_1 \sin\omega_1 t + \beta_2 \sin\omega_2 t\right]$$
$$= \beta_1 f_1 \cos(2\pi f_1 t) + \beta_2 f_2 \cos(2\pi f_2 t) \text{ Hz}$$

- The maximum of $f_i(t)$, in this case, is $\beta_1 f_1 + \beta_2 f_2$

- Suppose $\beta_1 = \beta_2 = 2$ and $f_2 = 10f_1$, then we see that $W = f_2 = 10f_1$ and

$$B = 2(W+\Delta f) = 2\left[10f_1+2(f_1+10f_1)\right] = 2(32f_1) = 64f_1$$



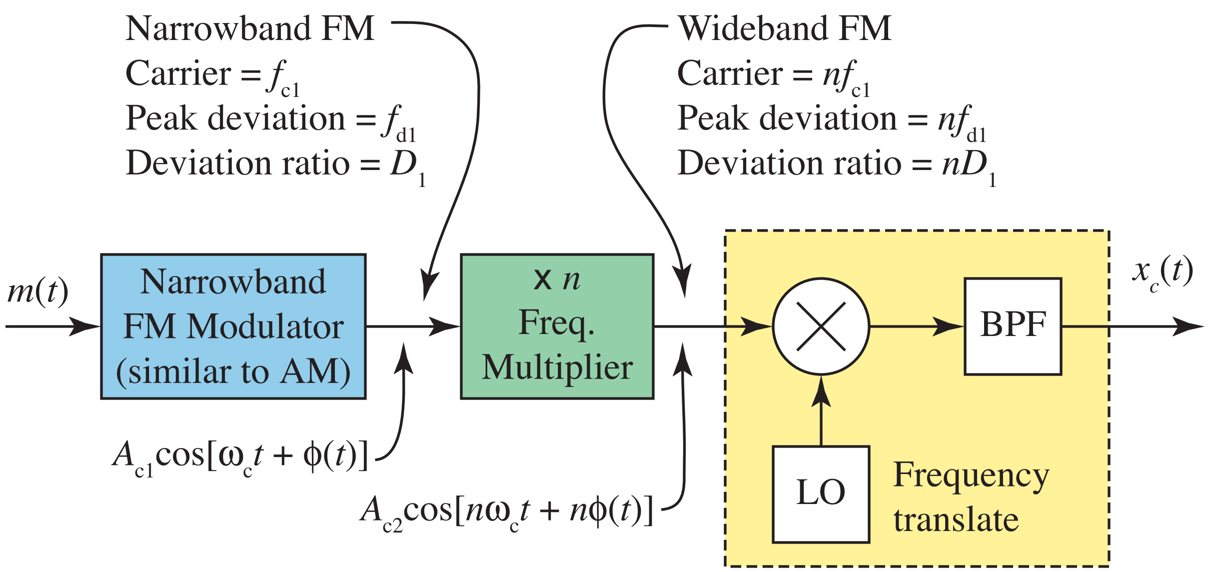
Example 4.6: **Bandlimited Noise PM and FM**

- This example will utilize simulation to obtain the spectrum of an angle modulated carrier

- The message signal in this case will be bandlimited noise having lowpass bandwidth of W Hz

- In Python/MATLAB we can generate Gaussian amplitude distributed white noise using `randn()` and then filter this noise using a high-order lowpass filter (implemented as a digital filter in this case)
- We can then use this signal to phase or frequency modulate a carrier in terms of the peak phase deviation, derived from knowledge of $\max[|\phi(t)|]$

4.1.5 Narrowband-to-Wideband Conversion



narrowband-to-wideband conversion

- Narrowband FM can be generated using an AM-type modulator as discussed earlier (without a VCO and very stable)
- A frequency multiplier, using say a nonlinearity, can be used to make the signal wideband FM, i.e.,

$$A_{c1}\cos[\omega_c t + \phi(t)] \xrightarrow{n\times} A_{c2}\cos[n\omega_c t + n\phi(t)]$$

so the modulator deviation constant of f_{d1} becomes nf_{d1}

4.1.6 Demodulation of Angle-Modulated Signals

- To demodulate FM we require a discriminator circuit, which gives an output which is proportional to the input frequency deviation

- For an ideal discriminator with input

$$x_r(t) = A_c \cos[\omega_c t + \phi(t)]$$

the output is

$$y_D(t) = \frac{1}{2\pi} K_D \frac{d\phi(t)}{dt}$$

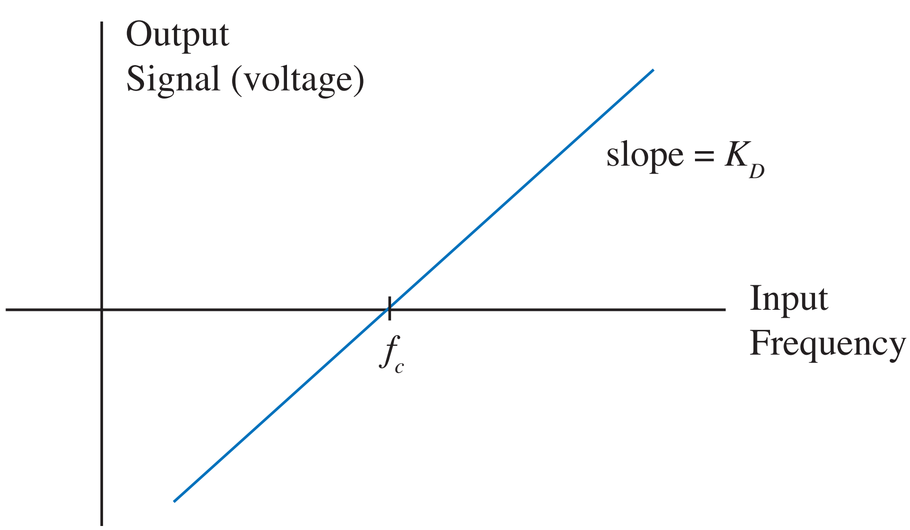
Ideal FM discriminator

- For FM

$$\phi(t) = 2\pi f_d \int^t m(\alpha)\, d\alpha$$

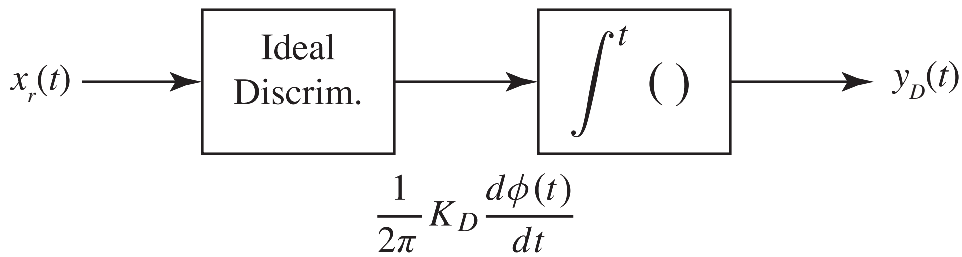
so

$$y_D(t) = K_D f_d m(t)$$



Ideal discriminator I/O characteristic

- For PM signals we follow the discriminator with an integrator

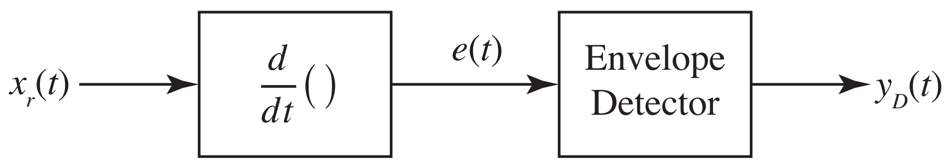


Ideal discriminator with integrator for PM demod

- For PM $\phi(t) = k_p m(t)$ so

$$y_D(t) = K_D k_p m(t)$$

- We now consider approximating an ideal discriminator with:



Ideal discriminator approximation

- If $x_r(t) = A_c \cos[\omega_c t + \phi(t)]$

$$e(t) = \frac{dx_r(t)}{dt} = -A_c\left(\omega_c + \frac{d\phi}{dt}\right)\sin\left[\omega_c t + \phi(t)\right]$$

- This looks like AM provided

$$\frac{d\phi(t)}{dt} < \omega_c$$

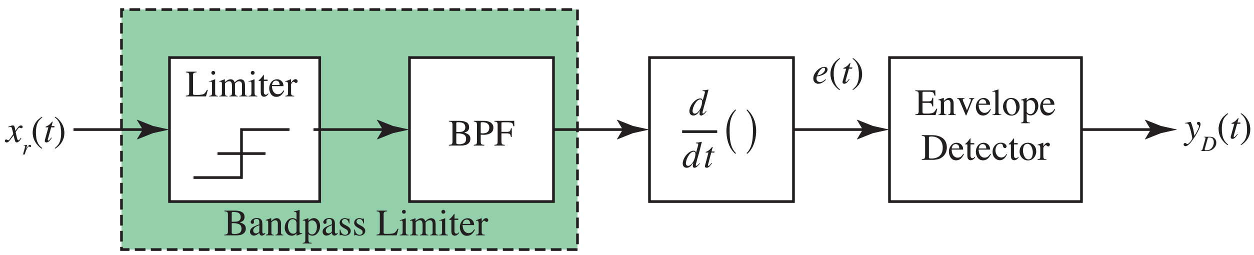
which is only reasonable

- Thus

$$y_D(t) = A_c \frac{d\phi(t)}{dt} = 2\pi A_c f_d m(t) \text{ (for FM)}$$

  - Relative to an ideal discriminator, the gain constant is $K_D = 2\pi A_c$

- To eliminate any amplitude variations on A_c pass $x_c(t)$ through a bandpass limiter



FM discriminator with bandpass limiter

- We can approximate the differentiator with a delay and subtract operation

$$e(t) = x_r(t) - x_r(t - \tau)$$

since

$$\lim_{\tau \to 0} \frac{e(t)}{\tau} = \lim_{\tau \to 0} \frac{x_r(t) - x_r(t - \tau)}{\tau} = \frac{dx_r(t)}{dt},$$

thus

$$e(t) \simeq \tau \frac{dx_r(t)}{dt}$$

- In a discrete-time implementation (DSP), we can perform a similar operation, e.g.

$$e[n] = x[n] - x[n - 1]$$

Example 4.7: **Complex Baseband Discriminator**

- A DSP implementation in MATLAB that works with complex baseband signals (complex envelope) is the following:

```
function disdata = discrim(x)
% function disdata = discrimf(x)
% x is the received signal in complex baseband form
%
% Mark Wickert

xI=real(x); % xI is the real part of the received signal
xQ=imag(x); % xQ is the imaginary part of the received signal
N=length(x); % N is the length of xI and xQ
b=[1 -1];   % filter coefficients
a=[1 0];    % for discrete derivative
der_xI=filter(b,a,xI);  % derivative of xI,
der_xQ=filter(b,a,xQ);  % derivative of xQ
% normalize by the squared envelope acts as a limiter
disdata=(xI.*der_xQ-xQ.*der_xI)./(xI.^2+xQ.^2);
```

- To understand the operation of `discrim()` start with a general angle modulated signal and obtain the complex envelope

$$\begin{aligned} x_c(t) &= A_c \cos(\omega_c t + \phi(t)) \\ &= \text{Re}\big\{A_c e^{j\phi(t)} e^{j\omega_c t}\big\} \\ &= A_c \text{Re}\big\{[\cos\phi(t) + j\sin\phi(t)] e^{j\omega_c t}\big\} \end{aligned}$$

- The complex envelope is

$$\tilde{x}_c(t) = \cos\phi(t) + j\sin\phi(t) = x_I(t) + jx_Q(t)$$

where x_I and x_Q are the in-phase and quadrature signals respectively

- A frequency discriminator obtains $d\phi(t)/dt$

- In terms of the I and Q signals,

$$\phi(t) = \tan^{-1}\left(\frac{x_Q(t)}{x_I(t)}\right)$$

- The derivative of $\phi(t)$ is

$$\begin{aligned}\frac{d\phi(t)}{dt} &= \frac{1}{1+(x_Q(t)/x_I(t))^2}\frac{d}{dt}\left(\frac{x_Q(t)}{x_I(t)}\right)\\ &= \frac{x_I(t)x_Q'(t) - x_I'(t)x_Q(t)}{x_I^2(t)+x_Q^2(t)}\end{aligned}$$

- In the DSP implementation $x_I[n] = x_I(nT)$ and $x_Q[n] = x_Q(nT)$, where T is the sample period

- The derivatives, $x_I'(t)$ and $x_Q'(t)$ are approximated by the *backwards difference* $x_I[n] - x_I[n-1]$ and $x_Q[n] - x_Q[n-1]$ respectively

- To put this code into action, consider a single tone message at 1 kHz with $\beta = 2.4048$

$$\phi(t) = 2.4048\cos(2\pi(1000)t)$$

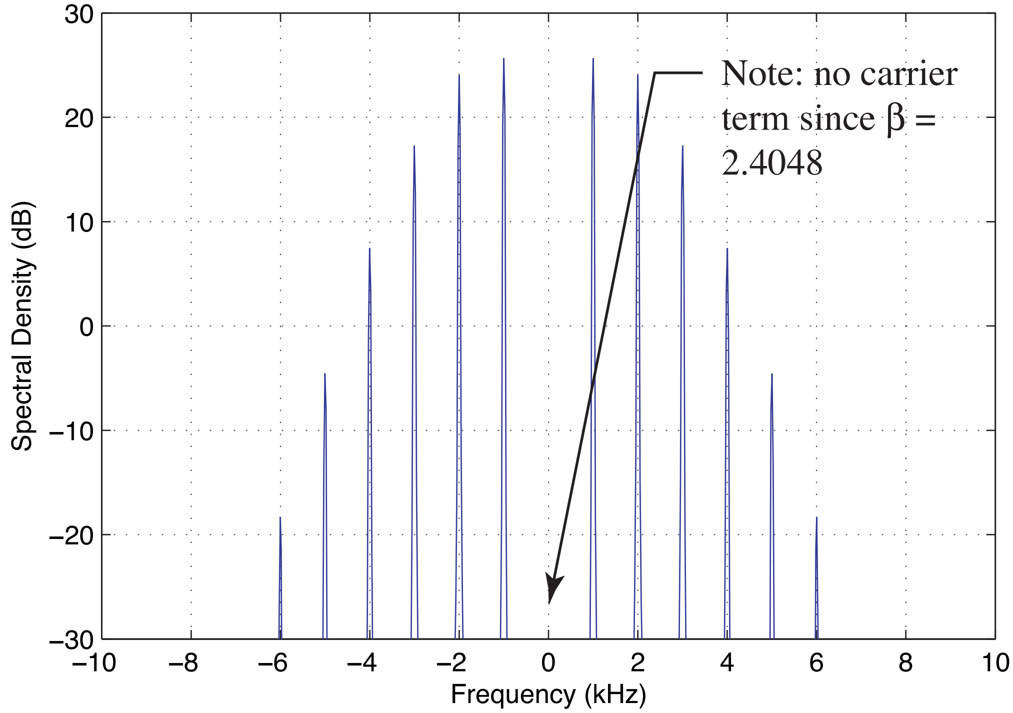
- The complex baseband (envelope) signal is

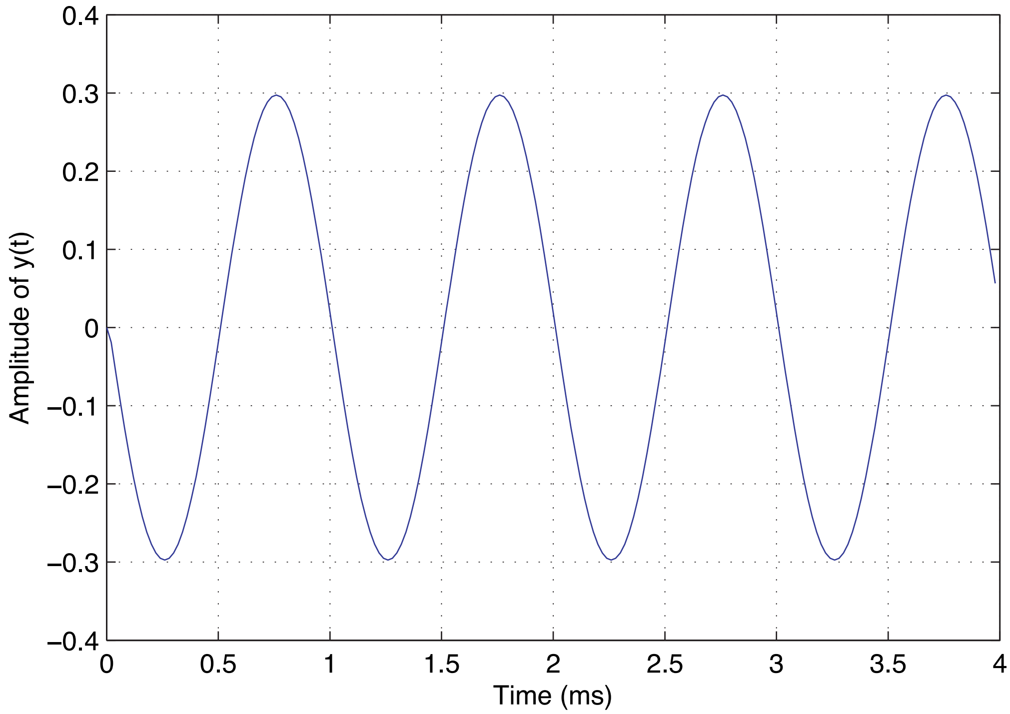
$$\tilde{x}_c(t) = e^{j\phi(t)} = e^{j2.4048\cos(2\pi(1000)t)}$$

- A MATLAB simulation that utilizes the function `Discrim()` is:

```
>> n = 0:5000-1;
>> m = cos(2*pi*n*1000/50000); % sampling rate = 50 kHz
>> xc = exp(j*2.4048*m);
>> y = Discrim(xc);
```

```
>> % baseband spectrum plotting tool using psd()
>> bb_spec_plot(xc,2^11,50);
>> axis([-10 10 -30 30])
>> grid
>> xlabel('Frequency (kHz)')
>> ylabel('Spectral Density (dB)')
>> t = n/50;
>> plot(t(1:200),y(1:200))
>> axis([0 4 -.4 .4])
>> grid
>> xlabel('Time (ms)')
>> ylabel('Amplitude of y(t)')
```

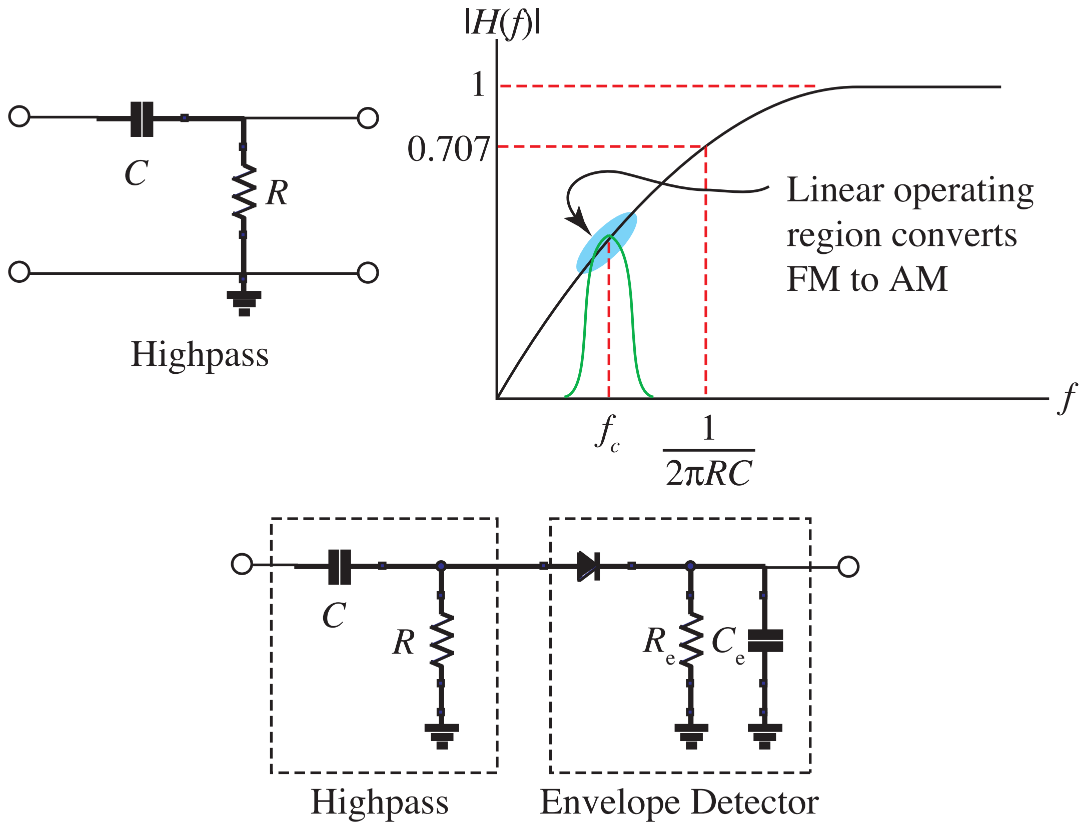




Baseband FM spectrum and demodulator output wavefrom

Analog Circuit Implementations

- A simple analog circuit implementation is an RC highpass filter followed by an envelope detector

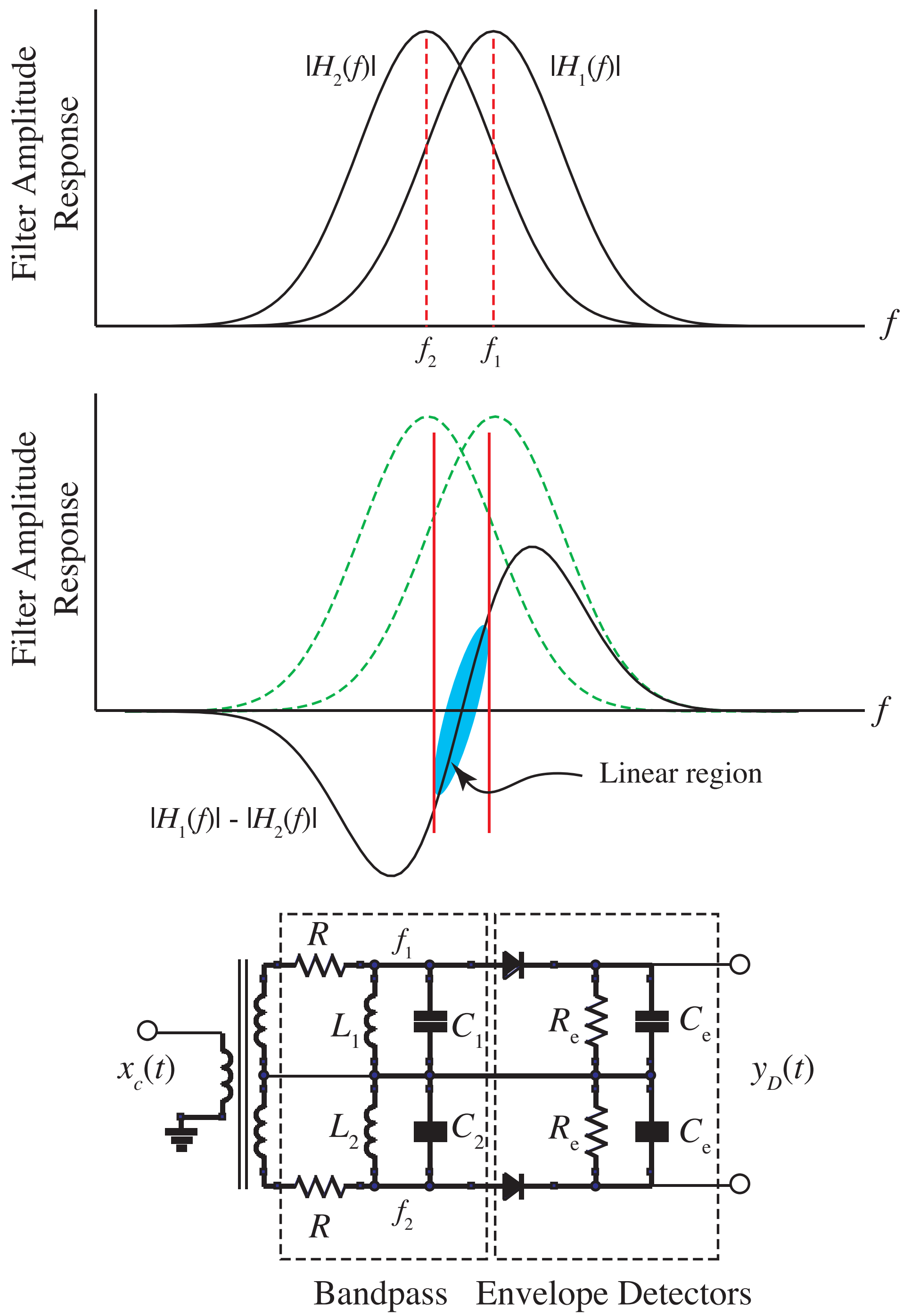


RC highpass filter + envelope detector discriminator (slope detector)

- For the RC highpass filter to be practical the cutoff frequency must be reasonable

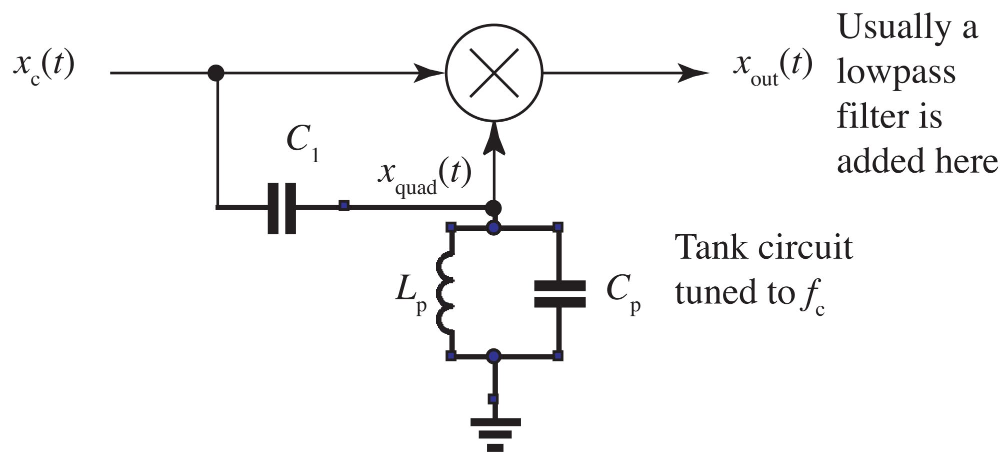
- Broadcast FM radio typically uses a 10.7 MHz IF frequency, which means the highpass filter must have cutoff above this frequency

- A more practical discriminator is the *balanced discriminator*, which offers a wider linear operating range



Balanced discriminator operation (top) and a passive implementation (bottom)

FM Quadrature Detectors



Quadrature detector schematic

- In analog integrated circuits used for FM radio receivers and the like, an FM demodulator known as a *quadrature detector* or quadrature discriminator, is quite popular
- The input FM signal connects to one port of a multiplier (product device)
- A quadrature signal is formed by passing the input to a capacitor series connected to the other multiplier input and a parallel tank circuit resonant at the input carrier frequency
- The quadrature circuit receives a phase shift from the capacitor and additional phase shift from the tank circuit
- The phase shift produced by the tank circuit is time varying in proportion to the input frequency deviation
- A mathematical model for the circuit begins with the FM input signal

$$x_c(t) = A_c[\omega_c t + \phi(t)]$$

- The quadrature signal is

$$x_{\text{quad}}(t) = K_1 A_c \sin\left[\omega_c t + \phi(t) + K_2 \frac{\phi(t)}{dt}\right]$$

where the constants K_1 and K_2 are determined by circuit parameters

- The multiplier output, assuming a lowpass filter removes the sum terms, is

$$x_{\text{out}}(t) = \frac{1}{2} K_1 A_c^2 \sin\left[K_2 \frac{d\phi(t)}{dt}\right]$$

- By proper choice of K_2 the argument of the sin function is small, and a small angle approximation yields

$$x_{\text{out}}(t) \simeq \frac{1}{2} K_1 K_2 A_c^2 \frac{d\phi(t)}{dt} = \frac{1}{2} K_1 K_2 A_c^2 K_D m(t)$$

4.2 Feedback Demodulators

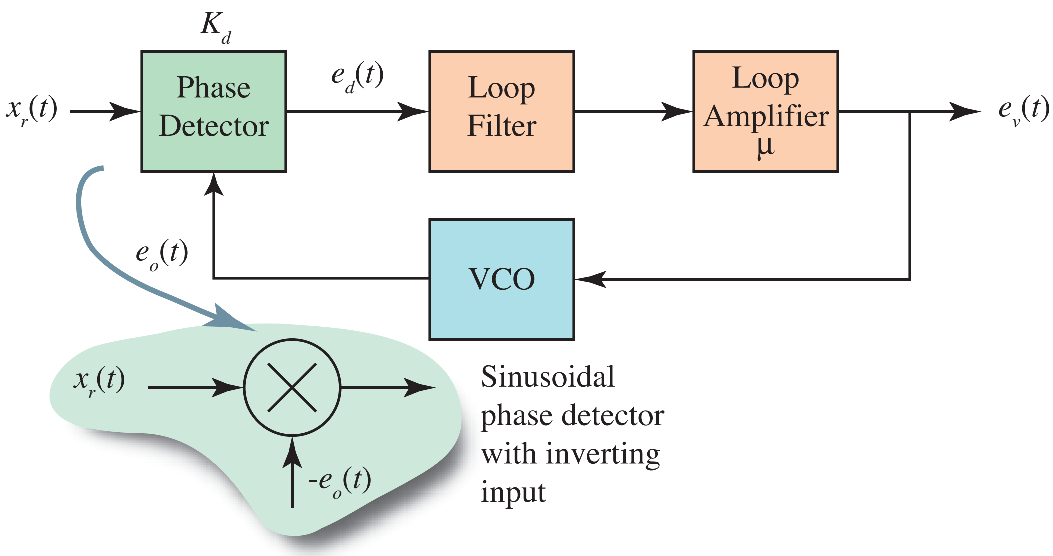
- The discriminator as described earlier first converts and FM signal to and AM signal and then demodulates the AM

- The *phase-locked loop* (PLL) offer a direct way to demodulate FM and is considered a basic building block by communication system engineers

4.2.1 Phase-Locked Loops for FM Demodulation

- The PLL has many uses and many different configurations, both analog and DSP based

- We will start with a basic configuration for demodulation of FM



Basic PLL block diagram

- Let

$$x_r(t) = A_c \cos\left[\omega_c t + \phi(t)\right]$$
$$e_o(t) = A_v \sin\left[\omega_c t + \theta(t)\right]$$

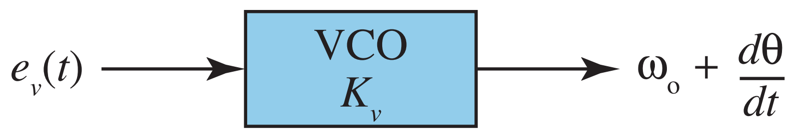
  – Note: Frequency error may be included in $\phi(t) - \theta(t)$

- Assume a sinusoidal phase detector with an inverting operation is included, then we can further write

$$e_d(t) = \frac{1}{2} A_c A_v K_d \sin\left[\phi(t) - \theta(t)\right]$$

  – In the above we have assumed that the double frequency term is removed (e.g., by the loop filter eventually)

- Note that for the *voltage controlled oscillator* (VCO) we have the following relationship

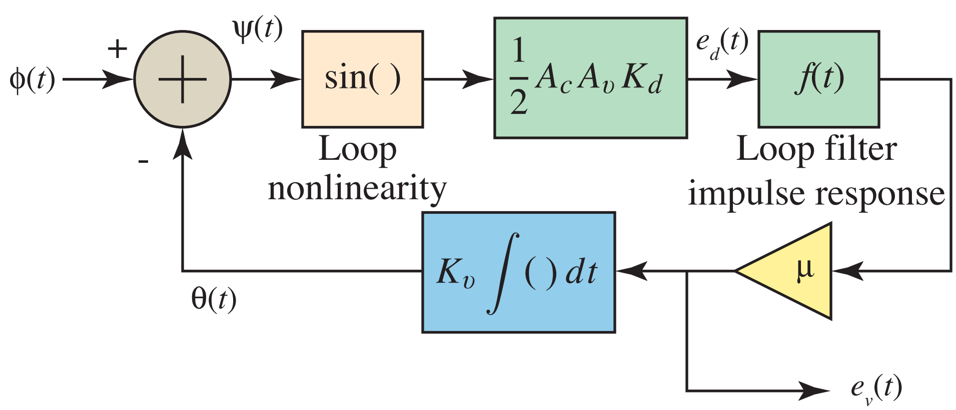


so

$$\frac{d\theta(t)}{dt} = K_v e_v(t) \text{ rad/s}$$

$$\Rightarrow \theta(t) = K_v \int^t e_v(\alpha)\, d\alpha$$

- In its present form the PLL is a nonlinear feedback control system



Nonlinear feedback control model

- To show tracking we first consider the loop filter to have impulse response $\delta(t)$ (a straight through connection or unity gain amplifier)

- The loop gain is now defined as

$$K_t \triangleq \frac{1}{2}\mu A_c A_v K_d K_v \text{ rad/s}$$

- The VCO output is

$$\theta(t) = K_t \int^t \sin[\phi(\alpha) - \theta(\alpha)]\, d\alpha$$

$$\text{or} \quad \frac{d\theta(t)}{dt} = K_t \sin[\phi(t) - \theta(t)]$$

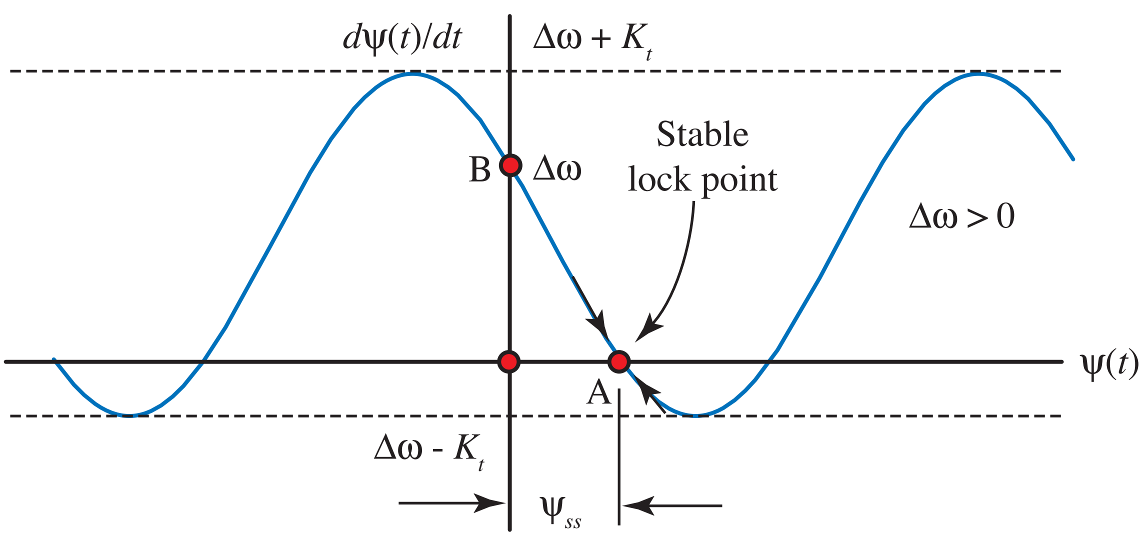
- Let $\psi(t) = \phi(t) - \theta(t)$ and apply an input frequency step $\Delta\omega$, i.e.,

$$\frac{d\phi(t)}{dt} = \Delta\omega\, u(t)$$

- Now, noting that $\theta(t) = \phi(t) - \psi(t)$ we can write

$$\frac{d\theta(t)}{dt} = \Delta\omega - \frac{d\psi(t)}{dt} = K_t \sin\psi(t),\ t \geq 0$$

- We can now plot $d\psi/dt$ versus ψ, which is known as a *phase plane plot*



Phase plane plot (1st-order PLL)

$$\frac{d\psi(t)}{dt} + K_t \sin\psi(t) = \Delta\omega u(t)$$

- At $t = 0$ the operating point is at B

$$\text{Since } dt \text{ is positive if } \frac{d\psi}{dt} > 0 \longrightarrow d\psi \text{ is positive}$$
$$\text{Since } dt \text{ is positive if } \frac{d\psi}{dt} < 0 \longrightarrow d\psi \text{ is negative}$$

  therefore the steady-state operating point is at A

- The frequency error is always zero in steady-state
- The steady-state phase error is ψ_{ss}
  - Note that for locking to take place, the phase plane curve must cross the $d\psi/dt = 0$ axis
- The maximum steady-state value of $\Delta\omega$ the loop can handle is thus K_t
- The *total lock range* is then

$$\omega_c - K_t \le \omega \le \omega_c + K_t \quad \Rightarrow 2K_t$$

  - For a *first-order* loop the *lock range* and the *hold-range* are identical

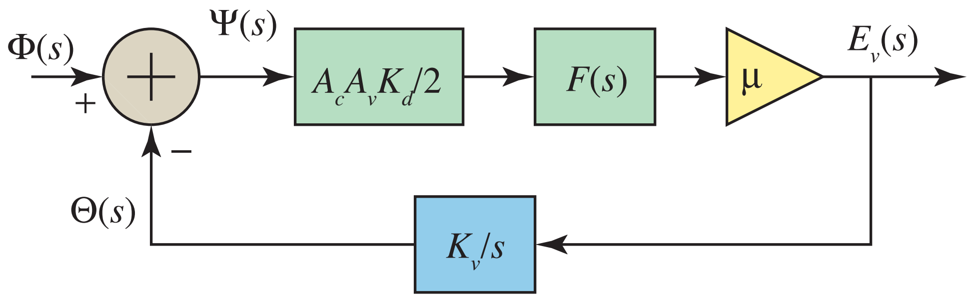
- For a given $\Delta\omega$ the value of ψ_{ss} can be made small by increasing the loop gain, i.e.,

$$\psi_{ss} = \sin^{-1}\left(\frac{\Delta\omega}{K_t}\right)$$

- Thus for large K_t the in-lock operation of the loop can be modeled with a fully linear model since $\phi(t) - \theta(t)$ is small, i.e.,

$$\sin[\phi(t) - \theta(t)] \simeq \phi(t) - \theta(t)$$

- The s-domain linear PLL model is the following



Linear PLL model

- Solving for $\Theta(s)$ we have

$$\Theta(s) = \frac{K_t}{s}\big[\Phi(s) - \Theta(s)\big]F(s)$$

$$\text{or} \quad \Theta(s)\left[1 + \frac{K_t}{s}F(s)\right] = \frac{K_t}{s}\Phi(s)F(s)$$

- Finally, the *closed-loop* transfer function is

$$H(s) \triangleq \frac{\Theta(s)}{\Phi(s)} = \frac{\frac{K_t}{s}F(s)}{1 + \frac{K_t}{s}F(s)} = \frac{K_t F(s)}{s + K_t F(s)}$$

First-Order PLL

- Let $F(s) = 1$, then we have

$$H(s) = \frac{K_t}{K_t + s}$$

- Consider the loop response to a frequency step, that is for FM, we assume $m(t) = Au(t)$, then

$$\phi(t) = Ak_f \int^t u(\alpha)\,d\alpha$$

$$\text{so} \quad \Phi(s) = \frac{Ak_f}{s^2}$$

- The VCO phase output is

$$\Theta(s) = \frac{Ak_f K_t}{s^2(K_t + s)}$$

- The VCO control voltage should be closely related to the applied FM message

- To see this write

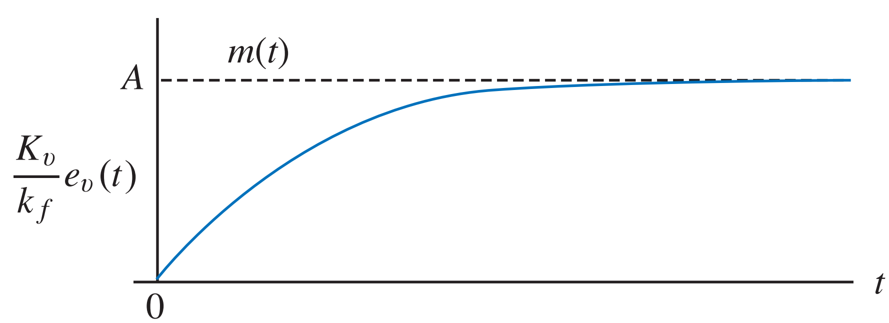
$$E_v(s) = \frac{s}{K_v}\Theta(s) = \frac{Ak_f}{K_v} \cdot \frac{K_t}{s(s + K_t)}$$

- Partial fraction expanding yields,

$$E_v(s) = \frac{Ak_f}{K_v}\left[\frac{1}{s} - \frac{1}{s + K_t}\right]$$

thus

$$e_v(t) = \frac{Ak_f}{K_v}\left[1 - e^{-K_t t}\right]u(t)$$



1st-Order PLL frequency step response at VCO input $K_v e_v(t)/k_f$

- In general,

$$\Phi(s) = \frac{k_f M(s)}{s}$$

so

$$E_v(s) = \frac{k_f M(s)}{s} \cdot \frac{s}{K_v} \cdot \frac{K_t}{s + K_t} = \frac{k_f}{K_v} \cdot \frac{K_t}{s + K_t} \cdot M(s)$$

- Now if the bandwidth of $m(t)$ is $W \ll K_t/(2\pi)$, then

$$E_v(s) \approx \frac{k_f}{K_v} M(s) \;\Rightarrow\; e_v(t) \approx \frac{k_f}{K_v} m(t)$$

- The first-order PLL has limited lock range and always has a nonzero steady-state phase error when the input frequency is offset from the quiescent VCO frequency

- Increasing the loop gain appears to help, but the loop bandwidth becomes large as well, which allows more noise to enter the loop

- Spurious time constants which are always present, but not a problem with low loop gains, are also a problem with high gain first-order PLLs

Example 4.8: **First-Order PLL Simulation Example**

- Tool such as Python (in Jupyter notebook), MATLAB, MATLAB with Simulink, VisSim/Comm, ADS, and others provide an ideal environment for simulating PLLs at the system level

- Circuit level simulation of PLLs is very challenging due to the need to simulate every cycle of the VCO

- The most realistic simulation method is to use the actual bandpass signals, but since the carrier frequency must be kept low to minimize the simulation time, we have difficulties removing the double frequency term from the phase detector output

- By simulating at baseband, using the nonlinear loop model, many PLL aspects can be modeled without worrying about how to remove the double frequency term
  - A complex baseband simulation allows further capability, but is only found in (`synchronization.py`)
- The most challenging aspect of the simulation is dealing with the integrator found in the VCO block (K_v/s)
- We consider a discrete-time simulation where all continuous-time waveforms are replaced by their discrete-time counterparts, i.e., $x[n] = x(nT) = x(n/fs)$, where f_s is the sample frequency and $T = 1/f_s$ is the sampling period
- The input/output relationship of an integration block can be approximated via the trapezoidal rule

$$y[n] = y[n-1] + \frac{T}{2}\big(x[n] + x[n-1]\big)$$

```
function [theta,ev,phi_error] = PLL1(phi,fs,loop_type,Kv,fn,zeta)
% [theta, ev, error, t] = PLL1(phi,fs,loop_type,Kv,fn,zeta)
%
%
% Mark Wickert, April 2007

T = 1/fs;
Kv = 2*pi*Kv; % convert Kv in Hz/v to rad/s/v

if loop_type == 1
    % First-order loop parameters
    Kt = 2*pi*fn; % loop natural frequency in rad/s
elseif loop_type == 2
    % Second-order loop parameters
    Kt = 4*pi*zeta*fn; % loop natural frequency in rad/s
    a = pi*fn/zeta;
else
```

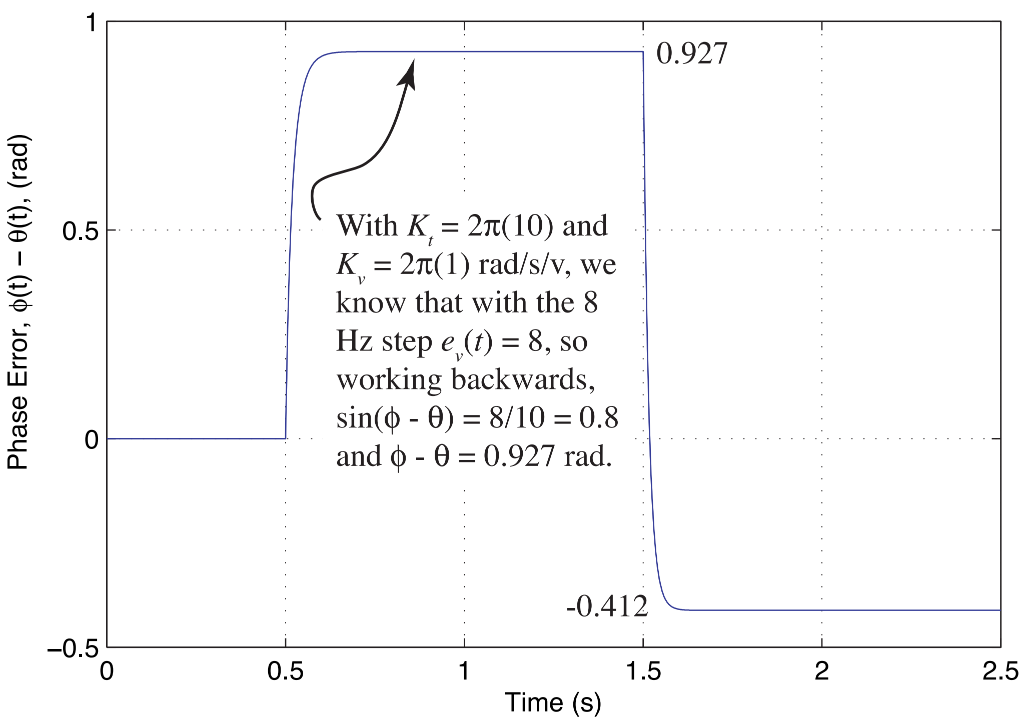
```
    error('Loop type must be 1 or 2');
end

% Initialize integration approximation filters
filt_in_last = 0; filt_out_last = 0;
vco_in_last = 0; vco_out = 0; vco_out_last = 0;

% Initialize working and final output vectors
n = 0:length(phi)-1;
theta = zeros(size(phi));
ev = zeros(size(phi));
phi_error = zeros(size(phi));

% Begin the simulation loop
for k = 1:length(n)
    phi_error(k) = phi(k) - vco_out;
    % sinusoidal phase detector
    pd_out = sin(phi_error(k));
    % Loop gain
    gain_out = Kt/Kv*pd_out; % apply VCO gain at VCO
    % Loop filter
    if loop_type == 2
        filt_in = a*gain_out;
        filt_out = filt_out_last + T/2*(filt_in + filt_in_last);
        filt_in_last = filt_in;
        filt_out_last = filt_out;
        filt_out = filt_out + gain_out;
    else
        filt_out = gain_out;
    end
    % VCO
    vco_in = filt_out;
    vco_out = vco_out_last + T/2*(vco_in + vco_in_last);
    vco_in_last = vco_in;
    vco_out_last = vco_out;
    vco_out = Kv*vco_out; % apply Kv
    % Measured loop signals
    ev(k) = vco_in;
    theta(k) = vco_out;
end
```

- To simulate a frequency step we input a phase ramp
- Consider an 8 Hz frequency step turning on at 0.5 s and a -12

Hz frequency step turning on at 1.5 s

$$\phi(t) = 2\pi\big[8(t-0.5)u(t-0.5) - 12(t-1.5)u(t-1.5)\big]$$

```
>> t = 0:1/1000:2.5;
>> idx1 = find(t>= 0.5);
>> idx2 = find(t>= 1.5);
>> phi1(idx1) =2*pi* 8*(t(idx1)-0.5).*ones(size(idx1));
>> phi2(idx2) = 2*pi*12*(t(idx2)-1.5).*ones(size(idx2));
>> phi = phi1 - phi2;
>> [theta, ev, phi_error] = PLL1(phi,1000,1,1,10,0.707);
>> plot(t,phi_error); % phase error in radians
```



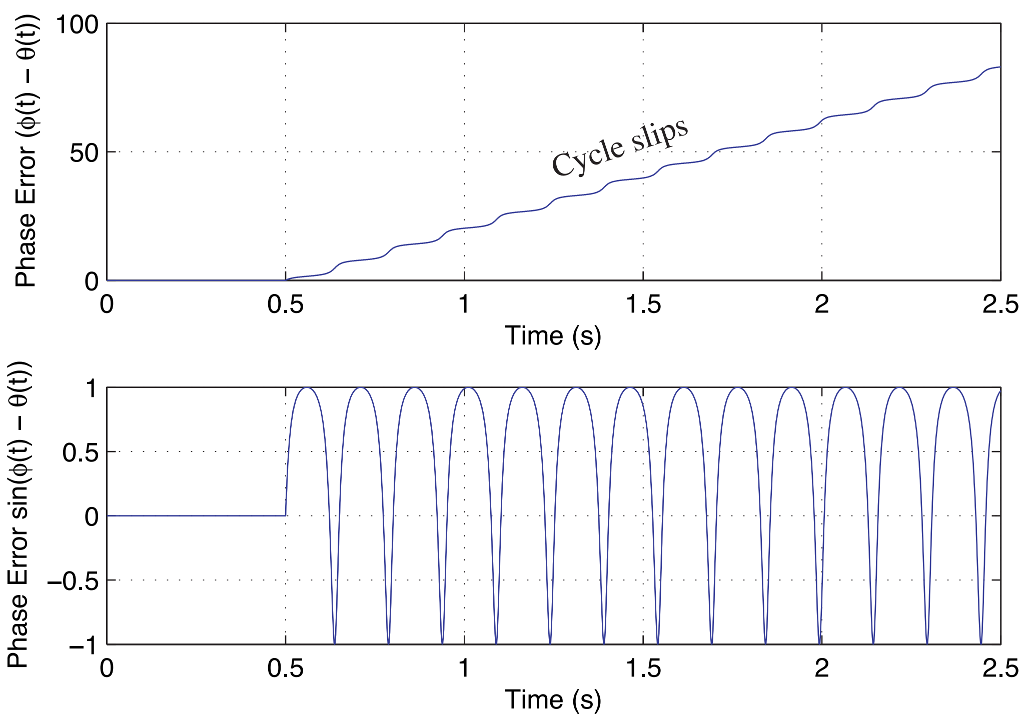
Phase error for input within lock range

- In the above plot we see the finite rise-time due to the loop gain being $2\pi(10)$
- This is a first-order lowpass step response

- The loop stays in lock since the frequency swing either side of zero is within the ± 10 Hz lock range

- Suppose now that a single positive frequency step of 12 Hz is applied, the loop unlocks and *cycle slips* indefinitely; why?

```
>> phi = 12/8*phi1; % scale frequency step from 8 Hz to 12 Hz
>> [theta, ev, phi_error] = PLL1(phi,1000,1,1,10,0.707);
>> subplot(211)
>> plot(t,phi_error)
>> subplot(212)
>> plot(t,sin(phi_error))
```



Phase error for input exceeding lock range by 2 Hz

- By plotting the true phase detector output, $\sin[\phi(t) - \theta(t)]$, we see that the error voltage is simply not large enough to pull the VCO frequency to match the input which is offset by 12 Hz

- In the phase plane plot shown earlier, this scenario corresponds to the trajectory never crossing zero

---

Second-Order Type II PLL

- To mitigate some of the problems of the first-order PLL, we can include a second integrator in the open-loop transfer function
- A common loop filter for building a second-order PLL is an integrator with lead compensation

$$F(s) = \frac{s + a}{s}$$

- The resulting PLL is sometimes called a *perfect second-order* PLL since two integrators are now in the transfer function
- In text Problem 4.28 you analyze the *lead-lag* loop filter

$$F(s) = \frac{s + a}{s + \lambda a}$$

  which creates an imperfect, or finite gain integrator, second-order PLL

- Returning to the integrator with phase lead loop filter, the closed-loop transfer function is

$$H(s) = \frac{K_t F(s)}{s + K_t F(s)} = \frac{K_t(s + a)}{s^2 + K_t s + K_t a}$$

- The transfer function from the input phase to the phase error $\psi(t)$ is

$$G(s) \stackrel{\Delta}{=} \frac{\Psi(s)}{\Phi(s)} = \frac{\Phi(s) - \Theta(s)}{\Phi(s)}$$

$$\text{or } G(s) = 1 - H(s) = \frac{s^2}{s^2 + K_t s + K_t a}$$

- In standard second-order system notation we can write the denominator of $G(s) = 1 - H(s)$ (and also $H(s)$) as

$$s^2 + K_t s + K_t a = s^2 + 2\zeta\omega_n s + \omega_n^2$$

  where

$$\omega_n = \sqrt{K_t a} = \text{natural frequency in rad/s}$$

$$\zeta = \frac{1}{2}\sqrt{\frac{K_t}{a}} = \text{damping factor}$$

- For an input frequency step the steady-state phase error is zero
  - Note the *hold-in range* is infinite, in theory, since the integrator contained in the loop filter has infinite DC gain
- To verify this we can use the final value theorem

$$\psi_{ss} = \lim_{s \to 0} s \left[\frac{\Delta\omega}{s^2} \cdot \frac{s^2}{s^2 + K_t s + K_t a} \right]$$

$$= \lim_{s \to 0} \frac{\Delta\omega s}{s^2 + K_t s + K_t a} = 0$$

- In exact terms we can find $\psi(t)$ by inverse Laplace transforming

$$\Psi(s) = \frac{\Delta\omega}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The result for $\zeta < 1$ is

$$\psi(t) = \frac{\Delta\omega}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t\right) u(t)$$

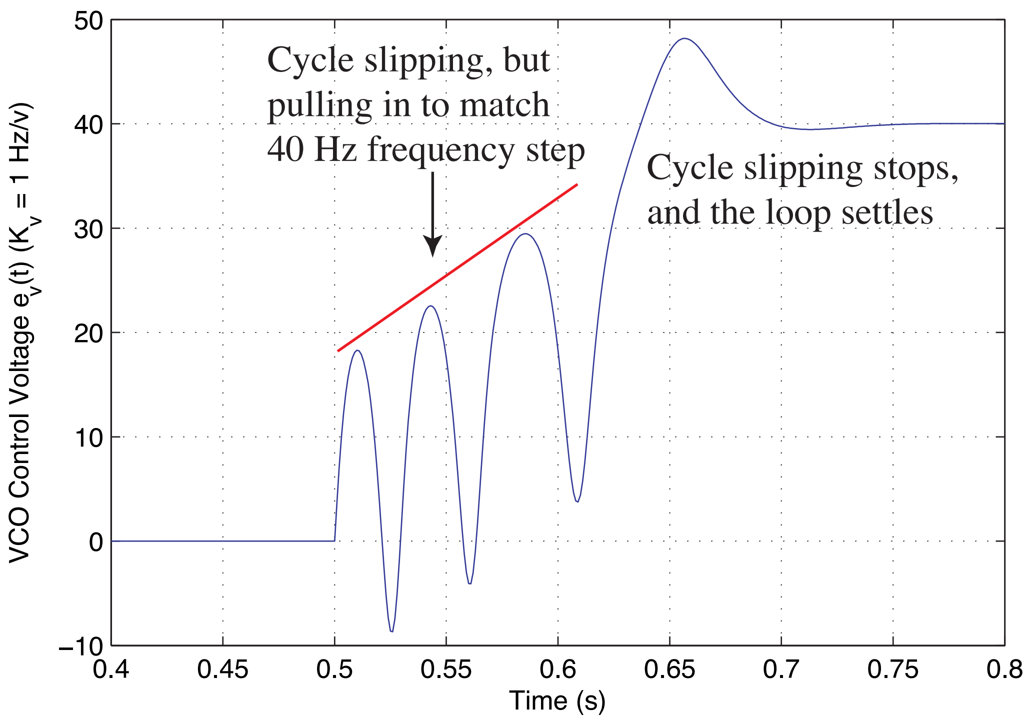
Example 4.9: Second-Order PLL Simulation Example

- As a simulation example consider a loop designed with $f_n = 10$ Hz and $\zeta = 0.707$

$$K_t = 2\zeta\omega_n = 2 \times 0.707 \times 2\pi \times 10 = 88.84$$
$$a = \frac{\omega_n}{2\zeta} = \frac{2\pi \times 10}{2 \times 0.707} = 44.43$$

- The simulation code of Example 4.9 includes the needed loop filter via a software switch
- The integrator that is part of the loop filter is implemented using the same trapezoidal formula as used in the VCO
- We input a 40 Hz frequency step and observe the VCO control voltage ($e_v(t)$) as the loop first slips cycles, gradually pulls in, then tracks the input signal that is offset by 40 Hz
- The VCO gain $K_v = 1$ v/Hz or 2π rad/s, so $e_v(t)$ effectively corresponds to the VCO frequency deviation in Hz

```
>> t = 0:1/1000:2.5;
>> idx1 = find(t>= 0.5);
>> phi(idx1) = 2*pi*40*(t(idx1)-0.5).*ones(size(idx1));
>> [theta, ev, phi_error] = PLL1(phi,1000,2,1,10,0.707);
>> plot(t,ev)
>> axis([0.4 0.8 -10 50])
```



VCO control voltage for a 40 Hz frequency step

Example 4.10: **Bandpass Simulation of FM Demodulation**

- Baseband PLL simulations are very useful and easy to implement, but sometimes a full bandpass level simulation is required
- The MATLAB simulation file `PLL1.m` is modified to allow passband simulation via the function file `PLL2.m`
- The phase detector is a multiplier followed by a lowpass filter to remove the double frequency term

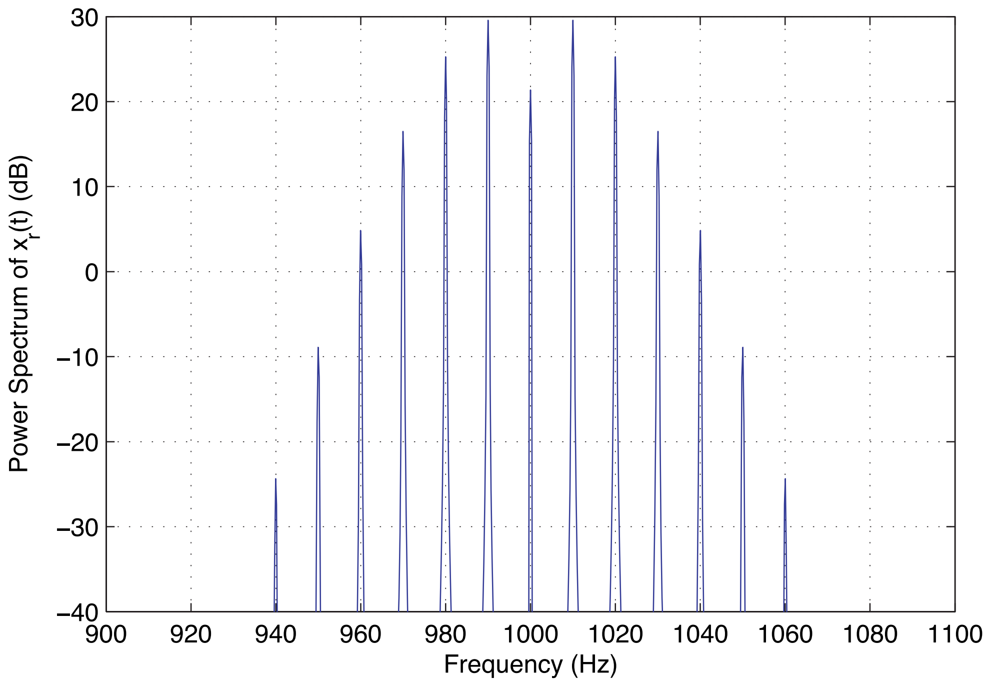
```
function [theta, ev, phi_error] = PLL2(xr,fs,loop_type,Kv,fn,zeta)
% [theta, ev, error, t] = PLL2(xr,fs,loop_type,Kv,fn,zeta)
%
```

```
%
% Mark Wickert, April 2007

T = 1/fs;
% Set the VCO quiescent frequency in Hz
fc = fs/4;
% Design a lowpass filter to remove the double freq term
[b,a] = butter(5,2*1/8);
fstate = zeros(1,5); % LPF state vector

Kv = 2*pi*Kv; % convert Kv in Hz/v to rad/s/v

if loop_type == 1
    % First-order loop parameters
    Kt = 2*pi*fn; % loop natural frequency in rad/s
elseif loop_type == 2
    % Second-order loop parameters
    Kt = 4*pi*zeta*fn; % loop natural frequency in rad/s
    a = pi*fn/zeta;
else
    error('Loop type musy be 1 or 2');
end

% Initialize integration approximation filters
filt_in_last = 0; filt_out_last = 0;
vco_in_last = 0; vco_out = 0; vco_out_last = 0;

% Initialize working and final output vectors
n = 0:length(xr)-1;
theta = zeros(size(xr));
ev = zeros(size(xr));
phi_error = zeros(size(xr));

% Begin the simulation loop
for k = 1:length(n)
    % Sinusoidal phase detector (simple multiplier)
    phi_error(k) = 2*xr(k)*vco_out;
    % LPF to remove double frequency term
    [phi_error(k),fstate] = filter(b,a,phi_error(k),fstate);
    pd_out = phi_error(k);
    % Loop gain
    gain_out = Kt/Kv*pd_out; % apply VCO gain at VCO
    % Loop filter
    if loop_type == 2
        filt_in = a*gain_out;
        filt_out = filt_out_last + T/2*(filt_in + filt_in_last);
```

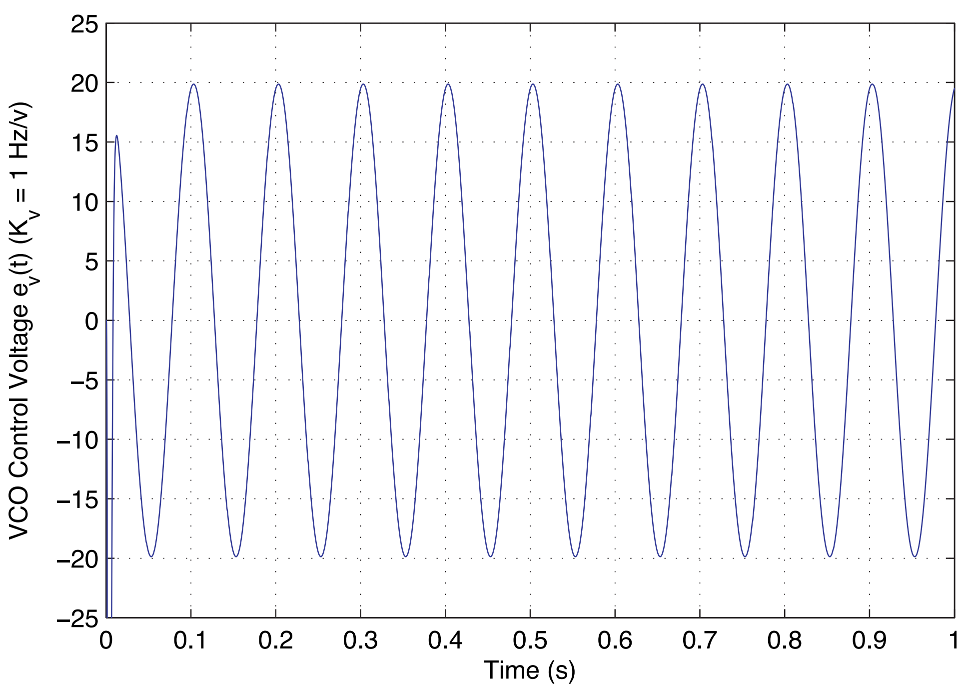
```
        filt_in_last = filt_in;
        filt_out_last = filt_out;
        filt_out = filt_out + gain_out;
    else
        filt_out = gain_out;
    end
    % VCO
    vco_in = filt_out + fc/(Kv/(2*pi)); % bias to quiescent freq.
    vco_out = vco_out_last + T/2*(vco_in + vco_in_last);
    vco_in_last = vco_in;
    vco_out_last = vco_out;
    vco_out = Kv*vco_out; % apply Kv;
    vco_out = sin(vco_out); % sin() for bandpass signal
    % Measured loop signals
    ev(k) = filt_out;
    theta(k) = vco_out;
end
```

- Note that the carrier frequency is fixed at $f_s/4$ and the lowpass filter cutoff frequency is fixed at $f_s/8$
- The double frequency components out of the phase detector are removed with a fifth-order Butterworth lowpass filter
- The VCO is modified to include a bias that shifts the quiescent frequency to $f_c = f_s/4$
- The VCO output is not simply a phase deviation, but rather a sinusoid with argument the VCO output phase
- We will test the PLL using a single tone FM signal

```
>> t = 0:1/4000:5;
>> xr = cos(2*pi*1000*t+2*sin(2*pi*10*t));
>> psd(xr,2^14,4000)
>> axis([900 1100 -40 30])
>> % Process signal through PLL
>> [theta, ev, phi_error] = PLL2(xr,4000,1,1,50,0.707);
>> plot(t,ev)
>> axis([0 1 -25 25])
```



Single tone FM input spectrum having $f_m = 10$ Hz and $\Delta f = 20$ Hz



Recovered modulation at VCO input, $e_v(t)$

General Loop Transfer Function and Steady-State Errors

- We have see that for arbitrary loop filter $F(s)$ the closed-loop transfer function $H(s)$ is

$$H(s) = \frac{K_t F(s)}{s + K_t F(s)}$$

and the loop error function $G(s) = 1 - H(s)$ is

$$G(s) = \frac{s}{s + K_t F(s)}$$

- In tracking receiver applications of the PLL we need to consider platform dynamics which give rise to a phase deviation of the received signal of the form

$$\phi(t) = \left[\pi R t^2 + 2\pi f_\Delta t + \theta_0\right] u(t)$$

which is a superposition of a phase step, frequency step, and a frequency ramp

- In the s-domain we have

$$\Phi(s) = \frac{2\pi R}{s^3} + \frac{2\pi f_\Delta}{s^2} + \frac{\theta_0}{s}$$

- From the final value theorem the loop steady-state phase error is

$$\psi_{\text{ss}} = \lim_{s \to 0} s \left[\frac{2\pi R}{s^3} + \frac{2\pi f_\Delta}{s^2} + \frac{\theta_0}{s}\right] G(s)$$

- If we generalize the loop filters we have been considering to the form

$$F(s) = \frac{1}{s^2}\left[s^2 + as + b\right] = 1 + \frac{a}{s} + \frac{b}{s^2}$$

we have for $G(s)$

$$G(s) = \frac{s^3}{s^3 + K_t s^2 + K_t a s + K_t b}$$

- Depending upon the values chosen for a and b, we can create a 1st, 2nd, or 3rd-order PLL using this $F(s)$

- The steady-state phase error when using this loop filter is

$$\psi_{\rm ss} = \lim_{s \to 0} \frac{s\big(\theta_0 s^2 + 2\pi f_\Delta s + 2\pi R\big)}{s^3 + K_t s^2 + K_t a s + K_t b}$$

- What are some possible outcomes for $\psi_{\rm ss}$?

4.2.2 PLL Frequency Synthesizers

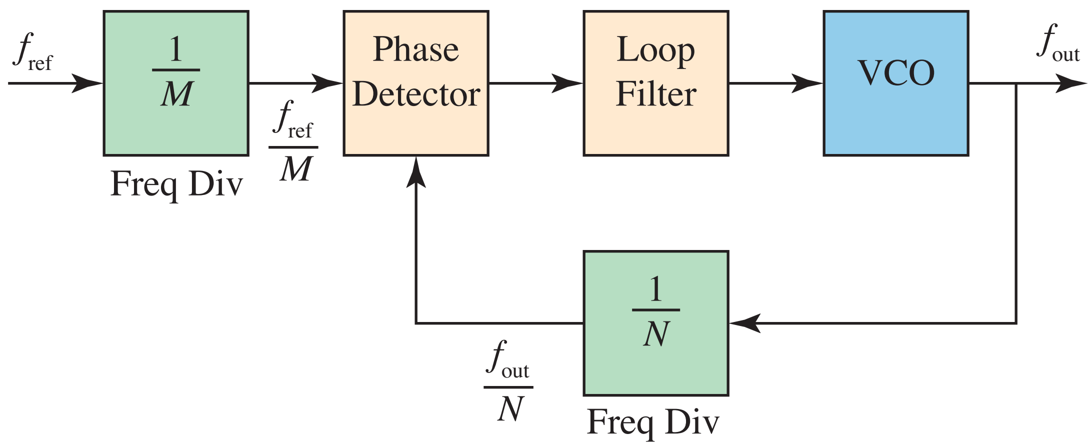
- A frequency synthesizer is used to generate a stable, yet programmable frequency source

- A frequency synthesizer is often used to allow digital tuning of the local oscillator in a communications receiver

- One common frequency synthesis type is known as *indirect synthesis*

- With indirect synthesis a PLL is used to create a stable frequency source

- The basic block diagram of an indirect frequency synthesizer is the following



Indirect frequency synthesis using a PLL

- When locked the frequency error is zero, thus

$$f_{out} = \frac{N}{M} \times f_{ref}$$

Example 4.11: A PLL Synthesizer for Broadcast FM

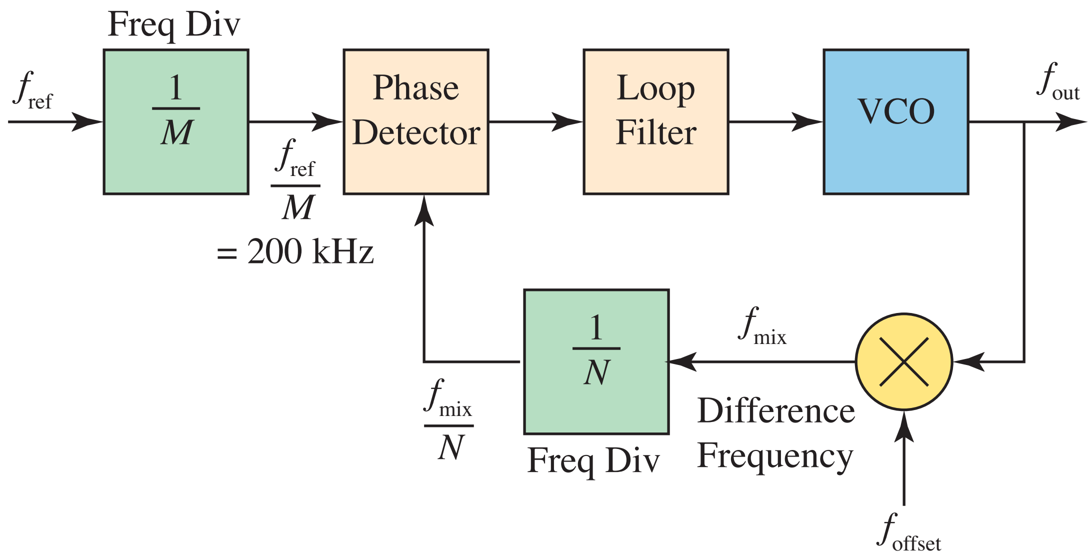
- In this example the synthesizer will provide the local oscillator signal for frequency converting the FM broadcast band from 88.1 to 107.9 MHz down to an IF of 10.7 MHz
- The minimum channel spacing should be 200 kHz
- We will choose high-side tuning for the LO, thus

$$88.1 + 10.7 \leq f_{LO} \leq 107.9 + 10.7 \text{ MHz}$$
$$98.8 \leq f_{LO} \leq 118.6 \text{ MHz}$$

  – The step size must be 200 kHz so the frequency step must be no larger than 200 kHz (it could be 100 kHz or 50 kHz)

- To reduce the maximum frequency into the divide by counter a frequency offset scheme will be employed

- The synthesizer with offset oscillator is the following



FM broadcast band synthesizer producing f_{LO} for $f_{IF} = 10.7$ MHz

- Choose $f_{offset} < f_{out}$ then $f_{mix} = f_{out} - f_{offset}$, and for locking

$$\frac{f_{ref}}{M} = \frac{f_{mix}}{N} \Rightarrow f_{out} = \frac{N f_{ref}}{M} + f_{offset}$$

  – Note that $F_{mix} = N f_{ref}/M$ and $f_{out} = f_{mix} + f_{offset}$, by virtue of the low side tuning assumption for the offset oscillator

- Let $f_{ref}/M = 200$ kHz and $f_{offset} = 98.0$ MHz, then

$$N_{max} = \frac{118.6 - 98.0}{0.2} = 103$$

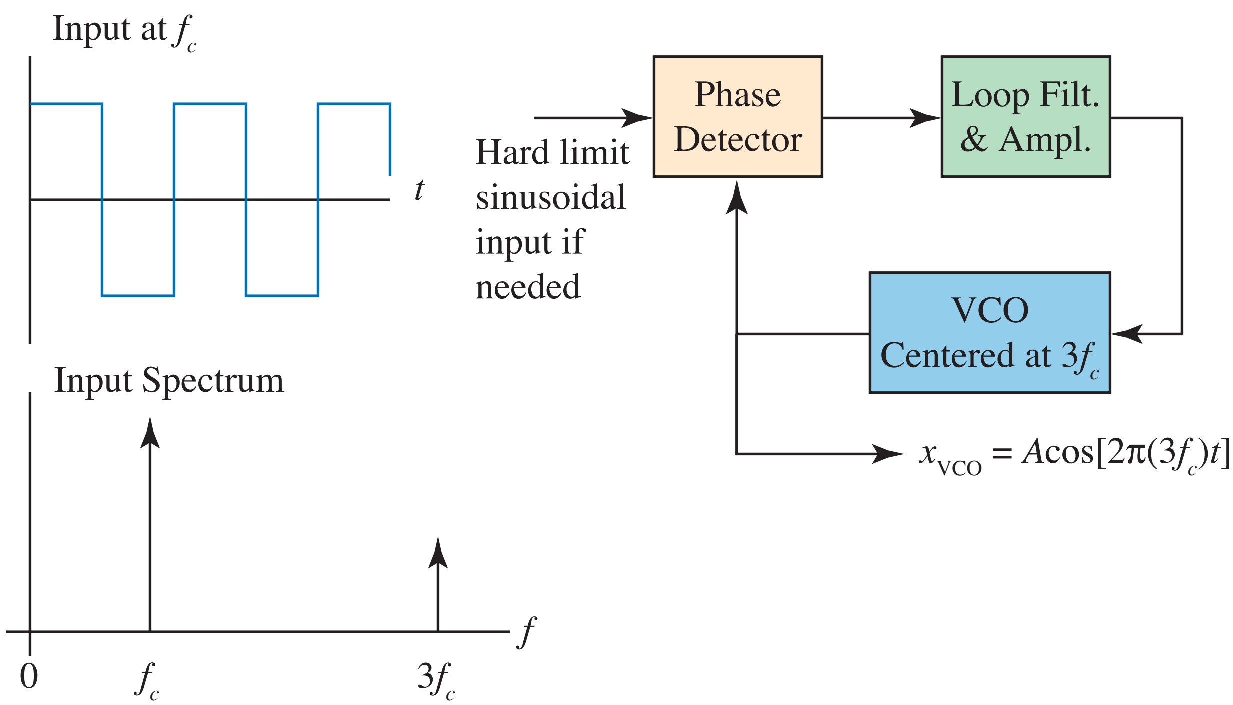
and

$$N_{min} = \frac{98.8 - 98.0}{0.2} = 4$$

- To program the LO such that the receiver tunes all FM stations step N from $4, 5, 6, \ldots, 102, 103$

Example 4.12: Simple PLL Frequency Multiplication

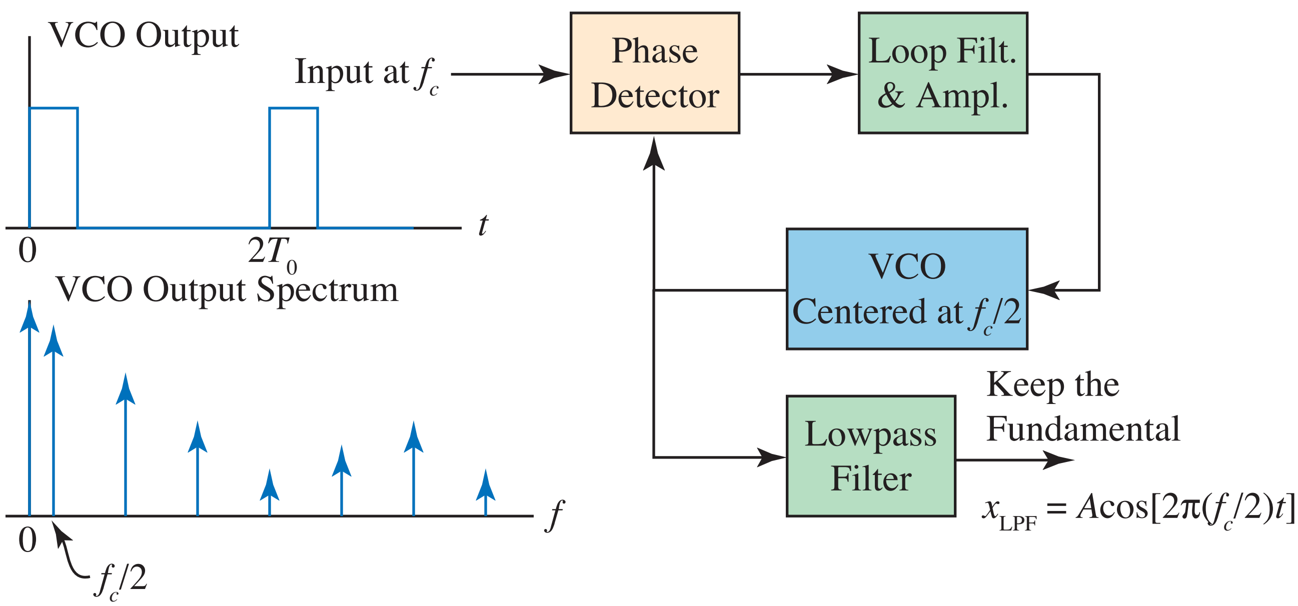
- A scheme for multiplication by three is shown below:



PLL as a Frequency Multiplier

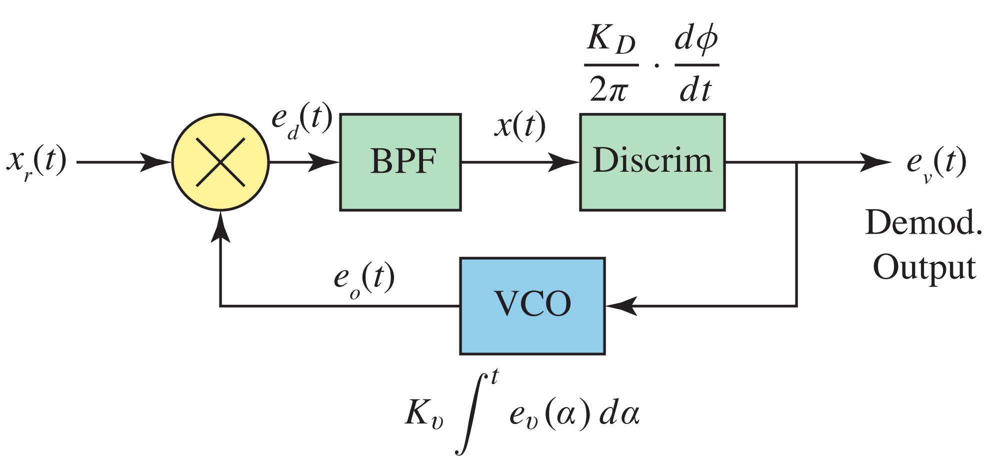
Example 4.13: Simple PLL Frequency Division

- A scheme for divide by two is shown below:



PLL as a Frequency Divider

4.2.3 Frequency-Compressive Feedback



Frequency compressive feedback PLL

- If we place a discriminator inside the PLL loop a compressing action occurs
- Assume that

$$x_r(t) = A_c \cos[\omega_c t + \phi(t)]$$

and

$$e_v(t) = A_v \sin\left[(\omega_c - \omega_o)t + K_v \int^t e_v(\alpha)\, d\alpha\right]$$

- Then,

$$e_d(t) = \frac{1}{2} A_c A_v \Bigg\{ \overbrace{\sin\left[(2\omega_c - \omega_o)t + \text{other terms}\right]}^{\text{blocked by BPF}} - \underbrace{\sin[\omega_o t + \phi(t) - K_v \int^t e_v(\alpha)\, d\alpha]}_{\text{passed by BPF}} \Bigg\}$$

so

$$x(t) = -\frac{1}{2} A_c A_v \sin\left[\omega_o t + \phi(t) - K_v \int^t e_v(\alpha)\, d\alpha\right]$$

- Assuming an ideal discriminator

$$e_v(t) = \frac{1}{2\pi} K_D \left[\frac{d\phi(t)}{dt} - K_v e_v(t)\right]$$

or

$$e_v(t)\left[1 + \frac{K_v K_D}{2\pi}\right] = \frac{K_D}{2\pi} \cdot \frac{d\phi(t)}{dt}$$

- For FM $d\phi(t)/dt = 2\pi f_d m(t)$, so

$$e_v(t) = \frac{K_D f_d}{1 + K_v K_D/(2\pi)} m(t)$$

which is the original modulation scaled by a constant

- The discriminator input must be

$$x(t) = -\frac{1}{2} A_c A_d \sin\left[\omega_o t + \frac{1}{1 + K_v K_D/(2\pi)} \phi(t)\right]$$

- Assuming that $K_v K_D/(2\pi) \gg 1$ we conclude that the discriminator input has been converted to a narrowband FM signal, which is justifies the name 'frequency compressive feedback'

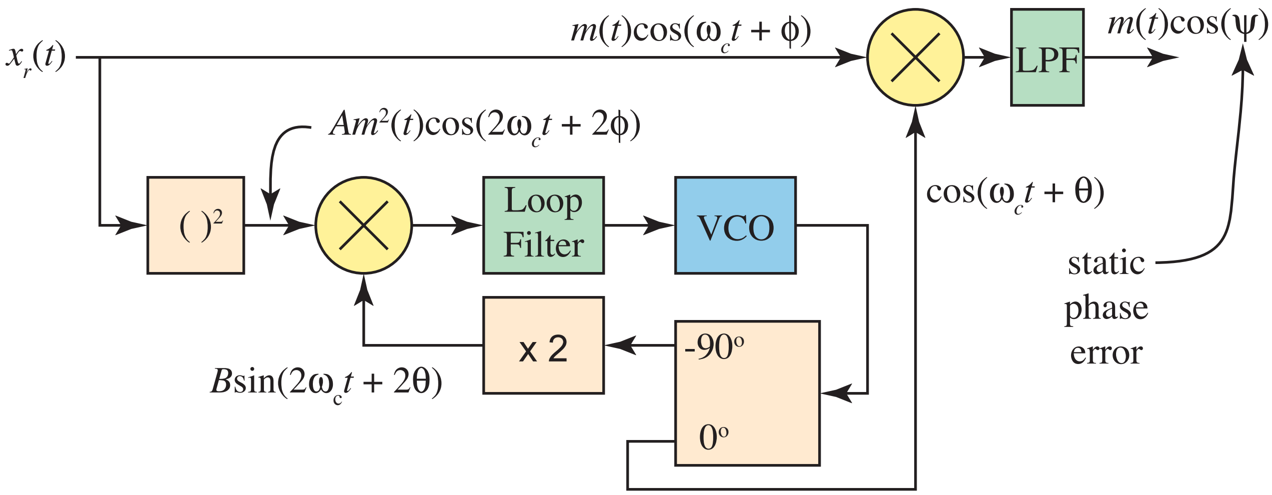
4.2.4 Coherent Carrier Recovery for DSB Demodulation

- Recall that a DSB signal is of the form

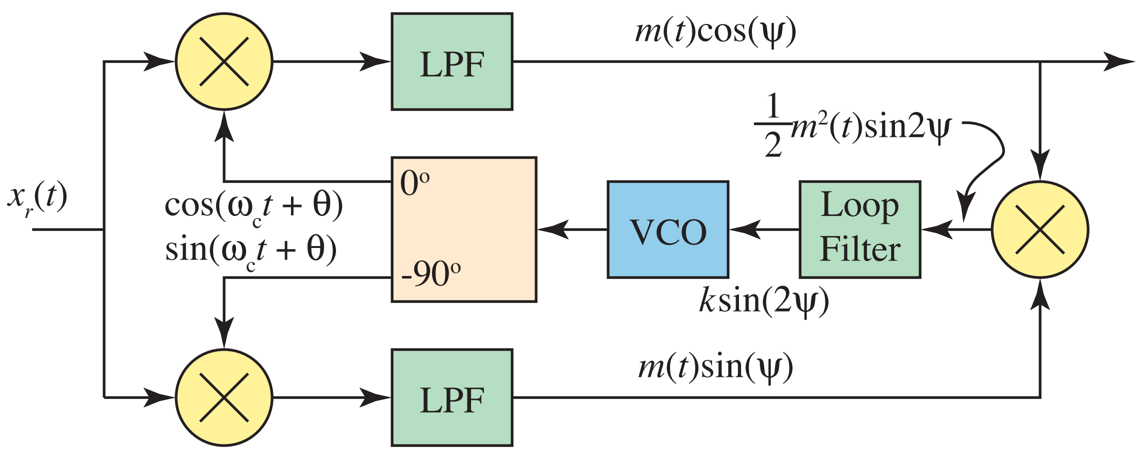
$$x_r(t) = m(t) \cos \omega_c t$$

- A PLL can be used to obtain a coherent carrier reference directly from $x_r(t)$

- Here we will consider the *squaring loop* and the *Costas loop*



Squaring Loop



Costas Loop

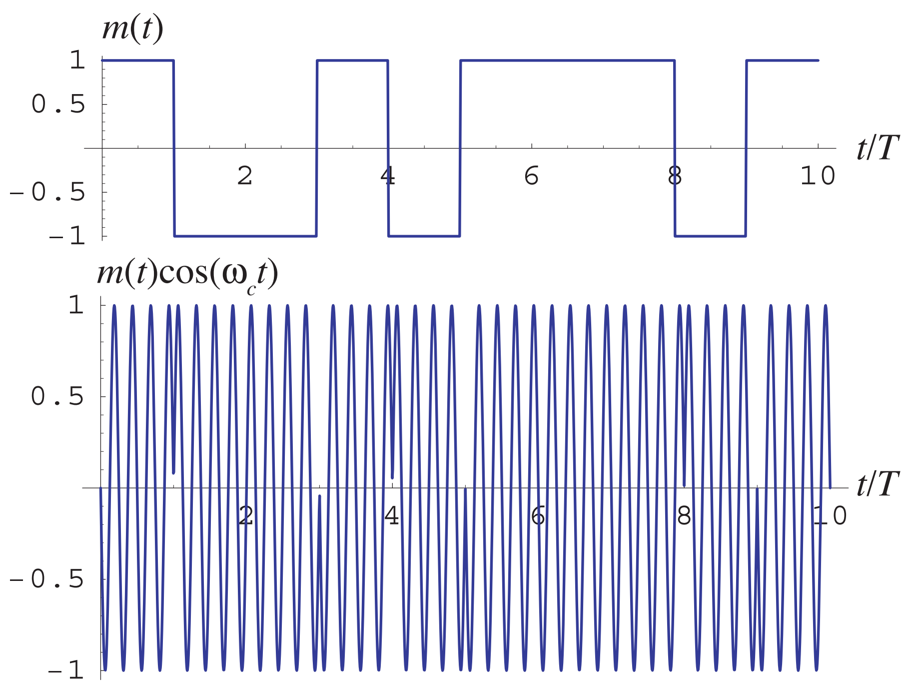
- Note: For both of the above loops $m^2(t)$ must contain a DC component
- The Costas loop or a variation of it, is often used for carrier recovery in digital modulation
- *Binary phase-shift keying* (BPSK), for example, can be viewed as DSB where

$$m(t) = \sum_{n=-\infty}^{\infty} d_n p(t - nT)$$

where $d_n = \pm 1$ represents random data bits and $p(t)$ is a pulse shaping function, say

$$p(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

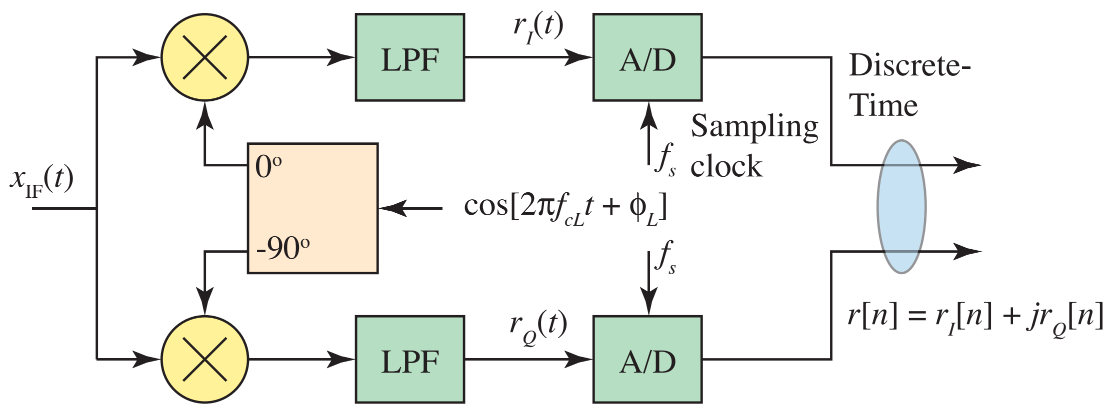
- Note that in this case $m^2(t) = 1$, so there is a strong DC value present



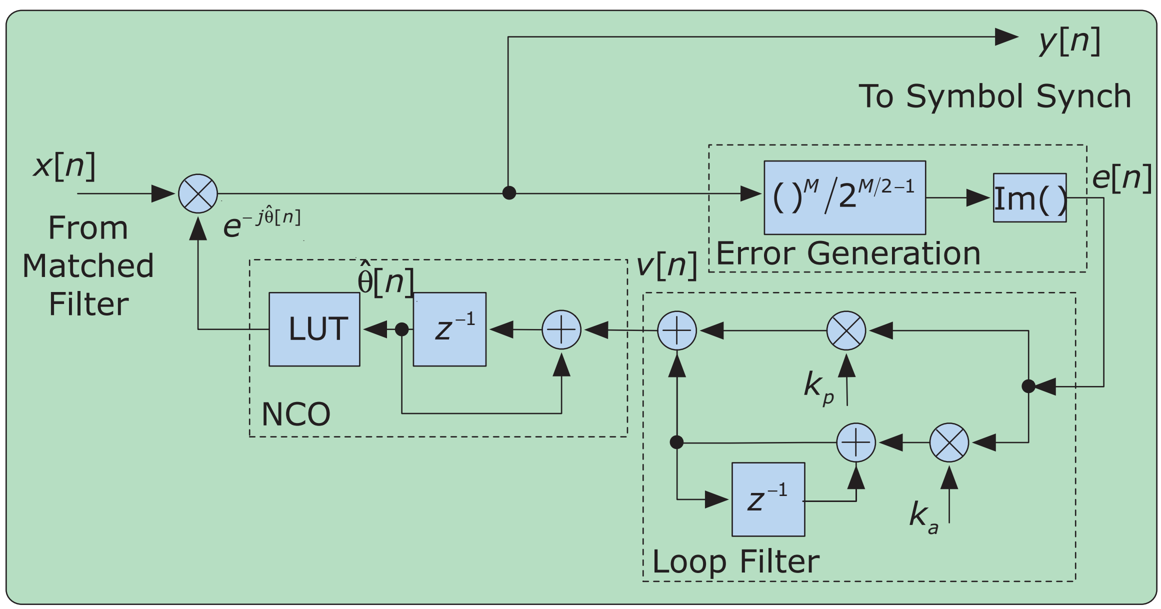
BPSK modulation

- Digital signal processing techniques are particularly useful for building PLLs

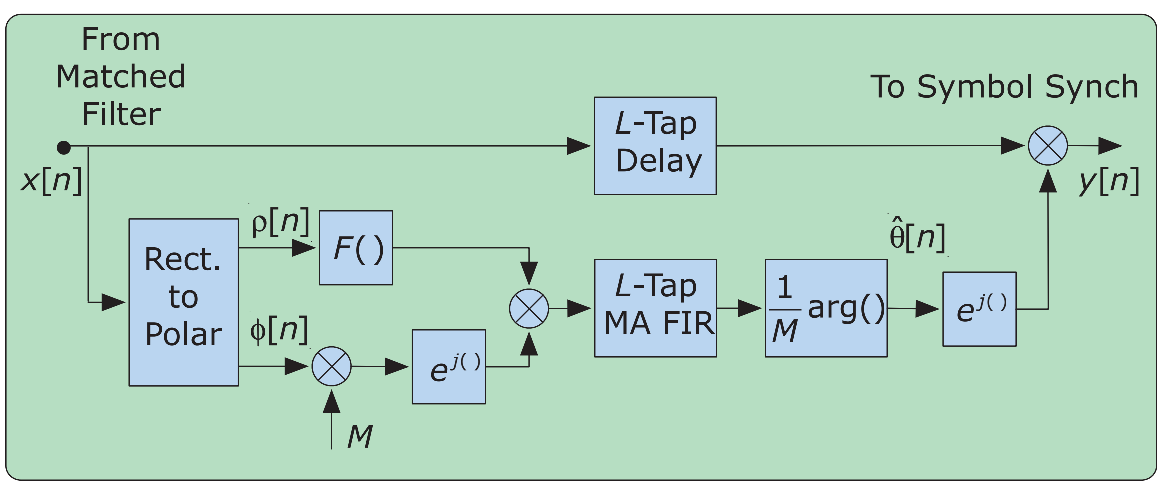
- In the discrete-time domain, digital communication waveforms are usually processed at complex baseband following some form of I-Q demodulation



IF to discrete-time complex baseband conversion



Mth-power digital PLL (DPLL) carrier phase tracking loop



Non-Data Aided (NDA) feedforward carrier phase tracking

4.3 Interference and Preemphasis

Interference is a fact of life in communication systems. A through understanding of interference requires a background in random signals analysis (Chapter 7 of the text), but some basic concepts can be obtained by considering a single interference at $f_c + f_i$ that lies close to the carrier f_c

4.3.1 Interference in Angle Modulation

- Initially assume that the carrier is unmodulated

$$x_r(t) = A_c \cos \omega_c t + A_i \cos(\omega_c + \omega_i)t$$

- In complex envelope form we have

$$x_r(t) = \text{Re}\Big\{[A_c + (A_i \cos \omega_i t + jA_i \sin \omega_i t)]e^{j\omega_c t}\Big\}$$

  with $\tilde{R}(t) = A_c + A_i \cos \omega_i t + jA_i \sin \omega_i t$

- The real envelope or envelope magnitude is, $R(t) = |\tilde{R}(t)|$,

$$R(t) = \sqrt{(A_c + A_i \cos \omega_i t)^2 + (A_i \sin \omega_i t)^2}$$

  and the envelope phase is

$$\phi(t) = \tan^{-1}\left[\frac{A_i \sin \omega_i t}{A_c + A_i \cos \omega_i t}\right]$$

- For future reference note that:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \overset{|x| \ll 1}{\simeq} x$$

- We can thus write that

$$x_r(t) = R(t)\cos\left[\omega_c t + \phi(t)\right]$$

- If $A_c \gg A_i$

$$x_r(t) \simeq \underbrace{(A_c + A_i\cos\omega_i t)}_{R(t)}\cos\left[\omega_c t + \underbrace{\frac{A_i}{A_c}\sin\omega_i t}_{\phi(t)}\right]$$

- Case of PM Demodulator: The discriminator recovers $d\phi(t)/dt$, so the output is followed by an integrator

$$y_D(t) = K_D\frac{A_i}{A_c}\sin\omega_i t$$

- Case of FM Demodulator: The discriminator output is used directly to obtain $d\phi(t)/dt$

$$y_D(t) = \frac{1}{2\pi}K_D\frac{A_i}{A_c}\frac{d}{dt}\sin\omega_i t = K_D\frac{A_i}{A_c}f_i\cos\omega_i t$$

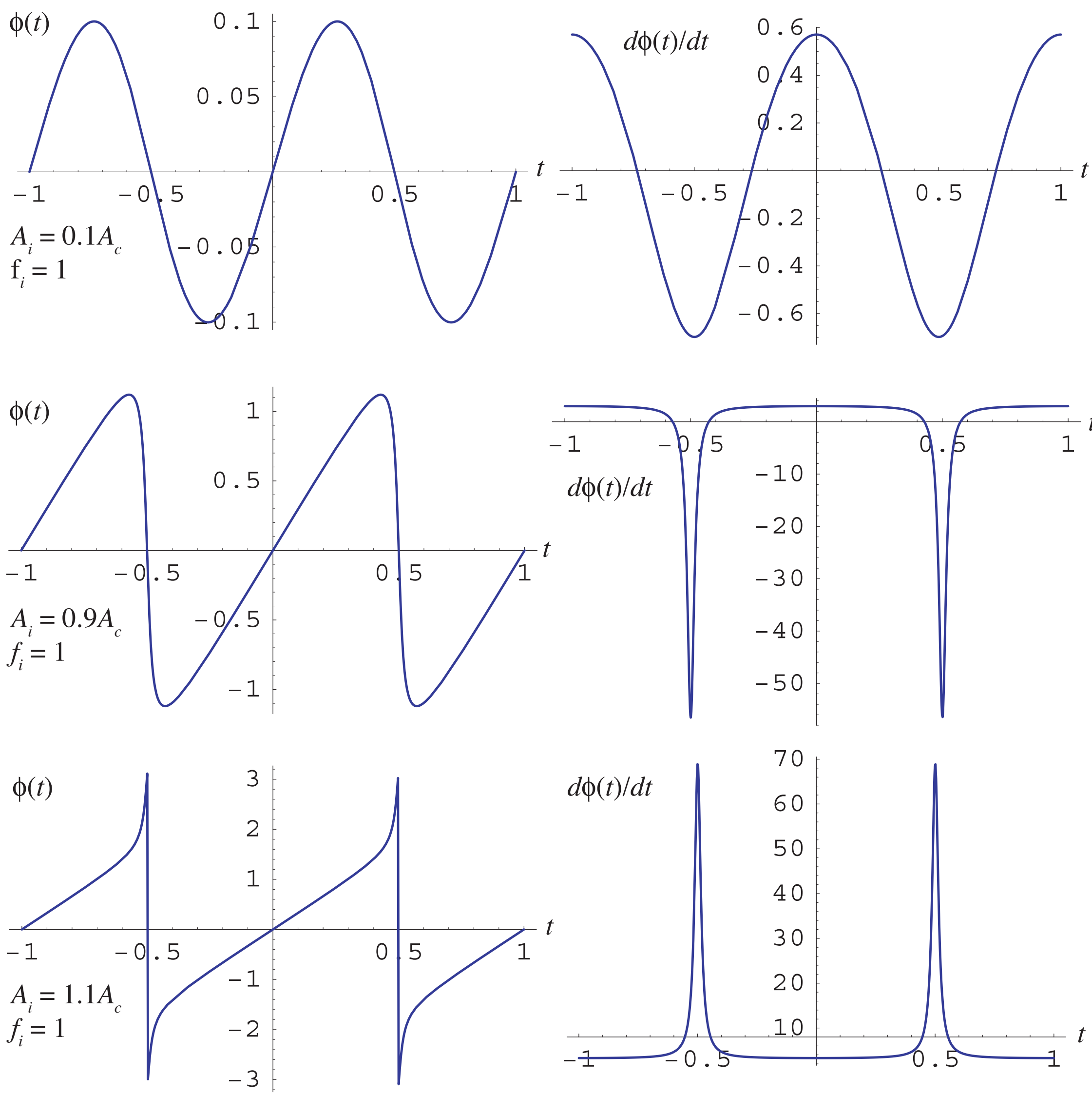
- We thus see that the interfering tone appears directly in the output for both PM and FM

- For the case of FM the amplitude of the tone is proportional to the offset frequency f_i

- For $f_i > W$, recall W is the bandwidth of the message $m(t)$, a lowpass filter following the discriminator will remove the interference

- When A_i is similar to A_c and larger, the above analysis no longer holds

- In complex envelope form
$$x_r(t) = \text{Re}\{[A_c + A_i e^{j\omega_i t}]e^{j\omega_c t}\}$$
- The phase of the complex envelope is
$$\phi(t) = \angle(A_c + A_i e^{j\omega_i t}) = \tan^{-1}\left[\frac{A_i \sin \omega_i t}{A_c + A_i \cos \omega_i t}\right]$$
- We now consider $A_i \approx A_c$ and look at plots of $\phi(t)$ and the derivative



Phase deviation and discriminator outputs when $A_i \approx A_c$

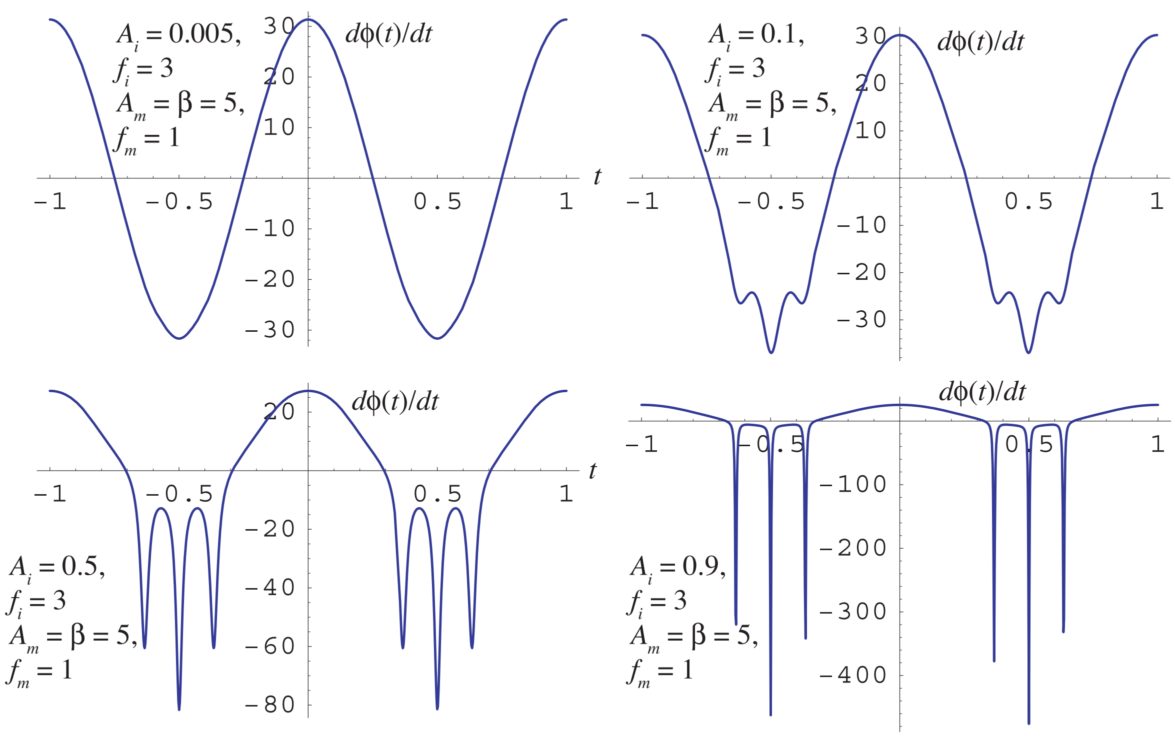
- We see that *clicks* (positive or negative spikes) occur in the discriminator output when the interference levels is near the signal level

- When $A_i \gg A_c$ the message signal is entirely lost and the discriminator is said to be operating *below threshold*

- To better see what happens when we approach threshold, apply single tone FM to the carrier

$$\phi(t) = \angle\big[A_c e^{jA_m \cos(\omega_m t)} + A_i e^{j\omega_i t}\big]$$

- Plot the discriminator output $d\phi(t)/dt$ with $A_m = 5$, $f_m = 1$, $f_i = 3$, and various values of A_i



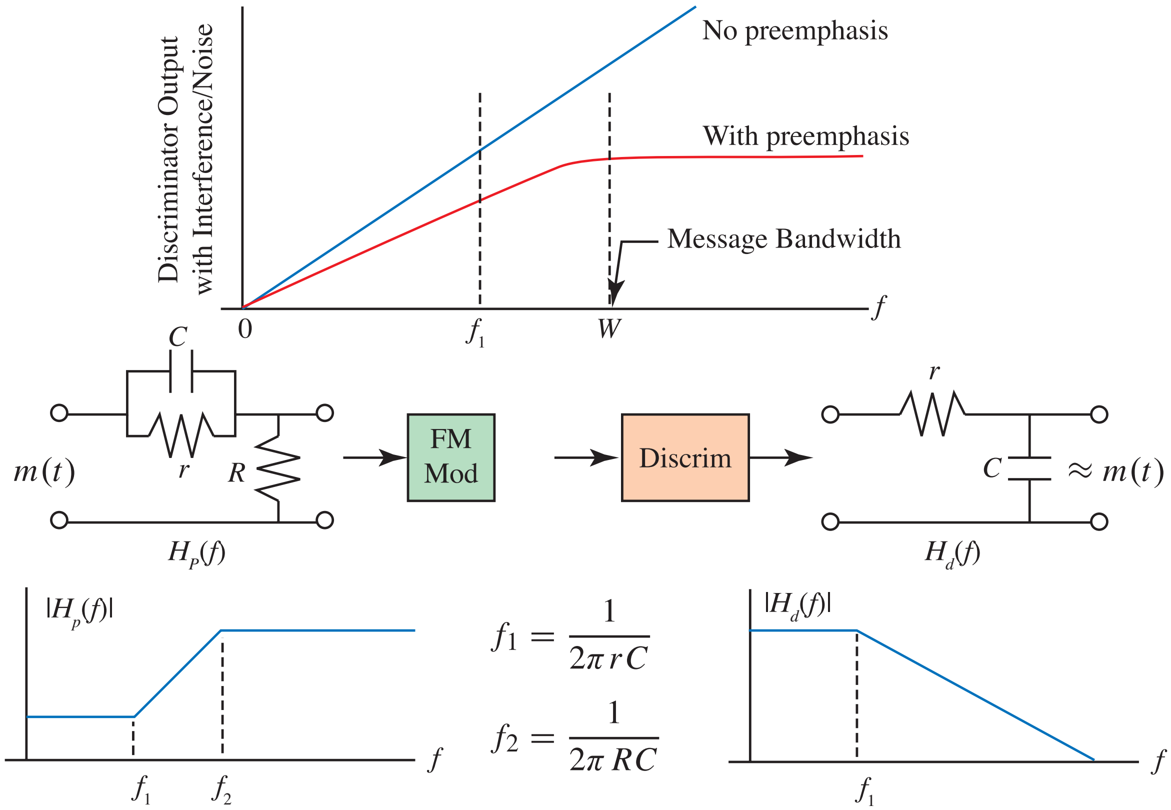
Discriminator outputs as A_i approaches A_c with single tone FM $\beta = 5$

4.3.2 The Use of Preemphasis in FM

- We have seen that when A_i is small compared to A_c the interference level in the case of FM demodulation is proportional to f_i

- The generalization from a single tone interferer to background noise (text Chapter 6), shows a similar behavior, that is wide bandwidth noise entering the receiver along with the desired FM signal creates noise in the discriminator output that has amplitude proportional with frequency (noise power proportional to the square of the frequency)

- In FM radio broadcasting a *preemphasis* boosts the high frequency content of the message signal to overcome the increased noise background level at higher frequencies, with a deemphasis filter used at the discriminator output to gain equalize/flatten the end-to-end transfer function for the modulation $m(t)$



FM broadcast preemphasis and deemphasis filtering

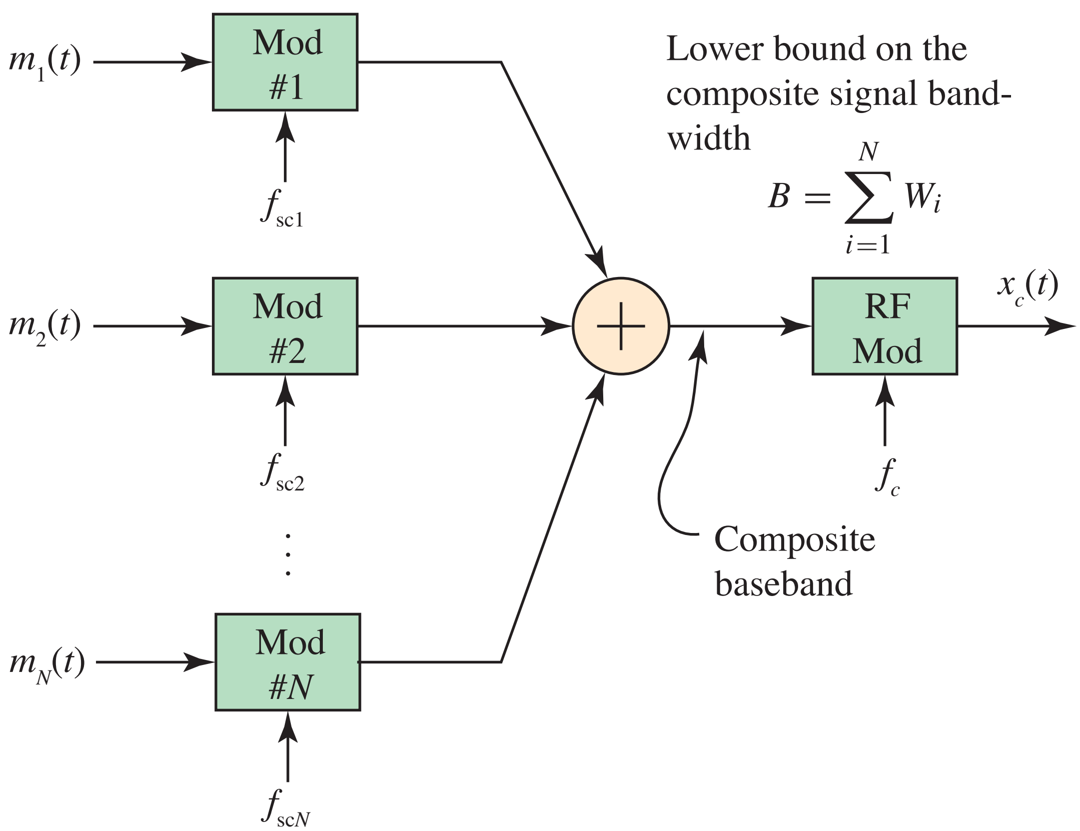
- The time constant for these filters is $RC = 75\ \mu\text{s}$ ($f_1 = 1/(2\pi RC) = 2.1$ kHz), with a high end cutoff of about $f_2 = 30$ kHz

4.4 Multiplexing

- It is quite common to have multiple information sources located at the same point within a communication system
- To simultaneously transmit these signals we need to use some form of multiplexing
- In this chapter we continue the discussion of multiplexing from Chapter 3 and investigate frequency-division multiplexing

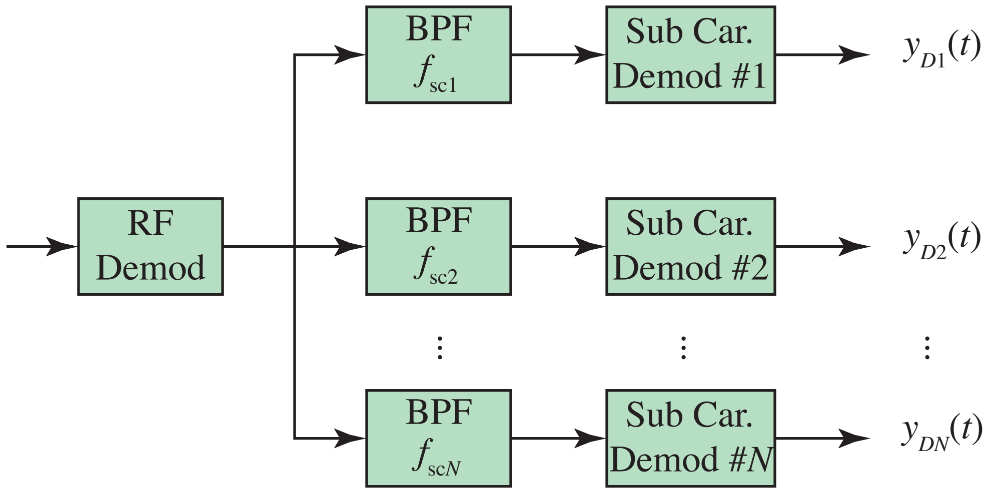
4.4.1 Frequency-Division Multiplexing (FDM)

- With FDM the idea is to locate a group of messages on different subcarriers and then sum then together to form a new baseband signal which can then be modulated onto the carrier



FDM transmitter

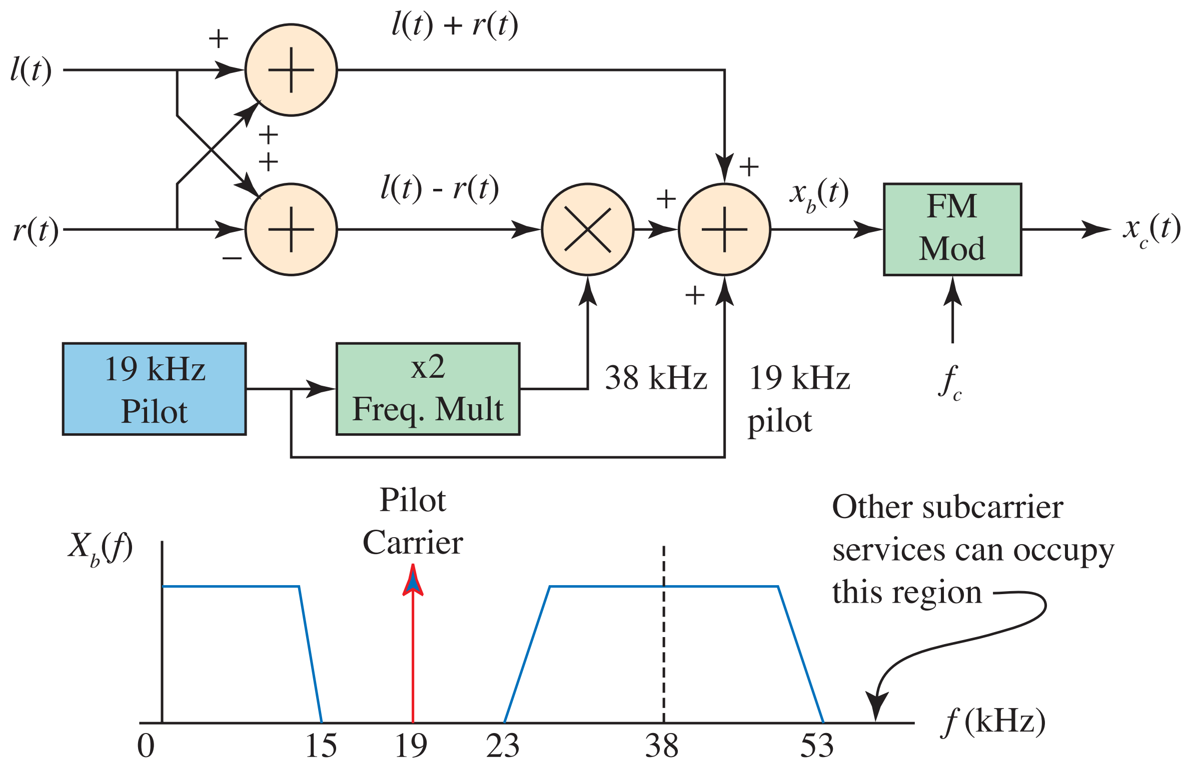
- At the receiver we first demodulate the composite signal, then separate into subcarrier channels using bandpass filters, then demodulate the messages from each subcarrier



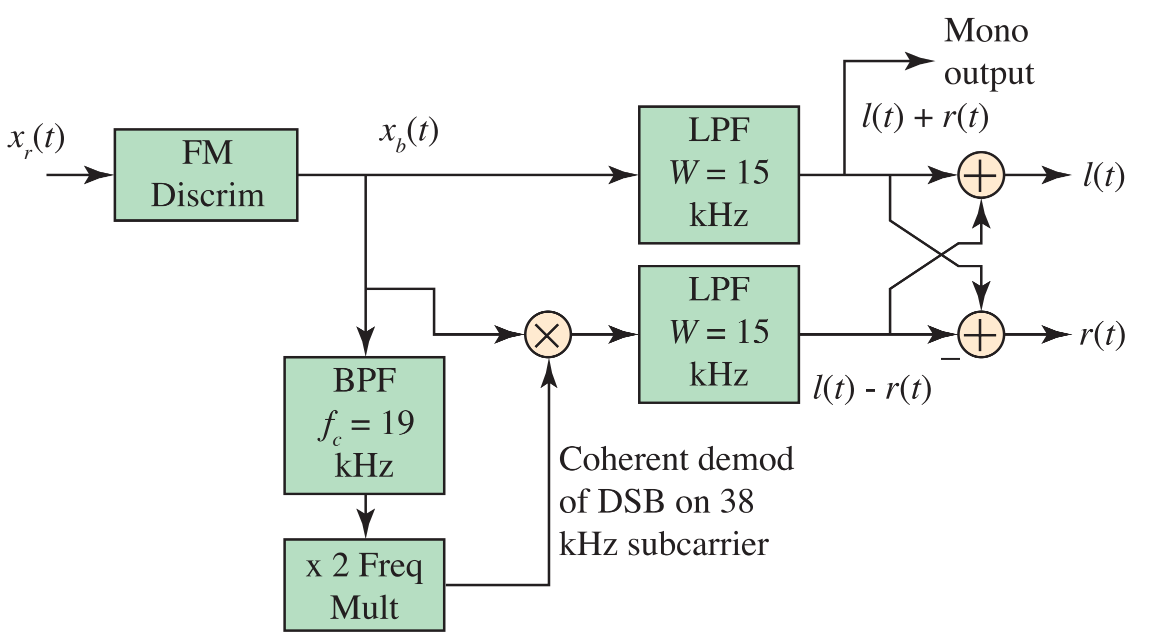
FDM receiver/demodulator

- The best spectral efficiency is obtained with SSB subcarrier modulation and no guard bands
- At one time this was the dominant means of routing calls in the *public switched telephone network* (PSTN)
- In some applications the subcarrier modulation may be combinations both analog and digital schemes
- The analog schemes may be combinations of amplitude modulation (AM/DSM/SSB) and angle modulation (FM/PM)

Example 4.14: FM Stereo

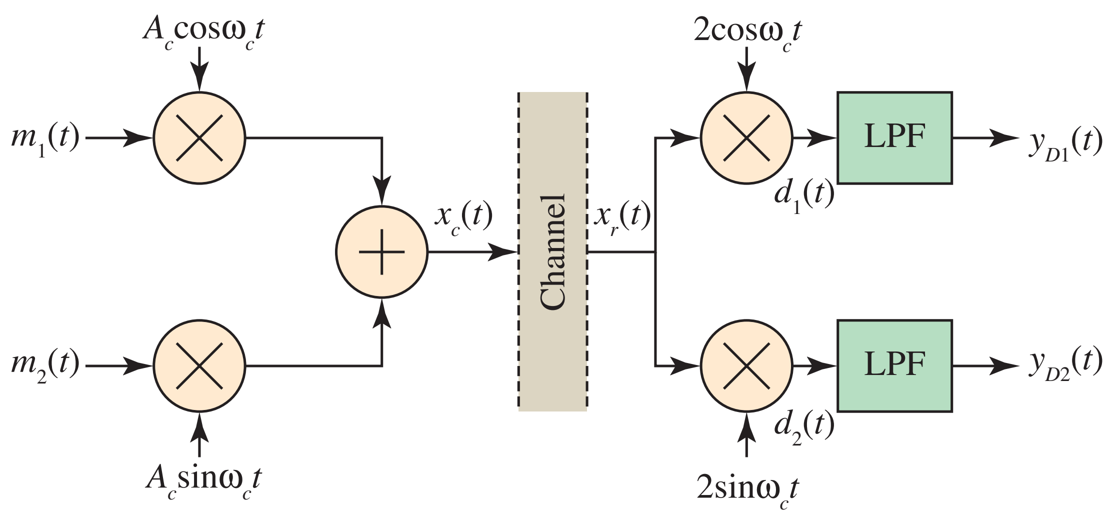


FM stereo transmitter



FM stereo receiver

4.4.2 Quadrature Multiplexing (QM)



QM modulation and demodulation

- With QM quadrature (sin/cos) carrier are used to send independent message sources
- The transmitted signal is

$$x_c(t) = A_c\big[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t\big]$$

- If we assume an imperfect reference at the receiver, i.e., $2\cos(\omega_c t + \theta)$, we have

$$\begin{aligned} d_1(t) &= A_c\big[m_1(t)\cos\theta - m_2(t)\sin\theta \\ &\qquad + \underbrace{m_1(t)\cos(2\omega_c t + \theta) + m_2(t)\sin(2\omega_c t + \theta)}_{\text{LPF removes these terms}}\big] \\ y_{D1}(t) &= A_c\big[m_1(t)\cos\theta + m_2(t)\sin\theta\big] \end{aligned}$$

- The second term in $y_{D1}(t)$ is termed *crosstalk*, and is due to the static phase error θ

- Similarly

$$y_{D2}(t) = A_c\big[m_2(t)\cos\theta - m_1(t)\sin\theta\big]$$

- Note that QM acheives a bandwidth efficiency similar to that of SSB using adjacent two subcarriers or USSB and LSSB together on the same subcarrier

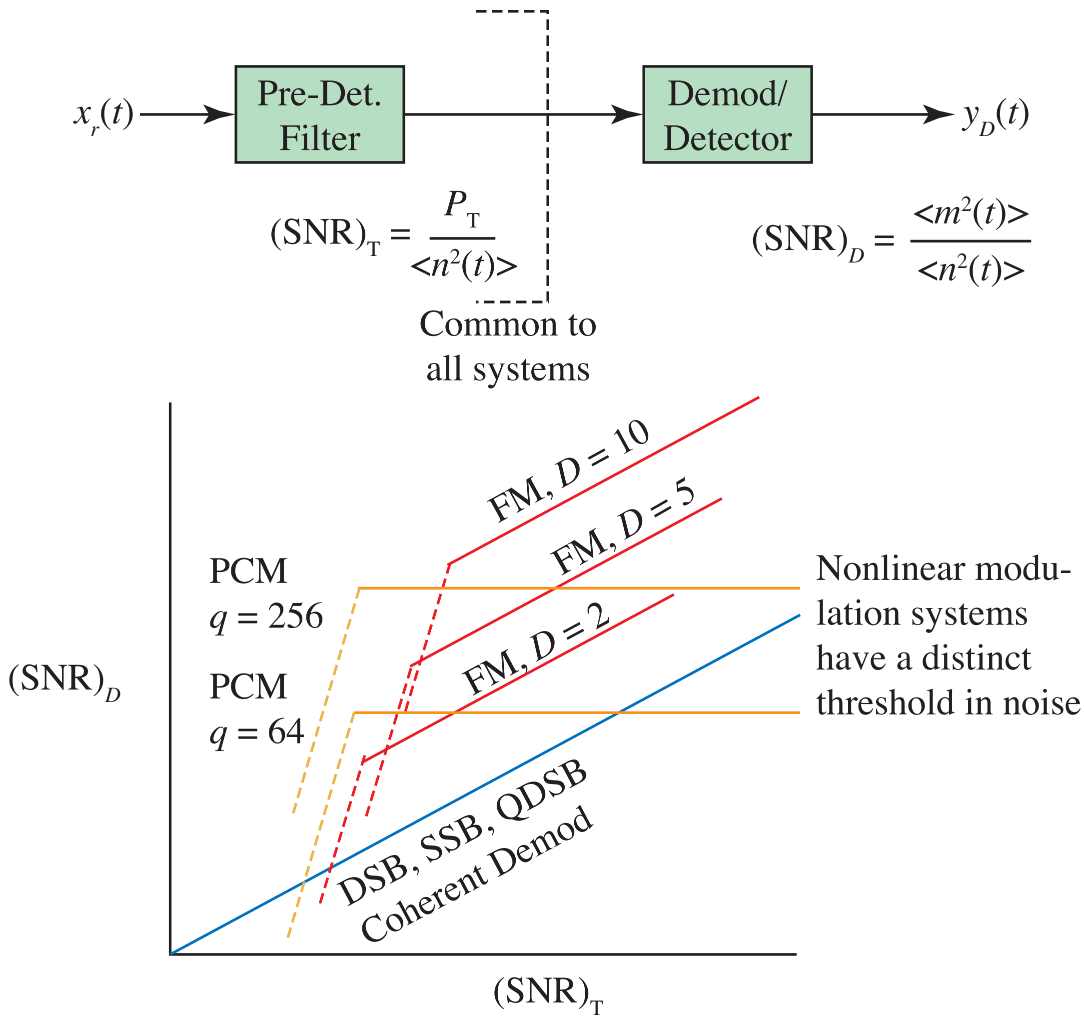
4.5 General Performance of Modulation Systems in Noise

- Regardless of the modulation scheme, the received signal $x_r(t)$, is generally perturbed by additive noise of some sort, i.e.,

$$x_r(t) = x_c(t) + n(t)$$

where $n(t)$ is a noise process

- The pre- and post-detection signal-to-noise ratio (SNR) is used as a figure or merit



General modulation performance in noise