Anita's Way to Add Fractions with Unlike Denominators

About Illustrations: Illustrations of the Standards for Mathematical Practice (SMP) consist of several pieces, including a mathematics task, student dialogue, mathematical overview, teacher reflection questions, and student materials. While the primary use of Illustrations is for teacher learning about the SMP, some components may be used in the classroom with students. These include the mathematics task, student dialogue, and student materials. For additional Illustrations or to learn about a professional development curriculum centered around the use of Illustrations, please visit mathpractices.edc.org.

About the *Anita's Way to Add Fractions with Unlike Denominators* Illustration: This Illustration's student dialogue shows the conversation among three students, who have already learned to use equivalent fractions to add two fractions with unlike denominators, exploring another method they thought of for adding two fractions with unlike denominators. They use a length model to think about the addition of the fractions and scale that representation to find a whole number distance, then relate that scaled distance back to the original distance that represents the sum of the two fractions.

Highlighted Standard(s) for Mathematical Practice (MP)
MP 1: Make sense of problems and persevere in solving them.
MP 2: Reason abstractly and quantitatively.
MP 3: Construct viable arguments and critique the reasoning of others.
MP 5: Use appropriate tools strategically.
MP 7: Look for and make use of structure.

Target Grade Level: Grades 5–6

Target Content Domain: The Number System, Ratios & Proportional Relationships

Highlighted Standard(s) for Mathematical Content

5.NF.A.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example,* $2/3 + 5/4 = 8/12 + 15/12 = 23/12$. *(In general,* $a/b + c/d = (ad + bc)/bd$.*)*

6.RP.A.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

Math Topic Keywords: fractions, unlike denominators, scaling, unit fractions

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Anita’s Way to Add Fractions with Unlike Denominators

Mathematics Task

Suggested Use
This mathematics task is intended to encourage the use of mathematical practices. Keep track of ideas, strategies, and questions that you pursue as you work on the task. Also reflect on the mathematical practices you used when working on this task.

How do we add two fractions with unlike denominators?

Anita's Way to Add Fractions with Unlike Denominators

Student Dialogue

Suggested Use
The dialogue shows one way that students might engage in the mathematical practices as they work on the mathematics task from this Illustration. Read the student dialogue and identify the ideas, strategies, and questions that the students pursue as they work on the task.

Students have already learned the procedure for using equivalent fractions to add two fractions with unlike denominators. They also have experience breaking a whole into unit fractions. In the Student Dialogue, Anita is wondering about a new way to add fractions with unlike denominators.

(1) Anita: Hey, Dana! Hey, Sam! Remember I told you I found a brand new way to figure out how to add fractions with unlike denominators? C'mon, lemme show you!

(2) Sam: It's lunch time! I'm hungry! Oh, sure, go ahead, Anita. You've really wanted to show us for a while.

(3) Anita: OK. For example, let's take $\frac{1}{4}+\frac{1}{6}$. The way we were taught to do it was to look for a common denominator and…

(4) Dana: Yes, we know. Equivalent fractions, all that stuff.

(5) Anita: But what if we were the very first people to need to figure out how to add $\frac{1}{4}+\frac{1}{6}$?

(6) Sam: But we're *not*! We *know* how to do it. The common denominator is 12 so we get $\frac{3}{12}+\frac{2}{12}=\frac{5}{12}$. Done! Now let's all go get lunch.

(7) Anita: No, wait! Yes, ***we*** can do it that way, but someone was the first to add fractions and she *didn't* already know how.

(8) Sam: Or *he* didn't.

(9) Anita: Whatever. How could someone *figure out* what to do if they hadn't already been given a rule?

(10) Dana: Another Anita adventure! OK, go on.

(11) Anita: Here's what I was thinking. I imagined walking $\frac{1}{4}$ mile and then another $\frac{1}{6}$ mile and…

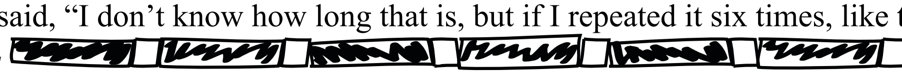
(12) Sam: Ah, and then you would lay out a ruler and measure how far you walked in fractions of a mile?

(13) Anita: Well, of course, the problem isn't about miles or inches—that's just what I made up to think about it—but yes, actually, I did imagine a ruler, at first, and I didn't even care that this imaginary ruler would have to be a mile long. But then something occurred to me. For this super-fancy ruler to work, somebody would already have had to figure out how to label it. *Our* problem needs only 12^{ths} of a mile, just like you said, Sam, but if we wanted a general method, that ruler would have to work for other problems, too. It would need 7^{ths} and 10^{ths} and 32^{nds} and 13^{ths} … and every fraction.

(14) Dana: *[intrigued]* OK, right, and that's not just *hard*; it's *impossible*! So what *did* you think up? Did you think of breaking each distance, the $\frac{1}{4}$ mile and the $\frac{1}{6}$ mile, into smaller parts that would nicely measure them both?

(15) Anita: You mean like finding 12^{ths} or 24^{ths} or some other fraction that both the $\frac{1}{4}$ and the $\frac{1}{6}$ can be converted to? That's the equivalent fraction method that we learned. It's fine, but I didn't do it because I couldn't imagine having *thought* of it on my own. Instead, I pictured the distance, like this. *[She draws*  *]*. The black one is $\dfrac{1}{6}$ and the white one is $\dfrac{1}{4}$. Never mind my terrible drawing. My sixth is *way* too small compared with the quarter but it really doesn't matter. Ignore the sizes and just use the colors. The problem is to figure out how long that *would* be if I drew it right.

(16) Sam: We know the *problem*! What's the *solution*?

(17) Anita: Then I said, "I don't know how long that is, but if I repeated it six times, like this *[Draws*  *],* then I would know the length!

(18) Dana: That helps?! Now you've just got six times who-knows-what!

(19) Sam: Actually, that's quite clever! The six blacks are a mile long, because each black was one-sixth of a mile. And the six whites are… Six quarter-miles, um, is a mile and a half. So the total length of this is $2\dfrac{1}{2}$ miles!

(20) Anita: Yup. And I could have stopped there, but that still felt too hard to work with, so I made the whole thing twice as long, an even 5 miles.

(21) Sam: Five is *odd*.

(22) Dana: *[groans at Sam]* Ha… ha… ha… *[Then, to Anita]* And you now have 12 copies of your original drawing.

(23) Anita: Right! And if, *12* copies of my distance makes 5 miles, then my distance is…

(24) Sam: Omigosh, Anita. That really *is* cool! I really *like* that! But now we do need to hurry to lunch!

(25) Dana: Oh! And I see how Anita's idea can work with *any* pair of fractions! But you're right, Sam, we'll have to talk about that later. Lunch period's almost over.

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Teacher Reflection Questions

Suggested Use
These teacher reflection questions are intended to prompt thinking about 1) the mathematical practices, 2) the mathematical content that relates to and extends the mathematics task in this Illustration, 3) student thinking, and 4) teaching practices. Reflect on each of the questions, referring to the student dialogue as needed. Please note that some of the mathematics extension tasks presented in these teacher reflection questions are meant for teacher exploration, to prompt teacher engagement in the mathematical practices, and may not be appropriate for student use.

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

2. Using Anita's way of adding fractions with unlike denominators, find the sum of the following. Don't bother reducing fractions or converting to mixed numbers, and don't worry about the *scale* of your sketches. Make the sketches "structurally correct" without worrying about whether their measurements are appropriate.

   A. $\frac{1}{4}+\frac{1}{3}$
   B. $\frac{1}{4}+\frac{2}{3}$
   C. $\frac{3}{4}+\frac{2}{3}$

3. Based on your experience using Anita's way to add fractions with unlike denominators in Question 2, explain, using words and algebraic language, how to use *that* method to add the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

4. Use the traditional algorithm (equivalent fractions) to add $\frac{a}{b}$ and $\frac{c}{d}$.

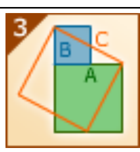
5. Compare Anita's way of adding fractions with the traditional algorithm (the equivalent-fractions method).

6. How might you support students who were using Anita's way to add $\frac{1}{4}+\frac{2}{3}$ and were struggling to figure out how to add the $\frac{2}{3}$ pieces together (since they aren't unit fractions)?

Mathematical Overview

Suggested Use
The mathematical overview provides a perspective on 1) how students in the dialogue engaged in the mathematical practices and 2) the mathematical content and its extensions. Read the mathematical overview and reflect on any questions or thoughts it provokes.

Commentary on the Student Thinking

Mathematical Practice	Evidence
 Make sense of problems and persevere in solving them.	In the Student Dialogue, students set aside the equivalent-fractions procedure they have learned and try to make sense of how one can add fractions with unlike denominators if they "were the very first people to need to figure [it] out" (line 5). They try "simpler forms of the original problem in order to gain insight into its solution," first by taking an abstract problem—add two fractions with unlike denominators—and choosing a concrete exemplar $\frac{1}{4}+\frac{1}{6}$ to work with, and then by creating and scaling a visual representation of that problem. Scaling the sum of the fractions first by 6 (line 17) and then by 2 (line 20) allows the students to figure out the solution of the scaled problem (line 20) before they go back to thinking about the solution to the original problem.
 Reason abstractly and quantitively.	MP2 describes how students should be able to "abstract a given situation …and manipulate the representing symbols" (i.e., decontextualize) and "pause as needed during the manipulation process in order to probe into the referents for the symbols involved" (i.e., contextualize). In this Student Dialogue, Anita (line 11) contextualizes the abstract problem as a sum of distances in order to reason about what $\frac{1}{4}$ and $\frac{1}{6}$ may represent. The way students contextualize fractions has implications for the way they proceed in "creating a coherent representation of the problem at hand." For example, in this case the students use the idea of distance to contextualize the fractions, leading them to draw pictures representing distance. If students had thought about the fractions in some other context, their pictures might have been something different, such as sectors of circle.
 Construct viable arguments and critique the reasoning of others.	Starting from the most basic shared assumptions, Anita derives a method for adding fractions with unlike denominators and constructs a coherent and logical argument to explain that method.

 Use appropriate tools strategically.	Students use no physical tools, but they do think about the implications of an imaginary ruler (lines 12–13). They make “sound decisions about when [this tool] might be helpful, recognizing both the insight to be gained and [its] limitations.” Anita points to the limitations of rulers: because we would need to know how to label the ruler beforehand, and we would need a ruler that would work for any two fractional distances, any such “ruler” would need to label infinitely many fractional denominations.
 Look for and make use of structure.	Students use the pictorial representation of distance (lines 15 and 17) and their knowledge of fractions to find a structure that could be used to solve the original problem. Anita scales the image by six (line 17), and Sam realizes that six black pieces (representing a sixth of a mile each) makes a whole mile because there are 6 sixths in 1. By scaling the original two distances, they are able to easily calculate a whole number total distance traveled (line 20) which can then lead to information about the original copy of 1 fourth plus 1 sixth (line 23).

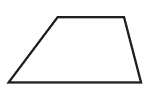
Commentary on the Mathematics

Perhaps the most striking piece of mathematics in this Student Dialogue is Anita’s starting question (line 5): what if we had to *invent* a piece of mathematics ourselves, not just learn a technique from someone who already knew it? This stance is novel enough that even after Sam and Dana agree to listen, it takes a while for them to understand that Anita is not trying to figure out why a known method works, but to figure out a method “from scratch,” from knowing *nothing* except the most basic assumptions. We don’t know what motivates her question—and, of course, she’s fictional—but the growth of mathematics as a discipline depends on thinking like this: what do I do when I’m faced with a genuinely new problem? This is an exceptional example of MP 1, and her argument is a clear illustration of MP 3.

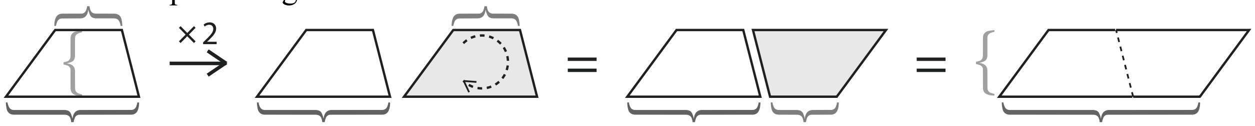
The particular details of Anita’s method for adding two non-integers—scaling the sum until the result *is* an integer and then dividing by that integer—works only when the two non-integers are commensurable. That is, it would not work for $\frac{1}{2}+\sqrt{2}$ (though it would work for any pair of rational numbers even if each member of the pair was multiplied by the same irrational number, as in $\frac{\pi}{2}+3.1\pi$).

But Anita uses a much deeper mathematical idea that works for a very wide variety of problems and appears in many guises: if you are faced with a situation you don’t know how to deal with, transform it to one that you do know how to deal with. Then after you’ve worked successfully with that new situation, undo the damage.

For example, in geometry, we might wonder how we’d ever find the area of an arbitrary trapezoid.



So we play with it. We notice that if we double the trapezoid and rotate one copy upside down, the two copies together seem to make a parallelogram. *That’s* a figure that we *do* know how to find the area of. Of course, at some point, we must take the steps to assure ourselves it really *does* make a parallelogram.



Because the parallelogram has twice the area of the trapezoid, we must cut that result in half. Writing "we double the trapezoid" makes it sound as if this is some obvious step. It is not, but this kind of thinking can *become* a "habit of mind"—a way of thinking that one begins to gravitate naturally to—when one encounters it often enough.

We might call this the "if-only-it-were…" method. I have a trapezoid. Darn! If only it were a parallelogram, I'd be all set. Wishing isn't enough, of course, so we see if we can *make* the thing we want from the thing we have.

The mathematical technique called "completing the square"—one method for deriving the quadratic formula—works the same way. If we want to find the value of x in $(x + 1)^2 = 9$, we have no trouble: take the square root of both sides and we get $x + 1 = \pm 3$. That says that either $x + 1 = 3$ or $x + 1 = -3$, so now we know that either $x = 2$ or $x = -4$. Alternatively, for students who struggle to understand why the ± sign appears, we can rewrite $(x + 1)^2 = 9$ as $(x + 1)^2 - 9 = 0$ and use "chunking" to produce the equation $\clubsuit^2 - 9 = 0$. This leads to $(\clubsuit + 3)(\clubsuit - 3) = 0$, where $\clubsuit + 3 = 0$ or $\clubsuit - 3 = 0$. Substituting $\clubsuit = x + 1$ back into the two equations helps us solve and get $x = 2$ or $x = -4$.

But what do we do when we are faced with something like $x^2 + 2x - 8 = 0$? Taking the square root of both sides doesn't help, because the left-hand side would be a mess. Even rewriting it as $x^2 + 2x = 8$ doesn't help. *If only at least the left-hand side were* a perfect square. So, how can we *make* it a perfect square? If we recognize that $x^2 + 2x + 1$ *is* a perfect square, $(x + 1)$ squared, then we can add 1 to both sides of our second equation to get $x^2 + 2x + 1 = 9$, rethink it as $(x + 1)^2 = 9$, and… Oh, we've solved that already!

Doing that same trick generically with $x^2 + bx + c = 0$, we'd see:

$$x^2 + bx = -c$$

To make the left side a perfect square, we must add $\left(\frac{b}{2}\right)^2$ to it, so we must add that to the right side, too.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$$

Then we can express the left side as the square we intended to make, and also clean up the right side.

$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$$

And now, when we take the square root of both sides, we get:

$$x + \frac{b}{2} = \pm\frac{\sqrt{b^2 - 4c}}{2}$$

Isolating x, this already looks quite familiar.

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Working even more generically with $ax^2 + bx + c = 0$ generates the quadratic formula.

Anita plays the same game. She has $\frac{1}{4} + \frac{1}{6}$ and wants to find a way to add them without using the hints and rules about fractions that she's been given. *If only they were integers*, she says to herself. How can she make them integers? She can multiply the whole thing by 6 to get $\frac{6}{4} + \frac{6}{6}$, which equals $1\frac{1}{2} + 1$. She could then claim her answer is $\frac{2\frac{1}{2}}{6}$, which would be correct, but she doesn't like that enough. So she doubles her sum once more to be $\frac{12}{4} + \frac{12}{6}$, which is 5. But, to get this, she has multiplied her original sum by 12, so she undoes that, giving her $\frac{5}{12}$ as a final result. Prior to grade 5, the CCSSM defines $\frac{5}{12}$ as a sum of unit fractions, $\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$, or the equivalent multiplication, $\frac{1}{12} \times 5$. Only in fifth grade are fractions introduced as the result of division integers; that is, $\frac{5}{12}$ is for the first time seen as the result of $5 \div 12$. This understanding is a critical element in the reasoning illustrated in this Student Dialogue. Students must know both that $\frac{5}{12}$ is one way to write the number that results from the computation $5 \div 12$ and (equivalently) that $5 \div 12$ is one interpretation of the notation $\frac{5}{12}$. The students in this Student Dialogue appear to take that for granted, but not all find it that easy.

The *if-only-it-were* technique often winds up being the key way of thinking when one is trying to do proofs. I have <*whatever*$_1$> and I want to prove <*whatever*$_2$> but I don't see how to get from here to there. If only one of those were…. What can I do to this thing in order to make it into something I can handle?

<u>Evidence of the Content Standards</u>

In the Student Dialogue, students are trying to figure out another way to add two fractions with unlike denominators even though they are capable of doing so using equivalent fractions (5.NF.A.1). In the process of developing a new way, they use a length model of $\frac{1}{4} + \frac{1}{6}$ and scale that representation (lines 15–20). In line 23, reasoning about ratio, students relate the length of the scaled diagram back to the length of the original diagram and task (6.RP.A.3). Note: The *Common Core* does not require students to use the lowest common denominator when adding and subtracting fractions with unlike denominators. However, students may use the lowest common denominator if helpful.

Anita's Way to Add Fractions with Unlike Denominators

Student Materials

Suggested Use
Student discussion questions and related mathematics tasks are supplementary materials intended for optional classroom use with students. If you choose to use the mathematics task and student dialogue with your students, the discussion questions can stimulate student conversation and further exploration of the mathematics. Related mathematics tasks provide students an opportunity to engage in the mathematical practices as they connect to content that is similar to, or an extension of, that found in the mathematics task featured in the student dialogue.

Student Discussion Questions

1. In the student dialogue, Anita says why it is impractical to invent a ruler that measures the sum of two fractional distances. Explain in your own words.

2. Anita sketches a picture of the distance $\frac{1}{4}+\frac{1}{6}$, then makes 6 copies, and then doubles *that* in order to get a whole number for the total distance. In general, what is the smallest number of copies needed to ensure that the total distance is a whole number?

3. At the end of the Student Dialogue, the students rush off to lunch before answering what is $\frac{1}{4}+\frac{1}{6}$. Finish *their* reasoning to find the sum of those two fractions. Explain.

Related Mathematics Task

1. Use Anita's way of thinking to find the following sums:

   A. $\frac{1}{4}+\frac{1}{3}$
   B. $\frac{1}{4}+\frac{2}{3}$
   C. $\frac{3}{4}+\frac{2}{3}$

2. In Problem 1, you calculated $\frac{3}{4}+\frac{2}{3}$ using Anita's method. Now use equivalent fractions to calculate $\frac{3}{4}+\frac{2}{3}$. How are the two methods alike? How are they different?

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Answer Key

Suggested Use
The possible responses provided to the teacher reflection questions are meant to be used as an additional perspective after teachers have thought about those questions themselves. The possible responses to the student discussion questions and related mathematics tasks are intended to help teachers prepare for using the student materials in the classroom.

Possible Responses to Teacher Reflection Questions

1. What evidence do you see of students in the dialogue engaging the Standards for Mathematical Practice?

   Refer to the Mathematical Overview for notes related to this question.

2. Using Anita's way of adding fractions with unlike denominators, find the sum of the following. Don't bother reducing fractions or converting to mixed numbers, and don't worry about the *scale* of your sketches. Make the sketches "structurally correct" without worrying about whether their measurements are appropriate.

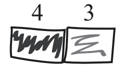
   A. $\frac{1}{4}+\frac{1}{3}$

   B. $\frac{1}{4}+\frac{2}{3}$

   C. $\frac{3}{4}+\frac{2}{3}$

   A. To compute $\frac{1}{4}+\frac{1}{3}$, picture the two distances shown below, where dark pieces represent $\frac{1}{4}$ and light pieces represent $\frac{1}{3}$.

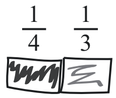
$\frac{1}{4}$ $\frac{1}{3}$



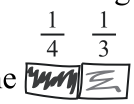
Four copies of this gives us 1 from the four dark pieces and $\frac{4}{3}$ from the four light pieces.

But that's still a bit annoying to work with, so we try to get rid of the thirds, too, by tripling that. Altogether, we wind up multiplying by 12.

Anita's Way to Add Fractions with Unlike Denominators

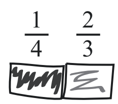
Twelve copies of the 

piece gives us this picture.

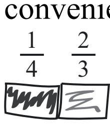
The twelve dark $\frac{1}{4}$-length pieces have a total length of 3; the twelve light $\frac{1}{3}$-length pieces have a total length of 4. So, twelve copies of the original dark/light piece have a total length of 7. One twelfth of that, $\frac{7}{12}$, is the length of one 

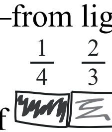
piece. That is, $\frac{1}{4}+\frac{1}{3}=\frac{7}{12}$.

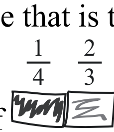
B. We can compute $\frac{1}{4}+\frac{2}{3}$ the same way. Because we don't care about scale, nothing needs to change except the labeling!



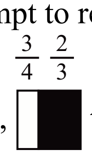
Of course, we could "get rid of the thirds" first, if we like, by tripling.

The six thirds work out nicely, but the total distance becomes $2\frac{3}{4}$, which isn't yet as convenient as it could be. So, we multiply *that* by 4, giving us 12 copies of the 

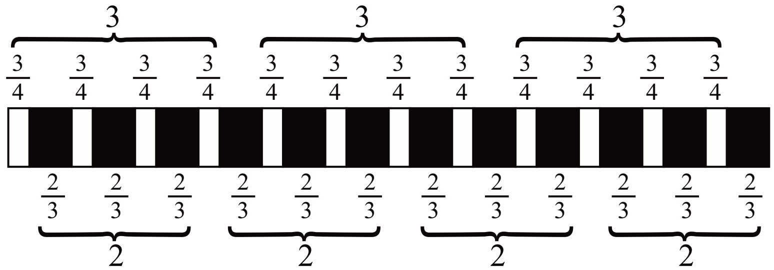
piece. The total length of all the dark/light pieces is 11—from light we get 8 and from dark we get 3. Because that is the length of 12 copies of 

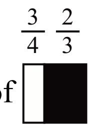
, we divide by 12 to get $\frac{11}{12}$ as the length of 

. That is, $\frac{1}{4}+\frac{2}{3}=\frac{11}{12}$.

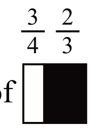
C. To calculate $\frac{3}{4}+\frac{2}{3}$, we use the same method. Again, the picture doesn't have to be even close to accurate, because the picture is not an attempt to represent the actual lengths, but the structure of the computation. In this case, 

the *drawing* is way off, the $\frac{3}{4}$ piece should be the larger of the two pieces, but it doesn't matter. Anita finesses the fact that we "can't add the unlike denominators" by combining only those fractions that have *like* denominators.



Again, 12 copies of 

are needed. Why 12? Where is that number coming from?

Because 12 copies of 

have a total length of 17, one copy has the length $\frac{17}{12}$.

That is, $\frac{3}{4}+\frac{2}{3}=\frac{17}{12}$.

Note that this problem is posed to students in the Related Mathematics Tasks.

3. Based on your experience using Anita's way to add fractions with unlike denominators in Question 2, explain, using words and algebraic language, how to use *that* method to add the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Anita would represent $\frac{a}{b}$ and $\frac{c}{d}$ as two distances, combined end to end into a single distance of $\frac{a}{b}+\frac{c}{d}$. Making b copies of this length ensures that all the $\frac{a}{b}$ pieces together produce a whole-number length. In general (unless we're lucky), we still need d copies of *that* length to ensure that the $\frac{c}{d}$ pieces combine to produce a whole-number length. By then, we have scaled the $\frac{a}{b}+\frac{c}{d}$ by bd. Represented algebraically, this is equivalent to $\left(\frac{a}{b}+\frac{c}{d}\right)bd$. Pursuing the algebra further, we get $\left(\frac{a}{b}+\frac{c}{d}\right)bd=\left(\frac{a\cancel{b}d}{\cancel{b}}+\frac{cb\cancel{d}}{\cancel{d}}\right)=ad+cb$. But this is bd times what we really wanted, so we divide that result by bd to get $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$. Note that when solving numerical problems, the final picture produced may or may not involve making bd copies. bd is the number of copies guaranteed to make the total distance a whole number; however, like in the Student Dialogue where 12 instead of 24 copies were made, you can make fewer (or more) copies of the two fractions.

Notice that in the first step—$\left(\frac{a}{b}+\frac{c}{d}\right)bd$—we are multiplying the sum we want to find by *bd*, and in the final step, we are remedying that by dividing by *bd*. That is, over the course of this whole process, we are multiplying the quantity we want to evaluate by $\frac{bd}{bd}=1$. The computation then looks like this:

$$\left(\frac{a}{b}+\frac{c}{d}\right)\frac{bd}{bd}=\left(\frac{a}{b}+\frac{c}{d}\right)bd\bullet\frac{1}{bd}=\left(ad+bc\right)\frac{1}{bd}=\frac{ad+bc}{bd}$$

4. Use the traditional algorithm (equivalent fractions) to add $\frac{a}{b}$ and $\frac{c}{d}$.

$\frac{a}{b}+\frac{c}{d}=$

$\frac{ad}{bd}+\frac{cb}{bd}=$

$\frac{ad+cb}{bd}$

5. Compare Anita's way of adding fractions with the traditional algorithm (the equivalent-fractions method).

Anita makes multiple copies of the two addends until she can assure herself that the sum produces a whole number, and then she divides that whole number by the number of copies she made. The equivalent-fractions method transforms each addend by cutting it into smaller pieces, integer multiples of which make each of the addends. That is, it transforms the two addends into like-denominator fractions so they can be added. Both methods are, in fact, identical when looked at algebraically. Anita's method essentially asks us to multiply the sum by the two denominators (or some other value that ensures a whole-number distance). When we take $\left(\frac{a}{b}+\frac{c}{d}\right)bd$ and distribute the bd, we get $\frac{ad}{bd}+\frac{cb}{bd}$, which is the same expression we get with the equivalent fractions method.

Note that in Anita's special-case example in the Student Dialogue, she *doesn't* multiply by "*bd*." The 24 is the product of the denominators. Though Anita's method generates a sum in the lowest terms, as if she were converting the fractions into equivalents with the least common denominator before adding, nothing of the kind was on Anita's mind. She is not apparently thinking about *any* common denominator, let alone the *least* common denominator. Her idea was to eliminate the fractions altogether, then add, and finally undo the change she had made. By eliminating *one* fraction (multiplying by 6) and then *evaluating the result* before tackling the remaining fraction, she got the "lowest terms" sum without even thinking about it. Had she deferred evaluation—one key element of MP 7—or started with fractions whose denominators had no common factor, and then analyzed the procedure (which Dana seems ready to do in line 25), her logic would generate the multiplication by the product of the denominators. In her special case, that would have been 24 instead of the 12 she wound up using.

The final result with both methods is the same (as expected). The only difference is the way we interpret the transformations: Anita pictures multiple copies of the intended sum, aggregating the like-denominator fractions and adding, then correcting the result by dividing at the end. The traditional method multiplies and divides at the very beginning. That is, with each fraction to be added, we multiply the numerator and denominator by the same integer. The result transforms the initial fractions to like-denominator equivalents, which can then be added.

6. How might you support students who were using Anita's way to add $\frac{1}{4}+\frac{2}{3}$ and were struggling to figure out how to add the $\frac{2}{3}$ pieces together (since they aren't unit fractions)?

   Perhaps start with the simpler case of $\frac{1}{3}$. In Anita's method, each three $\frac{1}{3}$ distances gives a total of 1. Since $\frac{2}{3}$ is twice as big as $\frac{1}{3}$, that must mean three $\frac{2}{3}$ distances will give us double the total, so 2. Alternatively, students might think about the multiplication problem $3\times\frac{2}{3}$. For some students, physical bars may be a good transition to help them organize their drawings.

Possible Responses to Student Discussion Questions

1. In the student dialogue, Anita says why it is impractical to invent a ruler that measures the sum of two fractional distances. Explain in your own words.

   Students realized that for every pair of fractions, the ruler would need labels, already marked, with just the right fractional denomination for the sum. That would require an infinite number of markings to account for all possible addends.

2. Anita sketches a picture of the distance $\frac{1}{4}+\frac{1}{6}$, then makes 6 copies, and then doubles *that* in order to get a whole number for the total distance. In general, what is the smallest number of copies needed to ensure that the total distance is a whole number?

The smallest number of copies needed is the lowest common multiple (LCM) of the denominators. Since both denominators can go into the LCM, that means the LCM times each fraction will yield a whole number. That does not mean that the LCM is necessarily needed to use Anita’s method. If you approach the scaling-up of the picture by first making enough copies to ensure a whole number for one fraction (e.g., 4 copies to turn $\frac{1}{4}$ into a whole number) and then making enough copies to ensure that the overall sum is a whole number (e.g., 2 copies to ensure that $2\frac{1}{2}$ is a whole number), then you will end up with the LCM number of copies. If, however, you start off by making copies of each fraction to ensure that each fraction yields a whole number (e.g., 4 copies for $\frac{1}{4}$ and 6 copies for $\frac{1}{6}$), you may or may not end up with the LCM number of copies.

3. At the end of the Student Dialogue, the students rush off to lunch before answering what is $\frac{1}{4}+\frac{1}{6}$. Finish *their* reasoning to find the sum of those two fractions. Explain.

Their reasoning shows that 12 copies of the distance $\frac{1}{4}+\frac{1}{6}$ gives a total distance of 5. To get the distance represented by *one* copy, we divide that 5 by 12. So $\frac{1}{4}+\frac{1}{6}$ must be $\frac{5}{12}$.

Possible Responses to Related Mathematics Task

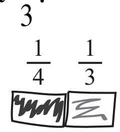
1. Use Anita’s way of thinking to find the following sums:

A. $\frac{1}{4}+\frac{1}{3}$

B. $\frac{1}{4}+\frac{2}{3}$

C. $\frac{3}{4}+\frac{2}{3}$

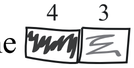
A. To compute $\frac{1}{4}+\frac{1}{3}$, picture the two distances shown below, where dark pieces represent $\frac{1}{4}$ and light pieces represent $\frac{1}{3}$.



Four copies of this gives us 1 from the four dark pieces and $\frac{4}{3}$ from the four light pieces.

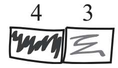
But that's still a bit annoying to work with, so we try to get rid of the thirds, too, by tripling that. Altogether, we wind up multiplying by 12.

Twelve copies of the $\frac{1}{4}$ $\frac{1}{3}$  piece gives us this picture.

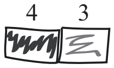
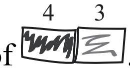
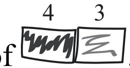
The twelve dark $\frac{1}{4}$-length pieces have a total length of 3; the twelve light $\frac{1}{3}$-length pieces have a total length of 4. So, twelve copies of the original dark/light piece have a total length of 7. One twelfth of that, $\frac{7}{12}$, is the length of one $\frac{1}{4}$ $\frac{1}{3}$  piece. That is, $\frac{1}{4}+\frac{1}{3}=\frac{7}{12}$.

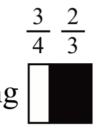
B. We can compute $\frac{1}{4}+\frac{2}{3}$ the same way. Because we don't care about scale, nothing needs to change except the labeling!

$\frac{1}{4}$ $\frac{2}{3}$

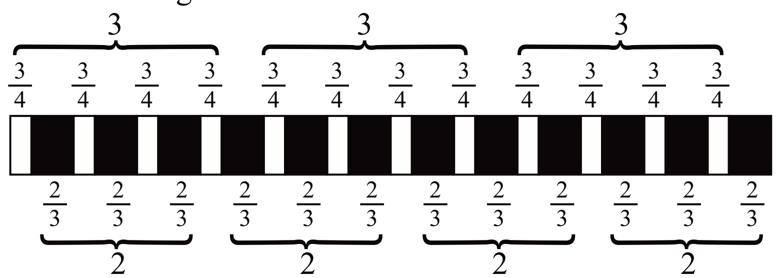


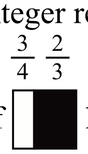
Of course, we could "get rid of the thirds" first, if we like, by tripling.

The six thirds work out nicely, but the total distance becomes $2\frac{3}{4}$, which isn't yet as convenient as it could be. So we multiply *that* by 4, giving us 12 copies of the $\frac{1}{4}$ $\frac{2}{3}$  piece. The total length of all the dark/light pieces is 11—from light we get 8 and from dark we get 3. Because that is the length of 12 copies of $\frac{1}{4}$ $\frac{2}{3}$ , we divide by 12 to get $\frac{11}{12}$ as the length of $\frac{1}{4}$ $\frac{2}{3}$ . That is, $\frac{1}{4}+\frac{2}{3}=\frac{11}{12}$.

C. To calculate $\frac{3}{4}+\frac{2}{3}$, we use the same method. Again, the picture doesn't have to be even close to accurate, because the picture is not intended to represent actual lengths but the structure of the computation. This drawing 

is *way* off—the white $\frac{3}{4}$ piece should be the larger of the two—but it doesn't matter.



As the diagram illustrates, Anita finesses the fact that we "can't add the unlike denominators" by combining only those fractions that have *like* denominators, getting integer results, adding *them*, and then correcting the outcome. Because the 12 copies of 

have a total length of 17, the length of 1 copy is $\frac{17}{12}$. That is, $\frac{3}{4}+\frac{2}{3}=\frac{17}{12}$.

2. In Problem 1, you calculated $\frac{3}{4}+\frac{2}{3}$ using Anita's method. Now use equivalent fractions to calculate $\frac{3}{4}+\frac{2}{3}$. How are the two methods alike? How are they different?

$$\frac{3}{4}+\frac{2}{3}=$$
$$\frac{3}{4}\times\frac{3}{3}+\frac{2}{3}\times\frac{4}{4}=$$
$$\frac{9}{12}+\frac{8}{12}=$$
$$\frac{17}{12}$$

In both methods, we have to multiply by 3 and 4. In the equivalent-fractions method, each fraction has its numerator and denominator multiplied by 3 or 4. Using Anita's way, we multiply the distance by 12 ($=3\times4$). Also using Anita's way, there were 9 white pieces (the $\frac{3}{4}$ distances) and 8 black pieces (the $\frac{2}{3}$ distances). These same numbers show up as the numerators of equivalent fractions of $\frac{2}{4}$ and $\frac{2}{3}$. Finally, both methods give the same final result of $\frac{17}{12}$. In the equivalent-fractions method, $\frac{17}{12}$ is the sum of two like fractions, while using Anita's way, the 17 represents the total length of 12 copies of $\frac{3}{4}+\frac{2}{3}$, so we divide 17 by 12 to get the length of 1 copy.