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HIGHER DIMENSIONAL STEINHAUS AND SLATER PROBLEMS VIA HOMOGENEOUS DYNAMICS

BY Alan HAYNES AND Jens MARKLOF

Abstract. – The three gap theorem, also known as the Steinhaus conjecture or three distance theorem, states that the gaps in the fractional parts of $\alpha, 2\alpha, \ldots, N\alpha$ take at most three distinct values. Motivated by a question of Erdős, Geelen and Simpson, we explore a higher-dimensional variant, which asks for the number of gaps between the fractional parts of a linear form. Using the ergodic properties of the diagonal action on the space of lattices, we prove that for almost all parameter values the number of distinct gaps in the higher dimensional problem is unbounded. Our results in particular improve earlier work by Boshernitzan, Dyson and Bleher et al. We furthermore discuss a close link with the Littlewood conjecture in multiplicative Diophantine approximation. Finally, we also demonstrate how our methods can be adapted to obtain similar results for gaps between return times of translations to shrinking regions on higher dimensional tori.

Résumé. – Le théorème des trois distances affirme que les intervalles entre les parties fractionnaires de $\alpha, 2\alpha, \ldots, N\alpha$ ont au plus trois longueurs distinctes. Motivés par une question de Erdős, Geelen et Simpson, nous explorons une variante en dimension supérieure, qui pose la question du nombre d'écarts entre les parties fractionnaires d'une forme linéaire. En utilisant les propriétés ergodiques de l'action diagonale sur l'espace des réseaux, nous prouvons que pour presque toutes valeurs des paramètres le nombre d'écarts distincts dans le problème en dimension supérieure est non-borné. Notre résultat améliore en particulier les travaux antérieurs de Boshernitzan, Dyson et Bleher et al. Nous discutons en outre le lien étroit avec la conjecture de Littlewood en approximation diophantienne multiplicative. Finalement, nous démontrons également comment nos méthodes peuvent être adaptées pour obtenir des résultats similaires pour les écarts entre les temps de retour de translations dans des régions contractantes sur les tores de plus grande dimension.

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1. Introduction

1.1. The Steinhaus problem

Let $\mathscr{D} \subset \mathbb{R}^d$ be a bounded convex set. For $\alpha \in \mathbb{R}^d$, define

$$S(\alpha, \mathscr{D}) = \{m \cdot \alpha \bmod 1 \mid m \in \mathbb{Z}^d \cap \mathscr{D}\} \subset \mathbb{R}/\mathbb{Z}, \tag{1.1}$$

and let $G(\alpha, \mathscr{D})$ be a number of distinct gaps between the elements of $S(\alpha, \mathscr{D})$. In other words, the set $S(\alpha, \mathscr{D})$ partitions $\mathbb{R}/\mathbb{Z}$ into intervals of $G(\alpha, \mathscr{D})$ distinct lengths.

In the classical case $d = 1$, the three gap theorem (also referred to as *Steinhaus conjecture* or *three distance theorem*) asserts that for all $\alpha \in \mathbb{R}$ and any interval $\mathscr{D}$, we have $G(\alpha, \mathscr{D}) \leq 3$. The first proofs of this remarkable fact were published in 1957 by Sós [25], in 1958 by Surányi [26], and in 1959 by Świerczkowski [27]. The theorem has been rediscovered repeatedly, and many authors have considered generalizations to various settings [1, 8, 9, 15, 18, 17, 19, 21, 22, 24, 28].

In this paper we are firstly interested in a higher dimensional version of the Steinhaus problem, which was previously studied by Geelen and Simpson [16], Fraenkel and Holzman [14], Chevallier [7], Boshernitzan [4, 5], Dyson [11], and Bleher, Homma, Ji, Roeder, and Shen [3]. For this problem our goal is twofold: to demonstrate the close connection between the multi-dimensional Steinhaus problem and the Littlewood conjecture, and to show how well known results from ergodic theory on the space of unimodular lattices in $\mathbb{R}^d$ can be used to shed new light on a question of Erdős as stated by Geelen and Simpson [16, Section 4].

Our first theorem describes the generic failure of the finite gap phenomenon in higher dimensions. Denote by $R\mathscr{D} = \{Rx \mid x \in \mathscr{D}\}$ the homothetic dilation of $\mathscr{D}$ by a factor of R. We say a sequence $0 < R_1 < R_2 < R_3 < \ldots$ is *subexponential* if

$$\lim_{i\to\infty} R_i = \infty, \qquad \lim_{i\to\infty} \frac{R_{i+1}}{R_i} = 1. \tag{1.2}$$

THEOREM 1. – *Let $d \geq 2$. There exists a set $P \subset \mathbb{R}^d$ of full Lebesgue measure, such that for every bounded convex $\mathscr{D} \subset \mathbb{R}^d$ with non-empty interior, every $\alpha \in P$, and every subexponential sequence $(R_i)_i$, we have*

$$\sup_i G(\alpha, R_i\mathscr{D}) = \infty \tag{1.3}$$

and

$$\liminf_i G(\alpha, R_i\mathscr{D}) < \infty. \tag{1.4}$$

A previous result in this direction is due to Bleher, Homma, Ji, Roeder, and Shen [3], who show in the case $d = 2$, and for a certain set of α, that

$$\sup_{R\geq 1} G(\alpha, R\mathscr{D}) = \infty, \tag{1.5}$$

where $\mathscr{D}$ is the triangle in $\mathbb{R}^2$ with vertices at $(0,0), (0,1)$, and $(1/2, 0)$. For purposes of comparison with Theorem 1, a careful computation shows that the size of the set of α to which the proof in [3] applies, has Hausdorff dimension $3/2$. (For the details of this computation, the reader may consult Lemma 6.1 of [18] and the paragraphs immediately following its proof.) Theorem 1 on the other hand admits a set of α of full Hausdorff dimension d.

In the case $d = 2$, for $\mathscr{D} = [0,1)^2$ a square, a folklore problem of Erdős (see the discussion at the end of [16]) asks whether eq. (1.5) holds whenever $1, \alpha_1, \alpha_2$ are $\mathbb{Q}$-linearly independent. The answer to this question is in fact, negative. As recorded in [3], this appears to have first been noticed in a private correspondence between Freeman Dyson and Michael Boshernitzan [4, 5, 11], who showed that (1.5) fails for badly approximable α.

We say that $\alpha \in \mathbb{R}^d$ is *badly approximable* if there is $c > 0$ such that $\|m \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} > c\|m\|^{-d}$ for all non-zero $m \in \mathbb{Z}^d$. Here $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{k \in \mathbb{Z}} \|x + k\|$ denotes the distance to the nearest integer.

THEOREM 2 (Boshernitzan and Dyson; Bleher, Homma, Ji, Roeder, and Shen).
Let $d \geq 2$. For every bounded convex $\mathscr{D} \subset \mathbb{R}^d$ with non-empty interior, and every badly approximable $\alpha \in \mathbb{R}^d$, we have

$$\sup_{R \geq 1} G(\alpha, R\mathscr{D}) < \infty. \tag{1.6}$$

We will see below that this statement is an immediate consequence of our dynamical interpretation of $G(\alpha, R\mathscr{D})$ combined with Dani's correspondence between badly approximable vectors and bounded orbits in the space of lattices.

Let us now turn to the connection between the Steinhaus problem and the Littlewood conjecture in multiplicative Diophantine approximation. The Littlewood conjecture states that for *every* $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\liminf_{n \to \infty} n \|n\alpha_1\|_{\mathbb{R}/\mathbb{Z}} \|n\alpha_2\|_{\mathbb{R}/\mathbb{Z}} = 0. \tag{1.7}$$

There is a higher dimensional version of this conjecture, that for any $d \geq 2$ and for *every* $\alpha \in \mathbb{R}^d$,

$$\liminf_{n \to \infty} n \|n\alpha_1\|_{\mathbb{R}/\mathbb{Z}} \cdots \|n\alpha_d\|_{\mathbb{R}/\mathbb{Z}} = 0. \tag{1.8}$$

Resolving the conjecture for $d = 2$ would imply the higher dimensional statement for all $d > 2$, but at present the conjecture has not been proved in full for any value of d. However, it is known that (1.8) holds for a set of α whose complement has Hausdorff dimension zero [12].

Consider the (in general non-homogeneous) dilation $\mathscr{D}_T = \{xT \mid x \in \mathscr{D}\}$ of $\mathscr{D}$, where $T = \operatorname{diag}(T_1, \ldots, T_d)$ is a diagonal matrix with expansion factors $T_i > 0$.

THEOREM 3. – *Let $d \geq 2$. Assume $\mathscr{D} \subset \mathbb{R}^d$ is bounded convex and contains the cube $[0,\epsilon)^d$ for some $\epsilon > 0$. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is such that*

$$\sup_{T_1, \ldots, T_d \geq 1} G(\alpha, \mathscr{D}_T) = \infty, \tag{1.9}$$

then

$$\liminf_{n \to \infty} n \|n\alpha_1\|_{\mathbb{R}/\mathbb{Z}} \cdots \|n\alpha_d\|_{\mathbb{R}/\mathbb{Z}} = 0. \tag{1.10}$$

Theorem 1 implies that eq. (1.9) holds for a set of α of full Lebesgue measure. We expect that there is a more concise characterisation of the set of exceptions, in analogy to the case of the Littlewood conjecture. But, unlike the Littlewood conjecture, eq. (1.9) is not true for all α. This is obvious for $\alpha \in \mathbb{Q}^d$. The following theorem gives a less trivial class of examples.