AP Calculus BC
Chapter 6 AP Problems
Answer Key

Antiderivatives

1.	E	1993	AB	#17	65%
2.	A	1993	AB	#38	82%
3.	A	1988	AB	#5	89%

Evaluating Definite Integrals

4.	A	1998	AB	#11	42%
5.	D	1985	AB	#1	89%
6.	D	1985	AB	#24	70%
7.	C	1998	AB	#3	71%
8.	E	1998	AB	#7	43%
9.	D	1988	AB	#17	72%
10.	E	1985	BC	#36	26%
11.	A	1985	AB	#22	66%
12.	C	1988	AB	#10	80%
13.	A	1998	AB	#20	69%
14.	A	1993	BC	#33	73%
15.	D	1985	AB	#27	25%
16.	C	1988	AB	#28	34%
17.	A	1988	AB	#13	70%
18.	B	1993	BC	#37	71%
19.	B	1993	AB	#28	13%
20.	E	1993	BC	#3	82%
21.		1988	AB	#6	

U-Substitutions

22.	D	1988	AB	#7	79%
23.	D	1988	AB	#14	67%
24.	A	1993	AB	#14	69%
25.	D	1988	BC	#2	89%
26.	C	1985	AB	#4	81%
27.	D	1985	AB	#32	56%
28.	C	1985	BC	#18	89%
29.	B	1998	BC	#8	55%
30.	A	1993	AB	#32	29%
31.	A	1985	BC	#7	57%
32.	E	1993	AB	#22	32%
33.	E	1985	BC	#28	35%
34.	A	1988	BC	#16	80%
35.	A	1993	BC	#7	81%
36.	B	1985	AB	#30	55%
37.	B	1988	AB	#19	58%
38.	E	1988	AB	#38	47%
39.	C	1998	AB	#88	55%
40.	B	1985	BC	#3	90%
41.	E	1998	AB	#82	24%
42.	A	1985	BC	#40	47%

Separation of Variables

43.	B	1993	AB	#33	14%
44.	C	1985	BC	#33	48%
45.	A	1985	BC	#44	52%
46.	C	1988	BC	#39	43%
47.	C	1993	BC	#13	34%
48.		1998	AB	#4	
49.		1985	BC	#4	
50.		2000	AB	#6	
51.		2002	BC	#5	FormB
52.		2001	AB	#6	
53.		2001	BC	#5	

Integration by Parts

54.	E	1985	AB	#7	62%
55.	C	1985	AB	#17	42%
56.	E	1988	AB	#26	59%
57.	E	1993	BC	#29	61%
58.	B	1993	AB	#43	46%
59.	B	1985	BC	#21	61%

Integration by Partial Fractions

60.	A	1985	BC	#12	60%
61.	D	1988	BC	#17	70%
62.	A	1998	BC	#4	61%

Slope Fields and Euler's Method

63.	C	1998	BC	#24	38%
64.		2000	BC	#6	
65.		1998	BC	#4	
66.		2002	BC	#5	
67.		2004	AB	#6	

Exponential Growth and Decay

68.	B	1993	AB	#42	30%
69.	A	1998	AB	#84	42%
70.	A	1988	BC	#43	44%
71.	C	1993	BC	#38	63%
72.	E	1998	BC	#26	20%
73.		1989	AB	#6	FRQ
74.		1996	BC	#3	FRQ
75.		1974	AB	#7	FRQ
76.		1987	BC	#1	FRQ
77.		2004	BC	#5	FRQ

Free Response – Problem #21
1988 AP Calculus AB Exam – Problem #6

6. Let f be a differentiable function, defined for all real numbers x, with the following properties.

(i) $f'(x) = ax^2 + bx$

(ii) $f'(1) = 6$ and $f''(1) = 18$

(iii) $\int_1^2 f(x)\,dx = 18$

Find $f(x)$. Show your work.

Solution	Distribution of Points
$a + b = 6$ $f''(x) = 2ax + b$ $2a + b = 18$ $\therefore\ a = 12,\ b = -6$ and $f'(x) = 12x^2 - 6x$	4: { 1: for $a + b = 6$ 1: for $f''(x)$ 1: for $2a + b = 18$ 1: for $a = 12,\ b = -6$
$f(x) = \int (12x^2 - 6x)\,dx$ $= 4x^3 - 3x^2 + C$ $\int_1^2 (4x^3 - 3x^2 + C)\,dx = x^4 - x^3 + Cx \Big\|_1^2$ $= (16 - 8 + 2C) - (1 - 1 + C)$ $= 8 + C = 18$ or $C = 10$ $\therefore f(x) = 4x^3 - 3x^2 + 10$	5: { 1: for an antiderivative of f' 1: for including C 1: for antiderivative of f 1: for evaluating integral and equating it to 18 1: for finding value of C

Free Response – Problem #48
1998 AP Calculus AB Exam – Problem #4

AB–4 1998

4. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2+1}{2y}$.

   (a) Find the slope of the graph of f at the point where $x = 1$.

   (b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

   (c) Find $f(x)$ by solving the separable differential equation $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

   (d) Use your solution from part (c) to find $f(1.2)$.

(a) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$

$\left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=4}} = \dfrac{3+1}{2\cdot 4} = \dfrac{4}{8} = \dfrac{1}{2}$

1: answer

(b) $y - 4 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x + \frac{7}{2}$

$f(1.2) - 4 \approx \frac{1}{2}(1.2 - 1)$

$f(1.2) \approx 0.1 + 4 = 4.1$

2 {
- 1: equation of tangent line
- 1: uses equation to approximate $f(1.2)$

(c) $2y\,dy = (3x^2+1)\,dx$

$\int 2y\,dy = \int (3x^2+1)\,dx$

$y^2 = x^3 + x + C$

$4^2 = 1 + 1 + C$

$14 = C$

$y^2 = x^3 + x + 14$

$y = \sqrt{x^3+x+14}$ is branch with point $(1, 4)$

$f(x) = \sqrt{x^3+x+14}$

5 {
- 1: separates variables
- 1: antiderivative of dy term
- 1: antiderivative of dx term
- 1: uses $y = 4$ when $x = 1$ to pick one function out of a family of functions
- 1: solves for y
  - 0/1 if solving a linear equation in y
  - 0/1 if no constant of integration

Note: max 0/5 if no separation of variables

Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation

(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$

1: answer, from student's solution to the given differential equation in (c)

Free Response – Problem #49
1985 AP Calculus BC Exam – Problem #4

4. Given the differential equation $\frac{dy}{dx} = \frac{-xy}{\ln y}$, $y > 0$.

(a) Find the general solution of the differential equation.

(b) Find the solution that satisfies the condition that $y = e^2$ when $x = 0$. Express your answer in the form $y = f(x)$.

(c) Explain why $x = 2$ is not in the domain of the solution found in part (b).

Solution	Distribution of Points
(a) $\frac{\ln y}{y}\,dy = -x\,dx$ $\frac{(\ln y)^2}{2} = -\frac{x^2}{2} + C$ or $(\ln y)^2 = -x^2 + C$ or $\ln y = \pm\sqrt{C - x^2}$ or $y = e^{\pm\sqrt{C - x^2}}$	(a) 5: { 1: for separating the variables 2: for antiderivative of $\frac{\ln y}{y}$ 1: for antiderivative of $-x$ 1: for general solution
(b) $(\ln e^2)^2 = 0 + C$ $C = 4$ $(\ln y)^2 = 4 - x^2$ $\ln y = \pm\sqrt{4 - x^2}$, but $x = 0, y = e^2 \Rightarrow \ln y = \sqrt{4 - x^2}$. Therefore, $y = e^{\sqrt{4-x^2}}$.	(b) 3: { 1: for using initial condition to solve general solution in part (a) for C 1: for substituting for C and solving for $\ln y$ 1: for the particular solution $y = f(x)$
(c) If $x = 2$, then $y = 1$, which causes $\frac{-xy}{\ln y}$ to be undefined. or If $x = 2$, then $\ln y = 0$, which causes $\frac{-xy}{\ln y}$ to be undefined.	(c) 1: for justification

Free Response – Problem #50
2000 AP Calculus AB Exam – Problem #6

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

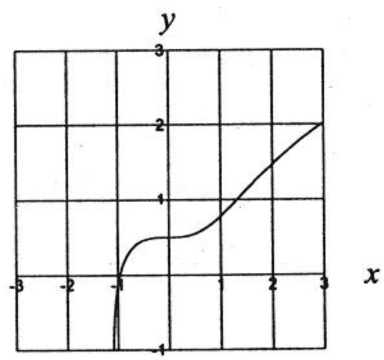
(a) $\frac{dy}{dx} = \frac{3x^2}{e^{2y}} \implies \int e^{2y}dy = \int 3x^2 dx \implies \frac{1}{2}e^{2y} = x^3 + C$

$f(0) = \frac{1}{2} \implies C = \frac{1}{2}e \implies \frac{1}{2}e^{2y} = x^3 + \frac{1}{2}e$

$\implies e^{2y} = 2x^3 + e \implies 2y = \ln\left(2x^3 + e\right) \implies y = \frac{1}{2}\ln\left(2x^3 + e\right)$ ☺

(b) $2x^3 + e > 0 \implies x^3 > -\frac{e}{2} \implies$ the domain $= \left\{x \,\middle|\, x > \sqrt[3]{-\frac{e}{2}}\right\}$ ☺

Since $2x^3 + e$ takes on all positive real values, the range $= \mathbb{R}$ (the set of reals)



Free Response – Problem #51
2002 AP Calculus BC Exam – Problem #5

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

Solutions

(a) If $y = -2$ is tangent to the graph at this point, then the slope of the tangent is 0.

$\frac{dy}{dx} = \frac{3-x}{y} \Rightarrow 0 = \frac{3-x}{y} \Rightarrow 0 = 3 - x \Rightarrow x = 3$

A small slope field about the point $(3, -2)$ indicates that the slopes are negative for the points $(2, -1)$, $(2, -2)$, and $(2, -3)$ while the slopes are positive for the points $(4, -1)$, $(4, -2)$, and $(4, -3)$. This indicates that the graph is decreasing to the left of $(3, -2)$ and increasing to the right of $(3, -2)$. As a result, the point $(3, -2)$ is a local minimum.

(b) $\frac{dy}{dx} = \frac{3-x}{y}$ and $g(6) = -4$.

$y\,dy = (3-x)dx \Rightarrow \int y\,dy = \int (3-x)dx$

$\frac{y^2}{2} = 3x - \frac{x^2}{2} + C$

$\frac{(-4)^2}{2} = 3(6) - \frac{6^2}{2} + C$

$8 = 18 - 18 + C$

$C = 8$

$\frac{y^2}{2} = 3x - \frac{x^2}{2} + 8$

Free Response – Problem #52
2001 AP Calculus AB Exam – Problem #6

The function f is differentiable for all real numbers. The point $\left(3,\frac{1}{4}\right)$ is on the graph of $y=f(x)$, and the slope at each point (x,y) on the graph is given by $\frac{dy}{dx}=y^2(6-2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3,\frac{1}{4}\right)$.

(b) Find $y=f(x)$ by solving the differential equation $\frac{dy}{dx}=y^2(6-2x)$ with the initial condition $f(3)=\frac{1}{4}$.

(a)
$$\frac{d^2y}{dx^2}=2y\frac{dy}{dx}(6-2x)-2y^2$$
$$=2y^3(6-2x)^2-2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3,\frac{1}{4}\right)}=0-2\left(\frac{1}{4}\right)^2=-\frac{1}{8}$$

3 :
- 2 : $\frac{d^2y}{dx^2}$
  - < −2 > product rule or chain rule error
- 1 : value at $\left(3,\frac{1}{4}\right)$

(b)
$$\frac{1}{y^2}dy=(6-2x)\,dx$$

$$-\frac{1}{y}=6x-x^2+C$$

$$-4=18-9+C=9+C$$

$$C=-13$$

$$y=\frac{1}{x^2-6x+13}$$

6 :
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $f(3)=\frac{1}{4}$
- 1 : solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Free Response – Problem #53
2001 AP Calculus BC Exam – Problem #5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

(a) Evaluate $\int_1^{\infty} -3xf(x)\,dx$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

(c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

(a) $\int_1^{\infty} -3xf(x)\,dx$

$= \int_1^{\infty} f'(x)\,dx = \lim_{b\to\infty} \int_1^{b} f'(x)\,dx = \lim_{b\to\infty} f(x)\Big|_1^b$

$= \lim_{b\to\infty} f(b) - f(1) = 0 - 4 = -4$

$2: \begin{cases} 1: \text{use of FTC} \\ 1: \text{answer from limiting process} \end{cases}$

(b) $f(1.5) \approx f(1) + f'(1)(0.5)$

$= 4 - 3(1)(4)(0.5) = -2$

$f(2) \approx -2 + f'(1.5)(0.5)$

$\approx -2 - 3(1.5)(-2)(0.5) = 2.5$

$2: \begin{cases} 1: \text{Euler's method equations or equivalent table} \\ 1: \text{Euler approximation to } f(2) \text{ (not eligible without first point)} \end{cases}$

(c) $\frac{1}{y}dy = -3x\,dx$

$\ln y = -\frac{3}{2}x^2 + k$

$y = Ce^{-\frac{3}{2}x^2}$

$4 = Ce^{-\frac{3}{2}}$; $C = 4e^{\frac{3}{2}}$

$y = 4e^{\frac{3}{2}}e^{-\frac{3}{2}x^2}$

$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition } f(1) = 4 \\ 1: \text{solves for } y \end{cases}$

Note : max 2/5 [1-1-0-0-0] if no constant of integration

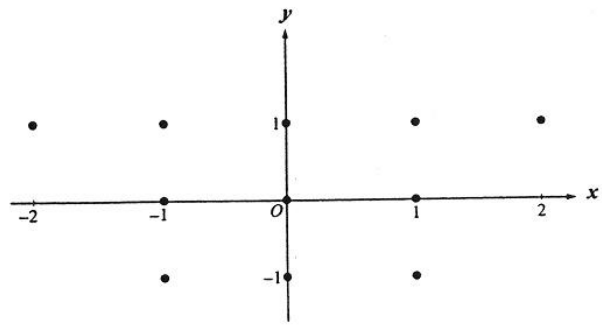
Note : 0/5 if no separation of variables

Free Response – Problem #64
2000 AP Calculus BC Exam – Problem #6

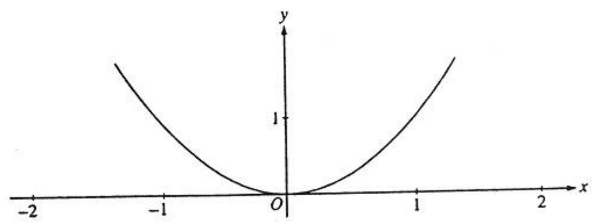
2000: BC-6

Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.



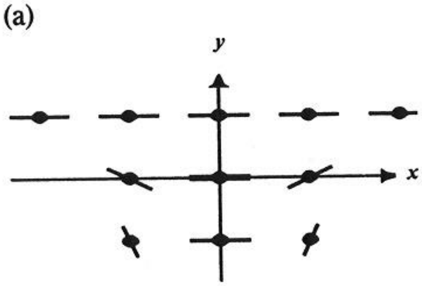
(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.

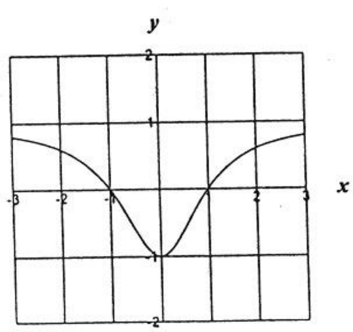


(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).

(a)





(b) Because when $y = 1$, $\frac{dy}{dx} = 0$ and in the given graph there is no horizontal tangent when $y = 1$.

(c) $\frac{dy}{dx} = x(y-1)^2 \Rightarrow \int \frac{1}{(y-1)^2}\,dy = \int x\,dx \Rightarrow$

$$\frac{-1}{(y-1)} = \frac{1}{2}x^2 + C \Rightarrow y = 1 - \frac{1}{\frac{1}{2}x^2 + C}$$

$$f(0) = -1 \Rightarrow C = \tfrac{1}{2} \Rightarrow y = 1 - \frac{2}{x^2+1}$$

(d) The range of $y = f(x)$ is $\{y \mid -1 \le y < 1\}$

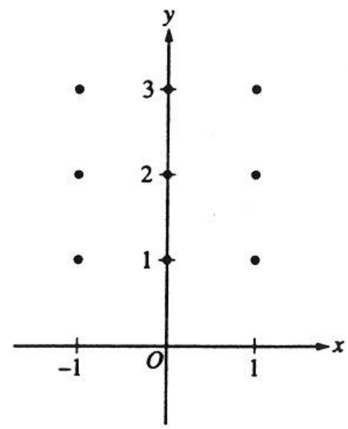
2000 BC6

Free Response – Problem #65
1998 AP Calculus BC Exam – Problem #4

1998: BC-4

Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

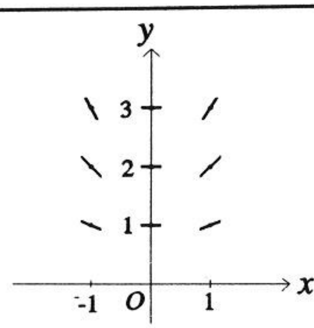
(a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

(a)



(b) Start at (0 , 3). $\frac{dy}{dx} = 0 \Longrightarrow \Delta y \approx \frac{dy}{dx}\Delta x = 0 \Longrightarrow f(.1) \approx 3 + 0 = 3.$

At (.1 , 3), $\frac{dy}{dx} = .15 \Longrightarrow \Delta y \approx (.15)(.1) = .015 \Longrightarrow f(.2) \approx 3 + .015 = 3.015$ ☺

(c) $\frac{dy}{dx} = \frac{xy}{2} \Longrightarrow \int \frac{1}{y}\,dy = \int \frac{x}{2}\,dx \Longrightarrow \ln|y| = \frac{1}{4}x^2 + C \Longrightarrow y = C_1 e^{\frac{x^2}{4}}$

$f(0) = 3 \Longrightarrow C_1 = 3 \Longrightarrow y = f(x) = 3e^{\frac{x^2}{4}}$ ☺

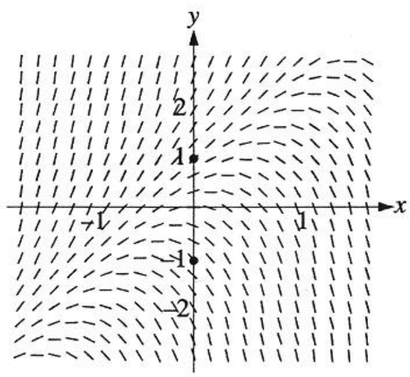
$f(0.2) = 3e^{.01} \approx 3.030$ ☺

1998 BC4

Free Response – Problem #66
2002 AP Calculus BC Exam – Problem #5

5. Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.
(Note: Use the slope field provided in the pink test booklet.)



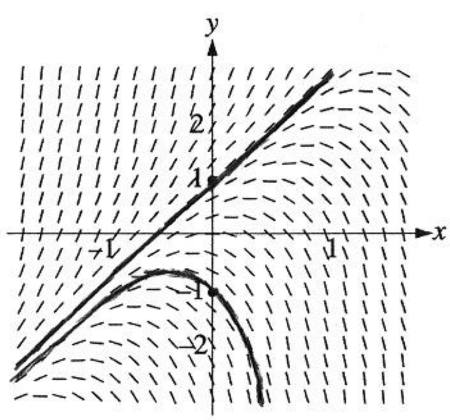
(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

Solutions

(a)



Free Response – Problem #67continued
2002 AP Calculus BC Exam – Problem #5

(b) $\frac{dy}{dx} = 2y - 4x$

At (0, 1) the slope of the tangent line is 2

$y - 1 = 2(x - 0)$

$y - 1 = 2x$

$y = 2x + 1$

$y(0.1) = 1.2$

At (0.1, 1.2) the slope of the tangent line is 2

$y - 1.2 = 2(x - 0.1)$

$y - 1.2 = 2x - 0.2$

$y = 2x + 1$

$y(0.2) = 1.4$

(c) The value of b for which $y = 2x + b$ is a solution to the differential equation is 1 as the solution that passes through (0, 1) is a straight line.

(d) The graph of g with initial condition $g(0) = 0$ has a local maximum as the slope of the tangent line at this point is zero. The slope field indicates that the graph is increasing to the left of (0, 0) but decreasing to the right of (0, 0).

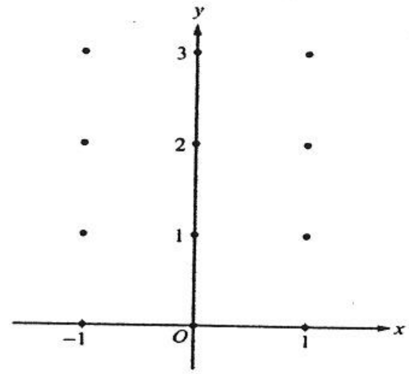
Free Response – Problem #67
2004 AP Calculus AB Exam – Problem #6

Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

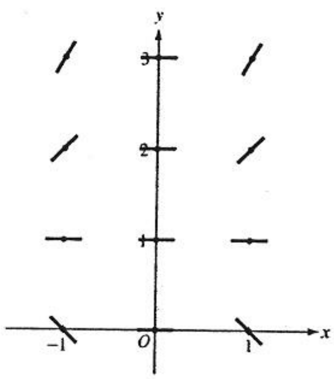
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the pink test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the *xy*-plane. Describe all points in the *xy*-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.



(a)



2 :
- 1 : zero slope at each point (x, y) where $x = 0$ or $y = 1$
- 1 :
  - positive slope at each point (x, y) where $x \neq 0$ and $y > 1$
  - negative slope at each point (x, y) where $x \neq 0$ and $y < 1$

(b) Slopes are positive at points (x, y) where $x \neq 0$ and $y > 1$.

1 : description

(c) $\frac{1}{y-1}dy = x^2 dx$

$\ln|y-1| = \frac{1}{3}x^3 + C$

$|y-1| = e^C e^{\frac{1}{3}x^3}$

$y - 1 = Ke^{\frac{1}{3}x^3}$, $K = \pm e^C$

$2 = Ke^0 = K$

$y = 1 + 2e^{\frac{1}{3}x^3}$

6 :
- 1 : separates variables
- 2 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition
- 1 : solves for y
  - 0/1 if y is not exponential

Note: max 3/6 [1-2-0-0-0] if no constant of integration
Note: 0/6 if no separation of variables