

AP Calculus BC: Study Guide



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Key Exam Details

The AP[®] Calculus BC exam is a 3-hour 15-minute, end-of-course test comprised of 45 multiple-choice questions (50% of the exam) and 6 free-response questions (50% of the exam).

The exam covers the following course content categories:

- Limits and Continuity: 4–7% of test questions
- Differentiation: Definition and Fundamental Properties: 4–7% of test questions
- Differentiation: Composite, Implicit, and Inverse Functions 4–7% of test questions
- Contextual Applications of Differentiation: 6–9% of test questions
- Analytical Applications of Differentiation: 8–11% of test questions
- Integration and Accumulation of Change: 17–20% of test questions
- Differential Equations: 6–9% of test questions
- Applications of Integration: 6–9% of test questions
- Parametric Equations, Polar Coordinates, and Vector-Valued Functions: 11–12% of test questions
- Infinite Sequences and Series: 17–18% of test questions

This guide offers an overview of the main tested subjects, along with sample AP multiple-choice questions that look like the questions you'll see on test day.

Limits and Continuity

About 4–7% of the questions on your exam will cover Limits and Continuity.

Limits

The limit of a function f as x approaches c is L if the value of f can be made arbitrarily close to L by taking x sufficiently close to c (but not equal to c). If such a value exists, this is denoted $\lim_{x \rightarrow c} f(x) = L$. If no such value exists, we say that the limit does not exist, abbreviated DNE.

Limits can be found using tables, graphs, and algebra.

Important algebraic techniques for finding limits include factoring and rationalizing radical expressions. Other helpful tools are given by the following properties.

Suppose $\lim_{x \rightarrow c} f(x) = L$, $\lim_{x \rightarrow c} g(x) = M$, $\lim_{x \rightarrow L} h(x) = N$, and a is any real number.

Then:

- $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$
- $\lim_{x \rightarrow c} [f(x) - g(x)] = L - M$
- $\lim_{x \rightarrow c} [af(x)] = aL$
- $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$, as long as $M \neq 0$
- $\lim_{x \rightarrow c} h(f(x)) = N$

For many common functions, evaluating limits requires nothing more than evaluating the function at the point c (assuming the function is defined at the point). These include polynomial, rational, exponential, logarithmic, and trigonometric functions.

Two special limits that are important in calculus are $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$.

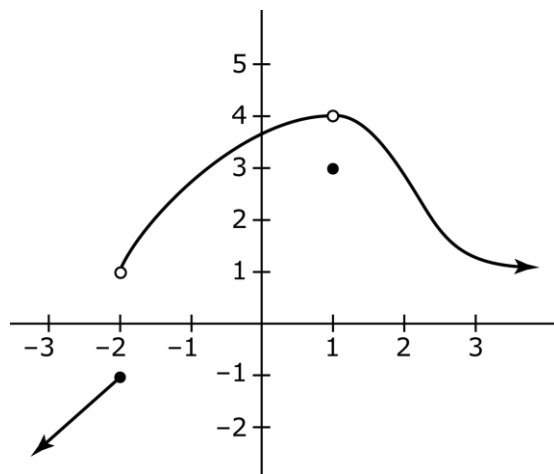
One-Sided Limits

Sometimes we are interested in the value that a function f approaches as x approaches c from only a single direction. If the values of f get arbitrarily close to L as x approaches c while taking on values greater than c , we say $\lim_{x \rightarrow c^+} f(x) = L$. Similarly, if x is taking on values less than c , we write $\lim_{x \rightarrow c^-} f(x) = L$.

We can now characterize limits by saying that $\lim_{x \rightarrow c} f(x)$ exists if and only if both $\lim_{x \rightarrow c^+} f(x)$ and $\lim_{x \rightarrow c^-} f(x)$ exist and have the same value. A limit, then, can fail to exist in a few ways:

- $\lim_{x \rightarrow c^+} f(x)$ does not exist
- $\lim_{x \rightarrow c^-} f(x)$ does not exist
- Both of the one-sided limits exist, but have different values

Example



The function shown has the following limits:

- $\lim_{x \rightarrow -2^-} f(x) = -1$
- $\lim_{x \rightarrow -2^+} f(x) = 1$
- $\lim_{x \rightarrow -2} f(x)$ DNE
- $\lim_{x \rightarrow 1^-} f(x) = 4$
- $\lim_{x \rightarrow 1^+} f(x) = 4$
- $\lim_{x \rightarrow 1} f(x) = 4$

Note that $f(1) = 3$, but this is irrelevant to the value of the limit.

Infinite Limits, Limits at Infinity, and Asymptotes

When a function has a vertical asymptote at $x = c$, the behavior of the function can be described using infinite limits. If the function values increase as they approach the asymptote, we say the limit is ∞ , whereas if the values decrease as they approach the asymptote, the limit is $-\infty$. It is important to realize that these limits do not exist in the same sense that we described earlier; rather, saying that a limit is $\pm\infty$ is simply a convenient way to describe the behavior of the function approaching the point.

We can also extend limits by considering how the function behaves as $x \rightarrow \pm\infty$. If such a limit exists, it means that the function approaches a horizontal line as x increases or decreases without bound. In other words, if $\lim_{x \rightarrow \pm\infty} f(x) = L$, then f has a horizontal asymptote $y = L$. It is possible for a function to have two horizontal asymptotes since it can have different limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$.

The Squeeze Theorem

The Squeeze Theorem states that if the graph of a function lies between the graphs of two other functions, and if the two other functions share a limit at a certain point, then the function in between also shares that same limit. More formally, if $f(x) \leq g(x) \leq h(x)$ for all x in some interval containing c , and if $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} g(x) = L$ as well.

Example

The sine function satisfies $-1 \leq \sin x \leq 1$ for all real numbers x , so $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ is also true for all real numbers x . Multiplying this inequality by x^2 , we obtain $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$. Now the functions on the left and right of the inequality, x^2 and $-x^2$, both have limits of 0 as $x \rightarrow 0$. Therefore, we can conclude that $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ also.

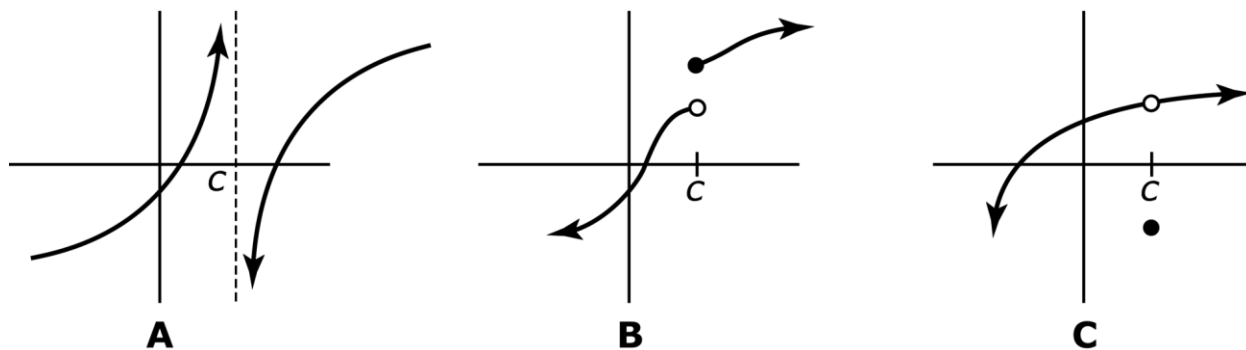
Continuity

The function f is said to be continuous at the point $x = c$ if it meets the following criteria:

1. $f(c)$ exists
2. $\lim_{x \rightarrow c} f(x)$ exists
3. $\lim_{x \rightarrow c} f(x) = f(c)$

In other words, the function must have a limit at c , and the limit must be the actual value of the function.

Each of the above criteria can fail, resulting in a discontinuity at $x = c$. Consider the following three graphs:



In graph A, the function is not defined at c . In graph B, the function is defined at c , but the limit as $x \rightarrow c$ does not exist due to the one-sided limits being different. In graph C, the function is defined at c and the limit as $x \rightarrow c$ exists, but they are not equal to each other.

The discontinuity in graph B is referred to as a jump discontinuity, since it is caused by the graph jumping when it reaches $x = c$. In contrast to this is the situation in graph C, where the discontinuity could be fixed by moving a single point; it occurs whenever the second condition above is satisfied and is called a removable discontinuity. If $\lim_{x \rightarrow c} f(x)$ exists, but f has a discontinuity at $x = c$ because it fails one of the other conditions, the discontinuity can be removed by defining or redefining $f(c)$ to be equal to the limit at that point.

A function is continuous on an interval if it is continuous at every point in the interval. The following categories of functions are continuous at every point in their respective domains:

- Polynomial
- Rational
- Power
- Exponential
- Logarithmic
- Trigonometric

If f is a piecewise-defined function with continuous component functions, then checking for continuity consists of checking whether it is continuous at its boundary points. Continuity at a boundary point requires that the functions on both sides of the point give the same result when evaluated at the point.

Intermediate Value Theorem

The Intermediate Value Theorem applies to continuous functions on an interval $[a, b]$. If d is any value between $f(a)$ and $f(b)$, then there must be at least one number c between a and b such that $f(c) = d$.

Example

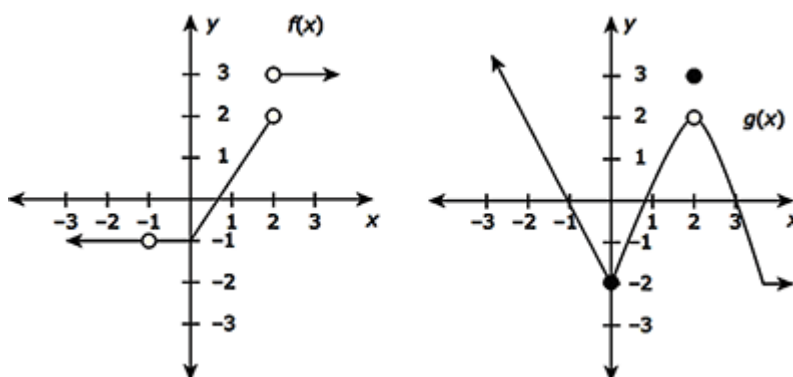
Consider $f(x) = e^x - 2$, which is continuous everywhere. We have $f(0) = e^0 - 2 = -1$, and $f(1) = e - 2$, which is certainly positive. If we take $d = 0$ in the statement of the theorem, then d is between $f(0)$ and $f(1)$. Therefore, the Intermediate Value Theorem guarantees at least one value c between 0 and 1 with the property that $f(c) = 0$. This value, of course, is $c = \ln 2$.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapter 1. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 2. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapter 2. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 2. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 1. New York, NY: W.H. Freeman.

Sample Limits and Continuity Questions

Consider the following graphs of f and g :



Compute $\lim_{x \rightarrow 2^+} [3f(x) - g(x)]$, provided the limit exists.

- A. Does not exist
- B. 4
- C. 6
- D. 7

Explanation:

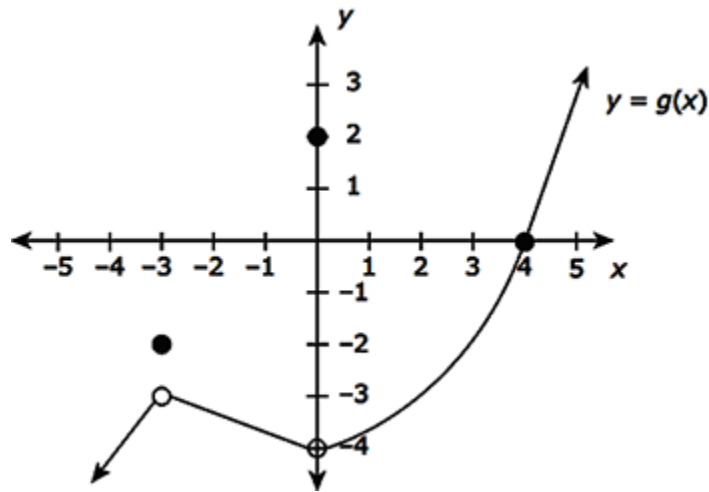
The correct answer is **D**. First, use linearity to write:

$$\lim_{x \rightarrow 2^+} [3f(x) - g(x)] = 3 \lim_{x \rightarrow 2^+} f(x) - \lim_{x \rightarrow 2^+} g(x).$$

Now, observe that $\lim_{x \rightarrow 2^+} f(x) = 3$ and $\lim_{x \rightarrow 2^+} g(x) = 2$. Substituting these values above yields

$$\lim_{x \rightarrow 2^+} [3f(x) - g(x)] = 3 \lim_{x \rightarrow 2^+} f(x) - \lim_{x \rightarrow 2^+} g(x) = 3(3) - 2 = 7.$$

Suppose that $f(x) = 2 \cos\left(\frac{\pi}{3}x\right)$ and the graph of $g(x)$ is given by



Compute $\lim_{x \rightarrow -3} f(x) \cdot g(x)$, provided it exists.

- A. -6
- B. 4
- C. 6
- D. Does not exist

Explanation:

The correct answer is C. Use the fact that the limit of a product is the product of the limits, provided they both exist independently, to compute:

$$\begin{aligned} \lim_{x \rightarrow -3} f(x) \cdot g(x) &= \left(\lim_{x \rightarrow -3} f(x) \right) \cdot \left(\lim_{x \rightarrow -3} g(x) \right) \\ &= \left(2 \cos\left(\frac{\pi}{3} \cdot (-3)\right) \right) \cdot (-3) \\ &= \left(2 \cos(-\pi) \right) \cdot (-3) \\ &= (-2)(-3) \\ &= 6 \end{aligned}$$

Which of the following limits does not exist?

A. $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$

B. $\lim_{x \rightarrow 1} \frac{|x-1|}{1-x}$

C. $\lim_{x \rightarrow 0^+} x^{2/5}$

D. $\lim_{x \rightarrow \pi^-} \sec x$

Explanation:

The correct answer is B. Recall that

$$|x-1| = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$$

So,

$$\frac{|x-1|}{1-x} = \begin{cases} 1, & x < 1 \\ -1, & x > 1 \end{cases}$$

Hence, $\lim_{x \rightarrow 1^+} \frac{|x-1|}{1-x} = -1$ while $\lim_{x \rightarrow 1^-} \frac{|x-1|}{1-x} = 1$. Since these values are different, it follows that

there is a jump at $x = 1$ and so, $\lim_{x \rightarrow 1} \frac{|x-1|}{1-x}$ does not exist.

Differentiation: Definition and Fundamental Properties

About 4–7% of the questions on your AP exam will cover Differentiation: Definition and Fundamental Properties.

Definition of the Derivative

The average rate of change of a function f over the interval from $x = a$ to $x = a + h$ is $\frac{f(a+h) - f(a)}{h}$. Alternatively, if $x = a + h$, this can be written $\frac{f(x) - f(a)}{x - a}$. When h is made smaller, so that it approaches 0, the limit that results is called the instantaneous rate of change of f at $x = a$, or the derivative of f at $x = a$, and is denoted $f'(a)$.

That is, $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, or equivalently, $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$.

If this limit exists, f is said to be differentiable at a . Graphically, $f'(a)$ represents the slope of the line tangent to the graph of $f(x)$ at the point where $x = a$. Therefore, the line tangent to $f(x)$ at $x = a$ is $y - f(a) = f'(a)(x - a)$.

If the function $y = f(x)$ is differentiable at all points in some interval, we can define a new function on that interval by finding the derivative at every point. This new function, called the derivative of f , can be denoted $f'(x)$, y' , or $\frac{dy}{dx}$, and is defined by $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

The value of the derivative at a particular point $x = a$ can then be denoted $f'(a)$ or $\left. \frac{dy}{dx} \right|_{x=a}$.

If f is differentiable at $x = a$, then it also must be continuous at $x = a$. In other words, if a function fails to be continuous at a point, it cannot possibly be differentiable at that point. Another way that differentiability can fail is via the presence of sharp turns or cusps in a graph.

Free Response Tip

When specific function values are given, the derivative at a point can be approximated by finding the average rate of change between surrounding points. For example, if you are given values of a function at $x = 3, 4,$ and 5 , then the derivative at 4 can be approximated by the average rate of change between 3 and 5 .

Basic Derivatives and Rules

There are several rules that can be used to find derivatives. Assume f and g are differentiable functions, and c is a real number.

- The constant rule: $\frac{d}{dx} c = 0$
- The power rule: $\frac{d}{dx} x^n = nx^{n-1}$, for any real number n
- The sum rule: $\frac{d}{dx} [f(x) + g(x)] = f'(x) + g'(x)$
- The difference rule: $\frac{d}{dx} [f(x) - g(x)] = f'(x) - g'(x)$
- The constant multiple rule: $\frac{d}{dx} [cf(x)] = cf'(x)$
- The product rule: $\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- The quotient rule: $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

As special cases of the power rule, note that $\frac{d}{dx}(cx) = c$, and $\frac{d}{dx} x = 1$.

In addition to these rules, the derivatives of some common functions are as follows:

e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapters 2 and 3. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 3. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapters 2 and 3. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 3. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 2. New York, NY: W.H. Freeman.

Sample Differentiation: Definition and Fundamental Properties Questions

Boyle's laws states that if a gas is compressed by constant temperature, the product of the volume and pressure remains constant; that is, $VP = K$, where K is a constant. Which of the following equals the instantaneous rate of change of pressure with respect to volume?

- A. $-K$
- B. 0
- C. $-KV^2$
- D. $-\frac{K}{V^2}$

Explanation:

The correct answer is D. Observe that since $VP = K$, it follows that $P = KV^{-1}$. Differentiating both sides with respect to V yields $\frac{dP}{dV} = -KV^{-2} = -\frac{K}{V^2}$.

An object moves along the curve $y = \frac{1}{x}$, starting at $x = \frac{1}{10}$. As it passes through the point (1,1) its x -coordinate increases at a rate of 2 inches per second. How fast is the distance between the object and the origin changing at this instance in time?

- A. 0 inches per second
- B. $4\sqrt{2}$ inches per second
- C. $\sqrt{2}$ inches per second
- D. -2 inches per second

Explanation:

The correct answer is A. The position of a point on this curve is of the form $(x, y) = (x, \frac{1}{x})$. So, the distance D between it and the origin is

$$D = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{(x-0)^2 + (\frac{1}{x}-0)^2} = \sqrt{x^2 + x^{-2}} .$$

While we could differentiate both sides with respect to t directly, the presence of the radical makes this inconvenient. So, we square both sides first to get $D^2 = x^2 + x^{-2}$ and *now* differentiate both sides with respect to t . Doing so yields:

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} - 2x^{-3} \frac{dx}{dt}$$

$$D \frac{dD}{dt} = x \frac{dx}{dt} - x^{-3} \frac{dx}{dt}$$

Note that when the object is at the point (1,1), we know that $x = 1$, $D = \sqrt{1^2 + 1^{-2}} = \sqrt{2}$, and

$\frac{dx}{dt} = 2$ inches per second. Substituting this information into the above equation yields

$$\sqrt{2} \frac{dD}{dt} = 1 \cdot 2 - 1 \cdot 2 = 0 .$$

So, $\frac{dD}{dt} = 0$.

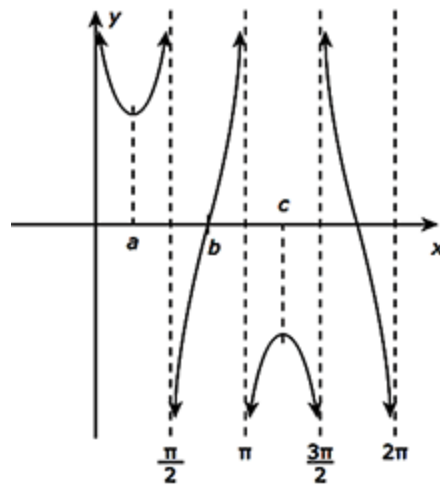
For how many values of x in the interval $[0, 2\pi]$ is the tangent line to the curve

$g(x) = \sec x + \csc x$ parallel to the line $x = y$?

- A. 0
- B. 2
- C. 3
- D. 4

Explanation:

The correct answer is B. First, use the graphing calculator to graph $g(x)$ on $[0, 2\pi]$:



Now, reasoning using the continuity of the graph and the vertical asymptotes reveals that there must be tangent lines to $g(x)$ with slope 1 at one x -value in each of the following intervals: $(a, \frac{\pi}{2})$ and (π, c) .

Differentiation: Composite, Implicit, and Inverse Functions

About 4–7% of the questions on your exam will cover the topic Differentiation: Composite, Implicit, and Inverse Functions.

Chain Rule

The chain rule makes it possible to differentiate composite functions. If $y = f(g(x))$, then the chain rule states that $y' = f'(g(x)) \cdot g'(x)$. In alternative notation, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

The chain rule can be extended to compositions of more than two functions by considering that $g(x)$ as described above may itself be a composition. If $y = f(g(h(x)))$, two applications of the chain rule yield $y' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$.

Implicit Differentiation and Inverse Functions

A function may sometimes be presented in implicit, rather than explicit, form. That is, it may not be given as $y = f(x)$, but rather as an equation that relates x and y to each other. In such cases, we say that y is implicitly defined as a function of x . *Implicit differentiation* is the process of finding the derivative $\frac{dy}{dx}$ for such functions, and it is accomplished by applying the chain rule.

Example

Consider the equation $y^3 + x^3 + xy = 5$. Differentiating both sides of the equation with respect to x , and remembering that we are assuming that y is, in fact, a function of x (so that the chain rule applies), we get

$$\begin{aligned}\frac{d}{dx}(y^3 + x^3 + xy = 5) &= \frac{d}{dx}(5) \\ 3y^2 \cdot \frac{dy}{dx} + 3x^2 + 1 \cdot y + x \cdot 1 \cdot \frac{dy}{dx} &= 0\end{aligned}$$

Note that differentiating xy required an application of the product rule, and that every time an expression in terms of y was differentiated, the derivative was multiplied by $\frac{dy}{dx}$. Now all of the terms with $\frac{dy}{dx}$ can be gathered on one side of the equation, and $\frac{dy}{dx}$ can be solved for:

$$3y^2 \cdot \frac{dy}{dx} + x \cdot \frac{dy}{dx} = -y - 3x^2$$

$$\frac{dy}{dx}(3y^2 + x) = -y - 3x^2$$

$$\frac{dy}{dx} = \frac{-y - 3x^2}{3y^2 + x}$$

This technique can also be applied to find the derivatives of inverse functions. Consider an invertible function f , with inverse f^{-1} . By definition this means that $f(f^{-1}(x)) = x$. Now, differentiating both sides with respect to x , we get $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$. Solving for $(f^{-1})'(x)$, we have $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Applying this rule to the inverse trigonometric functions, we can find the following derivatives:

$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccot } x$	$\frac{-1}{1+x^2}$
$\text{arcsec } x$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{arccsc } x$	$\frac{-1}{x\sqrt{x^2-1}}$

Higher Order Derivatives

The derivative f' of a function f is itself a function that may be differentiable. If it is, then its derivative is f'' , called the *second derivative* of f . The relationship of f' and f'' is identical to the relationship between f and f' . Similarly, the derivative of f'' is f''' , the third derivative of f . This process can continue indefinitely, as long as the functions obtained continue to be differentiable. After three, the notation changes, so that the 4th derivative of f is denoted $f^{(4)}$, and the n^{th} derivative is $f^{(n)}$.

If $y = f(x)$, then higher order derivatives are also denoted $y'', y''', y^{(4)}, \dots, y^{(n)}, \dots$, or

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}, \dots$$

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapter 3. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 3. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapter 3. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 3. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 3. New York, NY: W.H. Freeman.

Sample Differentiation: Composite, Implicit, and Inverse Functions Questions

Suppose $f(x)$ is a differentiable function, $f(2) = -\frac{\pi}{6}$, and $f'(2) = \frac{1}{2}$. If $y = \cos(f(x))$, compute $\frac{dy}{dx}$ (2).

- A. $-\sin\left(\frac{1}{2}\right)$
- B. $\cos\left(\frac{1}{2}\right)$
- C. $\frac{1}{4}$
- D. $\frac{1}{2}$

Explanation:

The correct answer is C. First, compute the derivative using the chain rule:

$$\frac{dy}{dx} = -\sin(f(x)) \cdot f'(x)$$

Now, substitute $x = 2$ and use the information provided:

$$\begin{aligned}\frac{dy}{dx}(2) &= -\sin(f(2)) \cdot f'(2) \\ &= -\sin\left(-\frac{\pi}{6}\right) \cdot \frac{1}{2} \\ &= -\left(-\frac{1}{2}\right) \cdot \frac{1}{2} \\ &= \frac{1}{4}\end{aligned}$$

What is the equation of the tangent line to $f(x) = \sin^{-1}(\sqrt{x})$ at $x = \frac{1}{4}$?

- A. $y - \frac{\pi}{3} = \frac{2\sqrt{3}}{3}\left(x - \frac{1}{4}\right)$
- B. $y - \frac{\pi}{6} = \frac{2\sqrt{3}}{3}\left(x - \frac{1}{4}\right)$
- C. $y - \frac{\pi}{6} = \frac{4\sqrt{15}}{15}\left(x - \frac{1}{4}\right)$
- D. $y - \frac{\pi}{3} = \frac{4\sqrt{15}}{15}\left(x - \frac{1}{4}\right)$

Explanation:

The correct answer is B. First, use the chain rule to compute $f'(x)$:

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

Now, evaluate this expression at $x = \frac{1}{4}$:

$$f'\left(\frac{1}{4}\right) = \frac{1}{\sqrt{1-\frac{1}{4}}} \cdot \frac{1}{2\sqrt{\frac{1}{4}}} = \frac{1}{\sqrt{\frac{3}{4}}} \cdot 1 = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}}{3}$$

This is the slope of the tangent line at $x = \frac{1}{4}$. To get the y -value of the point of tangency, compute $f\left(\frac{1}{4}\right)$:

$$f\left(\frac{1}{4}\right) = \sin^{-1}\left(\sqrt{\frac{1}{4}}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

So, using the point-slope form for the equation of a line, the equation of the tangent line is

$$y - \frac{\pi}{6} = \frac{2\sqrt{3}}{3}\left(x - \frac{1}{4}\right).$$

What is the slope of a line perpendicular to the tangent line to the curve defined implicitly by $x^2y^2 - xy = 42$ at the point $(2, -3)$?

- A. $\frac{3}{26}$
- B. $-\frac{2}{3}$
- C. $-\frac{26}{3}$
- D. $\frac{3}{2}$

Explanation:

The correct answer is B. This is the correct answer. Implicitly differentiate both sides with respect to x :

$$x^2 \cdot 2y \cdot y' + y^2 \cdot 2x - (xy' + y) = 0$$

$$2x^2y \cdot y' + 2xy^2 - xy' - y = 0$$

$$y'(2x^2y - x) = y - 2xy^2$$

$$y' = \frac{y - 2xy^2}{2x^2y - x}$$

$$y' = \frac{y(1 - 2xy)}{x(2xy - 1)}$$

$$y' = -\frac{y}{x}$$

So, the slope of the tangent line at the point $(2, -3)$ is $\frac{-(-3)}{2} = \frac{3}{2}$. Therefore, any line perpendicular to the tangent line at this point would have a slope equal to $-\frac{2}{3}$.

Contextual Applications of Differentiation

Around 6–9% of the questions on your AP exam will cover the topic Contextual Applications of Differentiation.

In any context, the derivative of a function can be interpreted as the instantaneous rate of change of the independent variable with respect to the dependent variable. If $y = f(x)$, then the units of the derivative are the units of y divided by the units of x .

Straight-Line Motion

Rectilinear (straight-line) motion is described by a function and its derivatives.

If the function $s(t)$ represents the position along a line of a particle at time t , then the velocity is given by $v(t) = s'(t)$. When the velocity is positive, the particle is moving to the right; when it is negative, the particle is moving to the left. The speed of the particle does not take direction into account, so it is the absolute value of the velocity, or $|v(t)|$.

The acceleration of the particle is $a(t) = v'(t) = s''(t)$. The velocity is increasing when $a(t)$ is positive and decreasing when $a(t)$ is negative. The speed, however, is only increasing when $v(t)$ and $a(t)$ have the same sign (positive or negative). When $v(t)$ and $a(t)$ have different signs, the particle's speed is decreasing.

Related Rates

Related rates problems involve multiple quantities that are changing in relation to each other. Derivatives, and especially the chain rule, are used to solve these problems. Though the problems vary widely with context, there are a few steps that usually lead to a solution.

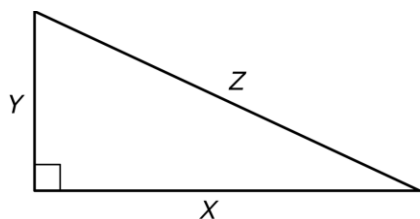
1. Draw a picture and label relevant quantities with variables.
2. Express any rates of change given in the problem as derivatives.
3. Express the rate of change you need to solve for as a derivative.
4. Relate the variables involved in the rates of change to each other with an equation.
5. Differentiate both sides of the equation with respect to time. This may involve applying many derivative rules but will always involve the chain rule.
6. Substitute all of the given information into the resulting equation.
7. Solve for the unknown rate.

Example

The length of the horizontal leg of a right triangle is increasing at a rate of 3 ft/sec, and the length of the vertical leg is decreasing at a rate of 2 ft/sec. At the instant when the horizontal leg is 7 ft and the vertical leg is 1 ft, at what rate is the length of the hypotenuse changing? Is it increasing or decreasing?

We will follow the steps given above.

1.



2. We are given $\frac{dx}{dt} = 3$ and $\frac{dy}{dt} = -2$

3. We need to find $\left. \frac{dz}{dt} \right|_{x=7, y=1}$

4. x , y , and z are related by the Pythagorean theorem: $x^2 + y^2 = z^2$

5. Differentiating both sides of the equation, and applying the chain rule (since all of the variables are functions of t), we get $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

6. After substituting all of the information we have, including $x = 7$, $y = 1$, and

$$z = \sqrt{7^2 + 1^2} = \sqrt{50}, \text{ the equation becomes } 2(7)(3) + 2(1)(-2) = 2(\sqrt{50}) \frac{dz}{dt}$$

7. Solving, we get $\frac{dz}{dt} = \frac{19}{\sqrt{50}}$. The length of the hypotenuse is increasing since its

derivative is positive, and it is doing so at a rate of $\frac{19}{\sqrt{50}}$ ft/sec

Linearization

The line tangent to a function at $x = c$ is the best possible linear approximation to the function near $x = c$. Because of this, the tangent line, seen as a function $L(x)$, is also called the linearization of the function at the given point.

Example

We can use the linearization of $f(x) = 3xe^{-x^2}$ at $x = 0$ to approximate the value of $f(0.1)$. To do this, we need to first find the derivative. Applying the product and chain rules, we get

$f'(x) = 3 \cdot e^{-x^2} + 3x \cdot e^{-x^2} \cdot -2x = 3e^{-x^2} - 6x^2e^{-x^2}$. The slope of the tangent line at $x = 0$ is

$f'(0) = 3e^0 - 6(0)e^0 = 3$. The function passes through the point $(0, f(0)) = (0, 0)$, so the tangent line is $y - 0 = 3(x - 0)$.

The linearization of f at $x = 0$ is $L(x) = 3x$, so the approximation of $f(0.1)$ is $L(0.1) = 3(0.1) = 0.3$. Note that the true value of $f(0.1)$ is approximately 0.297, so the linear approximation was an overestimate.

L'Hospital's Rule

When two functions f and g either both have limits of 0 or both have infinite limits, we say that the limit of their ratio is an indeterminate form, represented by $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Limits that result in one of these forms can be evaluated using L'Hospital's rule. The full statement of L'Hospital's rule is as follows: if $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ approaches $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$. In other words, when we encounter one of these indeterminate forms, we can take the derivative of each of the functions, and then reevaluate the limit.

Free Response Tip

Limits that require application of L'Hospital's Rule appear often in free response questions. Be careful not to confuse L'Hospital's Rule with the quotient rule. The derivative of the ratio is not being taken; rather, the derivative of the numerator and denominator are taken separately.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapter 4. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 4. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapter 4. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 4. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 4. New York, NY: W.H. Freeman.

Sample Contextual Applications of Differentiation Questions

Compute the limit, provided that it exists:

$$\lim_{x \rightarrow 1^+} \sqrt{10 + \frac{1-x}{\sin(x-1)}}$$

- A. 3
- B. $\sqrt{11}$
- C. $\sqrt{10}$
- D. Does not exist

Explanation:

The correct answer is A. First, use the properties of limits to simplify the problem:

$$\lim_{x \rightarrow 1^+} \sqrt{10 + \frac{1-x}{\sin(x-1)}} = \sqrt{\lim_{x \rightarrow 1^+} \left(10 + \frac{1-x}{\sin(x-1)} \right)} = \sqrt{\lim_{x \rightarrow 1^+} 10 + \lim_{x \rightarrow 1^+} \frac{1-x}{\sin(x-1)}}.$$

Observe that since the limit of a constant is the constant, $\lim_{x \rightarrow 1^+} 10 = 10$. The second limit is more delicate and can be handled one of two ways: using a known result or l'Hopital's rule (since it is indeterminate of the form $0/0$). We show the former:

$$\lim_{x \rightarrow 1^+} \frac{1-x}{\sin(x-1)} = \lim_{x \rightarrow 1^+} \frac{1}{\frac{\sin(x-1)}{-(x-1)}} = \frac{\lim_{x \rightarrow 1^+} 1}{\lim_{x \rightarrow 1^+} \left(\frac{\sin(x-1)}{-(x-1)} \right)} = \frac{1}{-\lim_{x \rightarrow 1^+} \frac{\sin(x-1)}{(x-1)}} = -1$$

Substituting these results into the initial equation yields $\lim_{x \rightarrow 1^+} \sqrt{10 + \frac{1-x}{\sin(x-1)}} = \sqrt{10-1} = \sqrt{9} = 3$.

Compute the limit: $\lim_{x \rightarrow 0} \frac{\cos(2x) - 1}{e^{3x} - (3x + 1)}$

- A. $-4/9$
- B. 0
- C. $4/9$
- D. ∞

Explanation:

The correct answer is A. Substituting in $x = 0$ directly shows that the limit is indeterminate of the form $0/0$. So, use l'Hopital's rule to compute the limit:

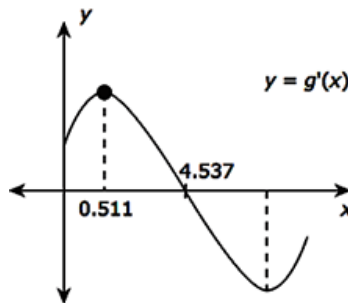
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos(2x) - 1}{e^{3x} - (3x + 1)} &= \lim_{x \rightarrow 0} \frac{-2 \sin(2x)}{3e^{3x} - 3} \quad (\text{still } 0/0, \text{ so apply l'Hopital's rule again!}) \\ &= \lim_{x \rightarrow 0} \frac{-4 \cos(2x)}{9e^{3x}} \quad (\text{no longer } 0/0, \text{ so substitute in } x = 0) \\ &= -\frac{4}{9} \end{aligned}$$

Let $g(x) = x^{2/3} \sin(\sqrt{x})$. Determine the smallest nonnegative value of c for which the tangent line to $g(x)$ at $x = c$ is horizontal.

- A. 0
- B. 0.511
- C. 4.4.537
- D. 12.654

Explanation:

The correct answer is C. Observe that $g'(x) = x^{2/3} \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + \sin(\sqrt{x}) \cdot \frac{2}{3} x^{-1/3}$. We must determine a value of c for which $g'(c) = 0$, which corresponds to an x -intercept of the graph of $g'(x)$. Use the graphing calculator to get the following:



Hence, the smallest such c value is approximately 4.511.

Analytical Applications of Differentiation

About 8–11% of the questions on your AP exam will cover Analytical Applications of Differentiation.

Mean Value Theorem

The Mean Value Theorem states that if f is continuous on $[a, b]$ and differentiable on (a, b) , then there is at least one point between a and b at which the instantaneous rate of change of f is equal to its average rate of change over the entire interval. In other words, there is at least one value c in the interval (a, b) for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Free Response Tip

When part of a free response question contains the phrase “explain why there must be a value...” you should immediately think of two theorems. If the function for which a value is being described is a derivative, consider the Mean Value Theorem first. If not, consider the Intermediate Value Theorem. In either case, make sure to justify why the theorem can be applied in terms of continuity and differentiability.

Intervals of Increase and Decrease and the First Derivative Test

When the derivative of a function is positive, the function increases, and when the derivative is negative, the function decreases. To find intervals on which a function is increasing or decreasing, then, it is necessary to solve for where its derivative is positive or negative. The procedure for doing this involves first finding the values, called critical points, at which the derivative is zero or undefined.

If f changes from increasing to decreasing at $x = c$, f has a local maximum at c . If it changes from decreasing to increasing at $x = c$, it has a local minimum at c . Taken together, local maximums and local minimums are referred to as local extrema.

The first derivative test summarizes these facts and describes the process of finding local maximums and minimums. Specifically, suppose $x = c$ is a critical point of f .

Then:

- If f' is positive to the left of c , and negative to the right of c , then f has a local maximum at c .
- If f' is negative to the left of c , and positive to the right of c , then f has a local minimum at c .
- If neither of the above conditions apply, f does not have a local extreme at c .

Example

Let $f(x) = x^5 - 3x^3$. To find the local extrema of f , we begin by finding the derivative, setting it to 0, and solving for x :

$$\begin{aligned} f'(x) &= 5x^4 - 9x^2 \\ 5x^4 - 9x^2 &= 0 \\ x^2(5x^2 - 9) &= 0 \\ x &= 0, \frac{3}{\sqrt{5}}, -\frac{3}{\sqrt{5}} \end{aligned}$$

Since f' is never undefined, these three values are the only critical points of f . These critical points divide the real number line into four intervals: $\left(-\infty, -\frac{3}{\sqrt{5}}\right)$, $\left(-\frac{3}{\sqrt{5}}, 0\right)$, $\left(0, \frac{3}{\sqrt{5}}\right)$, and $\left(\frac{3}{\sqrt{5}}, \infty\right)$. From each of these intervals we choose a point and use it to determine whether f' is positive or negative on the interval. Note that $\frac{3}{\sqrt{5}} \approx 1.34$.

Interval	$\left(-\infty, -\frac{3}{\sqrt{5}}\right)$	$\left(-\frac{3}{\sqrt{5}}, 0\right)$	$\left(0, \frac{3}{\sqrt{5}}\right)$	$\left(\frac{3}{\sqrt{5}}, \infty\right)$
Test point $x = a$	-2	-1	1	2
$f'(a)$	$f'(-2) = 44$	$f'(-1) = -4$	$f'(1) = -4$	$f'(2) = 44$
Conclusion	f' is positive, so f is increasing	f' is negative, so f is decreasing	f' is negative, so f is decreasing	f' is positive, so f is increasing

Examining the table, we see that f changes from increasing to decreasing at $x = -\frac{3}{\sqrt{5}}$, so f has a local maximum there. Also, f changes from decreasing to increasing at $x = \frac{3}{\sqrt{5}}$, so f has a local

minimum there. Note that at $x = 0$ f has neither a maximum nor a minimum, since the derivative does not change signs from the left to the right of the point.

Absolute Extrema

If $f(c) = M$ is the largest value that f attains on some interval I containing c , then M is called the global maximum of f on I . Similarly, if $f(c) = M$ is the smallest value that f attains on some interval I containing c , then M is called the global minimum of f on I .

There is no reason to expect that an arbitrary function has a global maximum or minimum value on a given interval. However, the Extreme Value Theorem guarantees that a function does have a global maximum and a global minimum on any closed interval on which it is continuous. On such an interval, both of the global extrema must occur at either a critical point or at an endpoint of the interval.

The candidate test gives a procedure for finding these global extrema on a closed interval $[a, b]$:

1. Check that f is continuous on $[a, b]$.
2. Find the critical numbers of f between a and b .
3. Check the value of f at each critical number, at a and at b .
4. The largest value found in the previous step is the global maximum, and the smallest value found is the global minimum.

Concavity and Inflection Points

The graph of a function f is concave up when its derivative f' is increasing, and it is concave down when f' is decreasing. Since the relationship of f'' to f' is the same as the relationship of f' to f , we can determine on which intervals f' is increasing (or decreasing) by checking where f'' is positive (or negative). Therefore, the criteria for f being concave up or down can be restated in terms of f'' : f is concave up when f'' is positive, and concave down when f'' is negative.

A point at which a function changes concavity (from up to down or down to up) is called a point of inflection. These can be found in a completely analogous manner to how local extrema are located using the first derivative test: find where the second derivative is 0 or undefined, and test points on either side to determine if concavity is changing.

Second Derivative Test







In addition to providing information about concavity and inflection points, the second derivative of a function can also help determine whether a critical point represents a relative maximum or minimum. Specifically, suppose f has a critical point at $x = c$.

Then:

- if $f''(c) > 0$, f has a local minimum at c .
- if $f''(c) < 0$, f has a local maximum at c .
- if $f''(c) = 0$, this test is inconclusive, and the first derivative test must be used.

Summary of Curve Sketching

The table below summarizes the behavior of a graph at $x = c$, depending on the values of $f'(c)$ and $f''(c)$.

	$f'(c) > 0$	$f'(c) < 0$	$f'(c) = 0$
$f''(c) > 0$			
$f''(c) < 0$			

Optimization

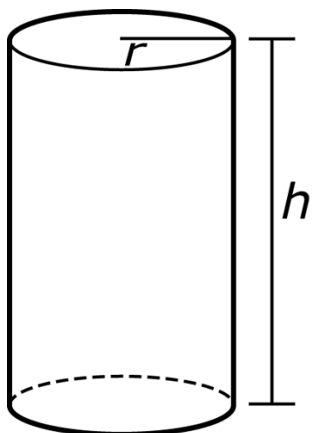
The techniques given for finding local and global extrema can be applied in a wide variety of application problems, known as optimization problems. The details of the procedure and strategy vary by context, but there are some nearly universal steps for such situations:

1. Draw a picture.
2. Write a function for the quantity to be optimized (maximized or minimized).
3. Rewrite the function from the previous step to be in terms of a single independent variable. This often involves using a secondary equation, called a constraint.
4. Determine the domain of interest.
5. Differentiate the function and find the relevant critical points.
6. Use the first derivative test, second derivative test, or candidates test to determine which of the critical points or endpoints represent the optimal solution.

Example

A manufacturer wants to construct a cylindrical container with a volume of 5 ft^3 . Using the steps notes, let us find the dimensions of the container that will minimize the amount of material used.

1.



2. The quantity to be optimized is the surface area of the container. In terms of r and h , the surface area is given by the function $S = 2\pi r^2 + 2\pi rh$.
3. As written, the function that gives the surface area depends on both r and h . However, since we know the volume of the cylinder is to be 5, and the volume formula is

$V = \pi r^2 h$, we have the constraint $\pi r^2 h = 5$. Solving for h gives $h = \frac{5}{\pi r^2}$. This can be

substituted into the function S : $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \left(\frac{5}{\pi r^2} \right) = 2\pi r^2 + \frac{10}{r}$.

Now S is written in terms of a single variable, r .

4. Considering the physical situation, it is clear that the domain of interest is $r > 0$. A cylinder cannot exist with $r \leq 0$.
5. Differentiating and setting to zero:

$$\frac{dS}{dr} = 4\pi r - \frac{10}{r^2}$$

$$4\pi r - \frac{10}{r^2} = 0$$

$$4\pi r^3 - 10 = 0$$

$$r^3 = \frac{5}{2\pi}$$

$$r = \sqrt[3]{\frac{5}{2\pi}}$$

The derivative is undefined at $r = 0$, but that is irrelevant since it is not in the domain.

6. The only critical point is $r = \sqrt[3]{\frac{5}{2\pi}}$, and the domain of r is $(0, \infty)$, so there are no endpoints. To justify that this critical point is indeed a minimum, we will use the second derivative test.

$$\frac{d^2S}{dr^2} = 4\pi + \frac{20}{r^3}. \text{ Evaluating at } r = \sqrt[3]{\frac{5}{2\pi}}, \text{ we have } \frac{d^2S}{dr^2} = 4\pi + \frac{20}{\left(\sqrt[3]{\frac{5}{2\pi}}\right)^3} = 12\pi. \text{ Since}$$

this is positive, the critical point is indeed a minimum, as desired.

Free Response Tip

As in the example provided, many applied optimization problems appear to have only one possible solution. Even when this is the case, make sure you include a justification for this solution being the desired optimal point. The Second Derivative Test is often the easiest way to do this—but keep the First Derivative Test and the Candidate Test in mind as well.

Implicitly Defined Curves

When a curve is defined implicitly in an equation involving x and y , the applications of derivatives discussed in this section still generally apply. As with explicitly defined functions, critical points are determined by examining where $\frac{dy}{dx} = 0$ or is undefined. However, the details

of finding where this occurs are often more complicated since the expression for $\frac{dy}{dx}$ usually involves both x and y . Second derivatives are often trickier to find as well. Two points are helpful:

- The derivative of $\frac{dy}{dx}$ with respect to x is the second derivative of y with respect to x . In other words, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.
- When the expression for $\frac{d^2y}{dx^2}$ involves $\frac{dy}{dx}$, it is usually possible to simplify by substituting a previously obtained expression for $\frac{dy}{dx}$.

Example

Suppose $x^2 + 2xy = 0$. The derivative $\frac{dy}{dx}$ can be found by differentiating with respect to x and

solving for $\frac{dy}{dx}$:

$$\frac{d}{dx}(x^2 + 2xy) = \frac{d}{dx}(0)$$

$$2x + 2 \cdot y + 2x \cdot 1 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x - y}{x}$$

To find the second derivative, differentiate both sides of this result with respect to x :

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-x - y}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\left(-1 - 1 \frac{dy}{dx}\right)(x) - (-x - y)(1)}{x^2}$$

Now substituting $\frac{-x - y}{x}$ for $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{\left(-1 - 1\left(\frac{-x - y}{x}\right)\right)(x) - (-x - y)(1)}{x^2} = \frac{-x + x + y + x + y}{x^2} = \frac{x + 2y}{x^2}$$

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapter 4. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 4. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapter 4. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 4. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 4. New York, NY: W.H. Freeman.

Sample Analytical Applications of Differentiation Questions

What is the absolute maximum value of $g(x) = x^2 e^{-2x}$ on $[-1, 2]$?

- A. -1
- B. 0
- C. e^2
- D. e^{-2}

Explanation:

The correct answer is C. You must first determine all the critical points on the interval $[-1, 2]$. Then evaluate $g(x)$ at each of them and identify the largest one as the absolute maximum. The endpoints -1 and 2 are automatically endpoints. To find the others, compute the derivative and find where it equals zero:

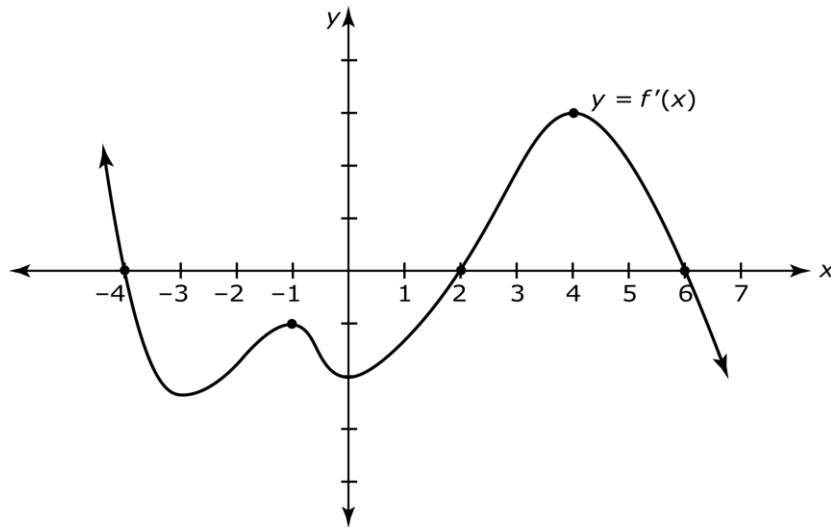
$$\begin{aligned}g'(x) &= x^2 e^{-2x}(-2) + 2x e^{-2x} \\ &= -2x e^{-2x}(x - 1)\end{aligned}$$

This is equal to zero when any of its factors equals zero, namely $x = 0$ and $x = 1$. So, the list of critical points is $-1, 0, 1,$ and 2 . Compute $g(x)$ at each of these:

$$g(-1) = e^2, \quad g(0) = 0, \quad g(1) = e^{-2}, \quad g(2) = 4e^{-4}$$

Observe that both $g(1)$ and $g(2)$ are less than 1. So, $g(-1) = e^2$ is the absolute maximum.

Suppose $f(x)$ is a twice differentiable function. The graph of $y = f'(x)$ is as follows:



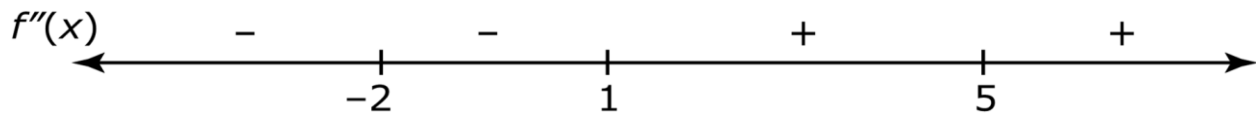
On what intervals is the graph of $y = f(x)$ concave down?

- A. $(-1, 0) \cup (2, 6)$
- B. $(-4, 2) \cup (6, \infty)$
- C. $(-\infty, -3) \cup (-1, 0) \cup (4, \infty)$
- D. $(-\infty, -4) \cup (2, 6)$

Explanation:

The correct answer is C. The graph of $y = f(x)$ is concave down on the intervals where the graph of $y = f'(x)$ is decreasing. This occurs when x is in the set $(-\infty, -3) \cup (-1, 0) \cup (4, \infty)$.

Suppose $f(x)$ is a function such that $f'(1) = 1$, $f''(-2) = f''(1) = 0$, and $f''(5)$ does not exist. Moreover, the sign of $f''(x)$ is given as follows:



Which of these statements must be true?

- (I) $f'(x)$ is increasing on $(1, 5)$.
- (II) $f(x)$ has a local maximum at $x = 5$.
- (III) $f(x)$ has an inflection point at $x = 1$.

- A. I and II only
- B. I only
- C. I and III only
- D. I, II, and III

Explanation:

The correct answer is C. I is true because $f''(x) = f'(x)' > 0$ means $f'(x)$ is increasing. (II) is false because $f(x)$ could have a vertical asymptote at $x = 5$; in such case, there is no local maximum at $x = 5$. III is true because the sign of $f''(x)$ changes on either side of $x = 1$ and the point $(1, f(1))$ must exist because $f'(1)$ is assumed to exist.

Integration and Accumulation of Change

Around 17–20% of the questions on your exam will cover Integration and Accumulation of Change.

Riemann Sums and the Definite Integral

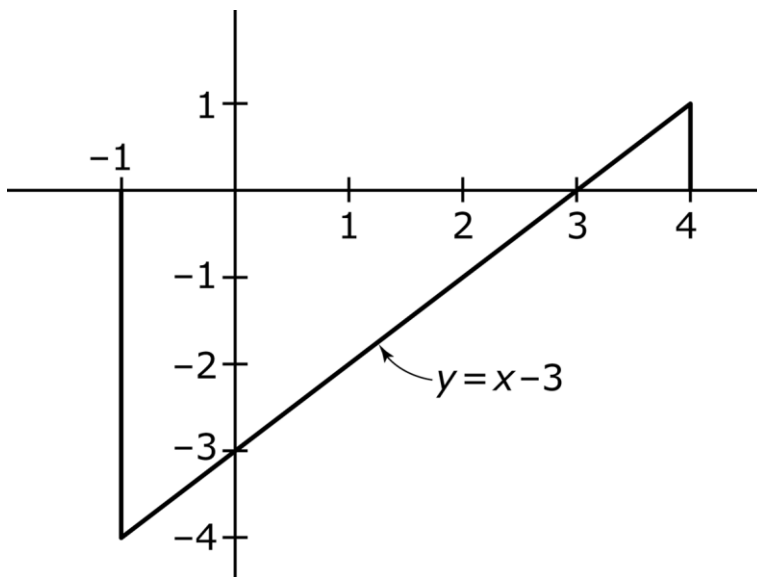
When a function represents a rate of change, the area between the graph of the function and the x -axis represents the accumulation of the change. If the area is above the x -axis, the accumulated change is positive, whereas if the area is below the x -axis, the accumulated change is negative.

More generally, the accumulation of a function on a closed interval $[a, b]$, represented graphically by the area between a function and the x -axis, is called the definite integral of the function on that interval, and is denoted $\int_a^b f(x) dx$.

For simple functions, the definite integral can often be evaluated geometrically.

Example

To evaluate $\int_{-1}^4 (x-3) dx$, draw a picture:



The area between the curve and the graph is divided into two triangles. The larger triangle has an area of 8. However, it is below the x -axis, so the accumulation is of negative values. Therefore, it contributes a value of -8 to the integral. The smaller triangle accumulates positive values and has an area of $\frac{1}{2}$. Together, we have $\int_{-1}^4 (x-3) dx = -8 + \frac{1}{2} = -\frac{15}{2}$.

Definite integrals can be approximated using a variety of sums, each term of which represents the area of a rectangle over a small subinterval. To begin, consider a function $f(x)$ over the interval $[a, b]$, and let n be the number of equally sized subintervals into which it is split. Then

$\Delta x = \frac{b-a}{n}$ is the width of each subinterval, and $x_i = a + i\Delta x$ is the left endpoint of the i^{th}

subinterval. If a rectangle is constructed on each subinterval so that its height is equal to the value of $f(x_i)$, the sum of the areas will be $(f(x_0) + f(x_1) + \dots + f(x_{n-1}))\Delta x$. This sum is called a left Riemann sum.

The notation $\sum_{i=1}^n a_i$ stands for the sum $a_1 + a_2 + \dots + a_n$. The left Riemann sum can be written

using this notation as $\sum_{i=0}^{n-1} f(x_i)\Delta x$. Other commonly used approximations are the right Riemann sum, and the midpoint Riemann sum, shown as follows:

Left Riemann sum	Right Riemann sum	Midpoint Riemann sum
$\sum_{i=0}^{n-1} f(x_i)\Delta x$	$\sum_{i=1}^n f(x_i)\Delta x$	$\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$

As n increases in size, each of these Riemann sums become a more accurate approximation of the definite integral. When the limit is taken as $n \rightarrow \infty$, any of these sums become equal to the definite integral. In other words, the integral of a function f over the interval $[a, b]$ can be

defined as $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$, provided this limit exists.

In fact, although $\Delta x = \frac{b-a}{n}$ is the most common way to divide an interval into subintervals, all of the above sums can be computed with potentially different Δx values for each subinterval. The limit of the sum is still equal to the definite integral.

Another expression that can be used to approximate the definite integral is a trapezoidal sum, which represents the areas of trapezoids, rather than rectangles, constructed over the subintervals. The trapezoidal sum is $\frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))$.

Free Response Tip

Free response questions often give values of a function in a table. A Riemann sum can be used to approximate its integral using the subintervals shown in the table, even if the intervals are not all the same length. The length of each subinterval is the distance between consecutive x -values, and the height of the rectangle on that subinterval is the y -value associated with either the left x (in case of a left Riemann sum) or the right x (in case of a right Riemann sum). In either case, your sum should have one fewer term than there are points given in the table.

Properties of the Definite Integral

The definite integral satisfies several properties:

- $\int_a^b c \, dx = c(b-a)$, for any constant c
- $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$
- $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$
- $\int_a^a f(x) \, dx = 0$
- $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$
- $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$

Example

Suppose $\int_1^7 f(x) \, dx = 9$ and $\int_1^4 f(x) \, dx = 12$. Find $\int_7^4 (f(x) - 3) \, dx$.

First, we have $\int_7^4 (f(x) - 3) \, dx = \int_7^4 f(x) \, dx - \int_7^4 3 \, dx$. The latter integral is simply $3(4 - 7) = -9$.

For the former:

$$\begin{aligned}\int_7^4 f(x) \, dx &= \int_7^1 f(x) \, dx + \int_1^4 f(x) \, dx \\ &= -\int_1^7 f(x) \, dx + 12 \\ &= -9 + 12 \\ &= 3\end{aligned}$$

The answer is then $\int_7^4 (f(x) - 3) dx = 3 - (-9) = 12$.

Accumulation Functions and the Fundamental Theorem of Calculus

A function can be defined in terms of a definite integral: $g(x) = \int_a^x f(t) dt$. The first part of the Fundamental Theorem of Calculus states that the derivative of this function at a given point is equal to the value of the function being accumulated. That is, $g'(x) = \frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$.

Since $\int_x^a f(t) dt = -\int_a^x f(t) dt$, we also have $\frac{d}{dx} \left(\int_x^a f(t) dt \right) = -\frac{d}{dx} \left(\int_a^x f(t) dt \right) = -f(x)$. If the upper limit of integration is a function of x , the chain rule can be applied along with the fundamental theorem.

Example

If $f(x) = \int_2^{x^2} \sin t dt$, then $f'(x) = \sin(x^2) \cdot 2x$

Antiderivatives and the Fundamental Theorem of Calculus

If $g'(x) = f(x)$, g is said to be an antiderivative of f . Note that if C is any constant, then $\frac{d}{dx} [g(x) + C] = g'(x) + 0 = f(x)$, so that $g(x) + C$ is also an antiderivative of f . In fact, all antiderivatives of a given function have this relationship with each other: they differ only by a constant. Every continuous function f has an antiderivative, since the function $g(x) = \int_a^x f(t) dt$ satisfies $g'(x) = f(x)$ and is therefore an antiderivative of f .

The second part of the Fundamental Theorem of Calculus states that if f is continuous on the interval $[a, b]$, and F is any antiderivative of f on that interval, then $\int_a^b f(x) dx = F(b) - F(a)$. This fact means that antiderivatives and integrals are very closely related. Because of this, an antiderivative is also called an indefinite integral, and is denoted $\int f(x) dx = F(x) + C$, where F is any antiderivative.

Basic Rules of Antiderivatives

Since finding an antiderivative is the inverse process of finding a derivative, the rules for derivatives can be reversed to find antiderivatives.

x^n	$\frac{1}{n+1}x^{n+1} + C$
e^x	$e^x + C$
$\frac{1}{x}$	$\ln x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\csc^2 x$	$-\cot x + C$

Integration by Substitution

Substitution, also known as change of variables, is a technique for finding antiderivatives and is analogous to the chain rule for derivatives. It works by noting that

$\int f'(g(x))g'(x)dx = f(g(x)) + C$. The technique, then, requires recognizing the $g(x)$ and $g'(x)$ in the expression being integrated.

If $u = g(x)$, then $du = g'(x)dx$, so the integral can be written $\int f(u)du = f(u) + C$.

When using this technique with definite integrals, it is important to translate the limits of integration to be in terms of the new function u .

Example

The integral $\int_{\pi/2}^{\pi} \sin^2 \theta \cos \theta d\theta$ can be evaluated by substituting $u = \sin \theta$. Then $du = \cos \theta d\theta$.

When $\theta = \frac{\pi}{2}$, $u = \sin \frac{\pi}{2} = 1$, and when $\theta = \pi$, $u = \sin \pi = -1$. The integral becomes

$$\int_1^0 u^2 du = \left[\frac{1}{3}u^3 \right]_1^0 = 0 - \frac{1}{3} = -\frac{1}{3}.$$

Integration by Parts

Integration by parts is another technique for finding derivatives and is the antidifferentiation analogue to the product rule for derivatives. The rule states that $\int u dv = uv - \int v du$. The key in applying it is choosing useful expressions for u and dv . The acronym LIPET can help you prioritize the expression chosen for u : Logarithm, Inverse trigonometric, Polynomial, Exponential, and Trigonometric. More generally, u should be an expression that will become simpler when differentiated, and dv should be an expression for which you can easily find an antiderivative.

Example

Consider $\int_0^1 xe^x dx$. By setting $u = x$ and $dv = e^x dx$, we can find $du = dx$ and $v = e^x$. Then, using the integration by parts rule (and temporarily ignoring the limits of integration), the integral becomes $\int xe^x = xe^x - \int e^x dx = xe^x - e^x$. The answer to the original integral is $[xe^x - e^x]_0^1 = (1e^1 - e^1) - (0e^0 - e^0) = 1$.

Note that it is possible that the integral $\int v du$ obtained when integrating by parts itself requires another integration by parts.

Other Integration Techniques

If the numerator of a rational function has a degree that is at least as high as the degree of the denominator, long division is often helpful in integration.

Example

Consider $\int \frac{x^3 + x}{x-1} dx$. Since the numerator has a higher degree than the denominator, long division can be applied to transform the integral. We get $\frac{x^3 + x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$, so that the answer is $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C$.

Another technique that can be useful for some integrals is completing the square.

Integration by Partial Fractions

If a rational function does not lend itself well to long division, it is possible that it can be decomposed into a sum of multiple terms, each of which can then be integrated. The AP exam only covers situations in which these terms have nonrepeating linear factors. The general technique is illustrated in the following example.

Example

The rational function $\frac{x+9}{x^2-3x-10}$ is not a candidate for long division, and the substitution

$u = x^2 - 3x - 10$ does not yield anything useful, since then $du = 2x - 3$. Instead, we will decompose it into partial fractions.

To begin, note that the denominator factors as $(x-5)(x+2)$. We would like to find A and B such that $\frac{x+9}{x^2-3x-10} = \frac{A}{x-5} + \frac{B}{x+2}$. Multiplying both sides of this equation by $(x-5)(x+2)$, we get $x+9 = A(x+2) + B(x-5)$. Substituting $x = -2$ gives $7 = -7B$, so $B = -1$, and substituting $x = 5$ gives $14 = 7A$, so $A = 2$. Thus, we have found that $\frac{x+9}{x^2-3x-10} = \frac{2}{x-5} - \frac{1}{x+2}$.

The integral of the original expression can now be found:

$$\int \frac{x+9}{x^2-3x-10} dx = \int \left(\frac{2}{x-5} - \frac{1}{x+2} \right) dx = 2 \ln|x-5| - \ln|x+2| + C$$

Improper Integrals

If an integral has $\pm\infty$ and has one or both of the limits of integration, it is called an improper integral. An improper integral of this type can be evaluated as the limit of a standard definite integral. If the limit exists, we say that the integral converges to the value of the limit; otherwise, the integral diverges.

Example

$\int_0^{\infty} e^{-x} dx$ is evaluated as $\lim_{a \rightarrow \infty} \int_0^a e^{-x} dx$. The antiderivative of e^{-x} is $-e^{-x}$, the expression becomes $\lim_{a \rightarrow \infty} [-e^{-x}]_0^a = -e^{-a} - (-e^0) = 0 + 1 = 1$. The integral converges and has a value of 1.

Another type of improper integral occurs when the integrand is unbounded within the interval of integration. This will usually be due to a vertical asymptote. Here too the integral is evaluated using the limits of standard integrals as the limits of integration approach the discontinuity.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapters 5-7. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapters 5 and 8. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapters 5 and 7. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapters 5 and 7. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapters 5 and 7. New York, NY: W.H. Freeman.

Sample Integration and Accumulation of Change Questions

Compute $\int \ln(\sqrt{x}) dx$.

- A. $\frac{2}{3}\sqrt{x} + C$
- B. $\frac{1}{2}x \ln x + C$
- C. $x \ln(\sqrt{x}) - \frac{1}{2}x + C$
- D. $\frac{1}{x} + C$

Explanation:

The correct answer is C. First, use the logarithm properties to write $\ln(\sqrt{x})$ as $\frac{1}{2} \ln x$. Then,

$\int \ln(\sqrt{x}) dx = \frac{1}{2} \int \ln x dx$. To compute $\int \ln x dx$, use the integration by parts formula

$\int u dv = uv - \int v du$ with

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

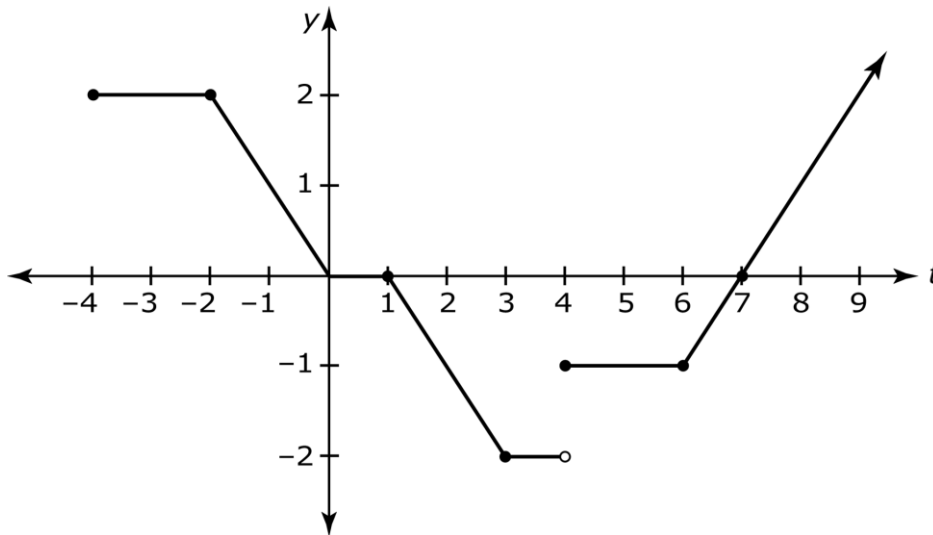
Doing so yields

$$\begin{aligned} \int \ln x dx &= x \ln x - \int x \cdot \frac{1}{x} dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + C \end{aligned}$$

So:

$$\begin{aligned}\int \ln(\sqrt{x}) dx &= \frac{1}{2} \int \ln x dx \\ &= \frac{1}{2}(x \ln x - x + C) \\ &= \frac{1}{2} x \ln x - \frac{1}{2} x + C \\ &= x\left(\frac{1}{2} \ln x\right) - \frac{1}{2} x + C \\ &= x \ln(\sqrt{x}) - \frac{1}{2} x + C\end{aligned}$$

The graph of $y = f(t)$ is given below:



Define the function $A(x) = \int_{-4}^x f(t) dt$, $x \geq -4$. At what x -value does $A(x) = 0$?

- A. 3
- B. 4
- C. 5
- D. 6

Explanation:

The correct answer is D. Use the geometric formulas for area of rectangles and triangles.

Observe that $A(0) = 2(2) + \frac{1}{2}(2)(2) = 6$ square units. So, $\int_{-4}^0 f(t) dt = 6$. Now you must determine a value of $x > 0$ such that the area that lies under the t -axis is 6 square units. To this end, observe that

$$\int_0^1 f(t) dt = \int_0^1 0 dt = 0$$

$$\int_1^3 f(t) dt = -\frac{1}{2}(2)(2) = -2$$

$$\int_3^4 f(t) dt = (-1)(2) = -2$$

$$\int_4^6 f(t) dt = (-1)(2) = -2$$

So, by interval additivity,

$$A(6) = \int_{-4}^6 f(t) dt = \int_{-4}^0 f(t) dt + \int_0^1 f(t) dt + \int_1^3 f(t) dt + \int_3^4 f(t) dt + \int_4^6 f(t) dt = 0.$$

If $f(x) = \int_{x^2}^1 \tan^{-1}(t) dt$, compute $f'\left(\frac{1}{\sqrt[4]{3}}\right)$.

A. $-\frac{\pi}{3\sqrt[4]{3}}$

B. $\frac{\sqrt{3}}{1+\sqrt{3}}$

C. $\frac{\sqrt{3}}{1+\sqrt{3}} - \frac{1}{2}$

D. $\frac{\pi}{4} - \frac{\pi}{3\sqrt[4]{3}}$

Explanation:

The correct answer is A. First write the integral so that the upper limit is the variable expression by flipping the limits and multiplying the integral by -1 .

$$f(x) = \int_{x^2}^1 \tan^{-1}(t) dt = -\int_1^{x^2} \tan^{-1}(t) dt$$

Now use the fundamental theorem of calculus with the chain rule to differentiate $f(x)$:

$$f'(x) = -\tan^{-1}(x^2) \cdot 2x$$

Now compute $f'\left(\frac{1}{\sqrt[4]{3}}\right)$:

$$\begin{aligned} f'\left(\frac{1}{\sqrt[4]{3}}\right) &= -\tan^{-1}\left(\left(\frac{1}{\sqrt[4]{3}}\right)^2\right) \cdot 2\left(\frac{1}{\sqrt[4]{3}}\right) \\ &= \frac{-2}{\sqrt[4]{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{-2}{\sqrt[4]{3}} \cdot \frac{\pi}{6} \\ &= \frac{-\pi}{3\sqrt[4]{3}} \end{aligned}$$

Differential Equations

About 6–9% of the questions on your exam will cover Differential Equations.

Introduction to Differential Equations

A differential equation is an equation that involves a function and one or more of its derivatives. The solution to a differential equation is a function that satisfies the equation

A differential equation may have infinitely many solutions parameterized by a constant; this is called the general solution to the equation. If additional information is given, the constant can be determined. This additional information comes in the form of an initial condition; that is, a value $f(x_0) = y_0$ that must be satisfied by the solution.

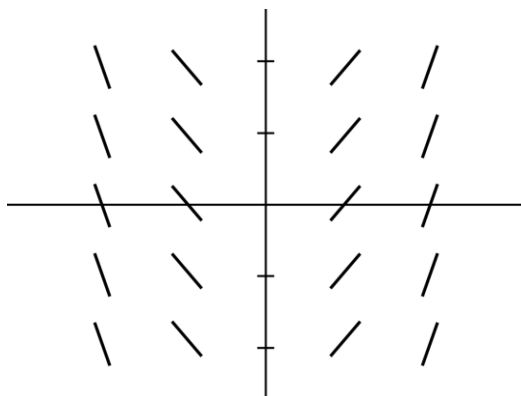
Example

Consider the differential equation $y'' = -y$ with initial condition $y\left(\frac{\pi}{2}\right) = -7$. Any function of the form $y = C \sin x$ is a general solution to this equation since $\frac{d^2}{dx^2}(C \sin x) = -C \sin x$. Using the initial condition given, we have $-7 = C \sin \frac{\pi}{2}$, so we can solve to find $C = -7$. The solution to the equation is $y = -7 \sin x$.

Slope Fields

A slope field is a graphical representation of a differential equation. At each of finitely many points in some section of a plane, a short line is drawn representing the slope of a function. This represents a differential equation whose solution is the function whose slopes are being drawn.

Example



The slope field shown represents the differential equation $\frac{dy}{dx} = 2x$. The solutions to this equation are the functions $y = x^2 + C$, as can be seen in the shapes formed by the slopes shown.

Euler's Method

Euler's Method is a technique for approximating a value of the solution to a differential equation given an initial condition. It works by following the slope defined by the differential equation for a series of short intervals, called steps, starting at the initial condition.

Given an initial point (x_0, y_0) , and step size Δx , successive points (x_n, y_n) for $n \geq 1$ are found using the equations $x_n = x_{n-1} + \Delta x$ and $y_n = y_{n-1} + \Delta x \cdot \frac{dy}{dx}\bigg|_{(x_{n-1}, y_{n-1})}$. This process continues until the desired point is reached.

Example

Given $\frac{dy}{dx} = 2xy + 1$ with initial condition $(0, 1)$, we will approximate $y(1)$ using two steps of equal size.

Since x has to move from 0 to 1 with 2 steps, we have $\Delta x = 0.5$. We have $(x_0, y_0) = (0, 1)$, so $x_1 = 0 + 0.5 = 0.5$ and $y_1 = 1 + 0.5(2(0)(1) + 1) = 1.5$. Continuing to the next step, we get $x_2 = 0.5 + 0.5 = 1$ and $y_2 = 1.5 + 0.5(2(0.5)(1.5) + 1) = 2.75$. Our estimate for $y(1)$ is 2.75.

Separation of Variables

A certain class of differential equations, called separable equations, can be solved using antidifferentiation. The technique requires separating the variables so that each is represented only on a single side of the equation. Integrating both sides of the equation then produces a general solution. If an initial condition is provided, it can be used to find a particular solution. When integrating, it is only necessary to include a constant C on one side of the equation.

Example

Consider the differential equation $\frac{dy}{dt} = 6y^2t$ with initial condition $y(1) = 1$. To solve this, we begin by separating the variables: $\frac{1}{y^2} dy = 6t dt$. Integrating both sides, we have:

$$\int \frac{1}{y^2} dy = \int 6t dt \Rightarrow -\frac{1}{y} = 3t^2 + C$$

This gives a general solution, although it is implicitly defined. We can solve for y to make it explicit, but it is often advisable to first use the initial condition to solve for C . In this case, substituting the initial values results in $C = -4$.

Using this value and solving for y , we can obtain the explicit solution:

$$-\frac{1}{y} = 3t^2 - 4$$

$$-y = \frac{1}{3t^2 - 4}$$

$$y = \frac{-1}{3t^2 - 4}$$

Exponential Models

Many applications of differential equations involve an exponential growth or decay model. This model occurs in any situation in which the rate of change of a quantity is proportional to the quantity. As an equation, this is represented by $\frac{dy}{dt} = ky$. This equation is easily solved using separation of variables, and the general solution is $y = y_0 e^{kt}$, where y_0 is the value of y when $t = 0$.

Example

The rate of growth in a bacteria culture is proportional to the number of bacteria present. A certain culture starts out with 200 bacteria, and after 2 hours there are 1,000. Let us find the number of bacteria present after 5 hours.

To begin, note that since this follows exponential growth with a starting value of 200, the population is modeled by the equation $y = 200e^{kt}$. To solve for k , use the fact that after 2 hours there are 1000 bacteria:

$$1000 = 200e^{2t}$$

$$5 = e^{2t}$$

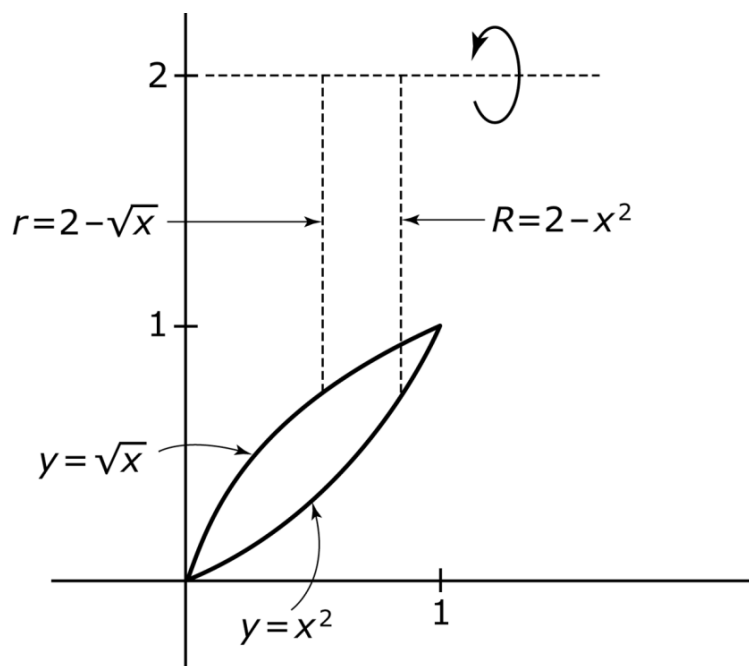
$$t = \frac{\ln 5}{2}$$

We can now rewrite the model as $y = 200e^{\frac{\ln 5}{2}t}$. The population after 5 hours is $200e^{\frac{5 \ln 5}{2}} \approx 11180$.

Logistic Models

The logistic model is a population model defined by the differential equation $\frac{dy}{dt} = ky(M - y)$, where M is called the carrying capacity. Alternatively, this can be written $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$.

Note that the value of k will be different depending on which form of the equation is used. The solution to this differential equation is called a logistic curve, and resembles the following graph:



The following facts about the logistic function $y(t)$ are important:

- $y(t)$ is increasing for all real numbers
- $y(t) < M$ for all real numbers t
- $\lim_{t \rightarrow \infty} y(t) = M$
- The y -coordinate of the inflection point of $y(t)$ is $\frac{M}{2}$. At this point, the function changes from concave up to concave down, and it is also the point of greatest growth rate.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapter 11. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 6. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapter 9. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 9. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 16. New York, NY: W.H. Freeman.

Sample Differential Equations Questions

What is the general solution of the differential equation $e^{2x}y'(x) = e^{4y(x)}$?

- A. $y(x) = \ln\left(\frac{1}{\sqrt[4]{2e^{-2x} + C}}\right)$
- B. $y(x) = -\frac{1}{4}\ln(e^{-2x} + C)$
- C. $y(x) = \ln\left(\sqrt[4]{-2e^{2x} + C}\right)$
- D. $y(x) = -\frac{1}{4}\ln(2e^{2x}) + C$

Explanation:

The correct answer is A. Separate variables by getting the y -terms on one side and the x -terms on the other. Then integrate both sides and solve for y :

$$\begin{aligned}
e^{2x}y'(x) &= e^{4y(x)} \\
e^{2x} \frac{dy}{dx} &= e^{4y} \\
e^{2x} dy &= e^{4y} dx \\
e^{-4y} dy &= e^{-2x} dx \\
\int e^{-4y} dy &= \int e^{-2x} dx \\
-\frac{1}{4}e^{-4y} &= -\frac{1}{2}e^{-2x} + C \\
e^{-4y} &= 2e^{-2x} + C \\
-4y &= \ln(2e^{-2x} + C) \\
y &= -\frac{1}{4}\ln(2e^{-2x} + C) \\
y &= \ln(2e^{-2x} + C)^{-1/4} \\
y &= \ln\left(\frac{1}{\sqrt[4]{2e^{-2x} + C}}\right)
\end{aligned}$$

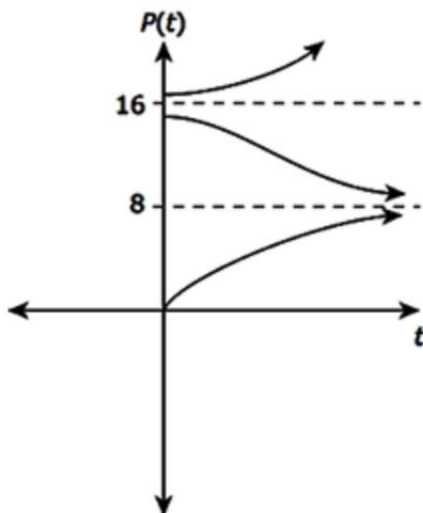
The rate at which a population of gray squirrels, $P(t)$, in a state park changes over time is governed by the differential equation $\frac{dP}{dt} = 5\left(2 - \frac{P}{4}\right)\left(1 - \frac{P}{16}\right)$. Here, $P(t)$ is measured in thousands of squirrels and t is measured in months. For which of the following initial population measurements $P(0) = P_0$ would the population of gray squirrels theoretically increase exponentially?

- A. $P(0) = 20$
- B. $P(0) = 16$
- C. $P(0) = 10$
- D. $P(0) = 4$

Explanation:

The correct answer is A. The equilibrium populations occur when $\frac{dP}{dt} = 0$. Setting the right side of the differential equation equal to zero yields $P = 8$ and $P = 16$.

The corresponding slope field for this equation would look like the following:



So, for any initial population $P(0)$ larger than 16, the population would increase exponentially. So, in particular, if $P(0) = 20$, this is the case.

Use Euler's method with time step $h = 0.25$ and $n = 2$ steps to approximate the solution of the initial-value problem $\begin{cases} y'(t) = y(t) + t \\ y(0) = 2 \end{cases}$ at time $t = 0.5$.

- A. 2.500
- B. 3.188
- C. 4.109
- D. 8.250

Explanation:

The correct answer is B. In the notation of Euler's method, we have:

$$y(t_1) = y_1 = y_0 + hf(t_0, y_0)$$

$$y(t_2) = y_2 = y_1 + hf(t_1, y_1)$$

where

$$y_0 = 2, h = 0.25,$$

$$t_0 = 0, t_1 = 0 + 0.25, t_2 = 0 + 2(0.25) = 0.5$$

$$f(t, y) = y + t$$

Observe that

$$y_1 = y(t_1) = 2 + 0.25(2 + 0) = 2.5$$

$$y_2 = y(t_2) = 2.5 + 0.25(0.25 + 2.5) = 3.1875 \approx 3.188$$

Applications of Integration

About 6–9% of the questions on your AP exam will cover Applications of integration.

Average Value

If f is continuous on the interval $[a, b]$, then the average value of f on that interval is

$\frac{1}{b-a} \int_a^b f(x) dx$. If f is nonnegative on the interval, the average value of the function has a simple graphical interpretation: it is the height a rectangle over the interval would have to be to have the same area as exists between the x -axis and the function.

Position, Velocity, and Acceleration

When a particle is moving along a straight line, its motion can be modeled using derivatives, as discussed earlier. The theory and techniques of integration now allow us to extend this description with the following two points:

- The displacement of the particle over the time interval $[t_1, t_2]$ is given by $\int_{t_1}^{t_2} v(t) dt$, where $v(t)$ is the velocity of the particle.
- The total distance traveled by the particle over the time interval $[t_1, t_2]$ is $\int_{t_1}^{t_2} |v(t)| dt$.
Recall that $|v(t)|$ is the speed of the particle at time t .

Accumulation Functions in Context

The net change of a quantity over an interval can be found by integrating the rate of change. This is an important fact that can be used in a variety of applications.

Example

A tank of water contains 53 gallons at 8:00 AM. Between 8:00 AM and 12:00 PM, water leaks from the tank at a rate of $L(t) = 3|\sin t|$, where t is the number of hours since 8:00 AM, and L is measured in gallons per hour. How much water is remaining in the tank at 12:00 PM?

To solve this, we need to consider two quantities: the amount of water that the tank has at 8:00 AM, and the total amount of water that leaks from the tank between the hours of 8:00 AM and 12:00 PM. The first quantity is given as 53. The second quantity is the accumulation of the rate of

leaking over the four hours. Therefore, the amount of water remaining in the tank at 12:00 PM is $53 - \int_0^4 3|\sin t| dt \approx 46$ gallons.

Free Response Tip

Pay attention to units in free response questions, as they are often required to be correct to receive full credit. Remember that the units of $\frac{dy}{dx}$ are the units of y over the units of x , and the units of $\int_a^b f(x)dx$ are the units of y times the units of x . When the units of y are a rate of change over time, and the units of x represent time, the units of the integral end up being equivalent to whatever quantity is changing.

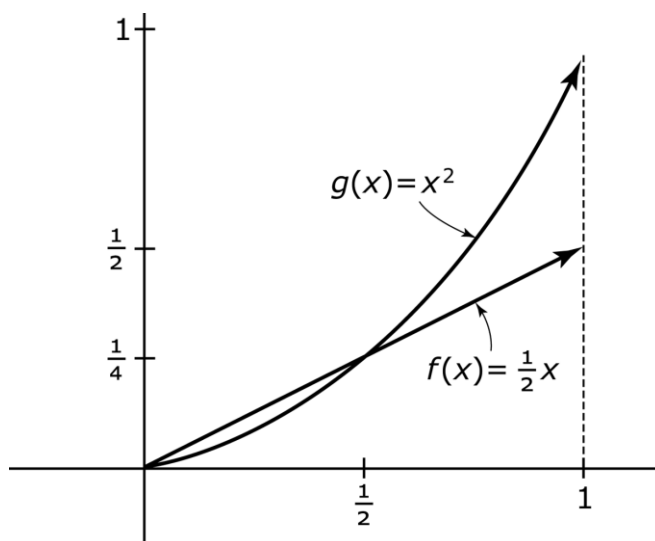
Area Between Curves

If $f(x) \geq g(x)$ on the interval $[a, b]$, then the area between f on g between a and b is

$\int_a^b [f(x) - g(x)] dx$. If the two functions intersect on an interval, the integral needs to be split into multiple subintervals, so that along each section the functions can be subtracted in the proper order.

Example

Let us find the area in the first quadrant bound by the graphs of $f(x) = \frac{1}{2}x$, $g(x) = x^2$, and the line $x = 1$. This is represented in the following graph:



The graphs cross at $x = \frac{1}{2}$, so we will need to evaluate the two regions separately. The first region has area $\int_0^{1/2} \left(\frac{1}{2}x - x^2 \right) dx = \frac{1}{48}$, and the second region has area $\int_{1/2}^1 \left(x^2 - \frac{1}{2}x \right) dx = \frac{5}{48}$.

The total area is $\frac{1}{48} + \frac{5}{48} = \frac{1}{8}$.

When curves are given as functions of y instead of x , the area between them can be found using the same technique. Instead of the integrand being the function on top minus the function on bottom, it is the function on the right minus the function on the left.

Volumes with Cross Sections

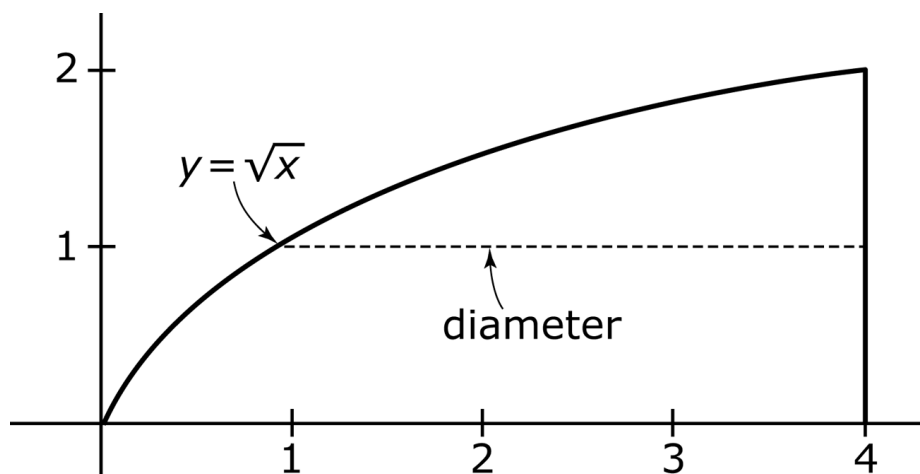
When a solid can be described in terms of its base and cross-sectional shapes, the volume of the solid can be computed by integrating the area of the cross sections along an appropriate interval. If the cross sections described are perpendicular to the x -axis, the volume is given by

$V = \int_a^b A(x) dx$, where $A(x)$ is the area of the cross section in terms of x . If the cross sections are perpendicular to the y -axis, the integral is with respect to y : $V = \int_a^b A(y) dy$.

Shapes commonly used as cross sections include squares, rectangles, right triangles, equilateral triangles, and semicircles.

Example

The base of a solid is the region in the first quadrant of the xy -plane bounded by $y = \sqrt{x}$ and the vertical line $x = 4$, shown as follows. Cross sections of the solid taken perpendicular to the y -axis are semicircles with the diameter lying in the region given.



To find the volume of this solid, we need to first find a formula for the area of each cross section and appropriate limits of integration.

Since the cross sections are perpendicular to the y -axis, the diameter of each semicircle is the horizontal distance between $y = \sqrt{x}$ and $x = 2$. Solving the square root function for x , we see that this distance is $4 - y^2$. The radius of the semicircle, which we need to calculate its area, is half of this, or $r = \frac{1}{2}(4 - y^2) = 2 - \frac{1}{2}y^2$. The area of the semicircle is

$$A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(2 - \frac{1}{2}y^2\right)^2 = \pi\left(2 - y^2 + \frac{1}{8}y^4\right).$$

The limits of integration are the range of y -values that span the region. The lower boundary is $y = 0$, and the upper boundary is $y = 2$. Therefore, the volume of the region is

$$V = \pi \int_0^2 \left(2 - y^2 + \frac{1}{8}y^4\right) dy = \frac{32\pi}{15}.$$

Free Response Tip

Free response volume questions often appear on the section that does not allow use of a calculator. In that case, the question is usually phrased as “set up, but do not evaluate, an integral that represents...” This means that you should NOT make any attempt at finding a numerical answer; rather, just leave the integral itself as your answer.

Solids of Revolution

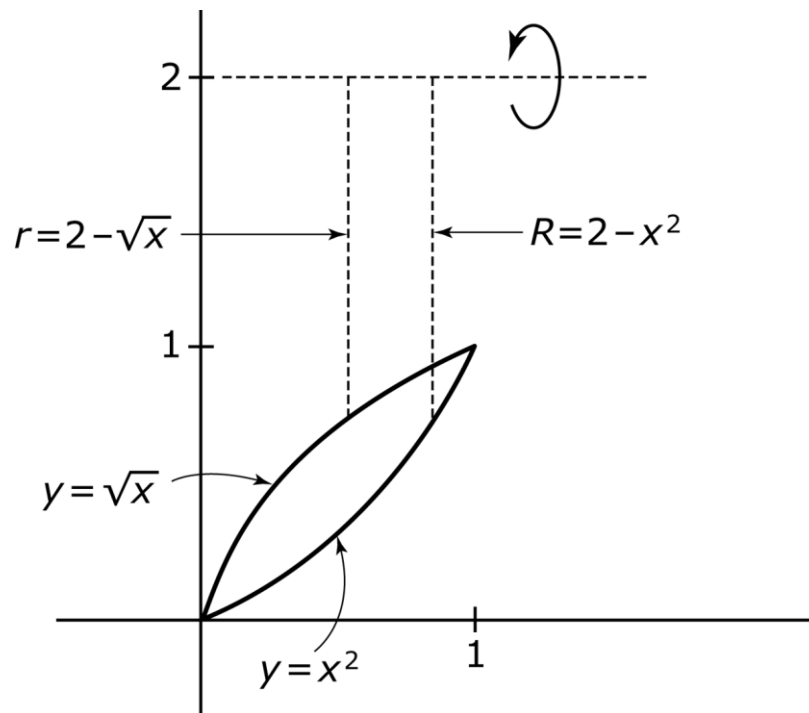
When a solid has circular or washer (ring-shaped) cross sections perpendicular to a vertical or horizontal line, it can be described as being obtained by revolving a region around that vertical or horizontal line. In this case, the volume can be calculated by a standard formula:

- If the axis of revolution is horizontal, and the cross sections are circles, the volume is $V = \pi \int_a^b r^2 dx$, where r is the radius in terms of x .
- If the axis of revolution is horizontal, and the cross sections are washers, the volume is $V = \pi \int_a^b (R^2 - r^2) dx$, where R is the radius of the outer circle, and r is the radius of the inner circle.

- If the axis of revolution is vertical, and the cross sections are circles, the volume is $V = \pi \int_a^b r^2 dy$, where r is the radius in terms of y .
- If the axis of revolution is vertical, and the cross sections are washers, the volume is $V = \pi \int_a^b (R^2 - r^2) dy$, where R is the radius of the outer circle, and r is the radius of the inner circle.

Example

The region bound by the graphs of $y = x^2$ and $y = \sqrt{x}$ is revolved around the line $y = 2$. The cross sections perpendicular to the x -axis are washers, with $R = 2 - x^2$ and $r = 2 - \sqrt{x}$, shown as follows:



The volume of the solid is $V = \pi \int_0^1 [(2 - x^2)^2 - (2 - \sqrt{x})^2] dx = \frac{31\pi}{30}$.

Free Response Tip

When volume questions do allow use of a calculator, you must show the integral you are evaluating in addition to the answer you obtained on your calculator. The numerical answer itself will not be sufficient to obtain full credit.

Arc Length

The length of a differentiable function $f(x)$ between $x = a$ and $x = b$ is $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$.

Example

The length of the arc traced by $f(x) = x^2$ on the interval $[1, 3]$ is $L = \int_1^3 \sqrt{1 + (2x)^2} dx \approx 8.27$.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapter 8. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 7. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapters 6 and 8. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapters 6 and 8. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 6. New York, NY: W.H. Freeman.

Sample Applications of Integration Questions

Which of the following expressions is equal to the area under the curve $g(x) = \ln x$ on the interval $[2,6]$?

A. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \ln\left(\frac{4i}{n}\right) \frac{4}{n}$

B. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \ln\left(\frac{4i}{n}\right) \frac{4i}{n}$

C. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \ln\left(2 + \frac{4i}{n}\right) \frac{4}{n}$

D. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\ln 2 + \ln\left(\frac{4i}{n}\right) \right] \frac{4i}{n}$

Explanation:

The correct answer is C. The strategy is to use the definition of the integral:

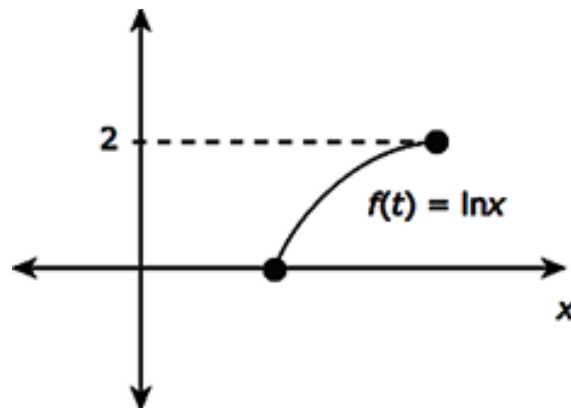
$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + \frac{b-a}{n} i\right) \cdot \frac{b-a}{n}$$

Assuming that the right endpoint of the i th subinterval is used as the sample point on this interval, and using $f(x) = \ln(x)$, $a = 2$, and $b = 6$ yields

$$\int_2^6 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \ln\left(2 + \frac{4i}{n}\right) \cdot \frac{4}{n}.$$

Finally, since the graph of $f(x) = \ln(x)$ on the interval $[2,6]$, this integral does indeed represent the area of the indicated region.

Which of the following integrals represents the length of the following curve:



A. $\int_1^{e^2} \sqrt{1+(\ln x)^2} dx$

B. $\int_0^2 \sqrt{1+\frac{1}{x}} dx$

C. $\int_0^2 \sqrt{1+(\ln x)^2} dx$

D. $\int_1^{e^2} \sqrt{1+\frac{1}{x^2}} dx$

Explanation:

The correct answer is D. The x -intercept of $f(x)$ is $(1,0)$ and $\ln x = 2 \Rightarrow x = e^2$. So, the interval of integration is $(1, e^2)$. The arc length formula for a curve $y = f(x)$ on an interval (a, b) is

$$\int_a^b \sqrt{1+(f'(x))^2} dx. \text{ Using that here with } a = 1, b = e^2, \text{ and } f'(x) = \frac{1}{x} \text{ yields the integral}$$

$$\int_1^{e^2} \sqrt{1+\left(\frac{1}{x}\right)^2} dx = \int_1^{e^2} \sqrt{1+\frac{1}{x^2}} dx.$$

Let $B(r)$ denote the number (measured in hundreds) of boulders with radius $\leq r$ meters in a small patch of a quarry. Geological data suggests that the radii are distributed according to the piecewise function

$$B'(r) = \begin{cases} 3.1r^{-1.9}, & r < 3 \\ 1.8r^{-2.1}, & r \geq 3 \end{cases}$$

Approximately how many boulders with radii between 1.3 and 3.4 meters are there in this small patch of quarry?

- A. 26
- B. 73
- C. 140
- D. 150

Explanation:

The correct answer is D. The accumulation of the number of boulders with radii between 1.3 and 3.4 meters is equal to the integral $\int_{1.3}^{3.4} B'(r)dr$. Since the integrand is piecewise-defined, we need to use interval additivity to write this integral as $\int_{1.3}^3 B'(r)dr + \int_3^{3.4} B'(r)dr$. Compute as follows:

$$\begin{aligned}
\int_{1.3}^3 B'(r)dr + \int_3^{3.4} B'(r)dr &= \int_{1.3}^3 3.1r^{-1.9} dr + \int_3^{3.4} 1.8r^{-2.1} dr \\
&= 3.1 \cdot \frac{r^{-0.9}}{-0.9} \Big|_{1.3}^3 + 1.8 \cdot \frac{r^{-1.1}}{-1.1} \Big|_3^{3.4} \\
&\approx \frac{3.1}{-0.9} (3^{-0.9} - 1.3^{-0.9}) - \frac{1.8}{1.1} (3.4^{-1.1} - 3^{-1.1}) \\
&\approx 1.50
\end{aligned}$$

So, there are about 150 such boulders.

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

Around 11–12% of the questions on your exam will cover the topic Parametric Equations, Polar Coordinates, and Vector-Valued Functions.

Parametric Equations

A curve in the xy -plane can be defined parametrically by a pair of functions $(x(t), y(t))$, where t ranges over some set of real numbers. The value of $\frac{dy}{dx}$ on such a curve is the slope of the line

tangent to the curve, and it is calculated by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. If $\frac{dy}{dt}$ and $\frac{dx}{dt} = 0$ at a particular point,

then the curve has a vertical tangent line at that point. The second derivative is obtained by

differentiating $\frac{dy}{dx}$ with respect to t , and then dividing by $\frac{dx}{dt}$: $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$. The length of

the curve defined by $(x(t), y(t))$ as t ranges from a to b is $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$.

If a parametric curve $(x(t), y(t))$ represents the motion of a particle moving in a plane, then the velocities in the horizontal and vertical directions are $x'(t)$ and $y'(t)$ respectively, and the corresponding accelerations are $x''(t)$ and $y''(t)$. The speed of the particle is $\sqrt{[x'(t)]^2 + [y'(t)]^2}$ so the integral for arc length given represents the distance traveled by the particle.

Vector-Valued Functions

Vector valued functions can be differentiated and integrated by performing these operations on each component. If a parametric curve $(x(t), y(t))$ represents the motion of a particle moving in a plane during the time interval $t = a$ to $t = b$, then the state and properties of the particle can be given in terms of vector valued functions.

- The position vector at time t is $\mathbf{r}(t) = \langle x(t), y(t) \rangle$
- The velocity vector at time t is $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$

- The displacement vector is $\left\langle \int_a^b x'(t) dt, \int_a^b y'(t) dt \right\rangle$
- The speed of the particle at time t is $s(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}$
- The distance traveled is $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$
- The acceleration vector of the particle is $\mathbf{a}(t) = \langle x''(t), y''(t) \rangle$

Polar Coordinates

A point in the plane can be represented by the polar coordinates (r, θ) , where r is the distance from the origin, and θ is the angle between the positive x -axis and a ray drawn to the point. The relationship between polar coordinates and the standard rectangular coordinates (x, y) is represented in four equations:

- $x = r \cos \theta$
- $y = r \sin \theta$
- $r^2 = x^2 + y^2$
- $\tan \theta = \frac{y}{x}$

The curve defined by a polar function defined by $r = f(\theta)$ is equivalent to the parametric curve given by $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$.

Therefore, the slope of the line tangent to a polar curve is $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

The area of the polar region bounded by the origin and the curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is $\int_a^b \frac{1}{2} r^2 d\theta$. Similarly, the area of the region between $r_1 = f(\theta)$ and $r_2 = g(\theta)$, where $r_1 < r_2$ between $\theta = \alpha$ and $\theta = \beta$ is $\int_a^b \frac{1}{2} (r_2^2 - r_1^2) d\theta$.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapters 4 and 8. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 10. Boston, MA: Cengage Learning.
- Stewart, et al. *Single Variable Calculus*. 9th edition. Chapter 10. Boston, MA: Cengage Learning.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 11. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapters 9 and 11. New York, NY: W.H. Freeman.

Sample Parametric Equations, Polar Coordinates, and Vector-Valued Functions Questions

Suppose a particle is moving in the xy -plane so that its position at time t is given by the vector-valued function $\vec{r}(t) = \langle x(t), y(t) \rangle$, where $x(t) = e^{-t}$, $y(t) = e^{3t}$ for $0 \leq t \leq \ln 2$ (measured in minutes). What are the components of the acceleration vector at $t = \ln 2$ minutes?

- A. $\langle -2, 54 \rangle$
- B. $\langle \frac{1}{2}, 72 \rangle$
- C. $\langle -\frac{1}{2}, 24 \rangle$
- D. $\langle \frac{1}{2}, 8 \rangle$

Explanation:

The correct answer is B. The acceleration vector we seek is equal to $\vec{r}''(\ln 2)$. Observe that

$$x(t) = e^{-t}, \quad y(t) = e^{3t}$$

$$x'(t) = -e^{-t}, \quad y'(t) = 3e^{3t}$$

$$x''(t) = e^{-t}, \quad y''(t) = 9e^{3t}$$

As such, $\vec{r}''(\ln 2) = \langle e^{-\ln 2}, 9e^{3\ln 2} \rangle = \langle e^{\ln 2^{-1}}, 9e^{\ln 2^3} \rangle = \langle \frac{1}{2}, 72 \rangle$.

An object moves along a path in the xy -plane so that it starts at $t = 0$ at the point $(1, 1)$ and moves with velocity vector $\vec{v}(t) = \langle 3, t^{\frac{3}{2}} \rangle$. What is the displacement on the time interval $(1, 8)$?

- A. $\langle 0, 7 \rangle$
- B. $\langle 21, \frac{381}{7} \rangle$
- C. $\langle 0, 15 \rangle$
- D. $\langle 0, \frac{7}{3} \rangle$

Explanation:

The correct answer is B. Displacement on a time interval (a, b) of an object with velocity vector $\vec{v}(t)$ is given by $\int_a^b \vec{v}(t) dt$. Here, we must compute $\int_1^8 \langle 3, t^{\frac{3}{2}} \rangle dt$. This is done componentwise, as follows:

$$\begin{aligned} \int_1^8 \langle 3, t^{\frac{3}{2}} \rangle dt &= \left\langle \int_1^8 3 dt, \int_1^8 t^{\frac{3}{2}} dt \right\rangle = \left\langle 3t, \frac{\frac{3}{2} t^{\frac{7}{2}}}{\frac{7}{2}} \right\rangle \Big|_1^8 \\ &= \left\langle 3 \cdot 8, \frac{\frac{3}{2}}{7} (8)^{\frac{7}{2}} \right\rangle - \left\langle 3 \cdot 1, \frac{\frac{3}{2}}{7} (1)^{\frac{7}{2}} \right\rangle \\ &= \left\langle 24, \frac{384}{7} \right\rangle - \left\langle 3, \frac{3}{7} \right\rangle = \left\langle 24 - 3, \frac{384}{7} - \frac{3}{7} \right\rangle = \left\langle 21, \frac{381}{7} \right\rangle \end{aligned}$$

Suppose the velocity vector of a particle moving in space is given by $\vec{v}(t) = \langle \cos(2t), 1, \sin(3t) \rangle$, $t \geq 0$. If the particle starts at the point $(3, -1, 1)$ at time $t = 0$, which of the following is its position vector $\vec{r}(t)$?

- A. $\vec{r}(t) = \langle \sin(2t) + 3, t - 1, 2 - \cos(3t) \rangle$, $t \geq 0$
- B. $\vec{r}(t) = \langle 3 - \sin(2t), t - 1, \cos(3t) \rangle$, $t \geq 0$
- C. $\vec{r}(t) = \langle \frac{1}{2} \sin(2t) + 3, t - 1, \frac{4}{3} - \frac{1}{3} \cos(3t) \rangle$, $t \geq 0$
- D. $\vec{r}(t) = \langle \frac{1}{2} \sin(2t), t, -\frac{1}{3} \cos(3t) \rangle$, $t \geq 0$

Explanation:

The correct answer is B. Since $\vec{v}(t) = \vec{r}'(t)$, this problem is equivalent to finding the vector $\vec{r}(t)$ satisfying

$$\begin{cases} \vec{r}'(t) = \langle \cos(2t), 1, \sin(3t) \rangle, t \geq 0 \\ \vec{r}(0) = \langle 3, -1, 1 \rangle \end{cases}$$

Integrate componentwise to find the general anti-derivative, using u -substitutions on the first and third components:

$$\vec{r}(t) = \left\langle \frac{1}{2} \sin(2t) + C_1, t + C_2, -\frac{1}{3} \cos(3t) + C_3 \right\rangle, t \geq 0$$

Now, use the fact that $\vec{r}(0) = \langle 3, -1, 1 \rangle$ to find the constants:

$$\vec{r}(0) = \left\langle C_1, C_2, -\frac{1}{3} + C_3 \right\rangle = \langle 3, -1, 1 \rangle$$

So, $C_1 = 3$, $C_2 = -1$, and $C_3 = \frac{4}{3}$. Therefore, $\vec{r}(t) = \left\langle \frac{1}{2} \sin(2t) + 3, t - 1, \frac{4}{3} - \frac{1}{3} \cos(3t) \right\rangle, t \geq 0$.

Infinite Sequences and Series

Finally, about 17–18% of the questions on your exam will cover Infinite Sequences and Series.

Convergent and Divergent Series

A sequence is a function $f(n)$ whose domain consists only of nonnegative integers. The notation $a_n = f(n)$ is often used. A sequence converges to L if $\lim_{n \rightarrow \infty} a_n = L$, and it diverges if this limit is infinite or does not exist.

An infinite series is a sum of the form $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$. The N^{th} partial sum of the series is the sum of its first N terms: $S_N = \sum_{n=0}^N a_n = a_0 + a_1 + \cdots + a_N$. If the sequence of partial sums S_N converges to S , we say that the series converges, and that it has sum S . If the sequence of partial sums diverges, the series diverges.

There are several tests that help in determining convergence or divergence of a series.

Test: n^{th} Term Test

Applies to: $\sum a_n$

Conclusion(s): If $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges.

Test: Integral Test

Applies to: $\sum a_n$, where $a_n = f(n)$ for some function f which is non-negative and decreasing on an interval of the form $[N, \infty)$ for some value N

Conclusion(s): The series converges if and only if the improper integral $\int_0^{\infty} f(x) dx$ converges.

Test: Direct Comparison Test

Applies to: $\sum a_n$ and $\sum b_n$ of non-negative terms, where $a_n \leq b_n$

Conclusion(s): If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges. If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Test: Limit Comparison Test

Applies to: $\sum a_n$ and $\sum b_n$ of non-negative terms, and $c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$.

Conclusion(s): If c is positive and finite, then either both series converge or both series diverge.

Test: Alternating Series Test

Applies to: $\sum (-1)^n a_n$ or $\sum (-1)^{n+1} a_n$, where a_n is non-negative

Conclusion(s): If a_n is decreasing and $\lim_{n \rightarrow \infty} a_n = 0$, then the series converges.

Notes: When an alternating series converges, we can also determine how far any of the partial sums are from the sum of the infinite series: if S_N is the N^{th} partial sum, and the series converges to S , then $|S - S_N| \leq a_{N+1}$. In other words, the error obtained from truncating the series of a particular term is no greater than the first omitted term. This fact is known as the alternating series error bound.

Test: Ratio Test

Applies to: $\sum a_n$, and $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Conclusion(s): If $L < 1$, the series converges. If $L > 1$, the series diverges.

Notes: If $L = 1$, the test is inconclusive.

Geometric Series and p -Series

A series of the form $\sum_{n=0}^{\infty} ar^n$ is called a geometric series. If $|r| < 1$, this series converges to $\frac{a}{1-r}$. If $|r| \geq 1$, the series diverges. It is important to note here that a is the first term of the series; if the series does not start with $n = 0$, its value should be determined by finding the first term.

A p -series is of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. This series converges if $p > 1$ and diverges if $p \leq 1$. When

$p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series, which diverges.

Absolute and Conditional Convergence

A series is said to be absolutely convergent if the series consisting of the absolute values of its terms converges. In other words, $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. If a series is absolutely convergent then it is also convergent. It is possible, however, for a series to be convergent but not absolutely convergent. In this case, the series is called conditionally convergent. Testing for absolute convergence is often done directly by replacing the terms of a sequence with their absolute values. If the ratio test is used for the original series, however, the absolute values do not need to be checked, as the test guarantees absolute convergence on its own.

Taylor Polynomials

The k^{th} Taylor polynomial $P_k(x)$ centered at $x = c$ for a function $f(x)$ is the k^{th} degree polynomial that best approximates f near $x = c$. The terms of this polynomial are of the form

$a_n(x-c)^n$, where $a_n = \frac{f^{(n)}(c)}{n!}$. For example, the first five Taylor polynomials are:

- $P_0(x) = f(c)$
- $P_1(x) = f(c) + f'(c)(x-c)$
- $P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$
- $P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3$
- $P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + \frac{f^{(4)}(c)}{24}(x-c)^4$

The Taylor polynomials of a function can be used to approximate the value of the function, in the same way that linearization was used. The 1st degree Taylor polynomial $P_1(x)$ is, in fact, the linearization of $f(x)$.

Example

Consider $f(x) = \cos x$ at $c = 0$. At 0, the values of f and its first few derivatives are as follows:

	$f^{(n)}(x)$	$f^{(n)}(0)$	
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-\frac{1}{2}$
3	$\sin x$	0	0
4	$\cos x$	1	$\frac{1}{24}$
5	$-\sin x$	0	0
6	$-\cos x$	-1	$-\frac{1}{720}$

The sixth Taylor polynomial is $P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

Error Bounds

If the terms of a Taylor polynomial form an alternating series, the alternating series error bound can be used to bound the difference between the Taylor polynomial approximation at some point near the center $x = c$ and the actual value of the function.

Example

The sixth Taylor polynomial for $f(x) = \cos x$ centered at $c = 0$ is

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6. \text{ This is the alternating series for which the } n^{\text{th}} \text{ term is}$$
$$(-1)^n \frac{x^{2n}}{(2n)!}.$$

Suppose we use $P_6(x)$ to approximate the value of $\cos 0.1$: $\cos 0.1 \approx P_6(0.1) \approx 0.99500$. The first omitted term is $(-1)^4 \frac{x^8}{8!}$, so this estimated value is off by at most $\frac{0.1^8}{8!} \approx 2.48 \times 10^{-13}$.

The Lagrange error bound gives an alternative way to calculate an error bound for any Taylor polynomial, even if it is not an alternating series. Suppose the n^{th} Taylor polynomial for f centered at c is used to approximate the value of $f(x)$. Let M be at least as large as the maximum value attained by $f^{(n+1)}$ on the interval between c and x . The difference between the estimated value $P_n(x)$ and the true value of the function $f(x)$ is at most $M \frac{|x-c|^{n+1}}{(n+1)!}$.

Free Response Tip

When applying the Lagrange error bound, enough information about $f^{(n+1)}$ must be available to obtain a value for M . This may be stated explicitly, implied by the increasing/decreasing nature of $f^{(n+1)}$, or related to the fact that the sin and cos functions are always bounded by 1.

Radius and Interval of Convergence

A power series centered at c is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$. Note that this includes a variable, x , so that the series is in fact a function of x . Therefore, the convergence or divergence of the series depends on x .

Despite this, there are only three possibilities:

- The series diverges for all x . In this case, we say that the radius of convergence of the series is ∞ .
- The series converges for only a single x . The radius of convergence is 0.
- The series converges only for all x within some interval centered at c . The radius of convergence is the distance between c and either endpoint of the interval.

The radius of convergence is usually found using the ratio test. Once this radius is known, it still remains to determine whether the series converges or diverges at the endpoints of the interval.

Example

Consider the series $\sum_{n=0}^{\infty} \frac{2^n}{n} (3x-2)^n$. To determine the interval of convergence, we will apply the ratio test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{2^{n+1}}{n+1} |3x-2|^{n+1} \div \frac{2^n}{n} |3x-2|^n \\ &= \lim_{n \rightarrow \infty} \frac{2^{n+1} |3x-2|^{n+1}}{n+1} \cdot \frac{n}{2^n |3x-2|^n} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{n+1} |3x-2| \\ &= 2|3x-2|\end{aligned}$$

The ratio test says this will converge when the limit is less than 1: $2|3x-2| < 1 \Rightarrow \left| x - \frac{2}{3} \right| < \frac{1}{6}$.

The radius of convergence is $\frac{1}{6}$, and the interval of convergence is tentatively $\frac{1}{2} < x < \frac{5}{6}$.

If $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{2^n}{n} \left(3 \cdot \frac{1}{2} - 2 \right)^n = \sum_{n=0}^{\infty} \frac{2^n}{n} \left(-\frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternating series test. If $x = \frac{5}{6}$, the series is $\sum_{n=0}^{\infty} \frac{2^n}{n} \left(3 \cdot \frac{5}{6} - 2 \right)^n = \sum_{n=0}^{\infty} \frac{2^n}{n} \left(\frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{1}{n}$, which is the divergent harmonic series.

Therefore, the full interval of convergence is $\left[\frac{1}{2}, \frac{5}{6} \right)$.

Taylor and Maclaurin Series

If the terms used in constructing the Taylor polynomials for a function f are extended infinitely, the resulting power series is called the Taylor series for f centered at c . When $c = 0$, this series is also called the Maclaurin series for f .

Some common Maclaurin series are shown in the following table:

$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$(-1, 1)$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$

Power series can be manipulated using algebraic techniques such as substitution, as well as differentiation and integration, to produce new series from previously known ones.

Example

We can derive the Maclaurin series for $\tan^{-1} x$ starting with the known series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Substituting $-x$ for x , we get $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, and further substituting x^2 for x gives

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$. Now integrating both sides with respect to x results in the desired series:

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Free Response Tip

When a free response question asks you to find a Taylor or Maclaurin series for a function, always begin by thinking about the series you already know. Use of the series for $\sin x$, $\cos x$, and e^x are especially common.

Suggested Reading

- Hughes-Hallett, et al. *Calculus: Single Variable*. 7th edition. Chapters 9 and 10. New York, NY: Wiley.
- Larson & Edwards. *Calculus of a Single Variable: Early Transcendental Functions*. 7th edition. Chapter 9. Boston, MA: Cengage Learning.
- *Single Variable Calculus 9e*, Stewart et al. Chapter 11.
- Rogawski, et al. *Calculus: Early Transcendentals Single Variable*. 4th edition. Chapter 10. New York, NY: Macmillan.
- Sullivan & Miranda. *Calculus: Early Transcendentals*. 2nd Edition. Chapter 8. New York, NY: W.H. Freeman.

Sample Infinite Sequences and Series Questions

For which of the following choices of c_n would the series $\sum_{n=1}^{\infty} \frac{c_n}{3n^4 + n}$ necessarily diverge using the Limit Comparison Test?

- A. $c_n = \frac{1}{n}$
- B. $c_n = 2\pi$
- C. $c_n = n^2 + 5n + 4$
- D. $c_n = 1 + n^3$

Explanation:

The correct answer is D. Consider the series $\sum_{n=1}^{\infty} \frac{1+n^3}{3n^4+n}$. Let $a_n = \frac{1+n^3}{3n^4+n}$ and $b_n = \frac{n^3}{3n^4} = \frac{1}{3n}$.

Observe that $\sum_{n=1}^{\infty} \frac{1}{3n}$ is divergent, being a constant multiple of the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\text{Also, } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1+n^3}{3n^4+n}}{\frac{1}{3n}} = \lim_{n \rightarrow \infty} \frac{3n+3n^4}{3n^4+n} = 1 > 0.$$

So, by the Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{1+n^3}{3n^4+n}$ must diverge.

Suppose $a_n > 0$, for all integers $n \geq 1$. Which of the following conditions ensures that the series

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges conditionally?

- A. $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$
- B. $\lim_{n \rightarrow \infty} a_n \neq 0$
- C. $\{a_n\}$ decreases to zero
- D. $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

Explanation:

The correct answer is C. This is the hypothesis of the Alternating Series Test and ensures that the series is conditionally convergent. Choice A is actually inconclusive, as seen by examples

using the ratio test. If $a_n = \frac{1}{n^2}$ the series converges, but for $a_n = n$ the series would diverge.

However, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ for both choices of a_n . Choice B would actually guarantee that the series

diverges. Choice D is insufficient; for instance, if $a_n = 1$ for all n , then $\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} (-1)^n$ diverges.

Which of the following is the MacLaurin series for $g(x) = \sin(x^2)$?

A. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, |x| < \infty$

B. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}, |x| < \infty$

C. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}, |x| < \infty$

D. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}, |x| < \infty$

Explanation:

The correct answer is D. Begin with the MacLaurin series for $\sin(u)$: $\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{(2n+1)!}$.

Now, substitute $u = x^2$ and simplify:

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}.$$