Limit of a Continuous Function

If *f(x)* is a continuous function for all real numbers, then $\lim_{x \to c} f(x) = f(c)$

Limits of Rational Functions

A. If *f(x)* is a rational function given by $f(x) = \frac{p(x)}{q(x)}$,such that $p(x)$ and $q(x)$ have no common factors, and *c* is a real number such that $q(c) = 0$, then

I. $\lim_{x \to c} f(x)$ does not exist

II. $\lim_{x \to c} f(x) = \pm\infty$ → x = c is a vertical asymptote

B. If *f(x)* is a rational function given by $f(x) = \frac{p(x)}{q(x)}$, such that reducing a common factor between $p(x)$ and $q(x)$ results in the agreeable function *k(x)*, then

$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{p(x)}{q(x)} = \lim_{x \to c} k(x) = k(c)$ → Hole at the point $(c, k(c))$

Limits of a Function as x Approaches Infinity

If *f(x)* is a rational function given by $(x) = \frac{p(x)}{q(x)}$, such that $p(x)$ and $q(x)$ are both polynomial functions, then

A. If the degree of *p(x) > q(x)*, $\lim_{x \to \infty} f(x) = \infty$

B. If the degree of *p(x) < q(x)*, $\lim_{x \to \infty} f(x) = 0$ → y = 0 is a horizontal asymptote

C. If the degree of *p(x) = q(x)*, $\lim_{x \to \infty} f(x) = c$, where *c* is the ratio of the leading coefficients.

→ y = c is a horizontal asymptote

Special Trig Limits

A. $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$ **B.** $\lim_{x \to 0} \frac{ax}{\sin ax} = 1$ **C.** $\lim_{x \to 0} \frac{1 - \cos ax}{ax} = 0$

L'Hospital's Rule

If results $\lim_{x \to c} f(x)$ or $\lim_{x \to \infty} f(x)$ results in an indeterminate form $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 0^0, 1^\infty, \infty^0\right)$, and $f(x) = \frac{p(x)}{q(x)}$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{p(x)}{q(x)} = \lim_{x \to c} \frac{p'(x)}{q'(x)} \quad \text{and} \quad \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{p(x)}{q(x)} = \lim_{x \to \infty} \frac{p'(x)}{q'(x)}$$

The Definition of Continuity

A function $f(x)$ is continuous at *c* if

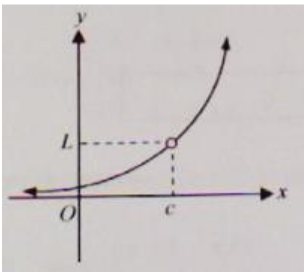
I. $\lim_{x \to c} f(x)$ exists

II. $f(c)$ exists

III. $\lim_{x \to c} f(x) = f(c)$

Types of Discontinuities

Removable Discontinuities (Holes)

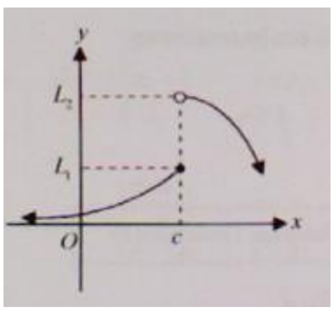


I. $\lim_{x \to c} f(x) = L$ (the limit exists)

II. $f(c)$ is undefined

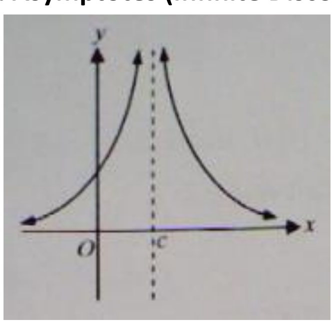
Non-Removable Discontinuities (Jumps and Asymptotes)

A. Jumps



$\lim_{x \to c} f(x) = DNE$ because $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

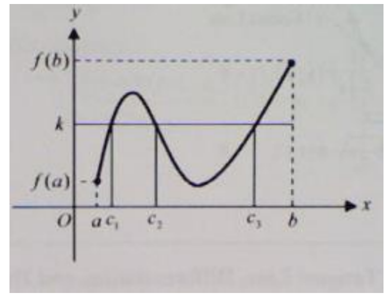
B. Asymptotes (Infinite Discontinuities)



$\lim_{x \to c} f(x) = \pm\infty$

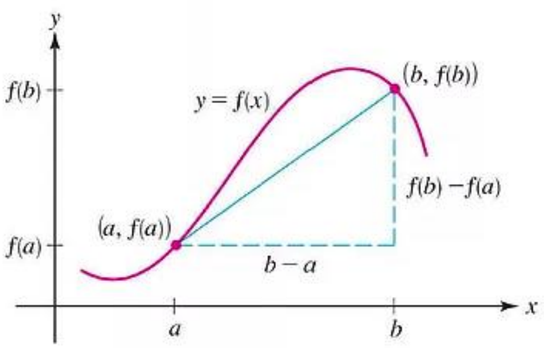
Intermediate Value Theorem

If *f* is a continuous function on the closed interval [a, b] and *k* is any number between *f(a)* and *f(b)*, then there exists at least one value of *c* on [a, b] such that *f(c) = k*. In other words, on a continuous function, if *f(a)< f(b)*, any y – value greater than *f(a)* and less than *f(b)* is guaranteed to exists on the function *f*.



Average Rate of Change

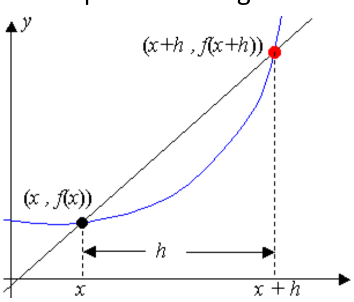
The average rate of change, *m*, of a function *f* on the interval [a, b] is given by the slope of the secant line.



$$m = \frac{f(b) - f(a)}{b - a}$$

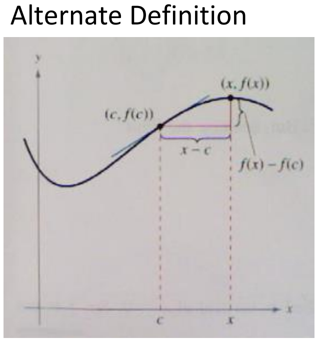
Definition of the Derivative

The derivative of the function *f*, or instantaneous rate of change, is given by converting the slope of the secant line to the slope of the tangent line by making the change is x, Δx or h, approach zero.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate Definition



$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Differentiability and Continuity Properties

A. If *f(x)* is differentiable at *x = c*, then *f(x)* is continuous at *x = c*.

B. If *f(x)* is not continuous at *x = c*, then *f(x)* is not differentiable at *x = c*.

C. The graph of *f* is continuous, but not differentiable at *x = c* if:

I. The graph has a cusp or sharp point at *x = c*

II. The graph has a vertical tangent line at *x = c*

III. The graph has an endpoint at *x = c*

Basic Derivative Rules

Given *c* is a constant,

1. Constant Rule $\frac{d}{dx}[c] = 0$
2. Constant Multiple Rule $\frac{d}{dx}[cf(x)] = cf'(x)$
3. Sum Rule $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
4. Difference Rule $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
5. Product Rule $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
6. Quotient Rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
7. Chain Rule $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$ **2.** $\frac{d}{dx}[\cos x] = -\sin x$ **3.** $\frac{d}{dx}[\tan x] = \sec^2 x$

4. $\frac{d}{dx}[\sec x] = \sec x \tan x$ **5.** $\frac{d}{dx}[\csc x] = -\csc x \cot x$ **6.** $\frac{d}{dx}[\cot x] = -\csc^2 x$

Derivatives of Inverse Trig Functions

1. $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$ **2.** $\frac{d}{dx}\left[\cos^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}}$ **3.** $\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$

4. $\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}$ **5.** $\frac{d}{dx}\left[\csc^{-1} x\right] = \frac{-1}{|x|\sqrt{x^2-1}}$ **6.** $\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}$

Derivatives of Exponential and Logarithmic Functions

1. $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$
2. $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}, (a > 0,\ a \neq 1)$
3. $\frac{d}{dx}[\ln x] = \frac{1}{x}$
4. $\frac{d}{dx}[\ln |x|] = \frac{1}{x}$
5. $\frac{d}{dx}[\log_a |x|] = \frac{1}{x \ln a}, (a > 0,\ a \neq 1)$
6. $\frac{d}{dx}[e^x] = e^x$
7. $\frac{d}{dx}[a^x] = a^x \ln a$

Explicit and Implicit Differentiation

A. Explicit Functions: Function y is written only in terms of the variable x $(y = f(x))$. Apply derivatives rules normally.

B. Implicit Differentiation: An expression representing the graph of a curve in terms of both variables x and y.

I. Differentiate both sides of the equation with respect to x. (terms with x differentiate normally, terms with y are multiplied by $\frac{dy}{dx}$ per the chain rule)

II. Group all terms with $\frac{dy}{dx}$ on one side of the equation and all other terms on the other side of the equation.

III. Factor $\frac{dy}{dx}$ and express $\frac{dy}{dx}$ in terms of x and y.

Tangent Lines and Normal Lines

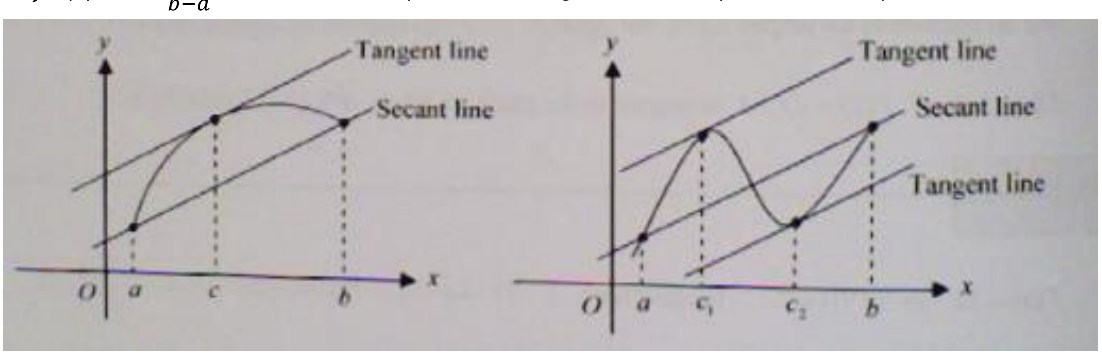
A. The equation of the tangent line at a point $(a, f(a))$: $y - f(a) = f'(a)(x - a)$

B. The equation of the normal line at a point $(a, f(a))$: $y - f(a) = -\frac{1}{f'(a)}(x - a)$

Mean Value Theorem for Derivatives

If the function *f* is continuous on the close interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number *c* between *a* and *b* such that

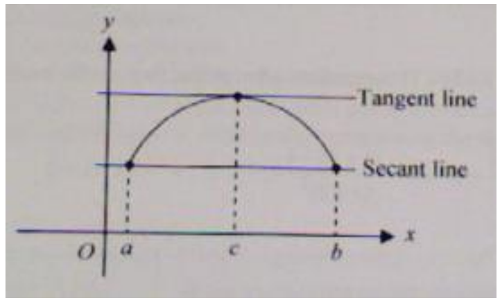
$f'(c) = \frac{f(b) - f(a)}{b - a}$ The slope of the tangent line is equal to the slope of the secant line.



Rolle's Theorem (Special Case of Mean Value Theorem)

If the function *f* is continuous on the close interval [a, b] and differentiable on the open interval (a, b), and *f(a) = f(b)*, then there exists at least one number *c* between *a* and *b* such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$$



Particle Motion

A velocity function is found by taking the derivative of position. An acceleration function is found by taking the derivative of a velocity function.

$x(t)$	Position	
$x'(t) = v(t)$	Velocity	* $\lvert v(t) \rvert = speed$
$x''(t) = v'(t) = a(t)$	Accleration	

Rules:

A. If velocity is positive, the particle is moving right or up. If velocity is negative, the particle is moving left or down.

B. If velocity and acceleration have the same sign, the particle speed is increasing. If velocity and acceleration have opposite signs, speed is decreasing.

C. If velocity is zero and the sign of velocity changes, the particle changes direction.

Related Rates

A. Identify the known variables, including their rates of change and the rate of change that is to be found. Construct an equation relating the quantities whose rates of change are known and the rate of change to be found.

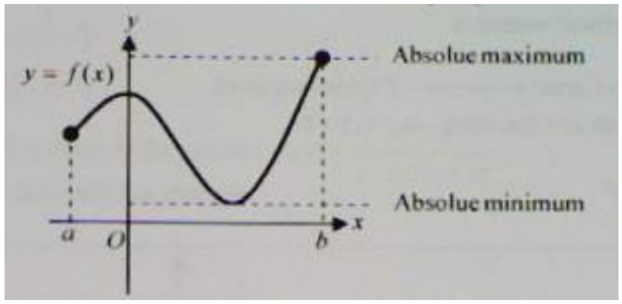
B. Implicitly differentiate both sides of the equation with respect to time. (Remember: DO NOT substitute the value of a variable that changes throughout the situation before you differentiate. If the value is constant, you can substitute it into the equation to simplify the derivative calculation).

C. Substitute the known rates of change and the known values of the variables into the equation. Then solve for the required rate of change.

*Keep in mind, the variables present can be related in different ways which often involves the use of similar geometric shapes, Pythagorean Theorem, etc.

Extrema of a Function

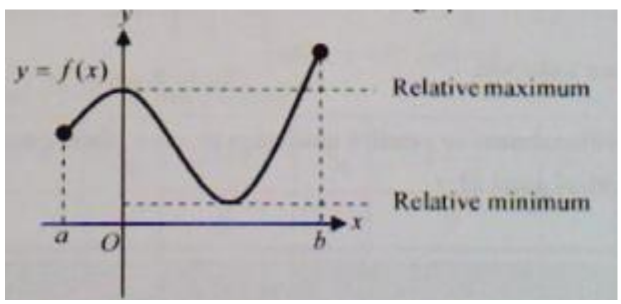
A. Absolute Extrema: An absolute maximum is the highest y – value of a function on a given interval or across the entire domain. An absolute minimum is the lowest y – value of a function on a given interval or across the entire domain.



B. Relative Extrema

I. Relative Maximum: The y-value of a function where the graph of the function changes from increasing to decreasing. Another way to define a relative maximum is the y-value where derivative of a function changes from positive to negative.

II. Relative Minimum: The y-value of a function where the graph of the function changes from decreasing to increasing. Another way to define a relative maximum is the y-value where derivative of a function changes from negative to positive.



Critical Value

When *f(c)* is defined, if $f'(c) = 0$ or f' is undefined at $x = c$, the values of the x – coordinate at those points are called critical values.

*If *f(x)* has a relative extrema at $x = c$, then *c* is a critical value of *f*.

Extreme Value Theorem

If the function *f* continuous on the closed interval [a, b], then the absolute extrema of the function *f* on the closed interval will occur at the endpoints or critical values of *f*.

*After identifying critical values, create a table with endpoints and critical values. Calculate the y – value at each of these x values to identify the extrema.

Increasing and Decreasing Functions

For a differentiable function *f*

A. If $f'(x) > 0$ in (a, b), then *f* is increasing on (a, b) → Tangent line has a positive slope

B. If $f'(x) < 0$ in (a, b), then *f* is decreasing on (a, b) → Tangent line has a negative slope

C. If $f'(x) = 0$ in (a, b), then *f* is constant on (a, b) → Tangent line has a zero slope (horizontal)

First Derivative Test

After calculating any discontinuities of a function *f* and calculating the critical values of a function *f*, create a sign chart for *f '*, reflecting the domain, discontinuities, and critical values of a function *f*.

A. If $f'(x)$ changes sign from negative to positive at $x = c$, then $f(c)$ is a relative minimum of *f.*

B. If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a relative maximum of *f.*

*If there is no sign change of $f'(x)$, there exists a shelf point

Concavity

For a differentiable function *f(x)*,

A. If $f''(x) > 0$, the graph of $f(x)$ is concave up

→ This means $f'(x)$ is increasing

B. If $f''(x) < 0$, the graph of $f(x)$ is concave down

→ This means $f'(x)$ is decreasing

Second Derivative Test

For a function *f(x)* that is continuous at *x = c*

A. If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.

B. If $f'(c) = 0$ and $f''^{(c)} < 0$, then $f(c)$ is a relative maximum.

* If $f'(c) = 0$ and $f''(c) = 0$, you must use the first derivative test to determine extrema

Point of Inflection

Let *f* be a functions whose second derivative exists on any interval. If *f* is continuous at *x = c*, *f ''(c) = 0* or *f ''(c)* is undefined, and *f ''(x)* changes sign at *x = c*, then the point $(c, f(c))$ is a point of inflection.

Optimization

Finding the largest or smallest value of a function subject to some kind of constraints.

A. Define the primary equation for the quantity to be maximized or minimized. Define a feasible domain for the variables present in the equation.

B. If necessary, define a secondary equation that relates the variables present in the primary equation. Solve this equation for one of the variables and substitute into the primary equation.

C. Once the primary equation is represented in a single variable, take the derivative of the primary equation.

D. Find the critical values using the derivative calculated.

E. The optimal solution will more than likely be found at a critical value from **D.** Keep in mind, if the critical values do not represent a minimum or a maximum, the optimal solution may be found at an endpoint of the feasible domain.

Derivative of an Inverse

If *f* and its inverse *g* are differentiable, and the point (c, f(c)) exists on the function *f* meaning the point (f(c), c) exists on the function *g*¸ then

$$\frac{d}{dx}[g(x)] = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{f'\left(f(c)\right)}$$

BC Only: Derivatives of Parametric Functions

If *f* and *g* are continuous functions of *t* on an interval, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations, providing the position in the coordinate plane, and *t* is called the parameter.

A. The slope of the curve at the point (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}\text{, provided } dx/dt \neq 0$$

B. The second derivative at the point (x, y) is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Antiderivatives

If $F'(x) = f(x)$ for all *x*, $F(x)$ is an antiderivative of *f*.

$$\int f(x) = F(x) + C$$

* The antiderivative is also called the Indefinite Integral

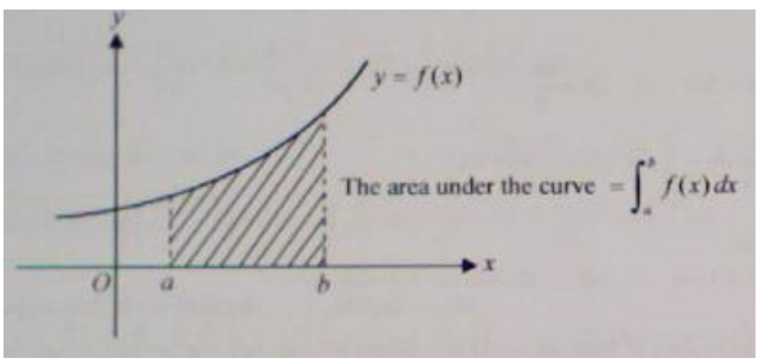
Basic Integration Rules

Let *k* be a constant.

$\int 0\,dx = C$	$\int k\,dx = kx + C$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\int \cos x\,dx = \sin x + C$
$\int \sin x\,dx = -\cos x + C$	$\int \sec^2 x\,dx = \tan x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \csc x \cot x\,dx = -\csc x + C$	

Definite Integrals (The Fundamental Theorem of Calculus)

A definite integral is an integral with upper and lower limits, *a* and *b*, respectively, that define a specific interval on the graph. A definite integral is used to find the area bounded by the curve and an axis on the specified interval (a, b).



If $F(x)$ is the antiderivative of a continuous function $f(x)$, the evaluation of the definite integral to calculate the area on the specified interval (a, b) is the First Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integration Rules for Definite Integrals

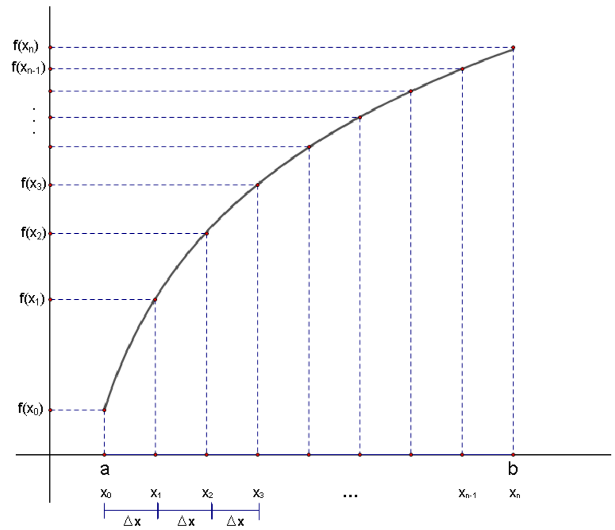
1. $\int_a^a f(x) = 0$
2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for $c \in (a,b)$ → *This means that *c* is a value of *x*, lying between *a* and *b*
3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, where f is an **even function.**
5. $\int_{-a}^a f(x)\,dx = 0$, where f is an **odd function.**

Riemann Sum (Approximations)

A Riemann Sum is the use of geometric shapes (rectangles and trapezoids) to approximate the area under a curve, therefore approximating the value of a definite integral.

If the interval [a, b] is partitioned into *n* subintervals, then each subinterval, Δx, has a width: $\Delta x = \frac{b-a}{n}$.

Therefore, you find the sum of the geometric shapes, which approximates the area by the following formulas:



A. Right Riemann Sum

$$Area \approx \Delta x\,[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})]$$

B. Left Riemann Sum

$$Area \approx \Delta x\,[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)]$$

C. Midpoint Riemann Sum

$$Area \approx \Delta x\left[f\left(x_{1/2}\right) + f\left(x_{3/2}\right) + f\left(x_{5/2}\right) + \cdots + f\left(x_{(2n-1)/2}\right)\right]$$

D. Trapezoidal Sum

$$Area \approx \frac{1}{2}\,\Delta x\,[f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Properties of Riemann Sums

A. The area under the curve is under approximated when

I.A Left Riemann sum is used on an increasing function.

II. A Right Riemann sum is used on a decreasing function.

III. A Trapezoidal sum is used on a concave down function.

B. The area under the curve is over approximated when

I.A Left Riemann sum is used on a decreasing function.

II. A Right Riemann sum is used on an increasing function.

III. A Trapezoidal sum is used on a concave up function.

Riemann Sum (Limit Definition of Area)

Let f be a continuous function on the interval [a, b]. The area of the region bounded by the graph of the function f and the x – axis (i.e. the value of the definite integral) can be found using

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

Where c_i is either the left endpoint ($c_i = a + (i-1)\Delta x$) or right endpoint ($c_i = a + i\Delta x$) and $\Delta x = (b-a)/n$.

Average Value of a Function

If a function f is continuous on the interval [a, b], the average value of that function f is given by

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Second Fundamental Theorem of Calculus

If a function f is continuous on the interval [a, b], let u represent a function of x, then

A. $\dfrac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$

B. $\dfrac{d}{dx}\left[\int_x^b f(t)dt\right] = -f(x)$

C. $\dfrac{d}{dx}\left[\int_a^{u(x)} f(t)dt\right] = f\big(u(x)\big)\cdot u'(x)$

Integration of Exponential and Logarithmic Formulas

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{u'}{u}\,du = \ln|u| + C$, where u is a differentiable function of x.
3. $\int \frac{1}{x-a}\,dx = \ln|x-a| + C$, where $a =$ constant.
4. $\int a^x\,dx = \frac{a^x}{\ln a} + C$
5. $\int e^x\,dx = \frac{e^x}{\ln e} + C = e^x + C$
6. $\int a^{u(x)}dx = \frac{a^{u(x)}}{(\ln a)u'} + C$
7. $\int e^{u(x)}dx = \frac{e^{u(x)}}{u'} + C$

Integration of Trig and Inverse Trig

1. $\int \cos x \, dx = \sin x + c$

2. $\int \sin x \, dx = -\cos x + C$

3. $\int \sec^2 x \, dx = \tan x + C$

4. $\int \csc^2 x \, dx = -\cot x + C$

5. $\int \sec x \tan x \, dx = \sec x + C$

6. $\int \csc x \cot x \, dx = -\csc x + C$

7. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

8. $\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + C$

9. $\int \frac{1}{1+x^2} dx = \arctan x + C$

10. $\int \frac{-1}{1+x^2} dx = \text{arc}\cot x + C$

11. $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \text{arc}\sec x + C$

12. $\int \frac{-1}{|x|\sqrt{x^2-1}} dx = \text{arc}\csc x + C$

If u is a differentiable function of x, then

1. $\int \frac{u'}{u} du = \ln|u| + C$

2. $\int \frac{u'}{\sqrt{1-u^2}} du = \arcsin u + C$

3. $\int \frac{u'}{1+u^2} du = \arctan u + C$.

BC Only: Integration by Parts

If *u* and *v* are differentiable functions of *x*, then

$$\int u \, dv = uv - \int v \, du$$

Tips: For your choice of the function *u*, make the selection following:

A. LIPET: Logarithmic, Inverse Trig, Polynomial, Exponential, Trig

B. LIATE: Logarithmic, Inverse Trig, Algebraic, Trig, Exponential

$*$ Comes from Integration by Parts. MEMORIZE $\int \ln x \, dx = x \ln x - x + C$

BC Only: Partial Fractions

Let *R(x)* represent a rational function of the form $R(x) = \frac{N(x)}{D(x)}$. If *D(x)* is a factorable polynomial, Partial Fractions can be used to rewrite *R(x)* as the sum or difference of simpler rational functions. Then, integration using natural log.

A. Constant Numerator

$$\int \frac{1}{x^2-5x+6}\,dx \quad \textbf{(Rule 1)}$$

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)}$$

$$\frac{A}{(x-3)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)} = \frac{(A+B)x-(2A+3B)}{(x-3)(x-2)}$$

Since

$$\frac{1}{(x-3)(x-2)} = \frac{(A+B)x-(2A+3B)}{(x-3)(x-2)},$$

$A+B=0$ and $2A+3B=-1 \Rightarrow A=1$ and $B=-1$.

$$\int \frac{1}{x^2-5x+6}\,dx = \int\left[\frac{1}{(x-3)} + \frac{-1}{(x-2)}\right]dx = \ln|x-3| - \ln|x-2| + C$$

B. Polynomial Numerator

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx \quad \textbf{(Rule 1)}$$

$$\frac{x^2+12x+12}{x^3-4x} = \frac{x^2+12x+12}{x(x+2)(x-2)} \Rightarrow \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} = \frac{A(x^2-4)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$\begin{cases} A+B+C=1 \\ 2C-2B=12 \\ -4A=12 \end{cases} \Rightarrow \begin{cases} A=-3 \\ B=-1 \\ C=5 \end{cases}$$

$$x^2+12x+12 = \frac{-3}{x} - \frac{1}{x+2} + \frac{5}{x-2}$$

$$I = \int\left(-\frac{3}{x} - \frac{1}{x+2} + \frac{5}{x-2}\right)dx = -3\ln|x| - \ln|x+2| + 5\ln|x-2| + C$$

BC Only: Improper Integrals

An improper integral is characterized by having a limits of integration that is infinite or the function *f* having an infinite discontinuity (asymptote) on the interval [a, b].

A. Infinite Upper Limit (continuous function)

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

B. Infinite Lower Limit (continuous function)

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

C. Both Infinite Limits (continuous function)

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_a^c f(x)dx + \lim_{b\to\infty} \int_c^b f(x)dx \text{, where } c \text{ is an } x \text{ value anywhere on } f.$$

D. Infinite Discontinuity (Let *x* = *k* represent an infinite discontinuity on [a, b])

$$\int_a^b f(x)dx = \lim_{x\to k^-} \int_a^k f(x)dx + \lim_{x\to k^+} \int_k^b f(x)dx$$

BC Only: Arc Length (Length of a Curve)

A. If the function $y = f(x)$is a differentiable function, then the length of the arc on [a, b] is

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

B. If the function $x = f(y)$is a differentiable function, then the length of the arc on [a, b] is

$$\int_a^b \sqrt{1 + [f'(y)]^2}\, dy$$

C. Parametric Arc Length: If a smooth curve is given by *x(t)* and *y(t)*, then the arc length over the interval $a \le t \le b$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Exponential Growth and Decay

When the rate of change of a variable *y* is directly proportional to the value of *y*, the function *y = f(x)* is said to grow/decay exponentially.

A. Differential Equation for rate of change: $\frac{dy}{dt} = ky$

B. General Solution: $y = Ce^{kt}$

I. If $k > 0$, then exponential growth occurs.

II. If $k < 0$, then exponential decay occurs.

BC Only: Logistic Growth

A population, *P*, that experiences a limit factor in the growth of the population based upon the available resources to support the population is said to experience logistic growth.

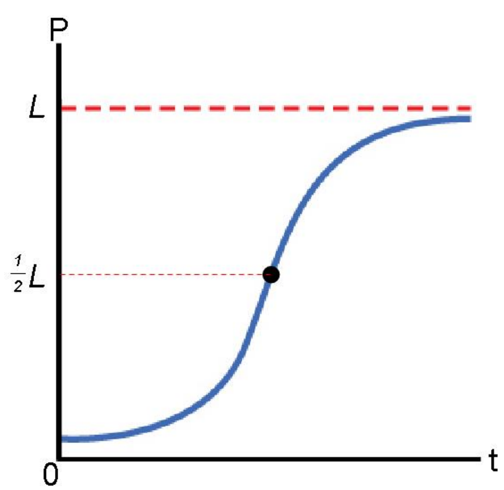
A. Differential Equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$

B. General Solution: $P(t) = \frac{L}{1+be^{-kt}}$

$P = \text{population}$ $k = \text{constant growth factor}$ $L = \text{carrying capacity}$ $t = \text{time,}$

$b = \text{constant (found with intital condition)}$

Graph



Characteristics of Logistics

I. The population is growing the fastest where $P = \frac{L}{2}$

II. The point where $P = \frac{L}{2}$ represents a point of inflection

III. $\lim_{t\to\infty} P(t) = L$

Area Between Two Curves

A. Let $y = f(x)$ and $y = g(x)$represent two functions such that $f(x) \geq g(x)$(meaning the function *f* is always above the function *g* on the graph) for every *x* on the interval [a, b].

$$\text{Area Between Curves} = \int_a^b [f(x) - g(x)]\, dx$$

B. Let $x = f(y)$ and $x = g(y)$represent two functions such that $f(y) \geq g(y)$(meaning the function *f* is always to the right of the function *g* on the graph) for every *y* on the interval [a, b].

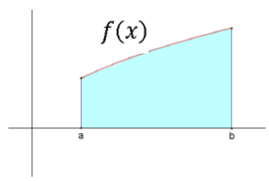
$$\text{Area Between Curves} = \int_a^b [f(y) - g(y)]\, dy$$

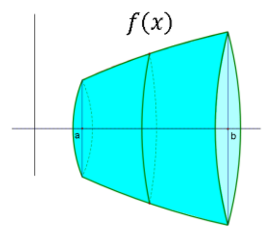
Volumes of a Solid of Revolution: Disk Method

If a defined region, bounded by a differentiable function *f*, on a graph is rotated about a line, the resulting solid is called a solid of revolution and the line is called the axis of revolution. The disk method is used when the defined region boarders the axis of revolution over the entire interval [a, b]

A. Revolving around the x – axis

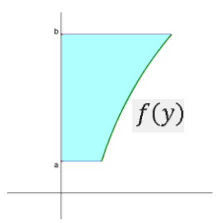
$$\text{Volume} = \pi \int_a^b \left(f(x)\right)^2 dx$$

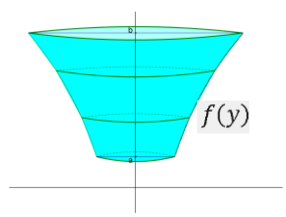




B. Revolving around the y – axis

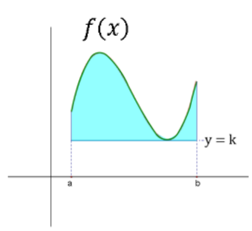
$$\text{Volume} = \pi \int_a^b \left(f(y)\right)^2 dy$$

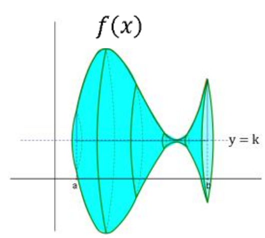




C. Revolving around a horizontal line *y* = *k*

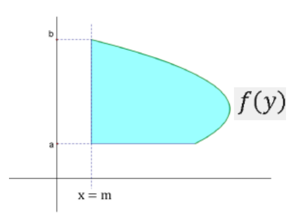
$$\text{Volume} = \pi \int_a^b (f(x) - k)^2 dx$$

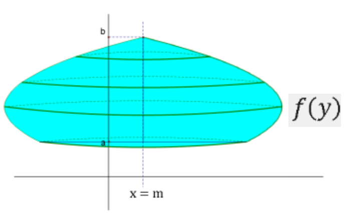




D. Revolving around a vertical line *x* = *m*

$$\text{Volume} = \pi \int_a^b (f(y) - m)^2 dy$$



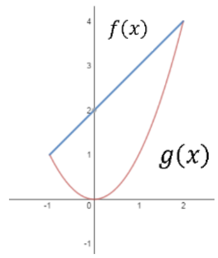


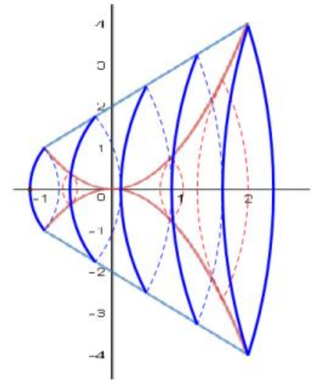
Volumes of a Solid of Revolution: Washer Method

If a defined region, bounded by a differentiable function *f*, on a graph is rotated about a line, the resulting solid is called a solid of revolution and the line is called the axis of revolution. The washer method is used when the defined region has space between the axis of revolution on the interval [a, b]

A. Revolving around the x – axis, where $f(x) \geq g(x)$(meaning the function *f* is always above the function *g* on the graph) for every *x* on the interval [a, b].

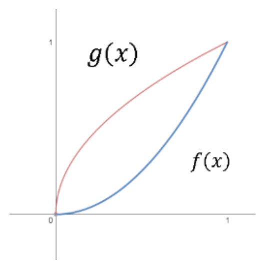
$$\text{Volume} = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

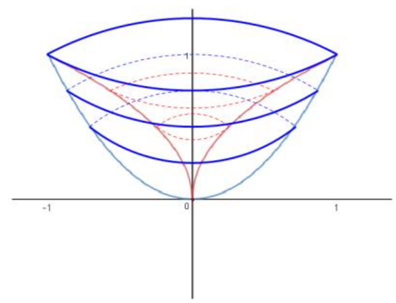




B. Revolving around the y – axis, where $f(y) \geq g(y)$(meaning the function *f* is always to the right of the function *g* on the graph)

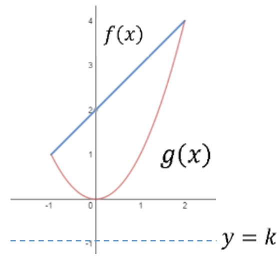
$$\text{Volume} = \pi \int_a^b ([f(y)]^2 - [g(y)]^2)dy$$

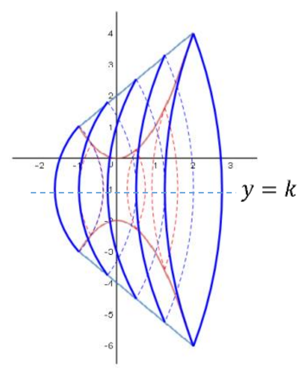




C. Revolving around a horizontal line *y = k*, where $f(x) \geq g(x)$(meaning the function *f* is always above the function *g* on the graph) for every *x* on the interval [a, b].

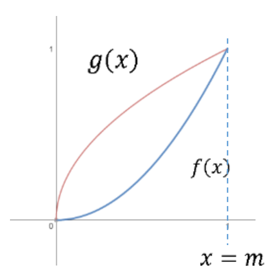
$$\text{Volume} = \pi \int_a^b ([f(x) - k]^2 - [g(x) - k]^2)dx$$

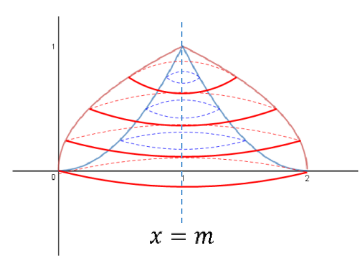




D. Revolving around a vertical line *x = m*, where $f(y) \geq g(y)$(meaning the function *f* is always to the right of the function *g* on the graph)

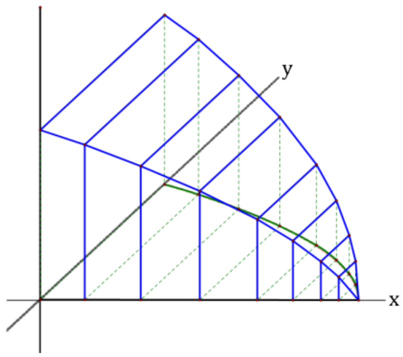
$$\text{Volume} = \pi \int_a^b ([f(y) - m]^2 - [g(y) - m]^2)dy$$

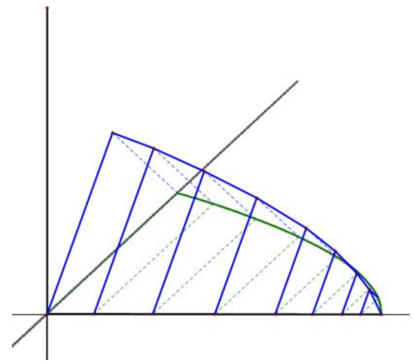


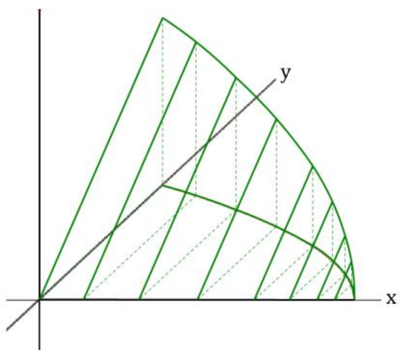


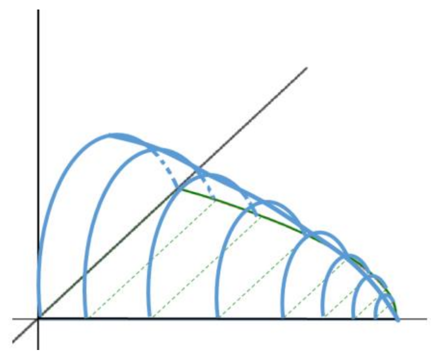
Volumes of Known Cross Sections

If a defined region, bounded by a differentiable function f, is used at the base of a solid, then the volume of the solid can be found by integrated using known area formulas.









For the cross sections perpendicular to the x – axis and a region bounded by a function f , on the interval [a, b], and the axis.

I. Cross sections are squares

$$\text{Volume} = \int_a^b [f(x)]^2 dx$$

II. Cross sections are equilateral triangles

$$\text{Volume} = \frac{\sqrt{3}}{4}\int_a^b [f(x)]^2 dx$$

III. Cross sections are isosceles right triangles with a leg in the base

$$\text{Volume} = \frac{1}{2}\int_a^b [f(x)]^2 dx$$

IV. Cross sections are isosceles right triangles with the hypotenuse in the base

$$\text{Volume} = \frac{1}{4}\int_a^b [f(x)]^2 dx$$

V. Cross sections are semicircles (with diameter in base)

$$\text{Volume} = \frac{\pi}{8}\int_a^b [f(x)]^2 dx$$

VI. Cross sections are semicircles (with radius in base)

$$\text{Volume} = \frac{\pi}{2}\int_a^b [f(x)]^2 dx$$

Differential Equations

A differential equation is an equation involving an unknown function and one or more of its derivatives

$\frac{dy}{dx} = f(x, y)$ ⟶ Usually expressed as a derivative equal to an expression in terms of *x* and/or *y*.

To solve differential equations, use the technique of separation of variables.

Given the differential equation $\frac{dy}{dx} = \frac{xy}{(x^2+1)}$

Step 1: Separate the variables, putting all *y's* on one side, with *dy* in the numerator, and all *x's* on the other side, with *dx* in the numerator.

$$\frac{1}{y} dy = \frac{x}{(x^2+1)} dx$$

Step 2: Integrate both sides of the equation.

$$\ln|y| = \frac{1}{2} \ln \sqrt{x^2+1} + C$$

Step 3: Solve the equation for y.

$$y = C\sqrt{x^2+1}$$

Given the differential equation $\frac{dy}{dx} = 2x^2$ with the initial condition $y(3) = 10$.

A. The general solution to a differential equation is left with the constant of integration, *C*, undefined.

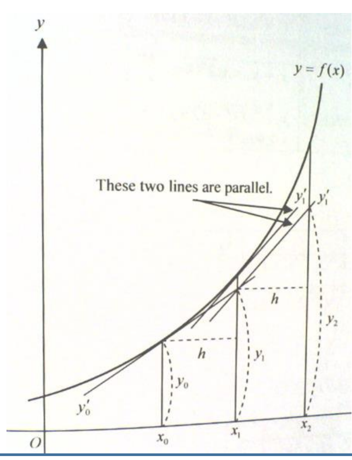
$$dy = 2x^2\, dx \quad \rightarrow \quad \int dy = \int 2x^2\, dx \quad \rightarrow \quad y = \frac{2}{3}x^3 + C$$

B. The particular solution uses the given initial condition to calculate the value of *C*.

$$10 = \frac{2}{3}(3)^3 + C \quad \rightarrow \quad C = -8 \quad \rightarrow \quad y = \frac{2}{3}x^3 - 8$$

BC Only: Euler's Method for Approximating the Solution of a Differential Equation

Euler's method uses a linear approximation with increments (steps), *h*, for approximating the solution to a given differential equation, $\frac{dy}{dx} = F(x, y)$, with a given initial value.



Process: Initial value (x_0, y_0)

$x_1 = x_0 + h \qquad y_1 = y_0 + h \cdot F(x_0, y_0)$

$x_2 = x_1 + h \qquad y_2 = y_1 + h \cdot F(x_1, y_1)$

$x_3 = x_2 + h \qquad y_3 = y_2 + h \cdot F(x_2, y_2)$

. .

. .

. .

* This process repeats until the desired y – value is given.

Slope Field

The derivative of a function gives the value of the slope of the function at each point *(x, y).* A slope field is a graphical representation of all of the possible solutions to a given differential equation. The slope field is generated by plugging in the coordinates of every point *(x, y)* into the differential equation and drawing a small segment of the tangent line at each point.

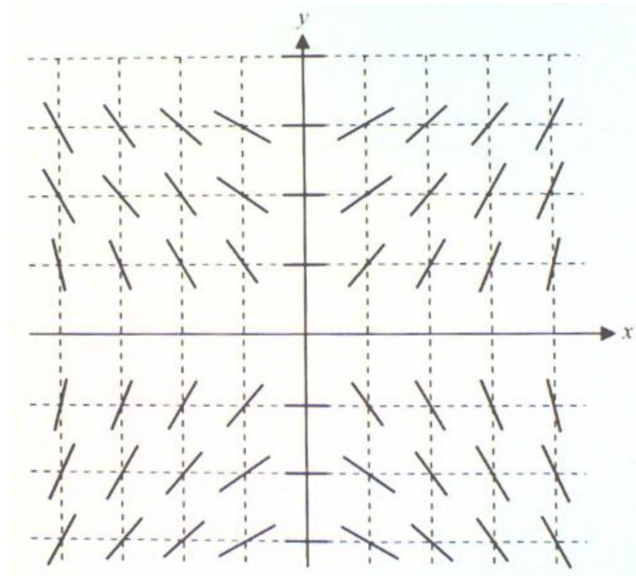
Given the differential equation $\frac{dy}{dx} = \frac{x}{y}$

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{0}{0}\ undefined$$

$$\left.\frac{dy}{dx}\right|_{(0,\pm 1)} = 0$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{1}{2}$$

*These are only three example points. You would do this for every point in the given region of the graph.



BC Only: Testing for Convergence/Divergence of a Series

• Sequence of Partial Sums	Given the series $\sum a_n = a_1 + a_2 + a_3 + \cdots$ The sequence of partial sums for the series is $S_1 = a_1 \quad S_2 = a_1 + a_2 \quad S_3 = a_1 + a_2 + a_3 \quad \ldots \quad S_n = a_1 + a_2 + a_3 + \cdots + a_n$ If $\lim_{n\to\infty} S_n = S$, then $\sum a_n$ converges to S.
• Nth Term	If the terms of a sequence do not converge to 0, then the series must diverge. **I.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. **II.** If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive.
• P – Series	The form of a p – series is $\sum \frac{1}{n^p}$ **I.** If $p > 1$, then the series converges. **II.** If $p < 1$, then the series diverges.

• Geometric Series	A geometric series is any series of the form $$\sum_{n=0}^{\infty} ar^n$$ **I.** If $\lvert r\rvert < 1$, then the series converges to $\frac{a}{1-r}$ *Series must be indexed at *n = 0* **II.** If $\lvert r\rvert > 1$, then the series diverges.
• Telescoping Series	A telescoping series is any series of the form $$\sum a_n - a_{n+1}$$ *Convergence and divergence is found using a sequence of partial sums *Partial decomposition may be used to break a single rational series into the difference of two series that form the telescoping series.
• Integral	If *f* is positive, continuous, and decreasing for $x \geq 1$, then $$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$ either both converge or both diverge.
• Alternating Series	A series, containing both positive terms, negative terms, and $a_n > 0$, of the form $$\sum_{n=1}^{\infty} (-1)^n \, a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \, a_n$$ The series' converge if both of the following conditions are met **I.** $a_{n+1} \leq a_n$ for all *n* **II.** $\lim_{n\to\infty} a_n = 0$
• Direct Comparison	When comparing two series, if $a_n \leq b_n$ for all *n*, **I.** If $\sum a_n$ diverges, then $\sum b_n$ diverges. **II.** If $\sum b_n$ converges, then $\sum a_n$ converges. *The convergence or divergence of the series chosen for comparison should be known

• Limit Comparison	If $a_n > 0$ and $b_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. *The convergence or divergence of the series chosen for comparison should be known *When choosing a series to compare to, disregard all but the highest powers (growth factor) in the numerator and denominator
• Root	Given a series $\sum a_n$ $\mathbf{I.}$ If $\lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$, then $\sum a_n$ converges. $\mathbf{II.}$ If $\lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$, then $\sum a_n$ diverges. $\mathbf{III.}$ If $\lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$, then the root test is inconclusive. *This is a test for absolute convergence
• Ratio	Given a series $\sum a_n$ $\mathbf{I.}$ If $\lim\limits_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$, then $\sum a_n$ converges. $\mathbf{II.}$ If $\lim\limits_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$, then $\sum a_n$ diverges. $\mathbf{III.}$ If $\lim\limits_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$, then the ratio test is inconclusive. *This is a test for absolute convergence

BC Only: Absolute vs Conditional Convergence

For a series, $\sum a_n$, with both positive and negative terms

$\mathbf{A.}$ If $\sum\limits_{n=1}^{\infty} \lvert a_n\rvert$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ also converges. $\sum\limits_{n=1}^{\infty} a_n$ is said to be absolutely convergent.

$\mathbf{B.}$ If $\sum\limits_{n=1}^{\infty} \lvert a_n\rvert$ diverges, but $\sum\limits_{n=1}^{\infty} a_n$ converges, $\sum\limits_{n=1}^{\infty} a_n$ is said to be conditionally convergent.

BC Only: Alternating Series Remainder Theorem

Given $\sum a_n$ is a convergent alternating series, the error associated with approximating the sum of the series by the first *n* terms is less than or equal to the first omitted term.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = S \approx S_n = a_1 - a_2 + \cdots + (-1)^{n+1} a_n \qquad \text{Error} = \lvert S - S_n\rvert \le \lvert a_{n+1}\rvert$$

BC Only: Power Series

A. Power Series Structure and Characteristics

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \longrightarrow \text{power series centered at } x = 0$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots \longrightarrow \text{power series centered at } x = c$$

A function *f* can be represented by a power series , where the power series converges to the function in one of three ways:

I. The power series only converges at the center *x = c.*

II. The power series converges for all real values of *x*.

III. The power series converges for some interval of values such that $|x-c| < R$, where *R* is the radius of convergence of the power series.

B. Interval of Convergence: Find this by applying the Ratio to the given series.

I. If $R = 0$, then the series converges only at *x = c.*

II. If $R = \infty$, then the series converges for all real values of *x*.

III. If the Ratio Test results in an expression of the form $|x-c| < R$, then the interval of convergence is of the form $c - R < x < c + R$.

*The convergence at the endpoints of the interval of convergence should be tested separately.

BC Only: Taylor and Maclaurin Series (specific power series)

If a function of *f* has derivatives of all orders at *x = c*, then the series is called a Taylor Series for *f* centered at *c*. A Taylor series centered at 0 is also known as a Maclaurin Series.

A. Maclaurin Series

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

B. Taylor Series

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + f'''(c)\frac{(x-c)^3}{3!} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(c)\frac{(x-c)^n}{n!}$$

BC Only: Common Series to MEMORIZE

Series	Interval of Convergence
A. $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \displaystyle\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
B. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$-\infty < x < \infty$
C. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \ldots + \dfrac{x^{2n}}{(2n)!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
D. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots + \dfrac{x^{2n+1}}{(2n+1)!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
E. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots + \dfrac{(-1)^n x^n}{n} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^n}{n}$	$-1 < x \leq 1$
F. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots + \dfrac{(-1)^n x^{2n+1}}{2n+1} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$-\infty < x < \infty$

BC Only: Lagrange Remainder of a Taylor Polynomial

When approximating a function $f(x)$ using an nth degree Taylor polynomial, $P_n(x)$, the associated error, $R_n(x)$, is bounded by

$$|R_n(x)| = |f(x) - P_n(x)| \leq \left| \frac{(x-c)^{n+1}}{(n+1)!} \cdot \max f^{(n+1)}(z) \right| \quad \text{where } c \leq z \leq x$$

BC Only: Polar Coordinates

A. The polar coordinates $(r, \theta)$of a point are related to the rectangular coordinates (x, y) as follows

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \tan\theta = \frac{x}{y}$$

B. If f is a differentiable function of θ (smooth curve), then the slope of the line tangent to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{f'(\theta)\sin\theta \ + \ f(\theta)\cos\theta}{f'(\theta)\cos\theta \ - \ f(\theta)\sin\theta}$$

C. If $r = f(\theta)$ is a smooth curve on the interval $[\alpha, \beta]$, where α and β are radial lines, then the area enclosed by the graph is

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$