AP Calculus AB

Notes 2018-2019

Arbor View HS

Name:____________________________

Period:___

Table of Contents

Notes #1-1

Date: _______

1.2 Finding Limits Graphically and Numerically (48)

Letter of recommendation: participate in class, stand out – in a good way!

$\lim\limits_{x \to c} f(x) = L$ * The limit (L) of $f(x)$ as x approaches c.

1) A penny: .01 =

2) Go ½ the distance each time over 10 ft.

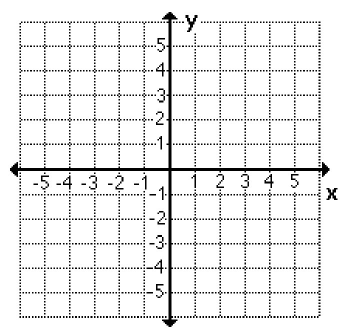
$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = ?$$

x	1.75	1.9	1.999	2	2.001	2.1	2.25
y				?			

Ex.1 Find $\lim\limits_{x \to 0} \frac{1 - \cos x}{x}$ numerically and graphically.

"hiccup" function

Ex.2 $\lim\limits_{x \to -2} f(x) = \begin{cases} 3 & x = -2 \\ -1 & x \neq -2 \end{cases}$?



* Existence at the point is irrelevant.

<u>Limits that fail to exist:</u>

1. $f(x)$ approaches different values from the left and right sides of c.

$\lim\limits_{x \to c^-} f(x) \neq \lim\limits_{x \to c^+} f(x)$

from the left from the right

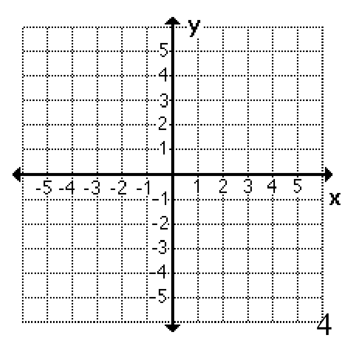
Diving board $f(x)$

Exists everywhere else.

$\lim\limits_{x \to 0} f(x) =$

$\lim\limits_{x \to 3} f(x) =$

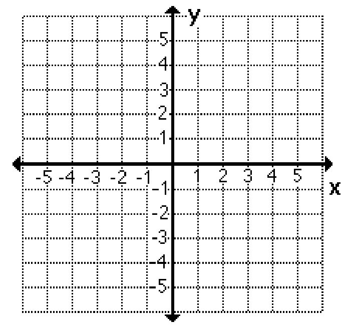
Ex.3 $\lim\limits_{x \to 1} f(x)$ if $f(x) = \begin{cases} 2 & x \geq 1 \\ 0 & x < 1 \end{cases}$



2. Unbounded Behavior

Ex.4 $\lim_{x \to 0} \frac{1}{|x|} =$

x	y



3. Oscillating Behavior

Ex.5 $\lim_{x \to 0} \tan\left(\frac{1}{x}\right) =$

<u>The Formal Definition of a Limit (52)</u>

We start with the informal: L is the limit of $f(x)$ as x approaches c if <u>$f(x)$ gets close to L</u> as <u>x gets close to c</u>.

$$\lim_{x \to c} f(x) = \mathrm{L}$$

The description is informal because the underlined phrases do not have precise meaning.

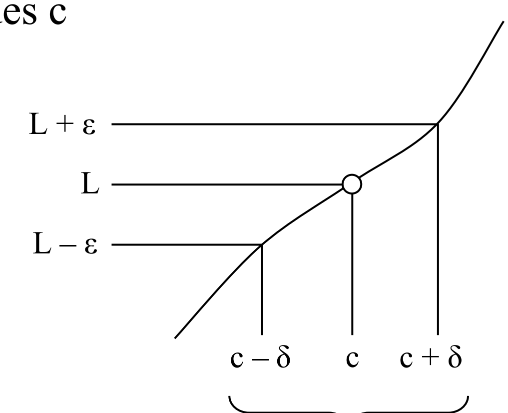
Lowercase Greek:
δ – delta (x)

ε – epsilon (y)

Alphabetical order.

ε-δ Definition of a Limit: for each ε > 0 there exists a δ > 0 such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

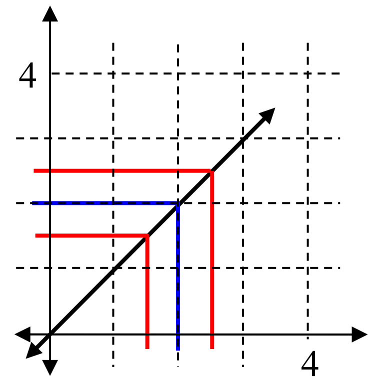
excludes c



$$0 < |x - c| < \delta$$

$y = x$

So for $f(x) = x$, $\lim\limits_{x \to 2} x$ if $\varepsilon < .5$, then $\delta < .5$ and so on….



Ex.6 Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.

a) $\lim\limits_{x \to 3} x$

b) $\lim\limits_{x \to 4} 2x - 6$

Ex.7 Find the limit L. Then use the ε-δ Definition to prove that the limit is L.

a) $\lim\limits_{x \to 1} 5x$

b) $\lim\limits_{x \to 4} 2x + 1$

Summary:

Ex.8 The sum of the prime divisors of 2010 is:

a) 211 b) 208 c) 76 d) 77 e) 78

Notes #1-2
Date:_______

1.3 Evaluating Limits Analytically (57)

What are we class?

Techniques of Finding Limits

1. Direct Substitution

See continuity in 1.4.

$$\lim_{x \to c} f(x) = f(c) \text{ if } f(x) \text{ is continuous at c.}$$

Ex.1 Find the limit:

a) $\lim_{x \to -2} 2x+7$

b) $\lim_{x \to 5} \sqrt[3]{x+22}$

c) $\lim_{x \to 3} \sin\frac{\pi x}{2}$

d) $\lim_{x \to 5} 4$

Composite functions see (59) Ex.4

$$\lim_{x\to c} g(x) = \text{L} \ \& \ \lim_{x\to L} f(x) = f(\text{L}), \text{ then } \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(\text{L})$$

Properties of Limits (57)

Ex.2-5 $\lim_{x \to 3} f(x) = 5$, $\lim_{x \to 8} f(x) = 4$ and $\lim_{x \to 3} g(x) = 8$

a. $\lim_{x \to 3} [f(x) + g(x)] =$

b. $\lim_{x \to 3} [f(x) - g(x)] =$

c. $\lim_{x \to 3} [f(x)]^2 =$

d. $\lim_{x \to 3} \frac{f(x)}{g(x)} =$

e. $\lim_{x \to 3} 6f(x) =$

f. $\lim_{x \to 3} \sqrt{g(x)} =$

g. $\lim_{x \to 3} f(g(x)) =$

h. $\lim_{x \to \pi} \sec x$

2. Dividing Out Technique $\left(\frac{0}{0}\right)$ = indeterminate form

Know the sum and difference of cubes!

$x^3 + 125$

Ex.6 Find the $\lim_{x \to c} \frac{8x^3 - 27}{2x - 3}$ for:

a) c = 0 b) c = 2

c) c = 1.5

Ex.7 Find the limit:

$64x^6 - 1$

a) $\lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5}$ b) $\lim_{\Delta x \to 0} \frac{3(x + \Delta x) - 3x}{\Delta x}$

3. Rationalizing Technique (the numerator)

Ex.8 Find the limit:

Don't multiply out the denominator.

a) $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}$ b) $\lim_{z \to 0} \frac{\sqrt{7 - z} - \sqrt{7}}{z}$

Memorize these!

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Ex.9 Find the limit using the Special Trig Limits (64)

a) $\lim_{x \to 0} \frac{1 - \cos^2 x}{3x}$ b) $\lim_{x \to 0} \frac{x}{\sin(5x)}$

Complex fraction

Ex.10 Evaluate the limit, if it exists: $\lim_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$.

a) $\frac{1}{4}$ b) $-\frac{1}{4}$ c) 1 d) -1 e) dne

$\tan\theta =$

Ex.11 Evaluate the limit, if it exists: $\lim_{x\to 1}\dfrac{\tan^{-1}x}{\sin^{-1}x+1}$.

a) 0 b) $\frac{1}{4}$ c) $\frac{1}{2}$ d) $\frac{\pi}{2}$ e) $\frac{\pi}{2\pi+4}$

<u>The Squeeze Theorem</u> aka Sandwich Theorem (63)

If $h(x) \le f(x) \le g(x)$ & $\lim\limits_{x\to c} h(x) = \mathrm{L}$ & $\lim\limits_{x\to c} g(x) = \mathrm{L}$

then $\lim\limits_{x\to c} f(x) = \mathrm{L}$.

Ex.12 Use the graph of $f(x) = x^2\sin\frac{1}{x}$ and the Squeeze Theorem to find $\lim\limits_{x\to 0} f(x)$ if $-x^2 \le f(x) \le x^2$.

2008 #5

Ex.13 $\lim_{x\to 0}\dfrac{5x^4+8x^2}{3x^4-16x^2}$

Summary:

(A) $-\frac{1}{2}$ (B) 0 (C) 1 (D) $\frac{5}{3}$ (E) dne

Notes #1-3
Date: _______

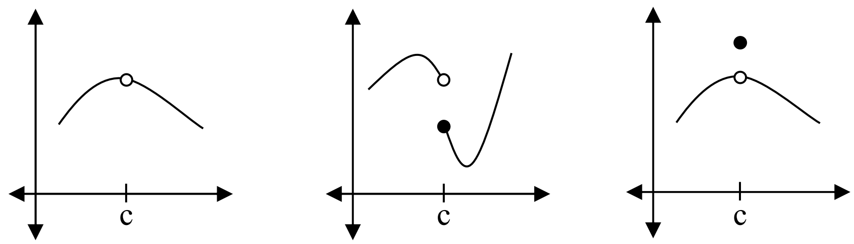
1.4 Continuity and One-Sided Limits (68)

Can be traced without lifting your pencil.

<u>A function is continuous (uninterrupted) at c if</u>:

1. *f*(c) is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

A discontinuity is <u>removable</u> if *f* can be made continuous by redefining *f*(c), i.e. the value at that single point.



Defined at c? ______ ______ ______

Limit at c? ______ ______ ______

Removable? ______ ______ ______

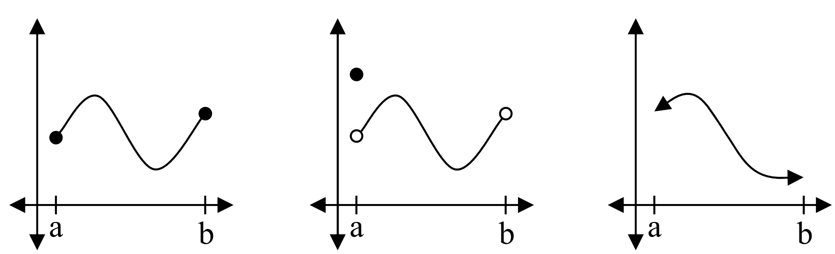
<u>A function is continuous:</u>

A) on an open interval if it continuous at each point in the interval.

B) everywhere if it is continuous on (-∞, ∞).

C) on a closed interval [a, b] if it is continuous on (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

+ from the right
– from the left

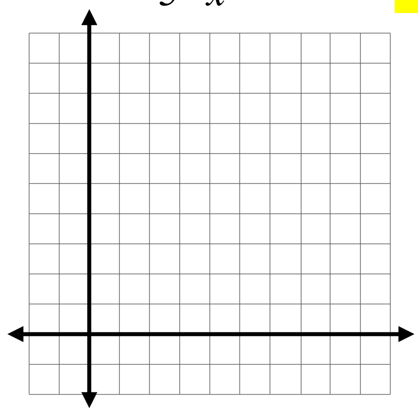


Match (A-C) _____ _____ _____

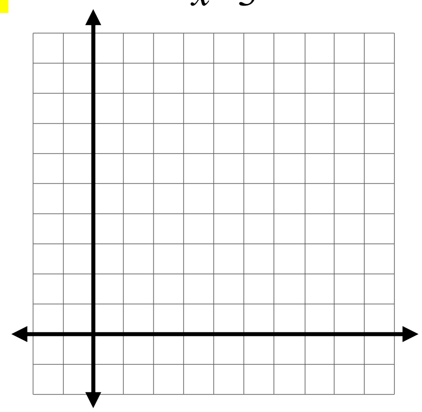
Discontinuities
Removable
Jump
Infinite

Ex.1 Discuss the continuity of the function:

a) $f(x)=\dfrac{1}{5-x}+2$



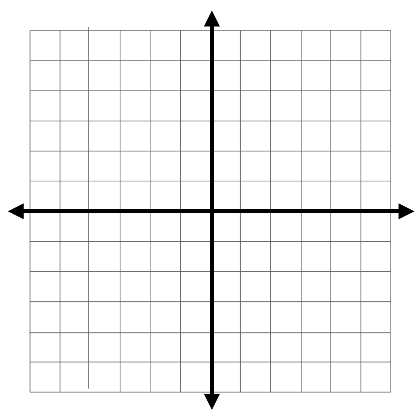
b) $f(x)=\dfrac{x^2-5x+6}{x-3}$



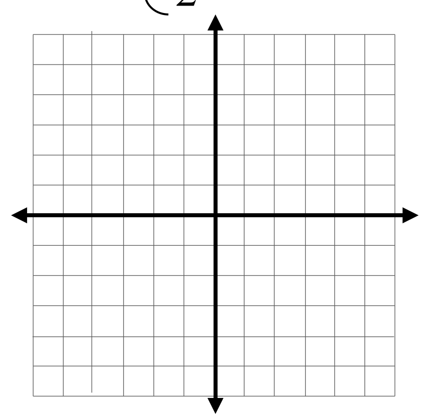
The Greatest Integer $f(x)$ is a Step function or staircase.

Graphing calc under catalog int(

c) $f(x)=[\![x]\!]$



d) $f(x)=\begin{cases} 2x-1 & x<2 \\ 5 & x=2 \\ \frac{1}{2}x+2 & x>2 \end{cases}$

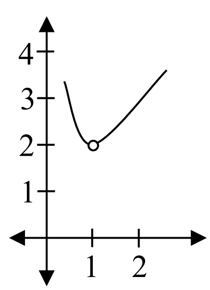


Ex.2 Find the limits from the graph of $f(x)$:

a) $\lim\limits_{x\to 1^-} f(x)$

b) $\lim\limits_{x\to 1^+} f(x)$

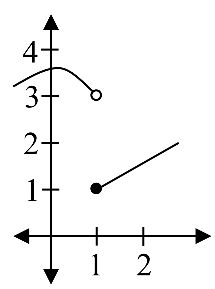
c) $\lim\limits_{x\to 1} f(x)$



d) $\lim\limits_{x\to 1^-} f(x)$

e) $\lim\limits_{x\to 1^+} f(x)$

f) $\lim\limits_{x\to 1} f(x)$



Ex.3 Find the limit of:

a) $\lim\limits_{x \to -5^+} \sqrt{x+5} =$ b) $\lim\limits_{x \to 3^-} \sqrt{2x-6} =$ c) $\lim\limits_{x \to 1^-} \dfrac{|x|}{x} =$

See Properties of Limits (57).

If f & g are continuous at c, then…

1. af
2. f +/– g
3. $f \cdot g$
4. $\dfrac{f}{g}$, g(c) $\neq 0$
5. f(g(x)), restrictions

are continuous at c.

(73)

Some functions are continuous at every point in their domain:

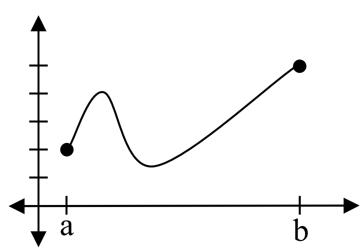
1. Polynomial
2. Rational
3. Radical
4. Trigonometric

Given #2 above and #1 & #4
$y = 2x^5 + 5 - \cos x$ is continuous.

Existence theorems tell you something exists, but do not give you a method for finding them.

The Intermediate Value Theorem (often 1 point on AP exam):
Suppose f(x) is continuous on [a, b]. If k is any number between f(a) and f(b), then there is at least one number c in [a, b] such that f(c) = k.

A person was 2 feet tall $f(a)$ when she was 1 year old (a) and she was 5 feet tall $f(b)$ when she was 14 years old (b). At some point (c) she must have been 4 feet tall (k) because human growth is continuous over an interval of time.



If f(a) = 2 and f(b) = 5, then there is a f(d) = 3 (three in this case) and a f(c) = 4 somewhere on [a, d].

Ex.4 Explain why the function has a zero in the given interval: $f(x) = x^3 + x^2 - 1$; [0, 1].

Ex.5 Use the Intermediate Value Theorem and the bisection method to estimate the zero of $f(x) = e^x - 3x$; [0, 1].

Ex.6 Find value(s) for a, so that the function

$$f(x)=\begin{cases} x^2-a^2x & x<2 \\ 4-2x^2 & x\geq 2 \end{cases}$$ is continuous.

Ex.7 Find values for a and b, so that the function

$$f(x)=\begin{cases} 5, & x\leq -4 \\ 2ax+b, & -4<x<3 \\ bx+5 & x\geq 3 \end{cases}$$ is continuous.

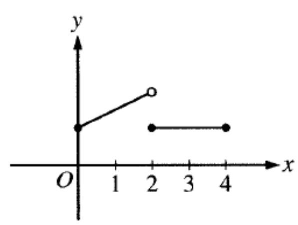
2008 #77

Ex.8 The figure shows the graph of a function f with domain $0 \leq x \leq 4$. Which of the limits exist?

I. $\lim\limits_{x\to 2^-} f(x)$

II. $\lim\limits_{x\to 2^+} f(x)$

III. $\lim\limits_{x\to 2} f(x)$



Graph of f

Summary:

(A) I (B) II (C) I & II (D)I & III (E) I, II, & III

Notes #1-4
Date:_______

1.5 Infinite Limits (80)

Infinite Limit: a limit in which $f(x)$ increases or decreases without bound as x approaches c.

It shows that $f(x)$ is unbounded, so the limit dne. The = sign is misleading.

The notation $\lim_{x\to 0}\left|\frac{1}{x}\right|=\infty$ does not mean that the limit exists!

Ex.1 Find the real number c that is not in the domain. Determine whether $f(x)$ approaches $-\infty$ or ∞ as x approaches c from the left and from the right.

÷ by 0

$\sqrt{-}$

a) $f(x)=\dfrac{3}{x-4}$

c =

$\lim_{x\to c^-} f(x)=$

$\lim_{x\to c^+} f(x)=$

b) $f(x)=\dfrac{1}{2-x}$

c =

$\lim_{x\to c^-} f(x)=$

$\lim_{x\to c^+} f(x)=$

c) $f(x)=\dfrac{2}{(x-3)^2}$

c =

$\lim_{x\to c^-} f(x)=$

$\lim_{x\to c^+} f(x)=$

d) $f(x)=\dfrac{-3}{(x+2)^2}$

c =

$\lim_{x\to c^-} f(x)=$

$\lim_{x\to c^+} f(x)=$

Vertical Asymptote: x = c, if $f(x)$ approaches $-\infty$ or ∞ as x approaches c from the left or the right.

<u>Degree</u>

N=D $y = \frac{A}{B}$

N < D y = 0

N > D no H.A.

The AP Calc test has 4 parts:

Multiple choice:
30 Qs in 60 minutes no calculator.

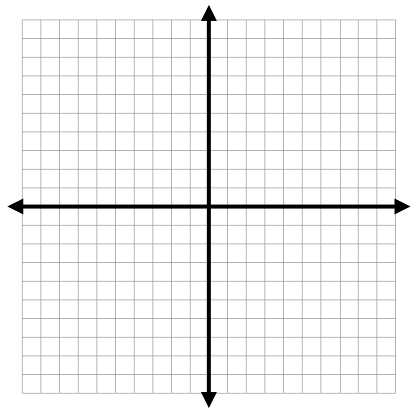
15 Qs in 45 minutes – some require a calculator.

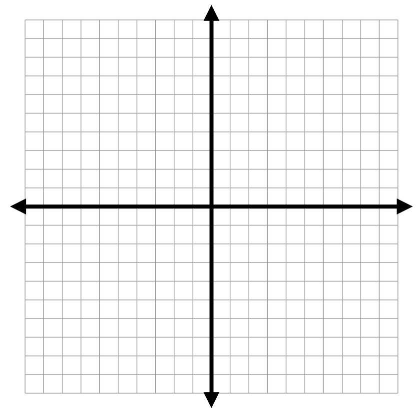
Free response:
2 Qs in 30 minutes some parts of Qs may require a calculator.

4 Qs in 60 minutes no calculator.

Ex.2 Find the vertical asymptotes and removable discontinuities of the functions:

a) $f(x)=\dfrac{3x^2+1}{x^2-9}$ b) $f(x)=\dfrac{x^2+2x-8}{x^2-4}$





<u>Past AP Type Problems Covered by this Chapter</u>:

Ex.3 Evaluate the limit: $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{2-x}$.

a) 5 b) 3 c) -3 d) -5 e) dne

Ex.4 Evaluate the limit, if it exists: $\lim\limits_{x\to 9}\dfrac{\sqrt{x-5}-2}{x-9}$.

a) $\dfrac{1}{4}$ b) $-\dfrac{1}{4}$ c) 1 d) 0 e) dne

Ex.5 How many vertical asymptotes exist for $f(x)=\dfrac{1}{2\sin^2 x-\sin x-1}$ in (0, 2π)?

a) 0 b) 1 c) 2 d) 3 e) 4

IVT

Ex.6 If $p(x)$ is a continuous function on [1, 3] with $p(1) \leq \text{K} \leq p(3)$ and c is in the closed interval [1, 3], then what must be true?

Ex.7 Identify the vertical asymptote(s) for $f(x) = \dfrac{x^2+3x-4}{x^2+x-2}$.

a) $x = -2, x = 1$ b) $x = -2$ c) $x = 1$

d) $y = -2, y = 1$ e) $y = -2$

GC

Ex.8 Find the limit: $\lim_{x\to 0} x \cdot \left(e^x + \dfrac{1}{x}\right)$.

a) 0 b) 1 c) 2 d) dne e) none

Ex.9 Is the function continuous at $x = 1$? Why or why not?

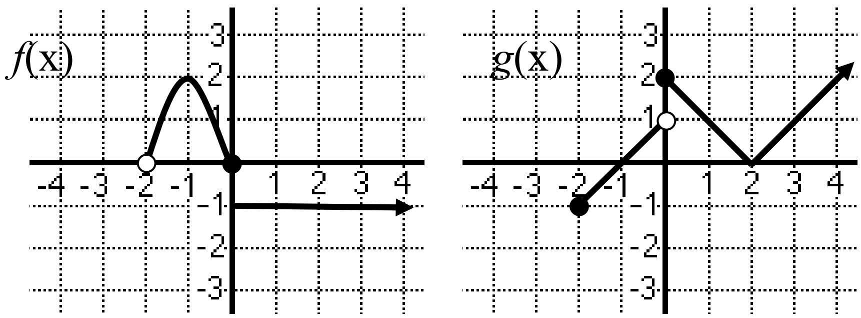
$$f(x) = \begin{cases} x^2 & \text{for } x \leq 1 \\ 2-x & \text{for } x > 1 \end{cases}$$

Ex.10 $f(x) = \begin{cases} \sin(2x), & x \leq \pi \\ 2x+k, & x > \pi \end{cases}$ what value of k will make this function continuous?

a) -2π b) $-\pi$ c) 0 d) π e) 2π

Free Response

Ex.10 Calculators may not be used. Use the graphs of $f(x)$ and $g(x)$ given below.



a) Is $f[g(x)]$ continuous at $x = 0$? Explain.

b) Is $g[f(x)]$ continuous at $x = 0$? Explain.

c) What is the $\lim_{x\to 1} f[g(x)]$? Explain.

d) If $h(x)=\begin{cases} f(x)+g(x), & -2 \le x < 0 \\ k\cdot g(x)f(x), & x \ge 0 \end{cases}$, what is k so that h(x) is continuous at $x = 0$?

Summary:

Notes #1-5
Date:______

3.5 Limits at Infinity (192)

End Behavior (left & right)

Horizontal Asymptote: $y = L$ if $\lim_{x \to -\infty} f(x) = L$ or

$\lim_{x \to \infty} f(x) = L$

Note: from this definition, the graph of a function of x can have at most two horizontal asymptotes – one to the right and one to the left.

$\frac{\infty}{\infty}$ indeterminate form ($\div$ numerator & denominator by the highest power of x in the denominator).

n: degree of numerator

d: degree of denominator

Guidelines for Finding Limits at Infinity of Rational Functions

- If n < d: the limit is 0.
- If n = d: limit $\frac{a_n}{b_d}$ (ratio of the leading coefficients).
- If n > d: the limit does not exist, we may write $\lim_{x \to \pm\infty} f(x) = \pm\infty$ to show that $f(x)$ increases or decreases without bound.

Ex.1 Find the limits: $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$

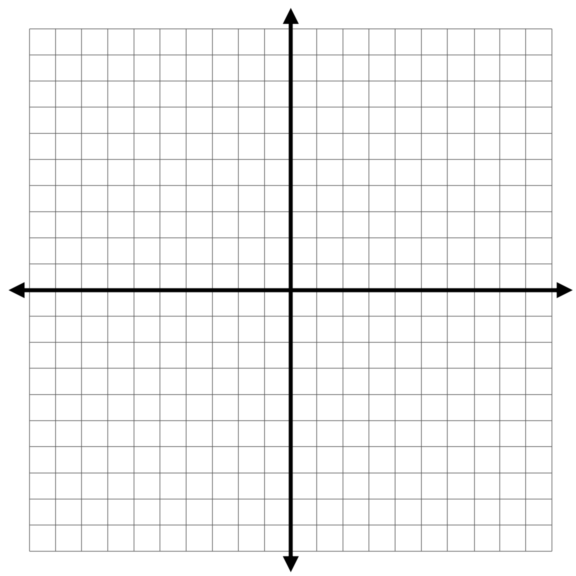
a) $f(x) = \dfrac{x-1}{x^3 + 3x^2}$ b) $g(x) = \dfrac{2x^2 + 6}{x^2 - 4}$

Ex.2 Find the limits: $\lim_{x \to -\infty} g(x)$ and $\lim_{x \to \infty} g(x)$

a) $h(x) = \dfrac{6x-1}{\sqrt{3x^2-5}}$ b) $g(x) = \dfrac{x^2-x}{x+1}$

Ex.3 Sketch the graph of the equation. Look for intercepts, symmetry, and asymptotes.

$g(x) = \dfrac{3x^2}{x^2-16}$



Ex.4 $\lim_{x \to \infty} \dfrac{(2x-1)(3-x)}{(x-1)(x+3)}$ is

2008 #77 (A) -3 (B) -2 (C) 2 (D) 3 (E) dne

Ex.5 $\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$

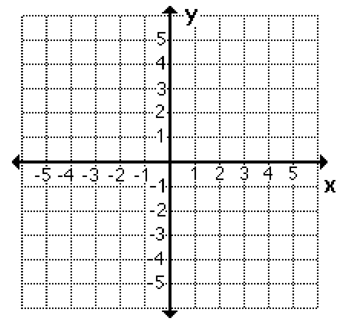
Summary:

Notes #1-6
Date:______

2.1 The Derivative and the Tangent Line Problem (94)

Tangent lines to a circle are a special case, because the radius is perpendicular to the tangent line.

Ex.1 Find the equation of a circle that has (0, 0) as its center and passes through (1, 2). Graph.

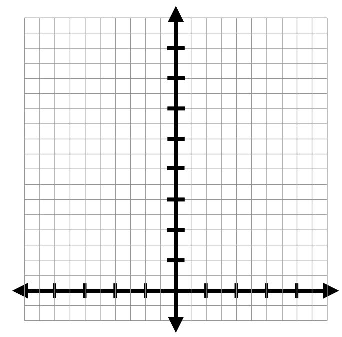


Ex.2 Find an equation of the tangent line to the circle that passes through (1, 2).

Leave your answer in point-slope form! Converting to slope-intercept form just creates more chances to make errors.

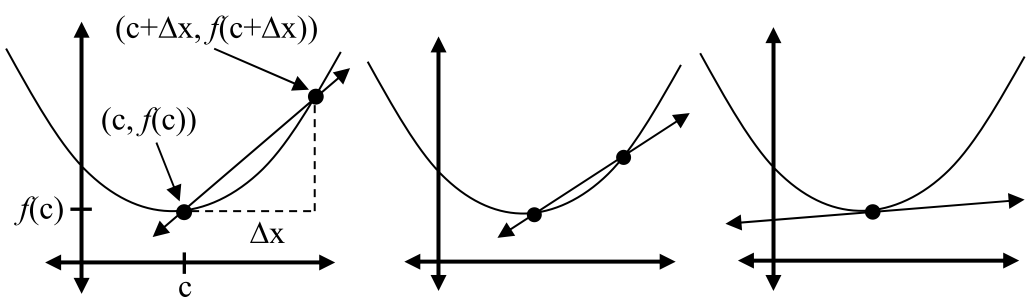
One way to find the tangent to a general curve is to use a secant line to approximate the slope.

Ex.3 Approximate the slope of $y = x^2 + 3$ at (1, 4).



Secant comes from the Latin secare, meaning to cut, and is not a reference to the trig function.

Calculus in Motion!



<u>Difference Quotient</u>
The quotient of two differences.

$$m_{\sec}: \frac{\Delta y}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{(c+\Delta x) - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x} \qquad m_{\tan}: \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

The smaller Δx the better.

Ex.4 Find the <u>slope</u> of the tangent line to the graph of $f(x) = 3x + 1$ at (3, 10).

<u>Stuff cancels</u>!

Ex.5 Find the formula for the slope of $f(x) = 2x^2$ and use it to find the equation of the tangent line at (-2, 8).

The derivative of a function f with respect to x:

limit definition of the derivative!

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ provided the limit exists.

"f prime of x"

$f'(x)$ is also a function of x.

Other notations (97):

	$D_x[y]$	"the derivative of y with respect to x"
	y'	"y prime"
	$\frac{dy}{dx}$	"$dy - dx$"
	$\frac{df}{dx}$	"the derivative of f with respect to x"
	$\frac{d}{dx}f(x)$	"$d - dx$ of f at x" or "the derivative of f at x"

derivative – slope of tangent line

AP Exam question

Ex.6 Let f be a function that is differentiable for all real numbers. The table below gives values of f for selected points in the closed interval $2 \le x \le 13$. Estimate $f'(4)$.

NOT regression!

x	2	3	5	8	13
y	1	4	-2	3	6

Box before simplify!

Ex.7 Find the derivative of $f(x) = \frac{1}{x}$ by the limit process.

conjugate

Ex.8 Find the derivative of $f(x) = \sqrt{x+3}$ by the limit process. Find the equation of the tangent line if $x = 1$.

The converse is not always true.

(103) #61-70

If f is differentiable at x = c, then f is continuous at x = c.

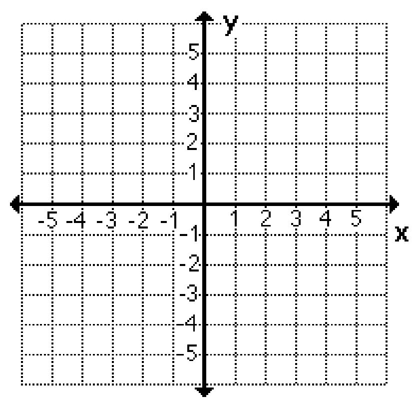
Alternative form of derivative: $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

Ex.9 Use the alternative form to find the derivative at $x = c$.

$f(x) = x^2$, $c = 3$

Ex.10 Find the $\lim\limits_{x \to -3} f(x)$.

Is $f(x) = |x+3|$ continuous?



Find the left & right derivatives of $f(x)$ at $x = -3$.

When is a function not differentiable at a point?

1) If it is not continuous at the point.
2) If the graph has a sharp turn at the point.
3) If there is a vertical tangent at the point.

Summary:

Evolution: $\dfrac{x}{\sin x} \to \dfrac{\sin(3x)}{x} \to \dfrac{4x}{\sin(5x)}$

Notes #1-7
Date:________

2.2 Basic Differentiation Rules (105)

Ex.1 Find the derivative of these constant functions:

a) $f(x) = -3$ b) $s(t) = 0$

The Constant Rule:

Ex.2 Find the derivative of these functions:

a) $y = x$ b) $f(x) = x^2$

1
1 1
1 2 1
1 3 3 1

c) $f(x) = x^3$ d) $y = x^{-1}$

The Power Rule:

Ex.3 Find the derivative of the function (rewrite):

a) $f(x) = x^7$ b) $y = \frac{1}{x^3}$ c) $g(x) = \sqrt[5]{x^3}$

Ex.4 Find the derivative of these functions using limits:

$y = 3x^2$

The Constant Multiple Rule:

Ex.5 Prove the Constant Multiple Rule:

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

derivative
slope of tangent line
rate of change

Ex.6 Find the slope of the graph of $f(x) = 2x^2$ when $x =$

a) -2 b) 0

Ex.7 Find the equation of the tangent line to the graph of $f(x) = \frac{1}{2}x^4$ when $x = -2$.

Ex.8 Using parentheses when differentiating:

Original	Rewrite	Differentiate	Simplify
a) $y = \frac{3}{7x^2}$			
b) $f(x) = \frac{5}{4x^{-3}}$			
c) $y = \frac{3}{2\sqrt[4]{x^3}}$			

The Sum & Difference Rules:

$$\frac{d}{dx}[f(x) + g(x)] =$$

$$\frac{d}{dx}[f(x) - g(x)] =$$

Derivative of Sin & Cos:

Memorize!!!

$$\frac{d}{dx}[\sin x] =$$

$$\frac{d}{dx}[\cos x] =$$

Ex.9 Find the derivative of the function:

a) $y = x(3x^5 - 2x^{-2})$

b) $f(x) = \frac{7 + x^4 - 2x^2}{3x^2}$

c) $f(x) = \frac{3\cos x}{4} + 2\sqrt{x}$

d) $y = -2x^2 - 3\sin x$

If in doubt, take the derivative and = to 0.

Ex.10 Find the derivative of the quadratic function $f(x) = ax^2 + bx + c$. Find the value of x that makes $f'(x) = 0$. The corresponding point on the graph $y = f(x)$ is special. Why?

Summary:

Notes #1-8
Date:______

2.2 Day 2 Rates of Change (109)

Utah State
Math 1210
Calculus 1
Exam 2

Ex.1 Prove the Difference Rule:

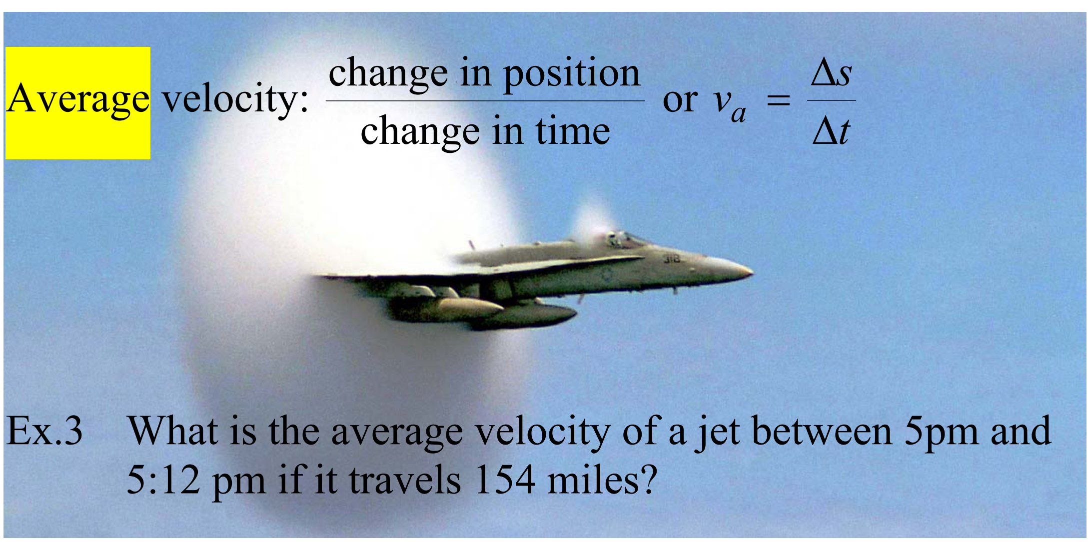
$\cos(x+y) =$

Ex.2 Prove the Derivative of $\cos x$:

$\sin(x+y) =$

$\cos(2x) =$

An unusual cloud might form as a plane accelerates to just break the sound barrier ($\boxed{\quad}$ mph at sea-level and 70° F in normal atmospheric conditions). A theory is that a drop in air pressure at the plane occurs so that moist air condenses there to form water droplets.



Average velocity: $\dfrac{\text{change in position}}{\text{change in time}}$ or $v_a = \dfrac{\Delta s}{\Delta t}$

Ex.3 What is the average velocity of a jet between 5pm and 5:12 pm if it travels 154 miles?

Average velocity: slope

Instantaneous vel: derivative

Free-fall Constants on the Earth

Acceleration due to gravity: $g = -32\dfrac{\text{ft}}{\sec^2}$ or $g = -9.8\dfrac{\text{m}}{\sec^2}$

Position Function: *s(t)*

Velocity Function: $v(t) = s'(t)$ Speed is the $|\text{velocity}|$.

The position of a free-falling object (neglecting air resistance) can be represented by: $s(t) = \dfrac{1}{2}gt^2 + v_0 t + s_0$

Ex.4 A gold coin is dropped from the top of the 1149 foot Stratosphere. **Indicate units of measure**.

a) Determine the position and velocity functions.

b) Determine the average velocity on [3, 7].

c) Find the instantaneous velocities when t = 3 & 7.

d) How long does it take to hit the ground?

e) Find the velocity at impact.

Derivative
1. formula-slope of tangent line
2. rate of change
3. velocity

Ex.5 Find the derivative of the area A of a circle with respect to its radius r.

$A(r) =$ $A'(r) =$

Ex.6 Find the derivative of the volume of a sphere with respect to its radius r.

AP Exam!

Ex.7 Evaluate using derivatives:

a) $\lim\limits_{h \to 0} \frac{(x+h)^7 - x^7}{h}$ b) $\lim\limits_{h \to 0} \frac{1}{h}\left(\sin\left(\frac{\pi}{6}+h\right)-\sin\left(\frac{\pi}{6}\right)\right)$

(114) #63

Ex.8 Find k such that the line $y = 5x - 4$ is tangent to the graph of the function: $f(x) = x^2 - kx$.

2008 #6

Ex.9 Which of the statements about f are true?

I. f has a limit at $x = 2$
II. f is continuous at $x = 2$
III. f is differentiable at $x = 2$

$$f(x)=\begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

(A) I (B) II (C) I & II (D) I & III (E) I, II, & III

Summary:

Notes #1-9
Date:_______

2.3 Product and Quotient Rules (117)

The algebra within the calculus can be more challenging than the calculus itself.

The Product Rule: $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

If $y = uv$, then $y' = uv' + u'v$.

Ex.1 Use the product rule to find $f'(x)$ if $f(x) = x \cdot x$.

How could we answer this question a different way?

Ex.2 Find the derivative of:
a) $k(x) = \sin x \cdot \cos x$ b) $f(x) = x^2 \cos x$

The Product Rule can be used with more than two functions:

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

Ex.3 $k(x) = x\sin x \cdot \cos x$

low d high minus high d low over low squared

The Quotient Rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}, \quad v\neq 0$

If y = $\dfrac{u}{v}$, then $y'=\dfrac{vu'-uv'}{v^2}$ or let y = uv^{-1}.

Ex.4 Find $k'(x)$:

a) $k(x)=\dfrac{9x^7}{x+1}$ b) $k(x)=\tan x$

You may be able to **rewrite** to avoid the Quotient Rule.

Ex.5 Find $f'(x)$ without using the Quotient Rule:

a) $f(x)=\dfrac{3x^{5/2}}{2x^2}$ b) $f(x)=\dfrac{2x^4-7x^3}{5x^2}$

Ex.6 Find the equation of the tangent line to the graph of f at the indicated point:

a) $y=\dfrac{x-4}{x^2+3}$ at $\left(2,-\dfrac{2}{7}\right)$

b) y = $(x^2-4x+2)(4x-1)$ when $x=1$.

When in doubt, find the derivative and set it = 0!

Ex.7 Determine the point(s) at which the graphs of the following functions have a horizontal tangent.

a) $y=\dfrac{x^2-3}{x^2+1}$ b) $y=\dfrac{x-1}{x^2+3}$

On AP Exam and a lot of our tests!

Ex.8 Use the information to find $f'(3)$:

$g(3) = 4$ $g'(3) = -2$ $h(3) = 3$ $h'(3) = \pi$

a) $f(x) = 4g(x) - \dfrac{1}{2}h(x) + 1$

b) $f(x) = g(x)h(x)$

c) $f(x)=\dfrac{g(x)}{2h(x)}$

d) $f(x)=\dfrac{g(x)-h(x)}{g(x)}$

Summary:

Notes #1-10
Date:_______

2.3 Day 2 Trigonometric & Higher-Order Derivatives (121)

Can we avoid the Quotient Rule here?

Ex.1 Find $k'(x)$ using the Quotient Rule $k(x) = \dfrac{2x+5}{3x}$.

Ex.2 Use an identity & the Prod. or Quot. Rule to find $\dfrac{d}{dx}[\sec x]$.

Tangent Function:
$$\frac{d}{dx}\tan x = \sec^2 x$$

Secant Function:
$$\frac{d}{dx}\sec x =$$

Notice that both cofunctions have negative derivatives.

Cotangent Function:
$$\frac{d}{dx}\cot x = -\csc^2 x$$

Cosecant Function:
$$\frac{d}{dx}\csc x = -\csc x \cot x$$

Ex.3 Find y':

a) $y = \dfrac{\sec x}{1+\tan x}$ b) $y = \dfrac{1+\sin x}{\cos x}$

Ex.4 Find the equation of the tangent line at the point:

a) $y = \tan x$, $(\pi/4, 1)$ b) $y = x \cdot \cos x$, $(\pi, -\pi)$

c) $y = \sec x - 2\cos x$, $(\pi/3, 1)$

(123) Higher-Order Derivatives $\frac{d^n y}{dx^n} = f^{(n)}(x)$

nth derivative: the derivative taken n times

$$y = 2x^4 - 5x^2 - 17 = f(x)$$

$$y' = \qquad = f'(x) = \frac{dy}{dx}$$

$$y'' = \qquad = f''(x) = \frac{d^2 y}{dx^2}$$

$$y''' = \qquad = f'''(x) = \frac{d^3 y}{dx^3}$$

$$y^{(4)} = \qquad = f^{(4)}(x) = \frac{d^4 y}{dx^4}$$

Ex.5 Find y'' for: $y = x \cdot \cos x$.

Ex.6 Find y'' for: $y = \dfrac{4x}{\sqrt{x+1}}$

Ex.7 Find $f^{(27)}(x)$ for $f(x) = \cos x$.

Position Function: $s(t)$ Speed $= |v(t)|$

Velocity Function: $s'(t) = v(t)$

Acceleration Function: $s''(t) = v'(t) = a(t)$

Throw pen up in the air and discuss $v(t)$ $a(t)$.

Speed increases when $v(t)$ & $a(t)$ have the same sign.

Speed decreases when $v(t)$ & $a(t)$ have the opposite sign.

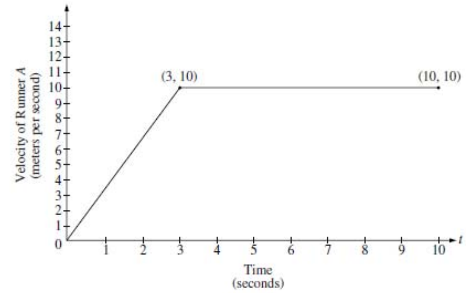
Ex.8 Find the velocity and acceleration when $t = 4$ sec.
$s(t) = t^3 - 6t^2 + 9t$ and s is in meters. Indicate units of measure.

Units!

Is the speed increasing or decreasing at $t = 4$ sec?

Ex.9

AP Calculus AB–2 / BC–2 **2000**



Two runners, A and B, run on a straight racetrack for $0 \leq t \leq 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

2008 #82

Ex.10 A particle moves along a straight line with velocity given by $v(t) = 7 - (1.01)^{-t^2}$ at time $t > 0$. What is the acceleration of the particle at time $t = 3$?

(A) -0.914 (B) 0.055 (C) 5.486 (D) 6.486 (E) 18.087

Summary:

Notes #1-11
Date:_______

2.4 Day 1 Chain Rule (127)

Ex.1 Find the derivative of $C(x) = (x^2 + 5)^3$.

Ex.2 Complete the table by decomposing the functions:

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
$y = \sqrt{x^2 + 1}$		
$y = \sin 6x$		
$y = (3x + 2)^5$		
$y = \tan^2 x$		

THE CHAIN RULE (the derivative of a composite function):

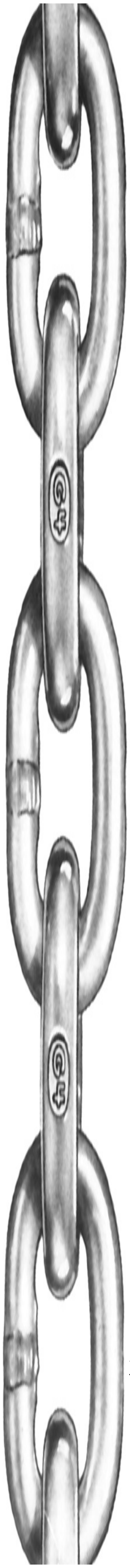
If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Or, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ where } \frac{dy}{du} \text{ is evaluated at } u = g(x)$$

Ex.3 Find $C'(x)$ of $C(x) = (x^2 + 5)^3$ using the Chain Rule.



"Outside-Inside" Differentiation:

Ex.4 Find y':

a) $y = (3x^2 + 1)^2$ b) $y = \sin(x^2 + x)$

Ex.5 Differentiate with respect to x:

a) $y = \dfrac{1}{(2x-3)^3}$ b) $y = \left(\dfrac{2x+1}{2x-1}\right)^5$

Power Chain Rule:

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Ex.6 Differentiate with respect to x:

a) $y = -5\sqrt{x^2 - 4x + 1}$ b) $y = \dfrac{-2}{\sqrt[4]{6x-1}}$

2008 #25 Ex.7 f is differentiable at $x = 2$, what is the value of c + d?

$$f(x) = \begin{cases} cx + d & \text{for } x \le 2 \\ x^2 - cx & \text{for } x > 2 \end{cases}$$

(A) -4 (B) -2 (C) 0 (D) 2 (E) 4

2003 AP Multiple Choice Questions

1. If $y = \left(x^3 + 1\right)^2$, then $\dfrac{dy}{dx} =$

(A) $(3x^2)^2$ (B) $2(x^3 + 1)$ (C) $2(3x^2 + 1)$

(D) $3x^2(x^3 + 1)$ (E) $6x^2(x^3 + 1)$

14. If y = $x^2 \sin 2x$, then $\dfrac{dy}{dx} =$

(A) $2x \cos 2x$ (B) $4x \cos 2x$ (C) $2x(\sin 2x + \cos 2x)$

(D) $2x(\sin 2x - x\cos 2x)$

(E) $2x(\sin 2x + x\cos 2x)$

From now on ask yourself, "Do I need to use the chain rule?"

Do (a) & (b):

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3. An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

(b) Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.
Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

(c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

(d) What is the position of the object at time $t = 4$?

Summary:

Notes #1-12
Date:_______

2.4 Day 2 Chain Rule (132)

Reminder:

Tangent Function:

$$\frac{d}{dx}\tan x = \boxed{}$$

Secant Function:

$$\frac{d}{dx}\sec x = \boxed{}$$

Cotangent Function:

$$\frac{d}{dx}\cot x = \boxed{}$$

Cosecant Function:

$$\frac{d}{dx}\csc x = \boxed{}$$

Ex.1 Differentiate with respect to x

a) $\sin(3x)$ b) $\tan\sqrt{x}$

c) $\csc(x^2 + x)$

Shrek

$\sin(x + x)$

Ex.2 Repeated use of the Chain Rule:

a) $\sin[1 + \tan(2x)]$ b) $\cos^2(3x)$

$\sin(2x) = 2\sin x\cos x$

Ex.3 Evaluate the derivative at the given point:

a) $y = (2x + 1)^{-3}$, $\left(1, \frac{1}{27}\right)$ b) $y = \frac{1}{x} + \sqrt{\sin x}$, $\left(\frac{\pi}{2}, \frac{2}{\pi} + 1\right)$

Ex.4 Find an equation of the tangent line to the graph of f at the given point or value:

a) $y = -\sqrt{x^2-4x+1}$, (4, -1)

b) $y = 2\tan x + \cos^2(2x)$, when $x = \pi$.

Ex.5 $g(2) = -3$, $g'(2) = -6$, $h(2) = 3$, $h'(2) = -2$, $g'(3) = 4$

Find the derivatives at $x = 2$.

a) $[g(x)]^3$

b) $f(x) = g(h(x))$

c) $g(x)h(x)$

d) $\dfrac{g(x)}{h(x)}$

Ex.6 Find the derivatives:

a) $y=\dfrac{x}{\sqrt{x^2-1}}$ b) $y=\sqrt{\dfrac{x}{4x-1}}$

Ex.7 Find the derivative of $y = \sin^3(2x)$.

2008 #8

Ex.8 If $f(x)=\cos(3x)$ then $f'\left(\frac{\pi}{9}\right)=$

(A) $\frac{3\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $-\frac{\sqrt{3}}{2}$ (D) $-\frac{3}{2}$ (E) $-\frac{3\sqrt{3}}{2}$

2008 #3

Ex.9 If $f(x)=(x-1)(x^2+2)^3$ then $f'(x)=$

(A) $6x(x^2+2)^2$ (B) $6x(x-1)(x^2+2)^2$

(C) $(x^2+2)^2(x^2+3x-1)$ (D) $(x^2+2)^2(7x^2-6x+2)$

(E) $-3(x-1)(x^2+2)^2$

Summary:

Notes #1-13
Date:______

2.5 Day 1 Implicit Differentiation (132)

W.1 Determine the derivative of $y = (3 - x)^4$.

Ex. Conic Sections

Implicitly-Defined Function: a function with multiple variables that is not solved for one of the variables

For example: $4(x-1)^2 + y^2 = 25$

Note: An explicitly-defined function is one that is written in function form, $y = f(x)$.

Implicit Differentiation: differentiating a function that is not written as an explicit formula.

Use the following steps:

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms with $\frac{dy}{dx}$ on one side of the equation.
3. Factor out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$.

$y = x^2$

$\frac{dy}{dx} = 2x\frac{dx}{dx}$

Note: When differentiating with respect to x, the derivative of x is $\frac{dx}{dx} = 1$, and the derivative of y is $\frac{dy}{dx}$.

Ex.1 Find $\frac{dy}{dx}$ by implicit differentiation $x^2 - xy + y^2 = 7$.

Note: There are many ways of writing the correct answer. Watch for these on multiple choice selections.

Ex.2 Find $\frac{dy}{dx}$ by implicit differentiation & <u>then evaluate</u> the derivative at the indicated point. $xy^2 + 2y^4 = x^2y$, (2, 1).

Ex.3 Calculate the derivative of y with respect to x of:
$\sin(x + y) = x + \cos y$

$\boxed{}$

2008 #16

Ex.4 If $\sin(xy) = x$, then $\frac{dy}{dx} =$

(A) $\frac{1}{\cos(xy)}$

(B) $\frac{1}{x\cos(xy)}$

(C) $\frac{1-\cos(xy)}{\cos(xy)}$

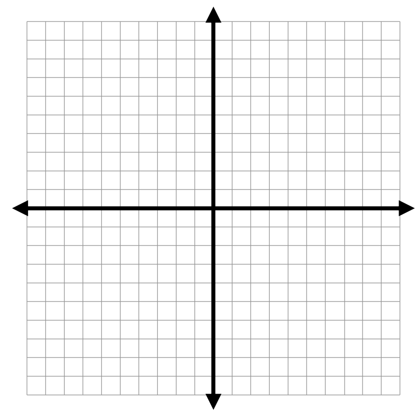
(D) $\frac{1-y\cos(xy)}{x\cos(xy)}$

(E) $\frac{y(1-\cos(xy))}{x}$

Ex.5 $x^2 + 4y^2 = 36$

a) Find two explicit functions.

b) Sketch the graphs of the equations and label the parts given by the corresponding explicit functions.



c) Differentiate the explicit functions.

d) Differentiate implicitly and show that the result is equivalent to part c.

e) Find the <u>points</u> at which the graph of the equation has a vertical or horizontal tangent line.

Summary:

Notes #1-14
Date:_______

2.5 Day 2 Implicit Differentiation (132)

Opposite reciprocal slope

The normal line at a point is $\perp$ to the tangent line at the point.

Ex.1 Find the tangent & normal line of $2xy + \pi \sin y = 2\pi$ at $\left(1, \frac{\pi}{2}\right)$.

Ex.2 Find $\frac{dy}{dx}$ if: $y = \frac{x^2}{2x+3y}$. Hint: rewrite.

Finding a Second Derivative

Ex.3 Find $\frac{d^2y}{dx^2}$ for $4y^3 = 9 - 5x^2$.

Differentiating again:

Now substitute $\frac{dy}{dx} = -\frac{5x}{6y^2}$ and simplify:

Eliminate the complex fraction:

Ex.4 Find $\frac{d^2y}{dx^2}$ if: $y = \frac{6}{x+y}$.

Using the Chain Rule with a Table of Values

Ex.5 Evaluate the derivatives using the table below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	2	3	1/3	−3
3	3	−4	2π	5

a) $f(g(x))$ at $x = 2$

b) $g(f(x))$ at $x = 2$

c) $\frac{1}{[g(x)]^2}$ at $x = 3$

d) $\sqrt{f(x)}$ at $x = 2$

Summary:

Notes #1-15
Date:_______

2.6 Day 1 Related Rates (144)

Related Rates Equation: an equation that relates the corresponding rates of two or more variables that are differentiable functions of time *t*

Steps to Solving a Related Rates Problem

1. Make a sketch (if possible). Name the variables and constants.
2. Write down the known information and the variable we are to find.
3. Write an equation that relates the variables.
4. Differentiate implicitly with respect to *t* using the chain rule.

   A common mistake is forgetting the chain rule. Be sure to use $\frac{dy}{dt}$ notation for derivatives, NOT y'.
5. Answer the question that was asked with correct units.

Ex.1 A 13-ft ladder is leaning (flush) against a wall. Suppose that the base of the ladder slides away from the wall at 3 ft/sec.

a) Find the rate at which the top of the ladder is moving down the wall at $t = 1$ sec.

b) Find the rate at which the area of the triangle is changing $t = 1$ sec.

Ex.2 A balloon rises at 15 feet per second from a point on the ground 45 feet from an observer. Find the rate of change of the angle of elevation when the balloon is 60 feet above the ground. Indicate units of measure. <u>**Hint**: you don't need to find θ</u>.

(146) #3 Sphere

Ex.3 Water runs into a conical tank at 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Note: When determining the units for the answer, use the units from the original problem. For example, if you are determining units for dh/dt, it would be the units for *h* (ft) over the units for *t* (min).

Ex.4 A particle is moving along the curve $y=\sqrt{x}$. As the particle passes through the point (4, 2), it's x-coordinate increases at a rate of 6 cm/s. How fast is the distance from the particle to the origin changing at this instant?

Ex.5 The radius of a sphere is decreasing at a rate of 2 cm/sec. At the instant when the radius of the sphere is 3 cm, what is the rate of change, in sq cm per second, of the surface area of the sphere?

(A) -108π (B) -72π (C) -48π (D) -24π (E) -16π

Summary:

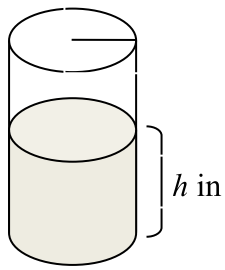
Notes #1-16
Date:______

2.6 Day 2 Related Rates (144)

Ex.1 A police cruiser, approaches a right-angled intersection from the north, chasing a car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the distance between them is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Ex.2 A searchlight is positioned 10 meters from a sidewalk. A person is walking along the sidewalk at a speed of 2 meters/sec. The searchlight rotates so that it shines on the person. Find the rate at which the searchlight rotates when the person is 25 meters from the searchlight.

Ex.3 A cylinder coffeepot has a radius of 5 inches. The depth of the coffee in the pot is h inches, where h is a function of time, in t seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. Find dh/dt.



(151) #35-36

Previous MC AP Question

Ex.4 A 5 ft tall woman walks at 4 ft/sec directly away from a 20 ft tall street light.

a) At what rate is the tip of her shadow moving?

b) At what rate is the length of her shadow changing?

Question 3

The wind chill is the temperature, in degrees Fahrenheit $(°F)$, a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is $32°F$, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

(a) Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

(b) Find the average rate of change of W over the interval $5 \le v \le 60$. Find the value of v at which the instantaneous rate of change of W is equal to the average rate of change of W over the interval $5 \le v \le 60$.

(c) Over the time interval $0 \le t \le 4$ hours, the air temperature is a constant $32°F$. At time $t = 0$, the wind velocity is $v = 20$ mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at $t = 3$ hours? Indicate units of measure.

Summary:

Notes #2-1
Date:______

3.1 Extrema on an Interval (160)

Global/absolute

Local/relative

Hidden behavior

(where, what)

Ex.1 Find the value of the derivative (if it exists) at the extremum (use a graphing calculator to identify).

$f(x) = x^3 - 9x^2 - 48x + 9$

Critical number

Critical Point: a point **in the** interior of the **domain** of a function f at which $f'(c) = 0$ or f is not differentiable. Let f be defined at c.

* Extreme values **only** occur at critical points and endpoints.

* A critical point is ***not necessarily*** an extreme value. For example, $y=x^3$ has a critical point at (0, 0) because $f'(0)=0$. However, $f(0)=0$ is **not** an extreme value.

(160)

Extreme Value Th. If f is cont on [a,b], then f has both a max and min value on the interval.

Always put:
$f'(x) = 0$

Ex.2 Find all the critical numbers of

a) $f(x)=\dfrac{2x^2}{x+2}$ b) $f(x) = (3x+1)^{\frac{2}{3}}$

Ch.3 MC! $f(c)$ not def
a) -4, -2, 0
b) -4, 0
c) -2, 0
d) -4, -2

Don't forget to evaluate the endpoints!
"closed"

Ex.3-6 Find the absolute extrema of:

Ex.3 $f(x) = 2x^3 - 3x^2 - 12x + 5$ on the interval [-2, 4]

Ex.4 $f(x) = 4x^{5/4} - 8x^{1/4}$ on the interval [0, 4]

Ex.5 $f(x) = \sin x \cos x$ on the interval $[0, 2\pi]$

Ex.6 $f(x) = \begin{cases} x^2 & x \leq 1 \\ 3x - 2 & x > 1 \end{cases}$ on the interval [-1, 3].

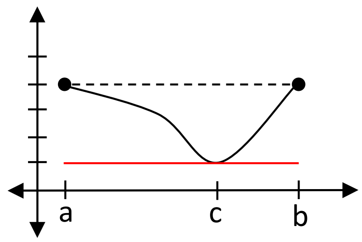
Summary:

Notes #2-2
Date:_______

3.2 Rolle's & Mean Value Theorems (168)

Specific

Rolle's Th Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there is at least one number c in (a, b) such that $f'(c) = 0$



Ex.1 Give two reasons why Rolle's Theorem might not apply to a function even though there exist a and b such that $f(a) = f(b)$.

Ex.2 Find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between them.

a) $f(x) = 2x\sqrt{x+3}$

b) $f(x) = 60x^2 - 130x - 80$

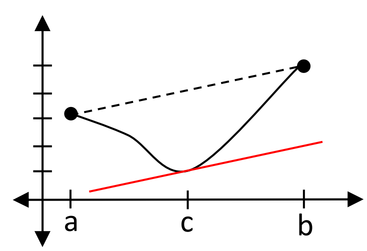
Ex.3 Determine whether Rolle's Th can be applied to *f* on the closed interval [0, 4]. If it can be, find all of the values of *c* in the open interval (0, 4) such that $f'(c)=0$.

a) $f(x)=\dfrac{x^2-4x}{x-2}$ b) $f(x)=\dfrac{x^2-4x}{x+2}$

MVT

Mean Value Theorem If *f* is continuous on the closed interval [*a*, *b*] and differentiable on the open interval (*a*, *b*), then there exists at least 1 number *c* in (*a*, *b*) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Ex.4 Sketch a graph where the Mean Value Theorem would not apply.

Ex.5 Can the Mean Value Theorem be applied to f on the interval $[a, b]$. If it can be, find all of the values of c in the open interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

a) $f(x) = x^3 - x^2 - x + 1$, $[0, 2]$

b) $f(x) = \sin x$, $[-\pi , 0]$

2003 #80

Ex.6 The function f is continuous for $-2 \leq x \leq 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

a) There exists c, where $-2 < c < 1$, such that $f(c) = 0$.

b) There exists c, where $-2 < c < 1$, such that $f'(c) = 0$.

c) There exists c, where $-2 < c < 1$, such that $f(c) = 3$.

d) There exists c, where $-2 < c < 1$, such that $f'(c) = 3$.

e) There exists c, where $-2 \leq c \leq 1$, such that $f(c) > f(x)$ for all x on the closed interval $-2 \leq x \leq 1$.

Summary:

Notes #2-3
Date:______

3.3 Increasing/Decreasing $f(x)$s and the 1st Derivative Test (174)

MUST USE CALCULUS!!! Must explain using $f'(x)$.

Don't use pronouns! Be specific, no "it" or "the derivative."

$f'(x) < 0$ $\quad f(x)$ is decreasing.

$f'(x) = 0$ $\quad f(x)$ is constant.

$f'(x) > 0$ $\quad f(x)$ is increasing.

*Be careful to distinguish between graphs of functions and graphs of derivatives!!!

$y = x,\ y = x^3,\ y = 2^x$

*Strictly monotonic: increasing/decreasing on entire interval.

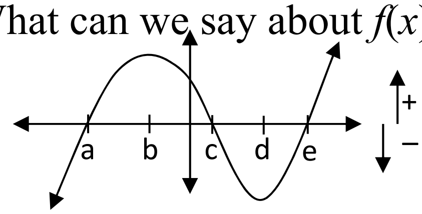
1st Derivative Test:

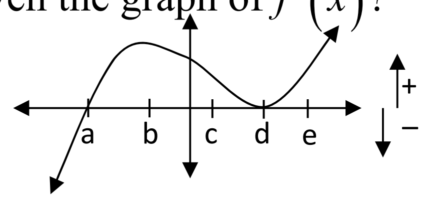
(176)

$f'(x)$ changes from + to – at c then $f(c)$ is a rel max

$f'(x)$ changes from – to + at c then $f(c)$ is a rel min

Ex.1 What can we say about $f(x)$ given the graph of $f'(x)$?

a) 

b) 

critical numbers & values not in the domain of f

Ex.2 Find the open intervals on which the function is increasing or decreasing and locate all extrema.

a) $f(x) = 2x^3 + 9x^2 - 24x - 10$

b) $f(x) = x^{5/3} - 3x^{2/3}$

c) $f(x) = x + 2\sin x$ over $[0, 2\pi]$

2017 #2 Customers remove bananas from a grocery store display at a rate modeled by $f(t)$ and store employees add bananas to the display at a rate modeled by $g(t)$.

$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right)$ and $g(t) = 3 + 2.4\ln(t^2 + 2t)$

(b) Find $f'(7)$. Using the correct units, explain the meaning of $f'(7)$ in the context of the problem.

(c) Is the number of pounds of bananas on the display increasing or decreasing at $t = 5$? Give a reason.

R.1 A liquid is cleared of sediment by allowing it to drain through a conical filter 16 cm high, with a radius of 4 cm at the top. The liquid is forced out of the cone at a constant rate of 2 cm^3 / min. At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep? Indicate units of measure.

R.2 a) $\lim\limits_{h \to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h}$ b) $\lim\limits_{x \to 4} \dfrac{\sqrt{x-3} - 1}{x-4}$

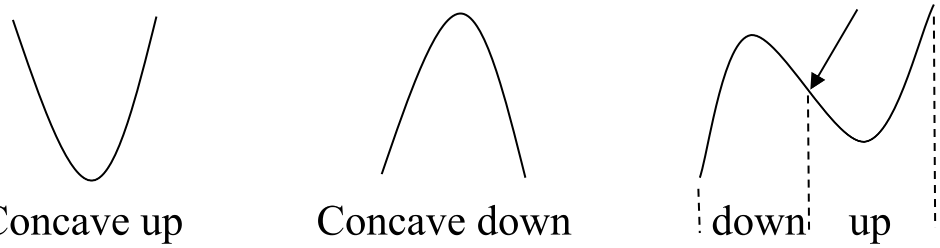
Speed increases when $v(t)$ & $a(t)$ have the same sign.
Velocity increases when $a(t) > 0$.

Summary:

Notes #2-4
Date:_______

3.4 Concavity and the 2nd Derivative Test (184)

Concavity: curving upward or downward



Concave up   Concave down   down up

What happens to the derivative of a concave up function as you move from left to right?

f' is increasing when

Test for Concavity: Let f be a function whose second derivative exists on an open interval I.

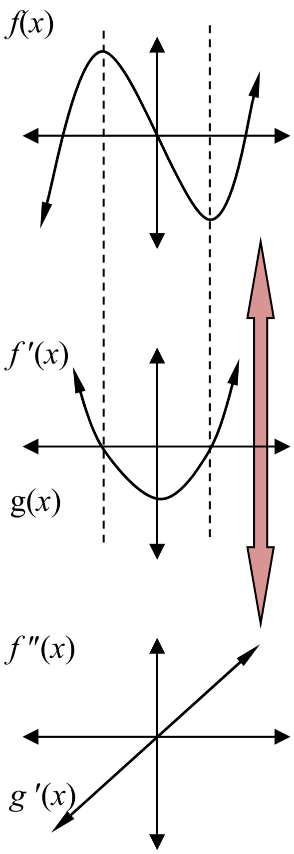
1. If $f''(x) > 0$ for all x in I, then the graph of f is concave up.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave down

Points of Inflection:
points where concavity **changes** (graph crosses its tangent line)
Tangent line must exist at the point!

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f is not differentiable at $x = c$.

Note: it is possible for the second derivative to be 0 at a point that is **not** a point of inflection. Be careful of asymptotes!

Ex.1 Determine the open intervals on which the graph is concave upward or concave downward.
a) $f(x) = 2x^3 + 9x^2 - 24x - 10$



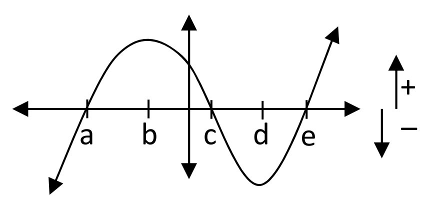
Must be able to connect from one to another.

b) $f(x)=\dfrac{2x}{x^2-4}$

Ex.2 Find any points of inflection.

a) $f(x) = 2x^3 + 9x^2 - 24x - 10$ b) $f(x)=\dfrac{2x}{x^2-4}$

Ex.3 The graph of $f'(x)$:



a) Find all of the intervals on which f is concave down.

b) Give all values of x for where f has points of inflection.

c) True or false: $f'(c) < f''(c)$

concave up

concave down

<u>The Second Derivative Test</u>: Let f be a function such that $f'(c) = 0$ and f'' exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

If $f''(c) = 0$, the test fails. Use the First Derivative Test.

Ex.4 Use the second derivative test.
a) $f(x) = 2x^3 + 9x^2 - 24x - 10$ b) $f(x) = 3x^5 - 5x^3$

Ex.5 Given the critical numbers: -4, -1/2 and 3 and $f''(x) = -6x^2 - 6x + 23$, use the Second Derivative Test to Determine which critical numbers, if any, give a relative maximum. Show work!

Summary:

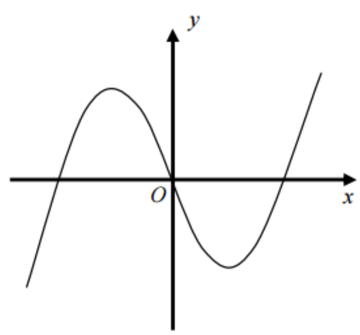
Notes #2-5
Date:_______

3.6 Summary of Curve Sketching (202)

W.1 Let $f''(x) = 12x^2 - 24x$ and let $f(x)$ have critical numbers 0 & 3. Use the Second Derivative Test to determine which of the critical numbers, if any, gives a relative max.

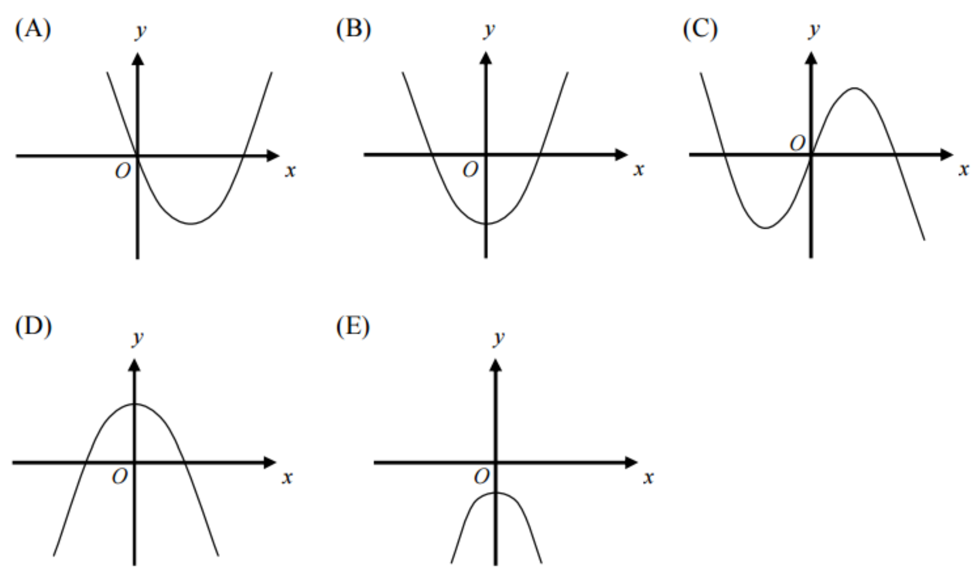
a) 0 b) 3 c) 0 & 3 d) none e) not enough info

W.2



Graph of f

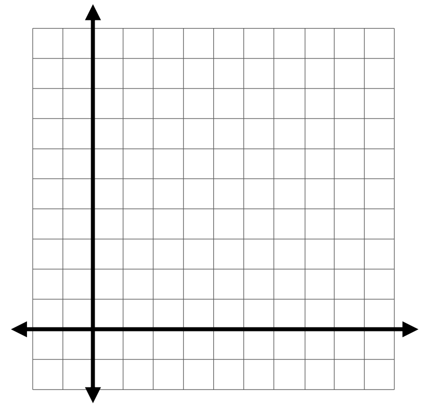
11. The graph of a function f is shown above. Which of the following could be the graph of f', the derivative of f?



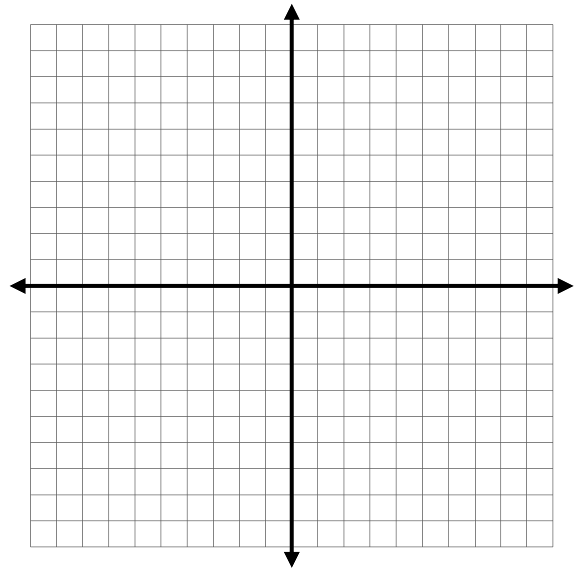
Hidden Behavior

Ex.1 Analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection and asymptotes.

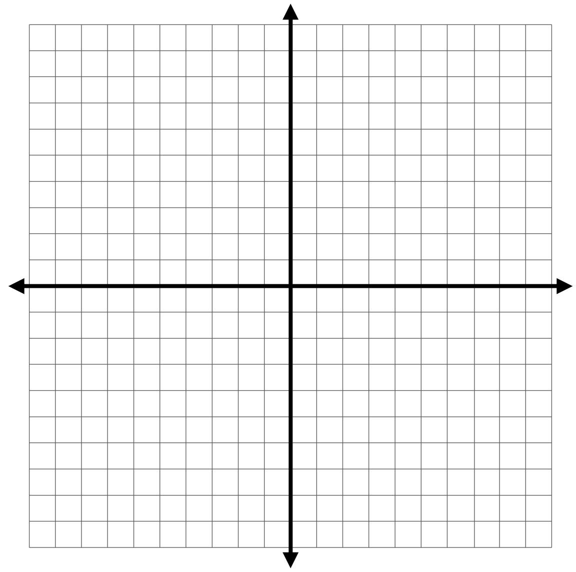
a) $f(x) = x + 2\sin x$ $[0, 2\pi]$



b) $f(x)=\dfrac{x}{\sqrt{x^2-4}}$



c) $f(x)=\dfrac{x^2-5x+4}{x-2}$



R.1 Which table could represent values of $f(x)$ if $f'(x)>0$ and $f(x)<0$.

a)

x	y
-1	7
0	4
1	2
2	1

b)

x	y
-1	7
0	6
1	4
2	1

c)

x	y
-1	1
0	2
1	4
2	7

d)

x	y
-1	1
0	4
1	6
2	7

Summary:

Notes #2-6
Date:_______

3.7 Optimization (211)

best

Optimization (Max/Min) Problem: a problem in which a quantity is to be maximized or minimized

Steps to Solving a Max/Min Problem

1. Assign symbols to all given quantities and quantities to be determined. Make a sketch.

2. Write a PRIMARY EQUATION for the quantity being maximized or minimized (use a capital letter).

Systems of eqations.

3. Reduce the primary equation to one having a single independent variable.

4. Determine the maximum/minimum using critical values.

Don't forget the endpoints of the domain!

5. Use the 1^{st} (or 2^{nd}) derivative test and choose the answer.

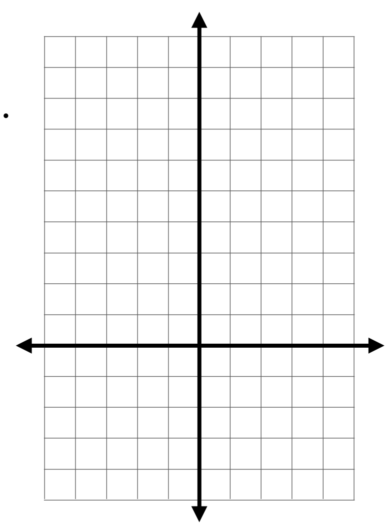
6. Answer the question asked. Include units with your answer.

Ex.1 Find two non-negative numbers whose sum is 16 and whose product is as large as possible.

(217) #20-21

Ex.2 An open rectangular box is made from a 4ft by 5ft piece of cardboard by cutting congruent squares from the corners and folding up the sides. How long should the sides of the square be to create the box of largest volume? Leave your answer in simple radical form.

Ex.3 Find the point on the parabola $y = 9 - x^2$ closest to point (3, 9).

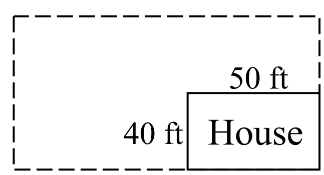


Ex.4 Oil is to be piped from an offshore oil well 5 km from a straight shoreline. It needs to piped 8 km down shore. The cost is $1,000,000/km under water and $500,000/km along the shore. Where should the pipeline be built?

Ex.5 You want to build a fence around your house as shown. You can afford 250 ft of fencing. What values for x & y will maximize the area enclosed?

Length of fencing:

L =



Area enclosed:

A =

Ex.6 In an apple orchard there are 30 trees per acre and the average yield is 400 apples per tree. For each additional tree planted per acre, the average yield per tree is reduced by 10 apples. How many trees per acre will maximize the crop?

Ex.7 A manufacturer wants to design an open box having a square base and a surface area of 147 in^2. What dimensions will produce a box with maximum volume?

Summary:

Notes #2-7
Date:_______

3.8 Newton's Method (222)

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c.

1. Make an initial estimate x_1 close to c. (Graph is helpful.)
2. Determine a new approximation $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
3. Repeat if necessary.

Iteration: each time you apply this process after the first

Ex.1 Calculate two iterations of Newton's Method to approximate a zero of $f(x) = 2x^4 - 4x$. Use $x_1 = 1.2$ as the initial guess.

n	x_n
1	
2	
3	

Ex.2 Use Newton's Method to approximate the zeros of $f(x) = 3\sqrt{x+2} - 5x$. Continue iterations until two successive approximations differ by less than 0.001.

n	x_n
1	
2	
3	

Condition for convergence

If $\left|\dfrac{f(x)f''(x)}{\left[f'(x)\right]^2}\right| < 1$, Newton's method yields convergence.

Top of (225)

Ex.3 Apply Newton's Method to $f(x) = x^{\frac{1}{5}}$ using $x_1 = 0.1$ as the initial guess.
Explain why the method fails.

n	x_n
1	
2	
3	

Ex.4 Apply Newton's Method to approximate the x-value of the point of intersection of the graphs of $f(x) = 3x^2 - 1$ and $g(x) = \sin x$ that is closest to 1. Continue the process until two successive approximations differ by less than 0.001. [Hint: Let $h(x) = f(x) - g(x)$.]

n	x_n
1	
2	
3	
4	

Smith (245)
Ex.1.6

Ex.5 Use Newton's Method to approximate $\sqrt[3]{7}$.

n	x_n
1	
2	
3	
4	

Summary:

Notes #2-8
Date:_______

3.9 Differentials (228)

Home
2^{nd} F1 is F6
2: New Problem

Linear Approximations

Write the equation of the tangent line at $(c, f(c))$ for a function f that is differentiable at c.

Point-slope:

or

The equation of the tangent line can be used to find the local linear approximation of the function close to c.

Ex.1 Find the equation of the tangent line T to the graph of $f(x) = \sqrt{x}$ at (1 , 1). Use this linear approximation to complete the table.

x	0.9	0.99	1	1.01	1.1
$f(x)$					
$T(x)$					

Differentials

Let $y = f(x)$ represent a function that is differentiable in an open interval containing x. The **differential of *x*** (denoted dx) is any nonzero real number. The **differential of *y*** (denoted dy) is $dy = f'(x)\,dx$.

$$\Delta y = f(c + \Delta x) - f(c) \approx f'(c)\,\Delta x.$$

Ex.2 Use the information to evaluate and compare dy & Δy.

$y = 3x^2 - 5 \qquad x = 2 \qquad \Delta x = dx = .1$

Error Propagation

$$\underbrace{f(x + \overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}} \Rightarrow f'(x)\,\Delta x$$

Test #2-4 & #2-5

Ex.3 Suppose that the side of a square is measured to be 10 inches with a measurement error of at most $\pm\frac{1}{32}$ in. Estimate the error in the computed area of the square.

Function	Derivative	Differential
$y = x^3$	$\frac{dy}{dx} = 3x^2$	$dy = 3x^2\, dx$

Ex.4 Find the differential dy of the given functions.

a) $y = x\cos x$

Leibniz notation

b) $y = (1 + 2x)^{-17}$

c) $y = \frac{1-x^3}{2-x}$

Very important, on a bunch of tests.

Ex.5 The function f is twice differentiable with $f(2) = 1$, $f'(2) = 4$, and $f''(2) = 3$. What is the tangent line approximation of $f(1.9)$? Is the estimate greater or less than the true value?

Consider concavity. Contrast with Reimann sum.

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$.

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

2017 #4

$\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$. $H(t) > 27$°C for all times $t > 0$.

(t, H)

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this to approximate the temperature at $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or overestimate of the temp at $t = 3$.

Summary:

Multiple Choice Questions: **Circle the best answer**.

1. If $f''(x)=x(x+1)(x-2)^2$, then the graph of f has inflection points when $x =$

(A) 1 (B) 2 (C) -1 and 0 (D) -1 and 3 (E) -1, 0, and 2

2. If g is a differentiable function such that $g(x)<0$ for all real numbers x, and if $f'(x)=(x^2-4)g(x)$, which of the following is true?

(A) f has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.

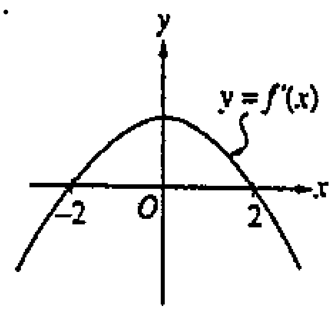
(B) f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.

(C) f has relative minima at $x = -2$ and at $x = 2$.

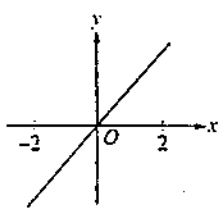
(D) f has relative maxima at $x = -2$ and at $x = 2$.

(E) It cannot be determined if f has any relative extrema.

3. The graph of the derivative of f is shown in the figure. Which of the following could be the graph of f?



(A)



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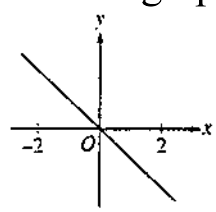
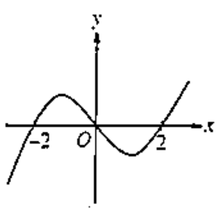
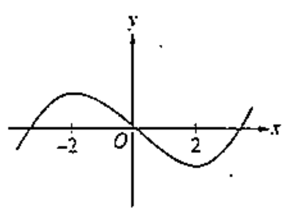


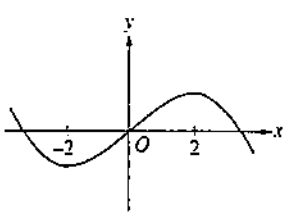
(C)



(D)



(E)



4.  If the derivative of f is given by $f'(x) = e^x - 3x^2$, at which of the following values of x does f have a relative maximum value?

(A) -0.46 (B) 0.20 (C) 0.91 (D) 0.95 (E) 3.73

5. Let $f(x) = \sqrt{x}$. If the rate of change of f at x = c is twice its rate of change at $x = 1$, then c =

(A) $\frac{1}{4}$ (B) 1 (C) 4 (D) $\frac{1}{\sqrt{2}}$ (E) $\frac{1}{2\sqrt{2}}$

6. The graph of a function f is shown. Which of the following statements about f is false?

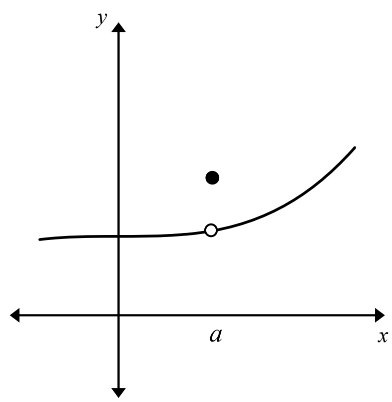
(A) f is continuous at $x = a$.

(B) f has a relative maximum at $x = a$.

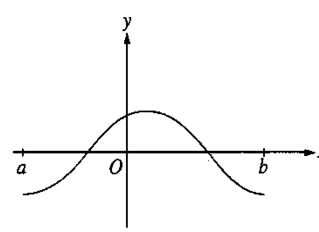
(C) $x = a$ is in the domain of f.

(D) $\lim_{x\to a^+} f(x)$ is equal to $\lim_{x\to a^-} f(x)$.

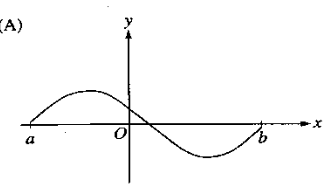
(E) $\lim_{x\to a} f(x)$ exists.



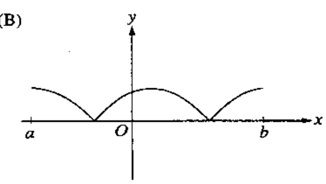
7. The graph of the function f is shown in the figure. Which of the following could be the graph of the derivative of f?



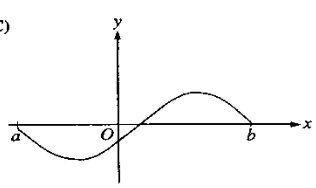
(A)



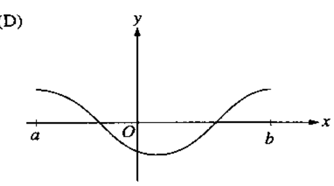
(B)



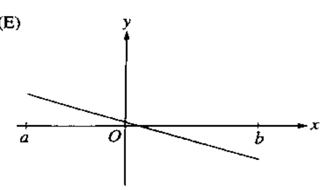
(C)



(D)



(E)



4.1 Antiderivatives and Indefinite Integration (242)

Notes #2-9
Date: _______

A process that is basically the "inverse" of differentiation.
We are going to undo derivatives.

$F(x)$	$f(x)$
x^3	
$x^5 + 4$	
$x^5 - 7$	
	$5x^4$
Antiderivatives	Derivatives

A function F is **an** antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Family of functions.

Ex.1 General Solutions — Differential Equations

a) $y =$ $\qquad$ $y' = -3$

b) $y =$ $\qquad$ $y' = 4x^3$

What is y if $\frac{dy}{dx} = f(x)$?

Use $y = \int \underbrace{f(x)}_{\text{Integrand}} \overbrace{dx}^{\text{Variable of Integration}} = F(x) + \underbrace{C}_{\text{Constant of Integration}}$

Basic Integration Rules: see (244). $\quad \int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

Ex.2 Describe the antiderivatives of x^4.

Rewriting Before Integrating (rewrite, integrate, simplify)

Ex.3 a) $\int \frac{1}{x^5} dx$

b) $\int \sqrt[3]{x} dx$

c) $\int 4 \cos x \, dx$

Integrating Polynomials

Ex.4 a) $\int 0 \, dx$

b) $\int (3x^6 - 2x^2 + 7x + 1) \, dx$

c) $\int (x + x^2) \, dx$

More Rewriting Before Integrating

Ex.5 $\int \frac{t^2 - 2t^4}{t^4} \, dt$

Ex.6 $\int \frac{\cos x}{\sin^2 x} \, dx$

Ex.7-8 Finding a Particular Solution

Ex.7 Solve the differential equation using the initial condition.

a) $f'(x) = (x+1)^2$, $f(-2) = 8$ b) $f'(x) = -\sin x$, $f(0) = 2$

Ex.8 $f''(x) = 60x^3$, $f'(1) = 17$, $f(-1) = 2$

Vertical Motion (Use $a(t)$ = -32 ft/sec^2 for acceleration due to gravity)

Ex.9 A stone is thrown vertically upward from a position of 144 feet above the ground with an initial velocity of 96 ft/sec.

a) Find the distance above the ground after t seconds.

b) How long does the stone rise?

c) When, and with what velocity, does it strike the ground? Speed?

Rectilinear Motion (a particle that can move either direction along a coordinate line)

Consider a particle moving along an s-axis where $s(t)$ is the position of the particle at time t, ______ is it's velocity, and ______ is it's acceleration.

Note: A particle moving in the negative direction ($v(t) < 0$) is speeding up if ***v*(*t*) & *a*(*t*) have the same sign**, **slowing down when opposite signs**

Ex.10 $s(t) = t^3 - 6t^2$, $0 \le t \le 8$, where s is measured in meters and t in seconds.

a) Find the velocity and acceleration of the particle.

b) Find the open t-interval(s) on which the particle is moving in the positive direction.

c) Find the velocity when the acceleration is 0.

d) Find the open t-interval(s) on which the particle is speeding up.

Ex.11 $s(t) = 2t^3 - 21t^2 + 60t + 3$, $0 \le t$, s is in meters and t in seconds.

a) Find the velocity and acceleration of the particle.

b) Find the open t-interval(s) on which the particle is moving in the positive direction.

c) Find the velocity when the acceleration is 0.

d) Find the open t-interval(s) on which the particle is slowing down.

4.2 Area (242)

Notes #2-10

Date: ______

Sigma Notation (Series: Summation)

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ can be written as: $\sum_{i=1}^{n} a_i = a_1+a_2+a_3+\ldots+a_n$

- i index of summation (j & k)
- a_i the ith term of the sum
- n upper bound of summation (the lower bound doesn't have to be 1)

Ex.1 Find the sum:

a) $\sum_{i=2}^{7}(2+3i)$

b) $\sum_{k=1}^{4}(-1)^k \cdot (2k)$

Ex.2 Use sigma notation to write the sum: $\sqrt{1+1^3}+\sqrt{2+2^3}+\cdots+\sqrt{n+n^3}$

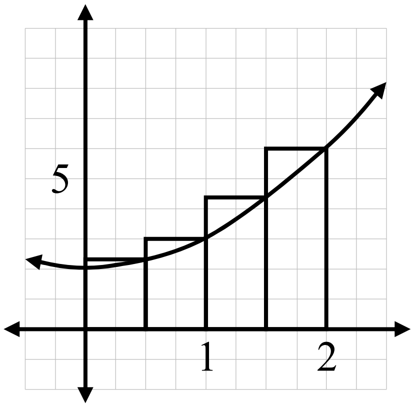
Summation Formulas:

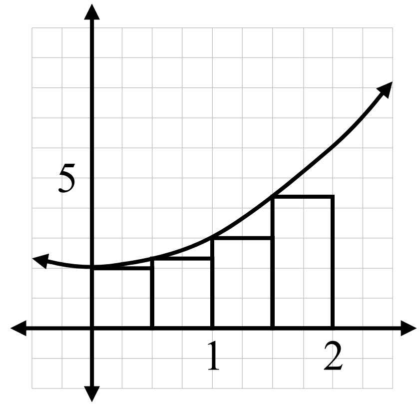
$\sum_{i=1}^{n} c = cn$ Powers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Ex.3 Evaluate the sum: $\sum_{k=1}^{30} k(k+1) =$

Area: we can approximate the area under a curve using the definition of the area of a rectangle $A = bh$.

Ex.4 Use upper and lower sums to approximate the area of the region bounded by the graph of $f(x) = x^2 + 2$, the x-axis, $x = 0$ and $x = 2$, using 4 subintervals.





Overestimate or underestimate and why? Consider increasing/decreasing.

The right endpoints are given by $\frac{2-0}{4}i$, where $i = 1, 2, 3, 4$.

Area =

We could use $A = \sum_{i=1}^{4} f\left(\frac{1}{2}i\right)\left(\frac{1}{2}\right) =$

The left endpoints are given by $\frac{2-0}{4}(i-1)$, where $i = 1, 2, 3, 4$.

We could repeat the process to find a lower approximation $s(n)$.

$s(n) <$ area of the region $< S(n)$

For large numbers of rectangles we need to generalize this result:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right)$$ called a Riemann Sum

c_i

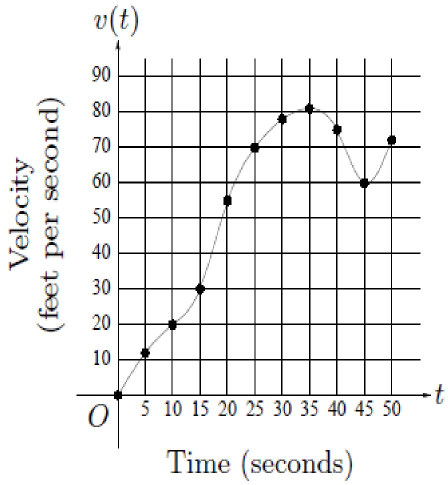
Ex.5 Find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6i+1}{n^2}$$

Ex.6 Use the limit process to find the area of the region between the graph of the function and the x-axis over the indicated interval.

a) $f(x) = 9 - x^2$, [0, 3] b) $f(x) = x^2$, [1, 4]

1998 Calculus AB Scoring Guidelines



t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72

3. The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \le t \le 50$, is shown above. A table of values for $v(t)$, at 5 second intervals of time t, is shown to the right of the graph.

   (a) During what intervals of time is the acceleration of the car positive? Give a reason for your answer.

   (b) Find the average acceleration of the car, in ft/sec^2, over the interval $0 \le t \le 50$.

   (c) Find one approximation for the acceleration of the car, in ft/sec^2, at $t = 40$. Show the computations you used to arrive at your answer.

   (d) Approximate $\int_0^{50} v(t)\,dt$ with a Riemann sum, using the midpoints of five subintervals of equal length. Using correct units, explain the meaning of this integral.

4.3 Riemann Sums and Definite Integrals (265)

Notes #2-11
Date: _______

Definite Integrals: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ = Area

read "the integral from a to b of f of x dx."

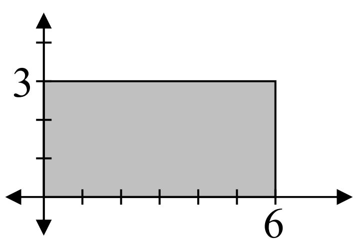
> Note that the integral symbol resembles an S, because an integral is a sum (Σ).

Ex.1 Evaluate the definite integral $\int_1^4 2dx$

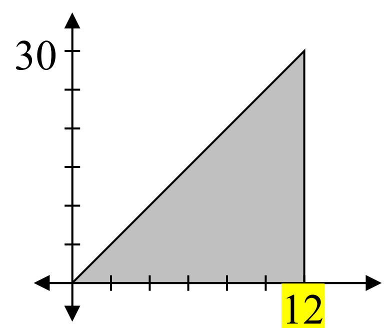
Ex.2 Express the limit as a definite integral on the interval [2, 3], where c_i is any point in the ith subinterval: $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(c_i^2 - 2c_i\right)\cdot\Delta x_i$

Ex.3 Set up a definite integral that yields the area of the region.

a)

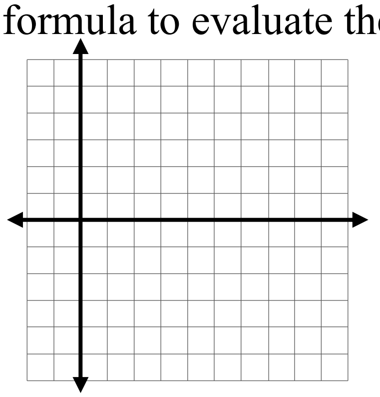


b)

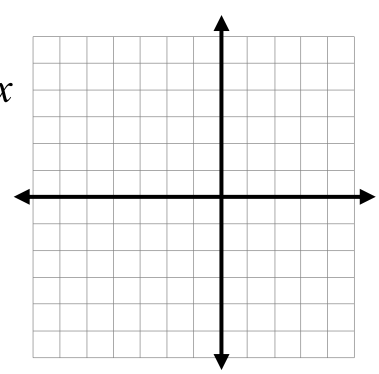


Ex.4 Sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

a) $\int_1^4 -2dx$

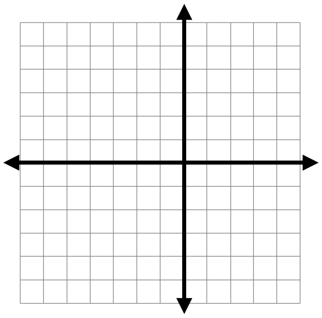


b) $\int_{-3}^{0} \sqrt{9-x^2}\,dx$

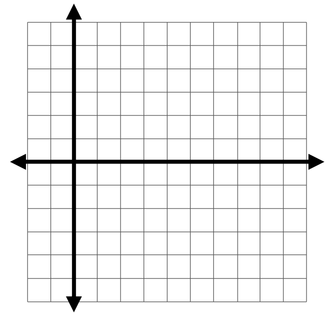


Ex.5 Sketch the region to evaluate the integral.

a) $\displaystyle\int_{-4}^{2} \frac{|x|}{x}\,dx$



b) $\displaystyle\int_{0}^{8} \frac{(25-x^2)}{5+x}\,dx$



RULES FOR DEFINITE INTEGRALS

$\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\displaystyle\int_a^a f(x)\,dx = 0$

$\displaystyle\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$, for any constant k

$\displaystyle\int_a^b \left(f(x)\pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Definite Integral: For any integrable function, $\displaystyle\int_a^b f(x)\,dx$ = (area above the x-axis) – (area below the x-axis).

Note: An integral can be negative, an area cannot!

Note: All continuous functions are integrable. A discontinuous function MAY be integrable.

Ex.6 Evaluate the integral using the given values. Properties (270).

$$\int_2^5 f(x)\,dx = 6,\ \int_{-1}^2 g(x)\,dx = -2,\ \int_{-1}^2 h(x)\,dx = 3,\ \int_{-1}^2 f(x)\,dx = -8$$

a) $\displaystyle\int_{-1}^{2} \left(g(x)-h(x)\right)dx$

b) $\displaystyle\int_{5}^{2} f(x)\,dx$

c) $\displaystyle\int_{-1}^{2} g(x)\cdot h(x)\,dx$

d) $\displaystyle\int_{-1}^{5} \left[2+7f(x)\right]dx$

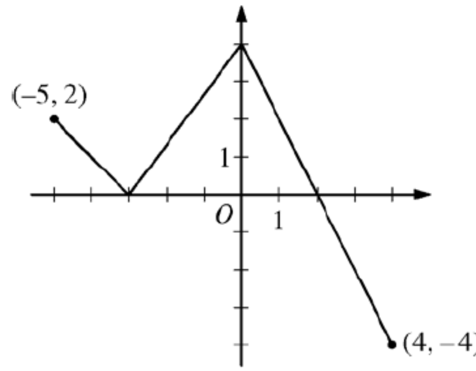
e) $\displaystyle\int_{-1}^{2} \left[4f(x)-2g(x)\right]dx$

2014

Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.



(a) Find $g(3)$.

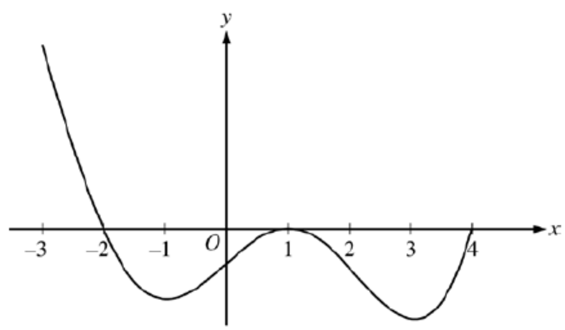
(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

2015 (d) For after 4.4:

Question 5

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.



Graph of f'

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

4.4 The Fundamental Theorem of Calculus (275) Notes #2-12

Date: _______

FTC: $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$ $\quad F(b) = F(a) + \int_a^b f(x)dx$

What about the + C?

$$F(b) + C - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$$

Ex.1 Evaluate the definite integral

a) $\int_0^2 (x^2 - 2x)dx$ b) $\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right)dx$

Ex.2 Find the area under the curve $f(x) = \sin x$ on the interval $[0, \pi]$.

The Mean Value Theorem for Integrals: $\int_a^b f(x)dx = f(c)(b-a)$

Ex.3 Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the indicated interval.

$f(x) = 3x^2$, $[0, 2]$

Average Value of a Function on an Interval $\frac{1}{b-a}\int_a^b f(x)dx$

(average value vs. average rate)

Average rate vs. Average value - depends on what they give you:

$\dfrac{x(b)-x(a)}{b-a}$ $\dfrac{\text{distance}}{\text{time}}$ $\dfrac{1}{b-a}$ time $\int v(t)dt$ distance

Ex.4 Find the average value of the function over the interval and all values of x in the interval for which the function equals it's average value.
$f(x)=x^2+1$ on the interval [1, 4].

Ex.5 A store gets1300 cases of candy every 30 days. x days after the shipment arrives, the inventory still on hand is $I(x)=1300-50x$. Find the average daily inventory. Then find the average daily holding cost if holding on to a case costs 3 cents a day.

Average Daily Inventory:

Average daily holding cost:

Ex.6 Water flows in and out of a storage tank. The net rate of change (rate in minus rate out) of water is $f(t) = 20(t^2 - 1)$ gallons per minute.

a) For $0 < t < 3$, determine when the water level is increasing.

b) If the tank has 200 gallons of water at time $t = 0$, determine how many gallons are in the tank at time $t = 3$.

Ex.7 $\int_2^x \left(3t^2+1\right)dt$

The Second Fundamental Theorem of Calculus (different variables)

If f is continuous on [a, b], and $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$, on [a, b].

Ex.8 Find $F'(x)$. $\frac{d}{dx}\int_2^x \left(7t^2+1\right)dt$

Special Case #1: What if the upper limit of integration is a variable expression other than x? We must use the **chain rule**.

Ex: Find $\frac{dy}{dx}$ if $y=\int_0^{x^3}\sqrt{1+\cos\left(t^2\right)}\,dt$.

(283) #3 and 2007 FR #3c

Special Case #2: What if the variable is the lower limit of integration? We must use the properties of integration to **switch** the limits of integration.

Ex: Find $\frac{d}{dx}\int_x^1 \frac{1}{t}dt$.

Special Case #3: What if there are variables in both the lower and upper limits of integration? Use the properties of integration to **split** them into two.

$$\frac{d}{dx}\int_x^{3x}\frac{1}{t}dt=$$

2017 #2 When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas at a rate modeled by

$$f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \leq 12,$$

where $f(t)$ is measured in pounds/hour and t is the number of hours after the store opened. After the store has been open for 3 hours store employees add bananas to the display at a rate modeled by

$$g(t) = 3 + 2.4\ln(t^2 + 2t) \text{ for } 0 < t \leq 12,$$

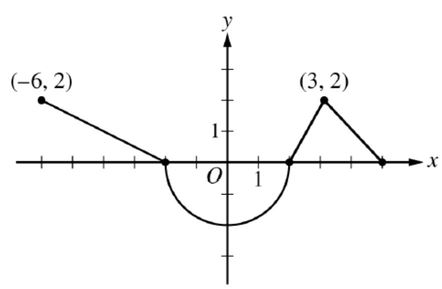
where $g(t)$ is measured in pounds/hour and t is the number of hours after the store opened.

(a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

(d) How many pounds of bananas are on the display table at time = 8?

2017 #3 f is differentialable on the closed interval [-6, 5] and $f(-2) = 7$.

(a) Find the values of $f(-6)$ and $f(5)$.



Graph of f'

(c) Find the absolute minimum value of f on the closed interval [-6, 5]. Justify your answer.

Slope Fields (Appendix pg. A6) Notes #2-13

Date: _______

Differential Equation (243): an equation containing a derivative, for example, $\frac{dy}{dx}=2y-\sin x$

Initial Value Problem: the problem of finding a function y of x when we are given its derivative and its value at a particular point

Initial Condition: the value of f for one value of x

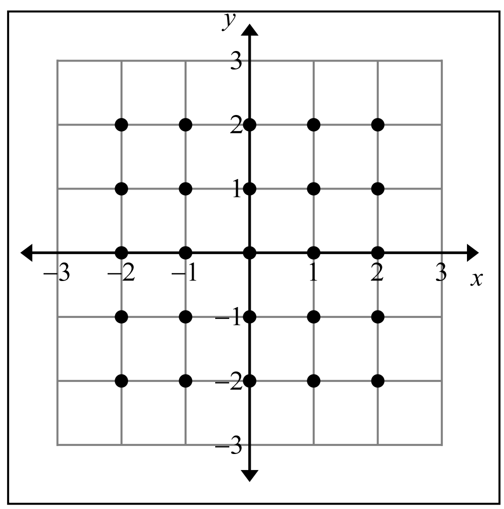
Ex.1 Find a particular solution to the differential equation $\frac{dy}{dx}=\cos x-2x$, for which $f(0) = 1$.

Drawing a Slope Field

- Evaluate the differential equation at various points, (x, y).
- At each of these points, (x, y), sketch a line segment with the slope found by evaluating the differential equation.

Ex.2 Draw a slope field for the differential equation. $\frac{dy}{dx}=x+y$

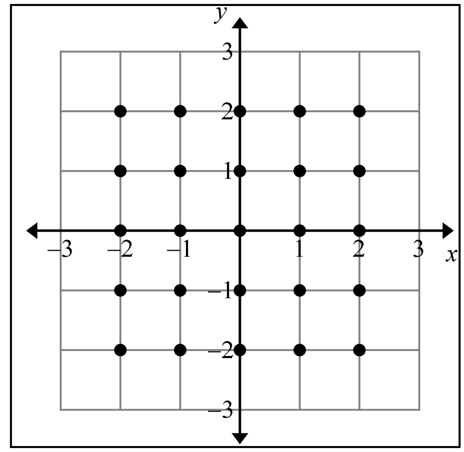
Make a table and choose values for (x, y).



Draw the line segments with slopes found in the table.

Ex.3 Draw a slope field for $\frac{dy}{dx} = x + 1$.

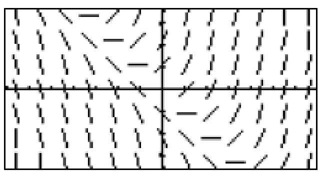
This differential equation, is **autonomous**. The slopes of the tangent lines in the field only depend upon one variable.



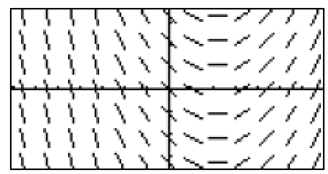
Matching Slope Fields to Differential Equations

Ex.4 Match the slope fields with their differential equations.

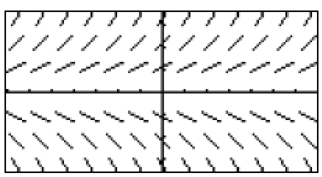
(A)



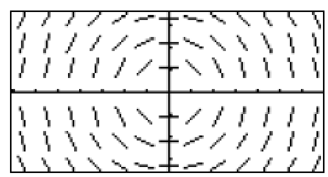
(B)



(C)



(D)



1. $\frac{dy}{dx} = 0.5x - 1$

2. $\frac{dy}{dx} = \frac{1}{2}y$

3. $\frac{dy}{dx} = -\frac{x}{y}$

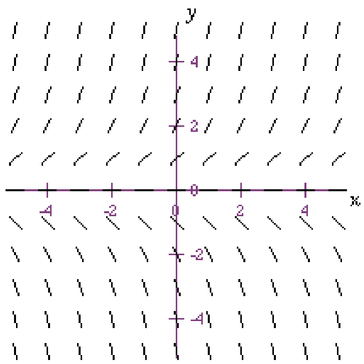
4. $\frac{dy}{dx} = x + y$

Slope Field (Direction Field): for the first order differential equation $\frac{dy}{dx} = f(x, y)$, a plot of short line segments with slopes $f(x, y)$ for lattice points (x, y) in the plane.

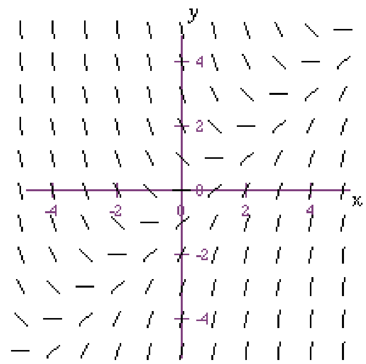
The solutions of the differential equations are certain functions. The differential equation defines the slope at the point (x, y) of the certain curve of the function that passes through this point. For each point (x, y), the differential equation defines a line segment with slope $f(x, y)$. We say that the differential equation defines the slope (or direction) field of the differential equation.

Sketching a Solution Curve in a Slope Field

Ex.5 Consider the slope fields below. Sketch at least 5 solution curves for each differential equation. Possible solution curves:



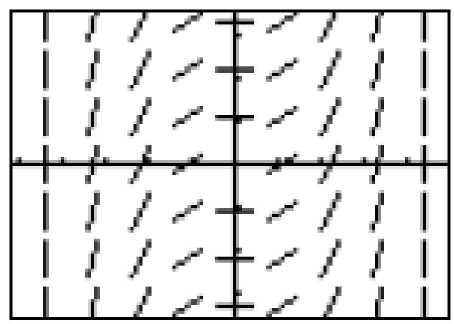
Slope Field for $\frac{dy}{dx} = y$



Slope Field for $\frac{dy}{dx} = x - y$

Note that the solution curves thread their way through the fields much like a leaf in a stream following the streamlines created by the current.

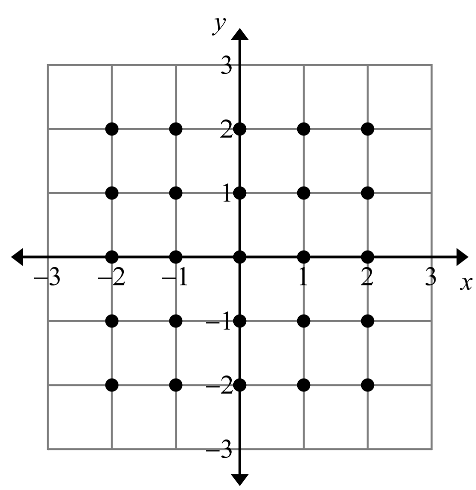
Ex.6 The slope field for a certain differential equation is shown. Which of the following could be a specific solution to that differential equation?



(A) $y = \sin x$ (B) $y = \cos x$

(C) $y = x^2$ (D) $y = \frac{1}{6}x^3$

Ex.7 Draw a slope field for the differential equation $\frac{dy}{dx} = 2x$. Then sketch a solution curve with initial condition $f(1) = -1$.



4.5 Integration by Substitution (288) Notes #2-14

Date: ______

W.1 Differentiate these functions:

a) $y = (3x + 2)^5 + 7$ b) $y = \sin 6x$

c) $y = \sqrt{x^2+1}$ d) $y = \tan^2 x$

$$\frac{d}{dx}\left[f(g(x))\right] = f'\left(g(x)\right)\cdot g'(x) \qquad \int f\left(g(x)\right)\cdot g'(x)\,dx = F(g(x)) + C$$

Let $u = g(x)$ and $du = g'(x)\,dx$, then $y = f(u) + C$ and $y' = f'(u)\cdot u'$ and $\int f\left(u\right)du = F(u) + C$.

Ex.1 Find:

a) $\int 15(3x+2)^4 dx$ b) $y = \int \frac{2x}{\sqrt{x^2+1}}dx$

Ex.2 Find:

a) $\int 6\cos 6x dx$ b) $\int 2\tan x \sec^2 x\, dx$

Ex.3 Find: $\int \cos 6x dx$

Ex.4 Find: $\int \frac{dx}{\left(\frac{1}{3}x-8\right)^5}$

Less apparent substitutions

Ex.5 Find: $\int x^2\sqrt{x-1}dx$

Ex.6 Find: $\int \cos^3(x)\cdot dx$

Change of Variables for Definite Integrals

Method 1: Convert the limits of integration to values in terms of u

Method 2: Leaving the limits of integration off and then convert back to a function in terms of x

Ex.7 Evaluate the definite integral. $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x dx$

Ex.8 Evaluate the definite integral. $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$

Integration of Even and Odd Functions

If f is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

If f is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

Ex.9 Evaluate:

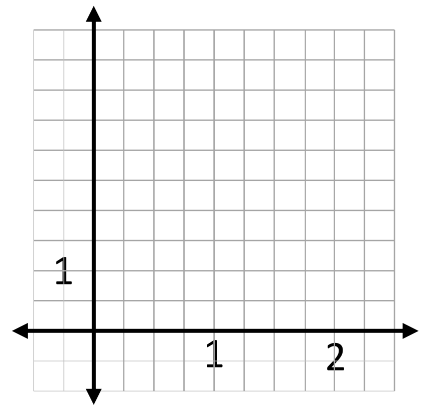
a) $\int_{-3}^{3} x^2 dx$

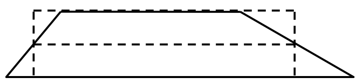
b) $\int_{-17}^{17} x^3 dx$

4.6 Numerical Integration (300) Notes #2-15

Date: _______

Ex.1 Use 4 trapezoids of equal heights to approximate the area under the curve $y = x^2$ on the interval [0, 2]. Then find the exact value. Draw the graph and sketch the trapezoids. (Note: Trapezoid I is a "special" trapezoid with one base equal to 0.)





Note: The bases of the trapezoids are the function values for each value of x.

Sum of the areas:

Note that all of the trapezoids have the same height. Also, trapezoids share a common base. So instead of finding each area individually, we could put them all together:

Exact value:

Is the trapezoidal approximation larger or smaller than the actual? Why?

Trapezoidal Rule: $\int_a^b f(x)dx \approx T_n = \frac{b-a}{2n}\left(y_0 + 2y_1 + 2y_2 + ... + 2y_{n-1} + y_n\right)$.

Note: Be careful! This only works if the trapezoids have a common height.

Ex.2 The table was created by recording the temperature every hour from noon until midnight. Use the trapezoidal rule to approximate the average temperature for the 12-hour period.
(average value vs. average rate)

Time	Noon	1	2	3	4	5	6	7	8	9	10	11	Midnight
Temperature	76	78	80	79	85	86	82	80	78	70	68	65	63

Average Temperature =

Ex.3 Find the approximations of $\int_a^b f(x)dx$. The function f is continuous on the interval [1, 7] and has these values:

x	1	2	4	6	7
$f(x)$	10	30	20	40	30

a) trapezoidal

b) midpoint

c) left Riemann

d) right Riemann

CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. For $0 \le t \le 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

   (a) Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

   (b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

   (c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

   (d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

WRITE ALL WORK IN THE EXAM BOOKLET.

11 perfect scores out of 104,600 in 2013.

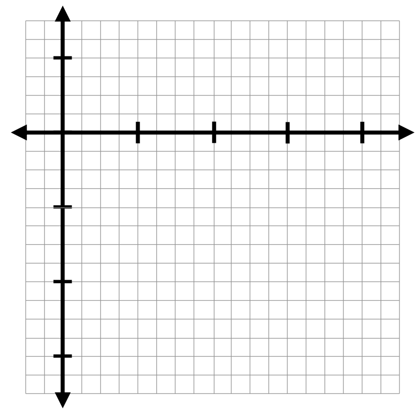
5.1 The Natural Log Function: Differentiation (314) Notes #3-1

Date: _______

The <u>natural logarithmic function</u> is defined by $\ln x = \int_1^x \frac{1}{t}\,dt$, $x > 0$.

The domain is the set of all positive real numbers.

Graph $\ln x$ using a slope field & the differential equation $\frac{dy}{dx} = \frac{1}{x}$. $\ln 1 = 0$



<u>Properties</u> (If a & b are positive and n is rational):

$\ln(ab) = \ln a + \ln b$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln(a^n) = n \cdot \ln a$ $\qquad$ $\ln 1 = 0$

Ex.1 Use the properties to expand:

a) $\ln x^4$ $\qquad$ b) $\ln\left(\frac{x}{yz}\right)$ $\qquad$ c) $\ln(ex^2)$

Use the properties to condense:

d) $\ln x + \ln 4$ $\qquad$ e) $\ln x - 3\ln(x+1)$ $\qquad$ f) $2\ln x - \ln(x+1) - \ln(x-1)$

Ex.2 Use the properties to approximate given $\ln 2 \approx 0.693$ and $\ln 3 \approx 1.099$.

a) $\ln 12$ $\qquad$ b) $\ln 27$ $\qquad$ c) $\ln\sqrt{18}$

The positive real number $e \approx 2.718281828459045\ldots$, such that $\ln e = \int_1^e \frac{1}{t}\,dt = 1$.

The Derivative of the Natural Logarithmic Function

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x},\ x > 0 \qquad\qquad \frac{d}{dx}\left[\ln u\right] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ u > 0$$

Ex.3 Find the derivative:

a) $\ln \frac{x}{3}$

b) $\ln \sqrt{\sin x}$

Ex.4 Find the derivative: $\ln (x^3 + 1)$

Ex.5 $\lim_{x \to 1} \frac{\ln(x+2) - \ln 3}{x-1}$

Ex.6 Find the derivative of $\ln \frac{\sqrt{x^2+1}}{(9x-4)^2}$

Logarithmic Differentiation (319)

Ex.7 Find the derivative of $f(x)=\dfrac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}$

2011

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

   (a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

   (b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

   (c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

   (d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

5.2 The Natural Log Function: Integration (324) Notes #3-2

Date: ______

$\int \frac{1}{x} dx = \ln|x| + C$ Properties of logarithms result in equivalent forms that may look different.

Ex.1 $\int \frac{1}{6x+1} dx$

Ex.2 $\int \frac{1}{x^{2/3}(x^{1/3}+1)} dx$

Ex.3 Find the area of the region:

$x = 1, x = e, y = 0, \ y = \frac{x+1}{x^2}$

Ex.4 Find:

$\int \frac{5x^4 - 3}{x^5 - 3x} dx$

<u>Using Long Division Before Integrating</u> (not likely on AP exam)

Ex.5 $\int \frac{x-1}{x+1} dx$

Ex.6 $\int \frac{x^3 + 2x^2 - 4}{x^2 - 2} dx$

Ex.7 Solve the differential equations:

a) $\dfrac{dy}{dx} = \dfrac{2x}{x^2+9}$

b) $\dfrac{dy}{dx} = \dfrac{3}{2x\ln(x^2)}$

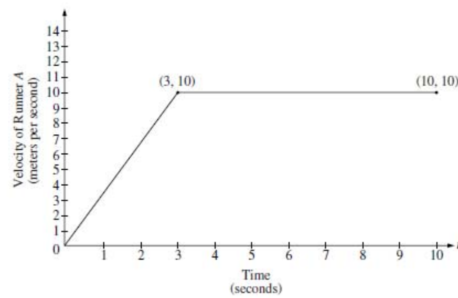
<u>Integrals of Trigonometric Functions (329)</u> 1: Memorized, 2: U-sub, or 3: Identity

Ex.8 $\int \tan x dx$ * Express 2 ways

Ex.9 $\int \sec\dfrac{x}{2}dx$

Ex.10 Find the average value of $f(x) = \sqrt{1+cot^2 x}dx$ on $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$.

Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.



(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

Catalog "a"
Type "abs("

5.3 Inverse Functions (332) Notes #3-3

Date: _______

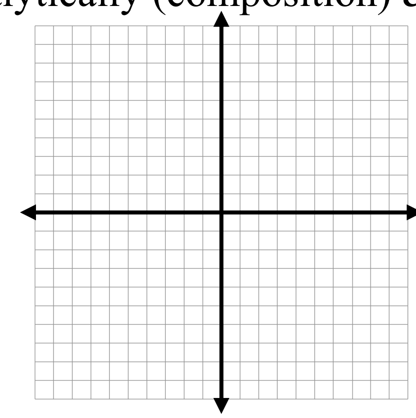
A function g is the inverse function [$f^{-1}(x)$ read "f inverse"] of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of f.

f^{-1} is not f to the -1 power, it's the symbol for inverse.

The domain of $f^{-1}(x)$ is equal to the range of f and the range of $f^{-1}(x)$ is equal to the domain of f.

Inverse functions are reflections over the line $y = x$.

Ex.1 Show the $f(x)$s are inverses of each other analytically (composition) & graphically. $f(x) = 2x+3$ and $g(x) = \frac{1}{2}x - \frac{3}{2}$.



Not every function has an inverse. **Horizontal Line Test**: a function has an inverse if and only if every horizontal line intersects the graph at most once (one-to-one, see P.3).

If a $f(x)$ is strictly monotonic (always decreasing or always increasing, see section 3.3) on its entire domain, then it's one-to-one and so has an inverse.

Ex.2 Is the function one-to-one (so has an inverse)? Use HLT or $f'(x)$.

a) $f(x) = 5 - 2x^3$

b) $f(x) = 5 + 2x - 2x^3$

Steps for finding an inverse relation:

1. Switch the x & y in the relation.
2. Solve for the variable (y).

Ex.3 Find the inverse function of:

a) $f(x)=\sqrt[5]{\frac{3x-1}{x-2}}$

b) $f(x)=\frac{2x}{x-3}$

Derivative of an Inverse Function:

$f(x)=\frac{x^2+3}{2}=$

$g(x)=\sqrt{2x-3}=$

$f'(x)=$

$g'(x)=$

(3, 6) on f $\Leftrightarrow$ (6, 3) on g

$f'(3)=$

$g'(6)=$

Note: The inverse of a function is the reflection over the line $y = x$, a change in y becomes a change in x, and a change in x becomes a change in y. Thus $\frac{dy}{dx}$ becomes $\frac{dx}{dy}$, which is why we use the reciprocal.

If g is the inverse of f then $g'(x)=\frac{1}{f'(g(x))}$ for (a, b) $g'(a)=\frac{1}{f'(b)}$

Ex.4 Let $f(x) = x^5 + x + 1$.

a) Use the derivative to determine whether the function is strictly monotonic.

b) Find $\left(f^{-1}\right)'(a)$ for a = 1.

2007

3. The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\, dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

5.4 Exponential f(x)s: Differentiation & Integration (341) Notes #3-4

Date: _______

$3^x = 7$

$e^5 = y$

$\log_4 6 = x$

$\ln x = 5$

common log: log x means $\log_{10} x$ ln e =

natural log: ln x means $\log_e x$ ln 1 =

e^x has a slope of 1 at (0, 1).

$f(x) = e^x$	$g(x) = \ln x$	Techniques
$y = e^x$	$y = \ln x$	1. Write in exp form
inverse $x = e^y$	inverse $x = \ln y$	2. Get in same base
rewrite $\ln x = y$	rewrite $y = e^x$	3. Take ln both sides

$e^{\ln 7} = n$ $\ln e^{x+2} =$

Ex.1 Solve

a) $2^x = 5$ b) $e^{\ln 2x} = 12$

Ex.2 Solve

a) $\ln(5x) = 8$ b) $\ln(3x+1)^2 = 8$

Derivatives of Exponential Functions

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}$$

Ex.3 a) Find $f'(x)$ for $f(x) = 3e^{x^2}$ b) $\frac{d}{dx}\left[xe^{2/x}\right]$

c) Find y' if $y = e^{\sqrt{x^2+1}}$ d) $\frac{d}{dx}\left[e^{\tan x}\right]$

Ex.4 Find the local extrema of $f(x) = e^{-x^2/2}$.

Ex.5 Find any points of inflection (if any exist) of $f(x) = xe^x$.

Integrating Exponential Functions

$\int e^x dx = e^x + C$ $\int e^u du = e^u + C$

Ex.6 $\int xe^{3x^2+1}dx$

Ex.7 $\int \frac{e^{3/x}}{x^2}dx$

Ex.8 a) $\int_0^4 \frac{1}{e^{2x}}dx$

b) Suppose the downward velocity of a sky diver is given by $v(t) = 30(1 - e^{-t})$ ft/s for the first 5 seconds of a jump. Compute the distance fallen.

Ex.9 Find the derivative of $f(x) = \ln\left(\frac{e^x + e^{-x}}{2}\right)$.

5.5 Bases Other than e and Applications (351) Notes #3-5

Date: _______

$\log_{16}8$ $\log_a x = n$

$\log 0.01 = x$

$2^{3x} = 50$

$a^r = e^{r\ln a}$

Ex.1 The half-life of carbon-14 is about 5730 years. Find two models that yield the fraction (A/A_0) of carbon-14 as a function of time and determine that fraction at 8,585 years.

Ex.2 Solve

a) $2^{-2x} = \dfrac{1}{32}$ b) $\log_4 (x + 3) + \log_4 x = 1$

<u>Derivatives of Exponential Functions</u> (use change of base 1st)

$$\frac{d}{dx}\left[a^x\right] = (\ln a)a^x \qquad \frac{d}{dx}\left[a^u\right] = (\ln a)a^u\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x} \qquad \frac{d}{dx}\left[\log_a u\right] = \frac{1}{(\ln a)u}\frac{du}{dx}$$

Ex.3 Find the derivative of:

a) $y = 4^w - 5\log_7 w$ b) $y = 2(3)^x + 5e^x$ c) $f(x) = 3^{2x^2}$

Integrating Exponential Functions

$$\int a^x dx = \frac{1}{\ln a} a^x + C \qquad\qquad \int e^u du = e^u + C$$

Ex.4 a) $\int 2^{-x} dx$ b) $\int \frac{2^x}{2^x+1} dx$ c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Ex.5 Find the derivative:

a) $y = e^5$

Logarithmic Differentiation

b) $y = x^{\ln x}$

Applications of Exponential Functions

Compound Interest:

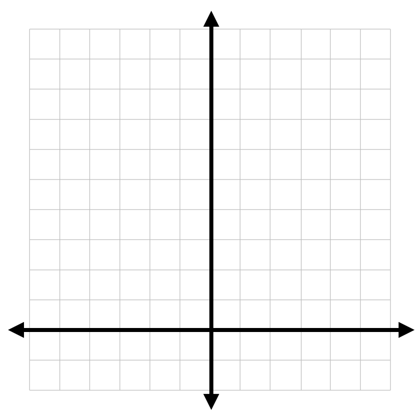
$$A = P\left(1+\frac{r}{n}\right)^{nt} \qquad\qquad A = Pe^{rt}$$

Ex.6 Calculate the balance when \$3000 is invested for 10 years, at 6% compounded:

a) quarterly b) weekly c) continuously

Ex.7 Estimate the maximum population for Dallas & for the year 2006 given the logistic function: $P(t)=\dfrac{1{,}301{,}642}{1+21.602e^{-0.05054t}}$ from 1900.

Ex.8 Graph by hand: $y = 2^{|x|}$.



5.6 Differential Equations: Grow and Decay (361)

Notes #3-6
Date: _______

<u>Differential Equations</u> (Separation of variables – 5.7)

1. Separate
2. Integrate
3. Evaluate - c

Ex.1 Solve the differential equation $y' = -4xy^2$, $y(0) = 1$.

<u>Exponential Growth & Decay Model</u>

<u>Law of Exponential Change</u>: If y is a differentiable function of t that changes at a rate proportional to the amount present $\left(\frac{dy}{dt}=ky\right)$ & $y > 0$, then $y = Ce^{kt}$.

C is the **initial value** (y_0) of y and k is the **proportionality constant** (**rate constant)**. $k > 0$ represents **exponential growth** & $k < 0$ represents **decay**.

Differentiate $y = Ce^{kt}$ with respect to t and verify $y' = ky$.

Ex.2 <u>The rate of change of y is proportional to y</u>. When $t = 0$, $y = 3$. When $t = 3$, $y = 5$. What is the value of y when $t = 4$?

Ex.3 Carbon-14 is radioactive and decays at a rate proportional to the amount present. Its half-life is 5730 years. If 10 grams were present originally, how long will it take to decay to 7.851 grams?

Ex.4 The world population was approximately 5.9 billion in 1998 and 6.9 billion in 2011. Approximately how many people were there in 1990?

Ex.5 (364)

Newton's Law of Cooling: The rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temp & the ambient (environmental) temp. $y(t)$ is the temp of the object at time t, and T_a is the ambient temp: $y'(t)=k[y(t)-T_a]$.

Ex.6 A hot potato at 100°C is put in a pan under running 20°C water to cool. After 6 minutes, the potato's temperature is found to be 40°C. How much longer will it take the potato to reach 25°C?

Ex.7 A cup of coffee is 180°F. After 2 minutes in a 70°F room, the coffee has cooled to 165°F. How much longer it will take to cool to 120°F?

5.7 Differential Eqs: Separation of Variables (369)

Notes #3-7
Date: _______

The order of a differential equation is determined by the highest-order derivative in the equation. ie $y^{(3)} = 4y$ is a third-order differential equation.

The general solution represents a family of curves - the solution curves.

General Solutions

Ex.1 Is the given function a solution of the differential equation $y' - y = e^{2x}$?

a) $y = e^{2x}$

b) $y = Ce^{x} + e^{2x}$

Separation of Varibales and Particular Solutions (Initial conditions)

1. Separate
2. Integrate
3. Evaluate (for C)

On almost every free response!

Ex.2 Find the particular solution of: $\frac{dy}{dx} = y + 1$, $y(0) = 1$

Ex.3 Find the particular solution of $y' = \frac{x^2 + 7x + 3}{y}$, given $y(0) = -2$.

Ex.4 Find the particular solution of:

Some exercises end up implicit in our text (372).

a) $\dfrac{dy}{dx} = \dfrac{2xy}{x^2+1}$, $y(2) = 10$

b) $(4y - \cos y)\dfrac{dy}{dx} - 3x^2 = 0$, $(0, 0)$

Ex.5 Find a curve in the xy-plane that passes through $(0, 3)$ and whose tangent line at point (x, y) has slope $2x/y^2$.

Ex.6 Find the particular solution to $\dfrac{dy}{dx} = \dfrac{1}{25}(y - 300)$ with $y(0) = 1400$.

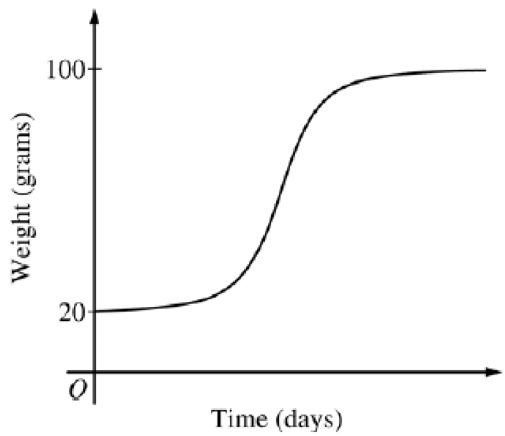
5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

2011 1.6/2.7
2012 2.9/4.3

5.8 Inverse Trig Functions - Differentiation (380) Notes #3-8

Date: _______

Function	Domain	Range
$y = \arcsin x$		
$y = \arccos x$		
$y = \arctan x$		
$y = \text{arccot}\, x$		
$y = \text{arcsec}\, x$		
$y = \text{arccsc}\, x$		

Ex.1 Evaluate:

a) $\arcsin \frac{\sqrt{2}}{2}$

b) $\arccos -\frac{\sqrt{2}}{2}$

c) $\sin^{-1} (\sin \pi/7)$

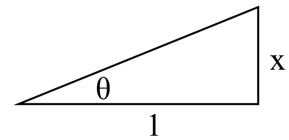
d) $\sin^{-1} 3$

e) $6\tan^{-1} x = \pi$

f) $\sin^{-1} (\sin x) = 6\pi/7$

Ex.2 Solve: $\arcsin(2x - 1) = \pi/6$

Ex.3 Use the triangle to answer the questions.



a) Find tan θ.

b) Find $\tan^{-1} x$.

c) Find the hypotenuse as a function of x.

d) Find sin ($\tan^{-1}$ (x)) as a ratio involving no trig $f(x)$s.

e) Find sec ($\tan^{-1}$ (x)) as a ratio involving no trig $f(x)$s.

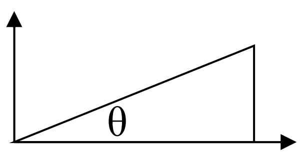
<u>Remember</u>: the inverse trig functions are angles! We don't need θ to solve.

Always think of the restricted domain! For arctan it is (-π/2, π/2). 1/5 is + so it is in QI.

Ex.4a cos (arctan 1/5)

$\downarrow$

θ



Ex.4b Find an algebraic expression equivalent to: sin [arccos (4x)]

<u>Derivatives of Inverse Trigonometric Functions</u> (383)

$$\frac{d}{dx}\left[\sin^{-1}u\right]=\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\cos^{-1}u\right]=-\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right]=\frac{u'}{1+u^2} \qquad \frac{d}{dx}\left[\cot^{-1}u\right]=-\frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[\sec^{-1}u\right]=\frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}\left[\csc^{-1}u\right]=-\frac{u'}{|u|\sqrt{u^2-1}}$$

Ex.5 Differentiate:

a) $y = \arcsin x^3$ b) $y = \operatorname{arcsec} e^x$

Derivative of Arcsine

$y = \sin^{-1} x$ can be rewritten as $\sin y = x$, and is differentiable on the open interval $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Using implicit differentiation, we have $\cos y \dfrac{dy}{dx} = 1$, or $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$.

We need the derivative in terms of *x*, so we will use a Pythagorean identity to replace $\cos y$.

$$\sin^2 y + \cos^2 y = 1 \;\Rightarrow\; \cos y = \sqrt{1 - \sin^2 y}$$

Note: We need only use the positive square root because $\cos y$ is positive on the interval $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Substituting, we have $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - \sin^2 y}}$, and since $\sin y = x$, we have $\boxed{\dfrac{d}{dx}\left[\sin^{-1} x\right] = \dfrac{1}{\sqrt{1 - x^2}}}$

Cofunctions of the Inverse Trigonometric Functions

Recall: $\cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1} x$ $\quad\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$ $\quad\csc^{-1} x = \dfrac{\pi}{2} - \sec^{-1} x$

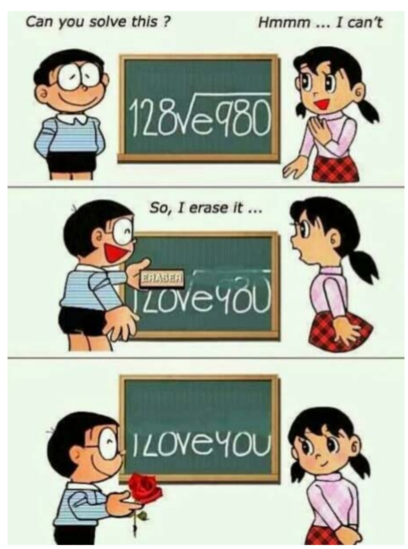
Derivatives of the Cofunctions of Inverse Trigonometric Functions (differentiate the equations shown above)

$$\frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\left[\cot^{-1} x\right] = -\frac{1}{1 + x^2}$$

$$\frac{d}{dx}\left[\csc^{-1} x\right] = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

So, the derivatives of the cofunctions are the *opposite* of the derivatives of their respective inverse trig functions.



5.9 Inverse Trig Functions - Integration (388)

Notes #3-9

Date: _______

$$\frac{d}{dx}\left[\sin^{-1}u\right]=\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\cos^{-1}u\right]=-\frac{u'}{\sqrt{1-u^2}}$$

<u>Integrals Involving Trig Functions</u>

$$\int \frac{u'}{\sqrt{a^2-u^2}}du = \sin^{-1}\frac{u}{a}+C$$

$$\int \frac{u'}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\frac{u}{a}+C$$

$$\int \frac{u'}{|u|\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\frac{|u|}{a}+C$$

Ex.1 Evaluate:

a) $\int \frac{dx}{\sqrt{4-x^2}}$ b) $\int \frac{dx}{2+9x^2}$ c) $\int \frac{dx}{x\sqrt{4x^2-9}}$

Ex.2 Find $\int \frac{dx}{\sqrt{e^{2x}-1}}$

Ex.3 Find $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$

Completing the Square

Ex.4 Find $\int \frac{dx}{x^2-4x+7}$

Ex.5 Find the area of the region bounded by the graph of $f(x)=\frac{1}{\sqrt{3x-x^2}}$ the x-axis and the lines $x=\frac{3}{2}$ and $x=\frac{9}{4}$.

Text: "Simplify: 2i<6u"

7.7 Indeterminate Forms and L'Hopital's Rule (530) Notes #3-11

Date: _______

Objectives: Recognize limits that produce indeterminate forms. (61) & (194)
Apply L'Hopital's Rule to evaluate a limit.

Limits that result in $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^∞, ∞^0 or $0 \cdot \infty$ when we attempt direct substitution are called **indeterminate forms**. Sometimes we can use the dividing out and rationalizing the numerator techniques to find these limits.

L'Hopital's Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ for $g'(x) \neq 0$ except possibly at c.

Warning: applying the rule to limits that are not indeterminate can produce errors. Always check direct substitution first!

Warning: only use for $\frac{0}{0}$ & $\frac{\infty}{\infty}$. **Warning**: not the quotient rule.

Ex.1-3 Evaluate the limits (a) using techniques from Ch. 1 & 3 and
(b) using L'Hopital's Rule.

Ex.1 $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} =$

Ex.2 $\lim_{x \to 0} \frac{\sin 2x}{x} =$

Ex.3 $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} =$

Ex.4 Evaluate the limit, using L'Hopital's Rule **if necessary**.

a) $\lim\limits_{x \to \pi/2} \dfrac{1-\sin x}{\cos x} =$

b) $\lim\limits_{x \to 2} \dfrac{x-2}{x-6} =$

c) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2} =$

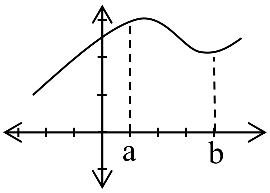
d) $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4} \right) =$

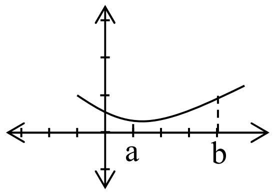
e) $\lim\limits_{x \to +\infty} \dfrac{x}{e^x} =$

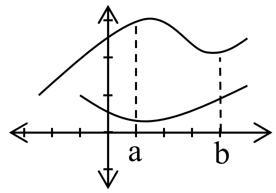
f) $\lim\limits_{x \to 0^+} \dfrac{\ln x}{\csc x} =$

6.1 Area of Region Between Two Curves (412)

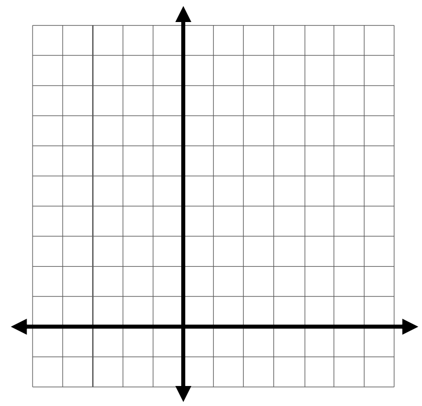
Notes #3-10
Date: _______





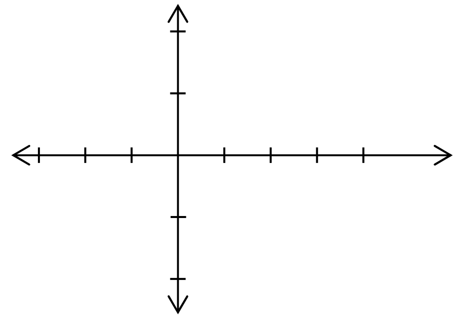


Ex.1 Find the area of the region bounded by the graphs of $y = x + 6$, $y = x^2 + 4$, $x = 0$ and $x = 1$.

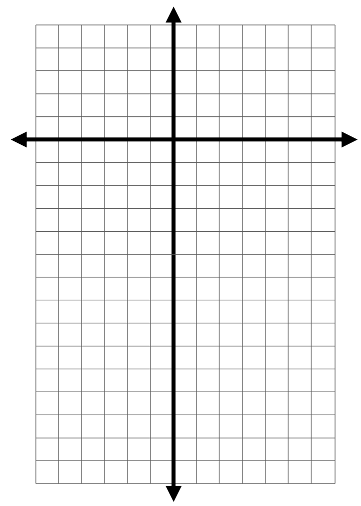


Ex.2 Find the area of the region bounded by the graphs of $f(x) = x + 6$ and $g(x) = x^2 + 4$.

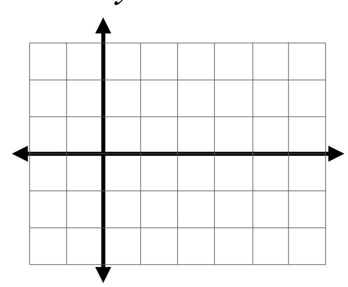
Ex.3 The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of those regions.



Ex.4 Find the area of the region between the graphs of $f(x) = x^2 - 5x - 7$ and $g(x) = x - 12$ over [-1, 5].

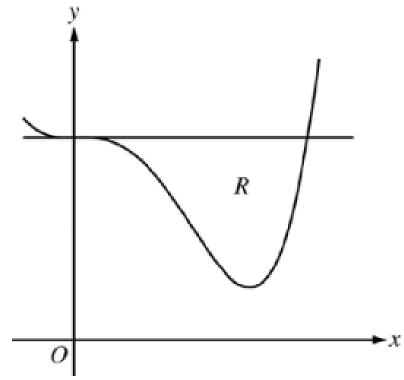


Ex.5 Find the area of the region bounded by the graphs of $x = y^2$ and $x = y + 2$. (Can be done two ways.)



Accumulation (417)

Question 2



Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

2017

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by

$x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$.

Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

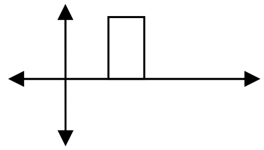
(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

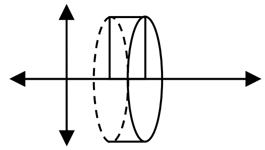
(d) Find the position of particle Q the first time it changes direction.

6.2 Volume the Disk Method (421)

Notes #3-12

Date: _______





Disk Method:

Volume of disk = (area of the disk)(width of the disk)

$V = \pi R^2 w$ $\Delta V = \pi R^2 \Delta x$ $\sum_{i=1}^{n} \pi \left[R(x_i) \right]^2 \Delta x$

$\pi \sum_{i=1}^{n} \left[R(x_i) \right]^2 \Delta x$ $\lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} \left[R(x_i) \right]^2 \Delta x$ $\pi \int_a^b \left[R(x) \right]^2 dx$

$n \to \infty$

Note: we can also do in terms of y if we are revolving around a vertical line.

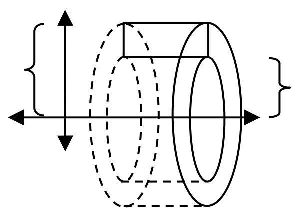
Ex.1 Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $[0, \pi]$ about the x-axis.

$y = 0$

Ex.2 Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.

* symmetry

Washer Method



V = $\pi R^2 w - \pi r^2 w$
V = $\pi(R^2 - r^2)w$

$$\pi\int_a^b \left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)dx$$

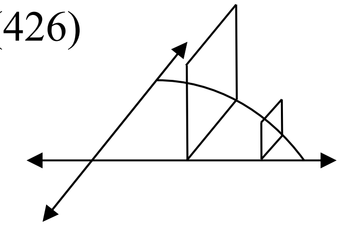
Ex.3 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$ about the x-axis. $y = 0$

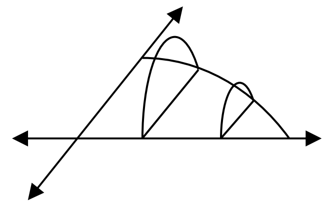
Ex.4 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$ about the y-axis. $x = 0$

Ex.5 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2$, $x = 2$ and $y = 0$ about the line $y = -1$.

Solids with Known Cross Sections (426)

$$\int_a^b (\text{Cross-sectional Area})\,dx$$



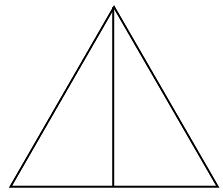


Square

$$\int_a^b [f(x)]^2\,dx$$

Semi-circle: varies!

$$\frac{\pi}{2}\int_a^b [\text{radius}]^2\,dx$$



Equilateral Triangle

$$\frac{\sqrt{3}}{4}\int_a^b [f(x)]^2\,dx$$

Ex.6-8 Find the volume of the solid.

Ex.6 The base is bounded by the circle $x^2 + y^2 = 4$. The cross sections are SQUARES, $\perp$ to the x-axis.

Ex.7 The base of a solid is the region between the x-axis and $y = 4 - x^2$. The vertical cross sections of the solid $\perp$ to the y-axis are SEMI-CIRCLES.

Ex.8 The base of a solid is bounded by $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$ and $x = 0$. The cross sections are EQUILATERAL Δs, $\perp$ to the x-axis.

2017 #1. A tank has a height of 10 feet.

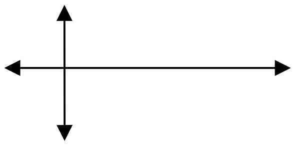
c) Based on the following model, find the volume of the tank. Indicate units of measure. The area, in square feet, of the horizontal cross section at height h ft is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$.

d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

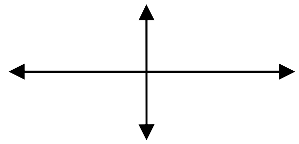
6.3 Volume the Shell Method (432)

Notes #3-13
Date: _______

Disk Method (rad $\perp$ rot)



Shell Method



Ex.1 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$, $x = 1$, $x = 4$ and the x-axis about the y-axis.
$x = 0$

Ex.2 Find the volume of the solid formed by revolving the region bounded by the graph of $x = e^{-y^2}$ and the y-axis ($0 \le y \le 1$) about the x-axis.
$y = 0$

Ex.3 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $x = 0$, $x = 1$ and $y = 0$ about the y-axis.
$x = 0$

Ex.4 Compute the volume of the solid obtained by rotating the area under $y = 9 - x^2$ over [0, 3] about the x -axis.

Ex.5 Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2$, $x = 2$ and $y = 0$ about the line $y = -1$.

See Ex.5 (436) for an example when the Shell Method is necessary.

Example: [1973 AP Calculus AB #35] The region in the first quadrant bounded by the graph of $y = \sec x$, $x = \frac{\pi}{4}$ and the axes is rotated about the x-axis. What is the volume of the solid generated?

A) $\frac{\pi^2}{4}$ B) $\pi - 1$ C) π D) 2π E) $\frac{8}{3}\pi$

Example: [1985 AP Calculus AB #45] The region enclosed by the graph of $y = x^2$ the line $x = 2$, and the x-axis is revolved about the y-axis. The volume of the solid generated is

A) 8π B) $\frac{32}{5}\pi$ C) $\frac{16}{3}\pi$ D) 4π

Example: [1985 AP Calculus BC #35] The region in the first quadrant between the x-axis and the graph of $y = 6x - x^2$ is rotated around the y-axis. The volume of the resulting solid of revolution is given by

A) $\int_0^6 \pi(6x - x^2)^2 dx$ B) $\int_0^6 2\pi x(6x - x^2) dx$ C) $\int_0^6 \pi x(6x - x^2)^2 dx$

D) $\int_0^6 \pi\left(3 - \sqrt{9-y}\right)^2 dy$ E) $\int_0^9 \pi\left(3 - \sqrt{9-y}\right)^2 dy$

Example: [1988 AP Calculus AB #30] A region in the first quadrant is enclosed by the graphs of $y = e^{2x}$, $x = 1$, and the coordinate axes. If the region is rotated about the y-axis, the volume of the solid that is generated is represented by which of the following integrals?

A) $2\pi\int_0^1 xe^{2x} dx$ B) $2\pi\int_0^1 e^{2x} dx$ C) $\pi\int_0^1 e^{4x} dx$

D) $\pi\int_0^e y \ln y \, dy$ E) $\frac{\pi}{4}\int_0^e \ln^2 y \, dy$

Example: [1988 AP Calculus BC #36] Let R be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \pi/2$. The volume of the solid obtained by revolving R about the x-axis is given by

A) $2\pi\int_0^{\frac{\pi}{2}} x\sin x dx$ B) $2\pi\int_0^{\frac{\pi}{2}} x\cos x dx$ C) $\pi\int_0^{\frac{\pi}{2}} (1-\sin x)^2 dx$

D) $\pi\int_0^{\frac{\pi}{2}} \sin^2 x\ dx$ E) $\pi\int_0^{\frac{\pi}{2}} \left(1-\sin^2 x\right) dx$

Example: [1988 AP Calculus AB #43] The volume of the solid obtained by revolving the region enclosed by the ellipse $x^2 + 9y^2 = 9$ about the x-axis is

A) 2π B) 4π C) 6π D) 9π E) $12\ \pi$

Example: [1993 AP Calculus AB #20] Let R be the region in the first quadrant enclosed by the graph of $y = (x+1)^{\frac{1}{3}}$ the line $x = 7$, the x-axis, and the y-axis. The volume of the solid generated when R is revolved about the y-axis is given by

A) $\pi\int_0^7 (x+1)^{\frac{2}{3}} dx$ B) $2\pi\int_0^7 x(x+1)^{\frac{1}{3}} dx$ C) $\pi\int_0^2 (x+1)^{\frac{2}{3}} dx$

D) $2\pi\int_0^2 x(x+1)^{\frac{1}{3}} dx$ E) $\pi\int_0^7 \left(y^3-1\right)^2 dy$

Example: [1993 AP Calculus BC #19] The shaded region R, shown in the figure below, is rotated about the y-axis to form a solid with a volume of 10 cubic inches. Of the following, which best approximates k?

A) 1.51
B) 2.09
C) 2.49
D) 4.18
E) 4.77