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2006–2007
Professional Development
Workshop Materials

Special Focus:
The Fundamental Theorem of Calculus

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Special Focus: The Fundamental Theorem of Calculus

Table of Contents

Important Note: The following set of materials is organized around a particular theme, or "special focus," that reflects important topics in the AP Calculus course. The materials are intended to provide teachers with resources and classroom ideas relating to these topics. The special focus, as well as the specific content of the materials, cannot and should not be taken as an indication that a particular topic will appear on the AP Exam.

Developing and Understanding the Fundamental Theorem of Calculus

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Numerous problems involving the Fundamental Theorem of Calculus (FTC) have appeared in both the multiple-choice and free-response sections of the AP Calculus Exam for many years. AP Calculus students need to understand this theorem using a variety of approaches and problem-solving techniques. Before 1997, the AP Calculus questions regarding the FTC considered only a limited number of variations. Traditional problems are often procedural: area, volume, or distance questions or those that check a student's mechanical ability to find the derivative of a function defined by an integral. In addition, we now assess the theorem from a wider variety of settings. These materials provide different ways that teachers can help students understand these settings. We hope the following chapters will help AP Calculus teachers think about how to organize and present the material on the FTC in ways that will encourage students to discover important principles, foster critical thinking, and build confidence.

The 2006–2007 *AP Calculus Course Description* includes the following item:

Fundamental Theorem of Calculus
- Use of the Fundamental Theorem to evaluate definite integrals.
- Use of the Fundamental Theorem to represent a particular antiderivative, and the analytical and graphical analysis of functions so defined.

Calculus texts often present the two statements of the Fundamental Theorem at once and refer to them as Part I and Part II, although even the most popular texts do not agree on which statement is Part I and which is Part II. In these materials, we refer to the two parts of the Fundamental Theorem of Calculus in the following manner:

Evaluation part of the FTC:

If f is continuous on $[a, b]$, and F is any antiderivative of f, then
$\int_a^b f(x)\,dx = F(b) - F(a)$.

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Antiderivative part of the FTC:

If f is continuous on $[a, b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for every x in $[a, b]$

In addition to explaining and expanding on these two parts, we showcase the approaches that several experienced calculus teachers successfully use in their classrooms. We discuss multiple approaches for introducing the FTC and include some calculator activities that will help students to learn important concepts that are connected to the FTC.

Experienced teachers agree that important topics require careful planning and subtle repetition of key concepts. The primary goal of these materials is to assist AP Calculus teachers in creating classroom activities that will help students to probe, discover, question, and master the FTC and its applications.

During the process of writing and editing these workshop materials, we found it necessary to articulate some organizing principles to present the work of individual writers in an order and manner that will be helpful to teachers of AP Calculus. J. T. Sutcliffe articulated these developmental stages, and we have used her suggestions to guide this work. We thought carefully about the stages through which many students pass between being introduced to a new concept and achieving competency with the concept. Although we strongly urge each teacher to create lesson plans that meet the needs of his or her students, we selected the following scenario to organize our materials.

Stage One: Exploring and Developing Interesting Results That Lead to Conjectures
Stage Two: Refining the Level of Understanding and Making Conjectures
Stage Three: Assessing and Improving Student Understanding and Confidence
Stage Four: Confirming Conjectures with a Formal Proof
Stage Five: Historical Background and More Advanced Examples

We encourage you to read through the materials for the first time in this order. (Several of the chapters include topics for more than one stage, but we have included each chapter in the first appropriate stage, rather than break up the thoughts of an individual writer. Editorial comments will guide you when certain parts of a chapter might be included in a stage that follows the one where it appears in these materials.) Then reread, reorganize, and create a scenario that works well for you and your students. We want you to have the freedom to determine when it is time to leave one stage, return to a previous stage, or move to a new stage. We hope these materials include both the calculus content and the pedagogical insights that will help you create meaningful activities for your students. And most of all, we extend our thanks for your dedication to and support of the AP Calculus program.

Special Focus: The Fundamental Theorem of Calculus

Stage One: Exploring and Developing Interesting Results That Lead to Conjectures

Introduction

Goals of Stage One

- Estimate distance traveled from a chart of positive velocity values
- Investigate properties of integral functions when the integrand is a positive constant

Articles in Stage One

- "Key Ideas That Help Students Understand the Fundamental Theorem" by Steve Olson
- "Functions Defined by an Integral" by Mark Howell

Keeping the context simple is an important first principle. When the context is simple, students can see and easily recognize the new and important concepts. Many teachers have found that velocity examples are particularly good for introducing the idea of the definite integral. Students are familiar with and have had practical experiences using velocity. The formula, distance = rate × time, is easy to remember and apply. At this stage, we use this formula to estimate the distance traveled based on a table of velocity values. We begin with positive velocity values in the first chapter, by Steve Olson. This article presents specific examples and gives some tips that will encourage students to explore and experiment with velocity to make conjectures about distance and displacement.

Mark Howell's "Functions Defined by a Definite Integral" considers functions of the form $f(x)=\int_a^x k\,dt$ when k is a positive constant, a is any constant, and $x \geq a$. The use of a positive constant k parallels the positive velocity values in the first article. Students will experiment with constant integrands and several lower limits to discover the relationship between the integrands and the slope of the linear functions defined by the definite integrals.

These two settings, positive velocity and integrals with constant integrands, are comfortable ones for students, and that makes it possible to informally introduce deep concepts.

Key Ideas That Help Students Understand the Fundamental Theorem

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I believe that we must focus on two important ideas as we help our students learn about the Fundamental Theorem of Calculus.

1. **The evaluation part of the Fundamental Theorem can and should be introduced to students so they can see that the result is reasonable.**
   Although students do not usually have difficulty in *applying* this part of the theorem, teachers can introduce this part in a way that helps students to believe that the result is almost "obvious."

2. **The antiderivative part of the Fundamental Theorem guarantees that a continuous function on an open interval has an antiderivative, whether or not a closed form of the antiderivative can be found.**
   Often students view this part simply as a way to take the derivative of a function that is defined by a definite integral. Teachers must help students understand that $F(x) = \int_a^x f(t)\,dt$ is the antiderivative of f whose graph contains the point $(a, 0)$. Thus, this part allows us to write the solution to an initial value problem when there is no known closed form antiderivative.

The discussion that follows illustrates how I try to deal with these issues in my classroom.

Evaluation Part of the Fundamental Theorem

In an attempt to help students not only understand this part, but to help make this part seem obvious, I have broken the development into three steps.

Evaluation Step 1: Understanding that partitioning a time interval and summing the products of the form (rate of change) × (subinterval width) for each subinterval gives an approximation for the total change

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I start by having my students complete the following exploration that uses a table of data comprised of nondecreasing speeds of a car at equally spaced moments in time to estimate the distance traveled.

A car is traveling so that its speed is never decreasing during a 10-second interval. The speed at various moments in time is listed in the table below.

Time in seconds	0	2	4	6	8	10
Speed in ft/sec	30	36	40	48	54	60

(a) Use the table to help explain why the best lower estimate for the distance traveled in the first 2 seconds is 60 feet.

(b) Use the table to help explain why the best upper estimate for the distance traveled in the first 2 seconds is 72 feet.

(c) Use the table to give the best lower estimate for the distance traveled in the first 10 seconds. An answer of 300 feet (which ignores some data in the table) is not correct.

(d) Use the table to give the best upper estimate for the distance traveled in the first 10 seconds. These sums of products that you have found in (c) and (d) are called Riemann sums.

(e) If you choose the lower estimate for your approximation of how far the car travels, what is the maximum amount your approximation could differ from the exact distance?

(f) Choose speeds to correspond with $t = 1, 3, 5, 7, 9$. Keep the nondecreasing nature of the above table and do not select the average of the consecutive speeds. Find new best upper and lower estimates for the distance traveled for these 10 seconds.

In parts (a) and (b), I am helping students enter the problem smoothly. If questions arise here, I can address them quickly. Most students will answer (a) by saying, "The minimum speed during the first 2 seconds is 30 ft/sec, so distance is at least $(30)(2) = 60$ feet." They will answer (b) by saying, "The maximum speed during the first 2 seconds is 36 ft/sec, so the distance is at most $(36)(2) = 72$ feet."

I want them to extend the ideas developed in the first two parts to each interval, thus getting the following answers to parts (c) and (d):

"The minimum speed for 10 seconds is $(30)(2)+(36)(2)+(40)(2)+(48)(2)+(54)(2)=416$ feet. The maximum speed for 10 seconds is $(36)(2)+(40)(2)+(48)(2)+(54)(2)+(60)(2)=476$ feet." I watch for those who might answer (c) with $(3)(10)=300$ feet and help them understand that even though this is a lower bound, it is not the greatest lower bound because they have not taken advantage of the data provided.

As we go over parts (c) and (d), I also introduce them to important vocabulary by pointing out that the sums of products they have just computed are known as Riemann sums.

Part (e) can sometimes give students trouble, but most will recognize that the exact distance is somewhere between 416 feet and 476 feet. This means that if you choose the lower estimate, 416, as an approximation, the maximum amount this approximation could differ from the exact distance is 60 feet.

Part (f) responses sometimes amaze students and lead to a lively discussion. Although their answers will vary based on the choice of speeds each student selects, the difference between the best upper and lower estimates will always be $(60-30)\Delta t$, or 30 in this case since $\Delta t = 1$.

At this point I have the students tell me their new upper and lower estimates, and I record them on the blackboard. It becomes obvious that the speeds that were chosen make a difference in the upper and lower estimates; however, in every case the maximum error has been halved. Students observe that the amount of error can be made as small as desired by choosing Δt sufficiently small. This is shown by expressing the error, E, as $E \leq \left| v(10) - v(0) \right| \cdot \Delta t = 30 \cdot \Delta t$, which approaches 0 as $\Delta t \to 0$.

After working with this example, the students work with textbook problems involving data for a rate over equally spaced time intervals. I am not in a rush to get to problems in which we work with a function defined by a formula. I want the students to experience adding products of a rate and a small amount of the independent variable (in most examples, this is a small amount of time).

Evaluation Step 2: Understanding that the limiting value of a Riemann sum is a definite integral

After the idea of summing products to approximate total change has had a few days to sink in, I begin a class by giving the students the formula $s(t) = t^2 + t + 20$ for the position of a car in feet as a function of time *t*, for $t=0$ to $t=10$ seconds, and have the students answer the following questions.

(a) What is the position of the car at $t=0$?

(b) What is the position of the car at $t=10$?

(c) What is the change in position of the car from time $t=0$ to time $t=10$?

(d) Were these questions easy or difficult for you?

Most students find these questions easy to answer and wonder why these questions were asked. I ask for their patience and suggest that we try to generalize the processes they have been using to answer recent textbook questions.

We start by reviewing the Step 1 exploration. This allows us to have a concrete example to refer to when necessary. Next, however, I guide them through a more general analysis of what we did. The details follow:

We began by estimating the change of position from time $t=0$ to $t=10$ by dividing the time interval into five subintervals, each with $\Delta t = 2$. Using general notation, we wrote

$$\text{Change in position} \approx v(t_1)\cdot \Delta t + v(t_2)\cdot \Delta t + v(t_3)\cdot \Delta t + v(t_4)\cdot \Delta t + v(t_5)\cdot \Delta t.$$

We can express this even more generally by partitioning the time interval into *n* pieces, writing

$$\text{Change in position} \approx v(t_1)\cdot \Delta t + v(t_2)\cdot \Delta t + \cdots + v(t_n)\cdot \Delta t, \text{ where } \Delta t = \frac{10}{n}.$$

Next, I usually ask how we make our estimate even more accurate, and almost always someone suggests a limit. We agree to write

$$\text{Change in position} = \lim_{n\to\infty}\left(v(t_1)\cdot \Delta t + v(t_2)\cdot \Delta t + \cdots + v(t_n)\cdot \Delta t\right).$$

At this point, I define a "new" notation for this limit of a Riemann sum. The new notation is $\int_0^{10} v(t)\,dt$, and we call $\int_0^{10} v(t)\,dt$ the definite integral of $v(t)$ from $t=0$ to $t=10$. The meaning of $\int_0^{10} v(t)\,dt$ is exactly the same as $\lim_{n\to\infty}\left(v(t_1)\cdot\Delta t + v(t_2)\cdot\Delta t + \cdots + v(t_n)\cdot\Delta t\right)$. We discuss the form of the definite integral and emphasize that $\int_0^{10} v(t)\,dt$ embodies everything that appears in the limit of the corresponding Riemann sum.

Evaluation Step 3: Understanding the statement of the evaluation part of the Fundamental Theorem

Now it is time to pull everything together. We currently have

$$\text{Change in position} = \lim_{n\to\infty}\left(v(t_1)\cdot\Delta t + v(t_2)\cdot\Delta t + \cdots + v(t_n)\cdot\Delta t\right) = \int_0^{10} v(t)dt.$$

Recall that $v(t) = s'(t)$, so change in position $= \int_0^{10} v(t)dt = \int_0^{10} s'(t)dt$. Earlier we saw that change in position $= s(10) - s(0)$. Therefore $\int_0^{10} s'(t)dt = s(10) - s(0)$.

My goal is for the students to not only see $\int_0^{10} s'(t)dt = s(10) - s(0)$ as a natural result, but also to generalize one more step to $\int_a^b s'(t)dt = s(b) - s(a)$.

Antiderivative Part of the Fundamental Theorem

In an attempt to help students understand that the antiderivative part of the Fundamental Theorem guarantees that a continuous function on a open interval has an antiderivative, whether or not a closed form of the antiderivative can be found, I break the development into two steps.

Antiderivative Step 1: Applying this part of the Fundamental Theorem to write the derivative of a function that is defined by a definite integral of the form $\int_k^x f(t)dt$, where *k* is a constant

Generally, textbooks do a good job developing this concept, and I do not need to create supplementary problems. I do like to use some of the excellent AP free-response questions that are most easily completed by using this antiderivative part of the Fundamental Theorem. One example follows.

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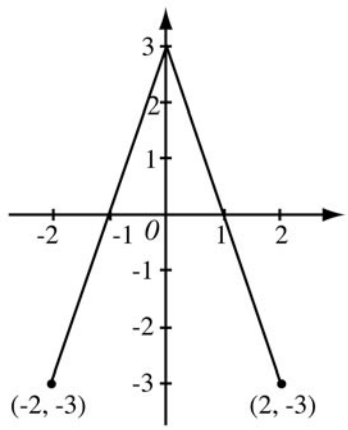
2002 AB4/BC4

Free-response question AB4/BC4 from the 2002 AP Exam gives an example of this application of the antiderivative part of the FTC. Students were not allowed to use a calculator on this question.



Graph of f

The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval (–2, 2) is g increasing? Explain your reasoning.

(c) For what values of x in the open interval (–2, 2) is the graph of g concave down? Explain your reasoning.

(d) Sketch the graph of g on the closed interval [–2, 2].

We will focus on parts (a) and (b) here. Students needed to use the Fundamental Theorem to determine that $g'(x) = f(x)$ in order to answer part (a). For part (b), some students attacked the problem by trying to make an area argument. Others used the fact that $g'(x) = f(x)$ and that from the given graph $f(x) > 0$ on $-1 < x < 1$, so f is increasing on the interval $-1 \le x \le 1$. The students who used this last argument were much more successful than those who tried an area argument.

Antiderivative Step 2: Writing an expression for the antiderivative of a function that is continuous on a open interval and that has a given value at a given point in the interval

The concept is very useful but is not (yet) typically emphasized in textbooks. We have already developed the idea that $\int_a^b f(t)\,dt = F(b) - F(a)$ if $F'(x) = f(x)$. Thus we can write the following relationship: $\int_a^x f(t)\,dt = F(x) - F(a)$ if $F'(x) = f(x)$. The concept I want students to see and understand is that $F(a) + \int_a^x f(t)\,dt = F(x)$ if $F'(x) = f(x)$. Students have no difficulty following the algebra that leads to the desired conclusion.

The following discussion illustrates how I try to help students understand the significance of looking at the Fundamental Theorem in this way. The first two of the following examples look quite similar, but students find them to be very different. My goal is for students to see how both examples can be solved using $F(a) + \int_a^x f(t)\,dt = F(x)$.

Example 1
Write the antiderivative F for $f(x) = -\sin x$ so that $F(0) = 3$.

Typically, students mentally run through the derivatives that they know or the antidifferentiation technique that might apply. For this problem, students quickly hit upon the following solution:

$$F'(x) = -\sin x \text{ and } F(0) = 3,$$
$$F(x) = \cos x + C,$$
$$3 = \cos 0 + C \Rightarrow C = 2,$$
$$F(x) = 2 + \cos x.$$

Solutions to similar problems appear in every calculus textbook.

Example 2
Write the antiderivative F for $f(x) = -\sin(x^2)$ so that $F(0) = 3$.

This problem appears to be very much like Example 1. Typically, the students attempt to come up with the formula of the function with the desired derivative. However, this time the student is not successful! There is no such “nice” function. A frequent response is, “I cannot do this problem!”

However, the use of the antiderivative part of the Fundamental Theorem gives the function directly:

$$F(x) = 3 + \int_0^x -\sin(w^2)\,dw.$$

It is a simple task to verify that $\boldsymbol{F}$ meets both of the given conditions, $F(0) = 3 - \int_0^0 \sin(w^2)\,dw = 3$ and $F'(x) = -\sin(x^2)$, directly from the statement of the Fundamental Theorem. If possible, the integral should be simplified. However, simplified or not, this is an expression for the function F. With the technology that is currently available, obtaining numerical values for F is not difficult.

When we try this technique for Example 1 (see below), we get the same result as before, since in this case the integral can be simplified:

$$F(x) = 3 + \int_0^x -\sin(w)\,dw,$$
$$F(x) = 3 + (\cos w)\Big|_0^x = 3 + (\cos x - \cos 0) = 2 + \cos x.$$

The scoring guidelines for AP free-response questions frequently use this approach. (See 2001 AB3/BC3, 2001 BC1, 2002 AB3, 2003 AB2, 2003 AB4/BC4, 2004 AB3, and 2004 BC3.) I believe that we will continue to see this application assessed on future AP Exams. A recent example using this technique follows.

Example 3: 2004 AB3
This application of the Fundamental Theorem was useful in solving part (d) of the 2004 AB3 free-response question. A calculator was allowed on this problem.

A particle moves along the y-axis so that its velocity $\boldsymbol{v}$ at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$. (Note $\tan^{-1} x = \arctan x$.)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

To answer part (d) of this question, many students tried to find the position function and evaluate it at $t = 2$. They tried to think of a function whose derivative is $\tan^{-1}(e^t)$, and they could not do so. Students need to expand their view of function types to include those defined by definite integrals. The students who knew to write $y(t) = -1 + \int_0^t v(w)\,dw$ and then computed $y(2) = -1 + \int_0^2 v(w)\,dw = -1.361$ found this to be an easy problem, and those who were not familiar with this process felt that it was an impossible problem.

Writing an expression for the antiderivative of a function that is continuous on a open interval and that has a given value at a given point in the interval—especially when the closed form of the antiderivative doesn't exist—demands a deep understanding that extends well beyond an ability to apply a procedure. It is a concept, however, that adds to my students' appreciation for the power of the Fundamental Theorem; thus I try to supplement the textbook problems with problems typical of those used on AP Examinations to assess this level of understanding.

Special Focus: The Fundamental Theorem of Calculus

Functions Defined by a Definite Integral

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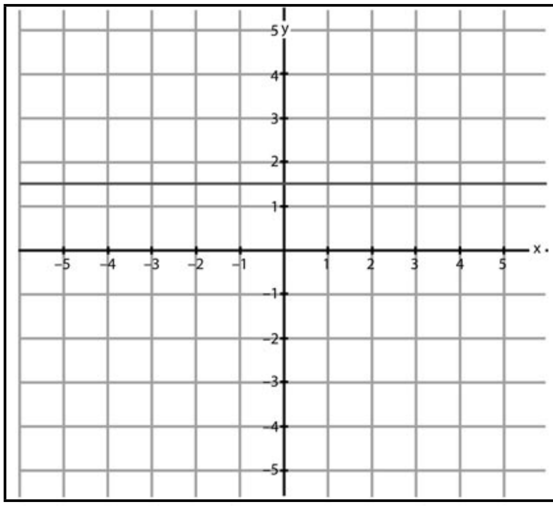
You are about to embark on a journey toward a great achievement in the history of mathematics: the Fundamental Theorem of Calculus. In preparation for that journey, it is essential that you become familiar with a new type of function, where the independent variable is a limit of integration.

Recall that a definite integral such as $\int_0^2 \frac{1.5}{\sqrt{1+t^2}}\,dt$ is a real number, in this case approximately 2.165. If you change the upper limit of integration from 2 to 5, you get a different real number that is $\int_0^5 \frac{1.5}{\sqrt{1+t^2}}\,dt \approx 3.469$. In fact, you could make that upper limit any real number and get a different value for the integral. If you let the limit of integration be the independent variable, you can define a function $g(x) = \int_0^x \frac{1.5}{\sqrt{1+t^2}}\,dt$.

Notice that if $x < 0$, then $g(x) = \int_0^x \frac{1.5}{\sqrt{1+t^2}}\,dt$ is negative. One way to see why is by recognizing that $\int_0^x \frac{1.5}{\sqrt{1+t^2}}\,dt = -\int_x^0 \frac{1.5}{\sqrt{1+t^2}}\,dt$. If $x < 0$, the value of $\int_x^0 \frac{1.5}{\sqrt{1+t^2}}\,dt$ is positive since the lower limit of integration is less than the upper limit, and the integrand is positive. Therefore $\int_0^x \frac{1.5}{\sqrt{1+t^2}}\,dt$ is negative.

In this activity, you will explore several functions of the form $\int_a^x k\,dt$, where a and k are constants.

First, consider the function $f_0(x) = \int_0^x 1.5\,dt$. For $x \geq 0$, the function f measures the area under the graph of $y = 1.5$ from 0 up to x. (The notation $f_0(x)$ is used to emphasize that f, 0, and x are the key elements in this definition: f is the integrand, 0 is the lower limit, and x is the upper limit.)



$y = 1.5$

When $x = 0$, the upper and lower limits of integration are the same.

1. What is $f_0(0)$? ___________________________

When $x = 1$, notice that you are finding the area of a rectangle with a base 1 unit long and a height of 1.5.

2. What is $f_0(1)$? ___________________________

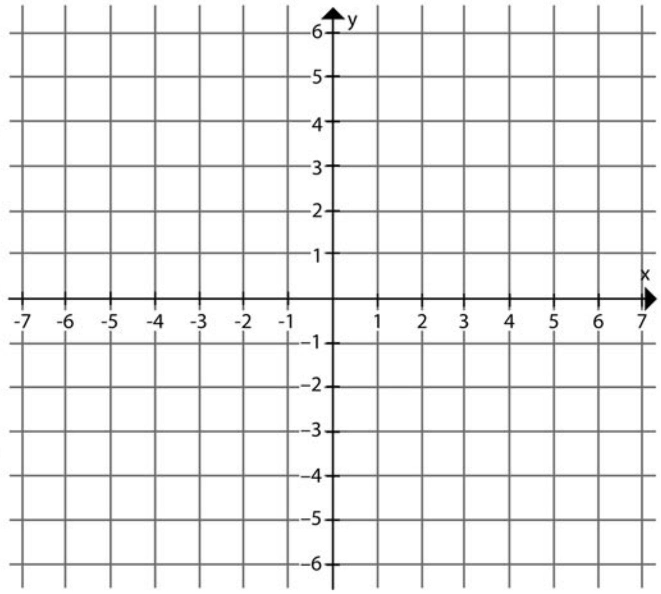
When $x = 2$, you are finding the area of a rectangle with a base 2 units long and a height of 1.5.

3. What is $f_0(2)$? ___________________________

4. Complete the table of values below.

x	$f_0(x) = \int_0^x 1.5\,dt$
0	
1	
2	
3	

5. Graph the ordered pairs $(x, f_0(x))$ on the axes below.

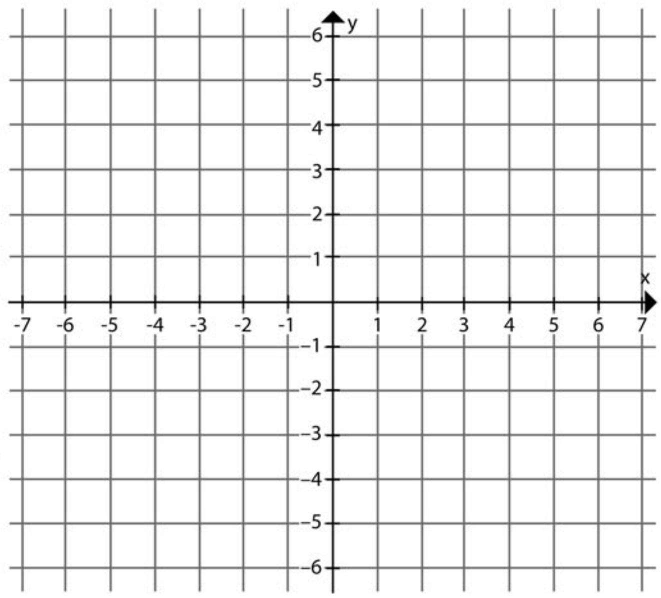


Now look at $g_0(x) = \int_0^x 0.5\,dt$. Notice that this function has a different integrand.

6. Complete the table of values below.

x	$g_0(x) = \int_0^x 0.5\,dt$
0	
1	
2	
3	

7. Graph the ordered pairs $(x, g_0(x))$ on the axes below.



Now, look at two additional functions. Each will be a modified version of f_0 or g_0 but using a different lower limit of integration. Call these functions $f_1(x) = \int_1^x 1.5\,dt$ and $g_1(x) = \int_1^x 0.5\,dt$.

8. Fill in the tables of values below.

x	$f_1(x) = \int_1^x 1.5\,dt$
1	
2	
3	
4	

x	$g_1(x) = \int_1^x 0.5\,dt$
1	
2	
3	
4	

9. Graph the ordered pairs $(x, f_1(x))$ on the same set of axes you used for number 5. Connect the dots on your graph corresponding to each function to produce two lines.

Extend your lines to include values of $x < 0$. Note that $f_0(-2) = \int_0^{-2} 1.5\,dt = -\int_{-2}^{0} 1.5\,dt$.

10. What is $f_0(-2)$? ____________________ What is $f_1(-2)$? ____________________

11. Graph the ordered pairs $(x, g_1(x))$ on the same set of axes you used for number 7. Connect the dots on your graph corresponding to each function to produce two lines.

Extend your lines to include values of $x < 0$. Note that $g_0(-2) = \int_0^{-2} 0.5\,dt = -\int_{-2}^{0} 0.5\,dt$.

12. What is $g_0(-2)$? ____________________ What is $g_1(-2)$? ____________________

13. What do the graphs of $y = f_0(x)$ and $y = f_1(x)$ have in common? ______________

14. What do the graphs of $y = g_0(x)$ and $y = g_1(x)$ have in common? ______________

Teacher Notes and Answers

Class Management
This activity takes less than one full class period to complete (30 to 35 minutes). Teachers should work through the activity once before doing it with students. Students could work independently or in groups.

Calculus Prerequisites
Before doing this activity, students should know:
- That a definite integral is a number
- The relationship between an integral and the area under the graph of a positive function

Calculator Prerequisites
A calculator is not required for this activity. If your students have never used their calculators to make a scatterplot, however, this activity could be used to introduce them to that functionality. This approach is illustrated in the answers.

Preliminary Comments
The primary purpose of this activity is to develop students' familiarity with and understanding of a function defined by a definite integral, where the independent variable of the function is a limit of integration. If $f(t) > 0$ and $a > 0$, then $\int_0^a f(t)\,dt$ represents the area between the graph of $y = f(t)$ and the t-axis from $t = 0$ to $t = a$. When $x > 0$, $A(x) = \int_0^x f(t)\,dt$ can be interpreted as the "area so far" function. It represents the area under the graph of f and above the t-axis from 0 up to the variable limit of integration, x.

A secondary purpose is to foreshadow the Fundamental Theorem. See the answers to questions 13 and 14.

The integrands in this activity are the simplest kind: constant functions. This enables students to evaluate the integrals by simply finding areas of rectangles. It would be useful at the end of the activity to ask students to express in words the connection they observe between the slope of the f and g functions and the integrand. For each unit that x increases, the area increases by an amount equal to the value of the integrand.

Extending These Ideas

If time permits, this activity can easily be extended by introducing simple linear functions as the integrands. Consider $f(x) = x,\ 2x,\ x+2,\ 2x+3$, or similar linear functions with negative slopes. Students can evaluate $f_0(x) = \int_0^x f(t)\,dt$ and $f_2(x) = \int_2^x f(t)\,dt$ for specific values of *x* by computing the areas of triangles and trapezoids.

Answers

1. 0

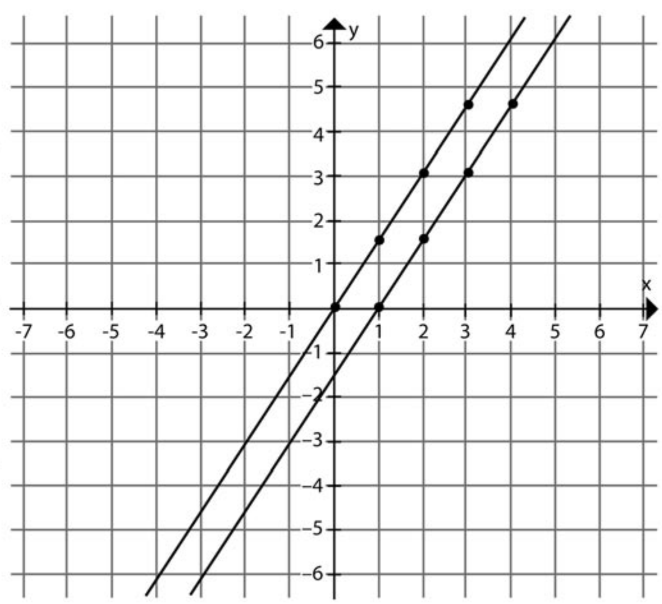
2. 1.5

3. 3

4.

x	$f_0(x) = \int_0^x 1.5\,dt$
0	0
1	1.5
2	3
3	4.5

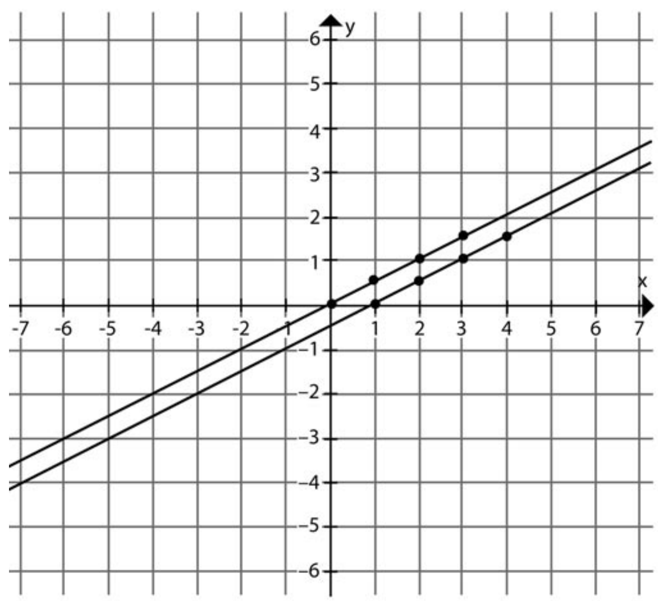
5.



6.

x	$g_0(x) = \int_0^x 0.5\,dt$
0	0
1	0.5
2	1
3	1.5

7.



8.

x	$f_1(x) = \int_1^x 1.5\,dt$
1	0
2	1.5
3	3
4	4.5

x	$g_1(x) = \int_1^x 0.5\,dt$
1	0
2	0.5
3	1
4	1.5

9. See answer to number 5.

10. $f_0(-2) = -3$, $f_1(-2) = -4.5$

11. See answer to number 6.

12. $g_0(-2) = -1$, $g_1(-2) = -1.5$

Questions 13 and 14 foreshadow the Fundamental Theorem. The functions f_0 and f_1 are antiderivatives of $y = 1.5$, while g_0 and g_1 are antiderivatives of $y = 0.5$. For each unit that x increases, $f_0(x)$ and $f_1(x)$ increase by 1.5. For each unit that x increases, $g_0(x)$ and $g_1(x)$ increase by 0.5. In a later activity, students will see that a similar result applies to any continuous integrand, not just constant ones.

13. The slope of each line is 1.5, the same as the integrand.

14. The slope of each line is 0.5, the same as the integrand.

Stage Two: Refining the Level of Understanding and Making Conjectures

Introduction

Goals of Stage Two

- Distinguish between displacement and distance traveled
- Investigate more complicated integral functions
- Discover and reinforce properties of the definite integral
- Use the FTC to solve initial value problems

Articles in Stage Two

- “From Riemann Sums to Net Change” by Ray Cannon
- “The Integral Function—Class Worksheet” by Benita Albert
- “The Integral Function—Teacher Notes” by J. T. Sutcliffe

Ray Cannon’s article, “From Riemann Sums to Net Change,” extends the simple setting of Stage One in several ways. The velocity is given by a formula, and the formula he considers, $v(t) = 80 - 32t$ on [0,3], gives both positive and negative velocity values. His exploration leads students to discover that definite integrals can give either displacement or distance traveled, depending on whether the integrand is velocity or speed.

“The Integral Function—Class Worksheet” encourages students to experiment with functions defined by definite integrals. Students will sketch graphs and investigate properties of functions of the form $F(x) = \int_a^x f(t)dx$ where f is given geometrically and not analytically. Students will change the constant lower limit in the definite integral and create new, although similar, functions. Reflecting on the similarities will reveal important properties of the definite integral. Some teachers will use “The Integral Function—Class Worksheet” in order to help students discover properties of the definite integral using a geometric approach. Other teachers may prefer to discuss properties of the definite integral first and use this activity to reinforce the concepts. “The Integral Function—Teacher Notes” gives teachers some classroom guidelines and hints on using “The Integral Function—Class Worksheet.”

In Steve Olson’s article in Stage One, the material in “Antiderivative Step 2” from the “Antiderivative Part of the Fundamental Theorem” section gives an important variation on the FTC that is helpful in solving initial value problems.

Stage Two focuses on a variety of contexts, and each one is more sophisticated than the Stage One settings. It is important to notice the similarities between the simple examples of Stage One and the new settings in Stage Two.

From Riemann Sums to Net Change

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Overview

Technology and the "reform movement" have allowed me to change my whole approach to the very introduction of the Riemann integral. It was always frustrating to draw a picture on the board of a Riemann sum over a partition with four or five subintervals (certainly never more than 10) and then "let n go to infinity" and act as if the students understood this. *Perhaps* the next step (depending on how pressed we were for time at this stage of the semester) would be to show that $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$ using some special summation formulas.

Now I am not interested in teaching a real analysis course and calling it Calculus I. Thus I am not offended if the course does not take full advantage of the definition of Riemann sums, especially in the degree of freedom in choosing the partition and elements in each subinterval. I am not so much interested when teaching Calculus I in filling in the logical necessities as I am with helping to provide the *beginnings* of understanding. The bottom line is that I like to start with partitioning the interval $[a, b]$ into n equal subdivisions, so $\Delta t_i = \frac{b-a}{n}$ for each i. In addition, in this article, I only consider two types of Riemann sums: the left-hand sum (LHS) and the right-hand sum (RHS). Furthermore, the discussion begins with considering functions that are monotonic on $[a, b]$. (Contrary to students' belief, the word is *not* monotonous.) In particular, the first examples use velocity to obtain displacement of position. I want students to become aware that $\int_a^b v(t)\,dt$ represents an area when $v(t) > 0$ and also realize from the beginning that the sums do not give area when $v(t)$ is negative. Students need immediate reinforcement that the concepts of "area under a curve" and "the definite integral" are distinct. *By using Riemann sums to present other applications of the integral first, I hope to motivate the definite integral as representing the net change of a function.*

The starting point is to compute the position of a rock that is thrown into the air. I do this as the "inverse" to the problem that motivated the derivative, namely instantaneous velocity. Now "up" will be the positive direction, so the acceleration due to gravity is negative. We know from applications with derivatives we have talked about previously in the course that the acceleration is constant, and the rock slows down at a rate of 32 ((feet per second) per second) as it travels upward, so we let $v(t)=80-32t$. The initial velocity is 80 ft/sec. The question is, can we tell where the rock is at various times?

This investigation has two main goals:

1. Using an integrand that is negative on some of the interval in a natural setting

2. Looking at the integral with a variable upper limit of integration

The Particular Investigation
Where is the rock after one second? How far has it traveled? (These are actually two different questions, but that point need not be made here; it will show up in the discussion later.)

We first get the bounds that the rock must travel at least 48 feet (the minimum velocity is 48 ft/sec) and at most 80 feet (the maximum velocity is 80 ft/sec). Let *H* stand for distance traveled. By taking the average of our two bounds, we have

$$H = 64 \text{ feet} \pm 16 \text{ feet}.$$

The next step is difficult to get students to suggest. To get a better estimate, we have to sample the velocity at more than just two points. We divide the interval [0,1] into two subintervals.

Using minimum velocities, we get

$$64 \cdot \frac{1}{2} + 48 \cdot \frac{1}{2} \le H \le 80 \cdot \frac{1}{2} + 64 \cdot \frac{1}{2}.$$

So *H* is between 56 feet and 72 feet. Again we take the average and see

$$H = 64 \text{ feet} \pm 8 \text{ feet}.$$

When we divide the interval up into four equal pieces, we get

$$H = 64 \text{ feet} \pm 4 \text{ feet}.$$

When we divide the interval up into eight equal pieces, we get

$$H = 64 \text{ feet} \pm 2 \text{ feet}.$$

There is a pattern here. When we double the number of subintervals, the error in our estimation is cut in half! So as n gets bigger, RHS and LHS are getting close to a common number.

Note: For these computations, we use the built-in commands of our calculators, and compute

RHS = sum(seq$\left(f(t)\cdot\frac{b-a}{n}, t, a+\frac{b-a}{n}, b, \frac{b-a}{n}\right)$), and

LHS = sum(seq$\left(f(t)\cdot\frac{b-a}{n}, t, a, b-\frac{b-a}{n}, \frac{b-a}{n}\right)$).

We call that common number "the definite integral of 80–32t from 0 to 1" and write this number as $\int_0^1 (80-32t)\,dt$. In general, when the function f is defined on the interval $[a,b]$ we write "$\int_a^b f(t)\,dt$ for the definite integral of $f(t)$ from a to b." Here we have discovered the definition of the definite integral in the context of doing a problem about distance traveled.

Returning to our example, we see that RHS and LHS are both converging to 64. We have seen this numerically, and now if we look at the graph of $v(t)$ (see Figure 1), we note that the area under the graph corresponds to the distance traveled, since 64 is the area of the trapezoid.

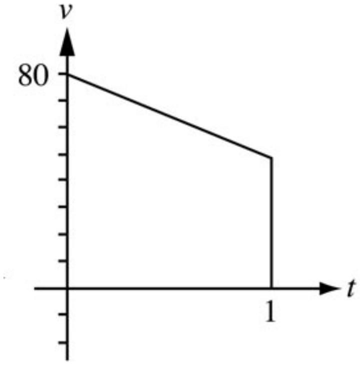

Figure 1: The graph of $y = 80 - 32t$ for $0 \le t \le 1$

We can now write that after 1 second, distance traveled = $\int_0^1 (80-32t)\,dt$ = 64 feet. Where is the rock after 2 seconds? We use the calculator with 100 subintervals to compute the average of RHS and LHS. Using our definite integral notation, we obtain $\int_0^2 (80-32t)\,dt$ = 96 feet. (Again, we confirm geometrically that the area under the graph, shown in Figure 2, is 96.)

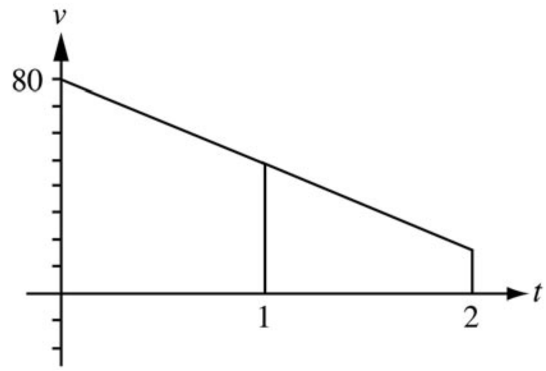

Figure 2: The graph of $v = 80 - 32t$ for $0 \le t \le 2$

After 2 seconds, distance traveled = $\int_0^2 (80-32t)\,dt = 96$ feet.

The next question is to find out where the rock is after 3 seconds. So we again change the upper limit of integration and use the calculator to compute Riemann sums to approximate $\int_0^3 (80-32t)\,dt$ (see Figure 3). We get $\int_0^3 (80-32t)\,dt = 96$ feet.

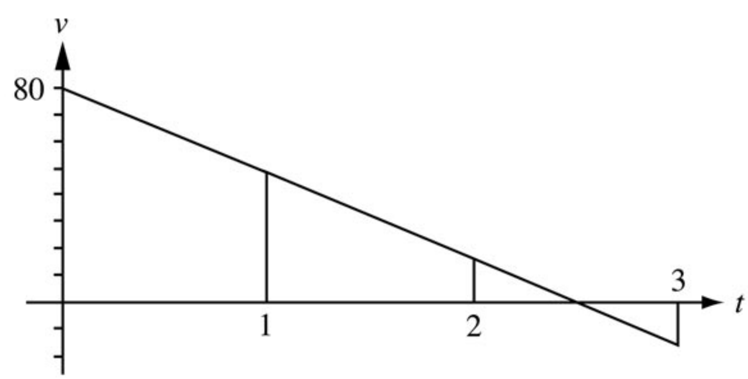

Figure 3: The graph of $v = 80 - 32t$ for $0 \le t \le 3$

This is silly; the rock has obviously moved during the third second. How can

$\int_0^2 (80-32t)\,dt = \int_0^3 (80-32t)\,dt$? What's going on? We have to talk about the motion of the rock. The rock reaches its maximum height when t = 2.5 seconds and then starts back down. The number $\int_0^3 (80-32t)\,dt$ does not represent distance traveled, but rather the height of the rock. What do we mean by "the height of the rock"? How do we measure height? What we have really computed is the height of the rock above its initial height:

$$\int_0^3 (80-32t)\,dt = h(3)-h(0)\,.$$

But now recognize $80-32t$ as $v(t)$, the velocity. What is the relation between velocity and height? Velocity is the derivative of the height function. We have

$$\int_0^3 (80-32t)\,dt = \int_0^3 v(t)\,dt = \int_0^3 h'(t)\ dt = h(3)-h(0)\,.$$

The last part of this equation is one form of the FTC! (We have had three instances of the equation $\int_0^x (80-32t)dt = h(x)-h(0)$.)

Look again at the graph of $y=80-32t$ on $[0,3]$. Remember that positive velocity means the rock is going up, and negative velocity means the rock is going down. To find the net change in height, we must add the area above the x-axis, which is 100, and subtract the area below the x-axis, which is 4. Observe that $96=100-4$. To find the distance traveled, we add the area above the axis and the area below the axis; the distance the rock traveled in the first 3 seconds is $\int_0^3 |80-32t|\,dt = 104$ feet; to get distance traveled, we must integrate the speed, not the velocity.

Let's look at some other interpretations of net change. Suppose $g(t)$ is equal to the amount of water flowing into (out of) a container, with a positive (negative) value, which means that water is coming in (going out). If the flow rate is constant, then flow rate × time = amount. This is the same setup as rate × time = distance. Thus Riemann sums give the approximate change in the amount of water, and so $\int_a^b g(t)\,dt$ represents the net change in the amount of water over the time interval as t goes from a to b.

More generally, if we think of the integrand as an instantaneous rate of change, say f', then the definite integral gives the net change in the value of f. That is,

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

This is a version of the evaluation part of the FTC, and even though what we have presented is not a proof, it does give us some understanding of what the FTC is saying: **The definite integral accumulates net amount of change when the integrand is the instantaneous rate of change.**

We now return to my second goal: investigating the integral with a variable upper limit of integration. The preceding development has led us to the conclusion that

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

By replacing b with x we have

$$\int_a^x f'(t)\,dt = f(x) - f(a).$$

Notice the bonus that results. If we take the derivative of both sides, we get

$$\frac{d}{dx}\int_a^x f'(t)\,dt = f'(x).$$

(The derivative of $f(a)$ is zero since $f(a)$ is a number.) This is the antiderivative version of the FTC, in the case where the integrand is a continuous derivative. Here again, we have not presented a formal proof, but we have made significant progress in helping our students begin to understand this part of the FTC. This approach can be used to help students understand what the formulas are saying when we get into the topic of derivatives of functions defined by integrals.

The Integral Function—Class Worksheet

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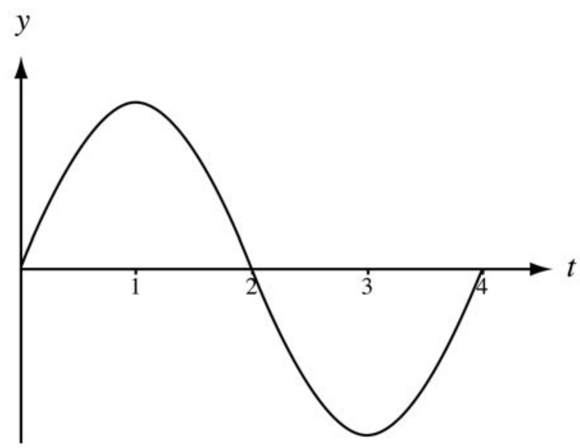
The graph of the function $y = f(t)$ is shown below. The function is defined for $0 \le t \le 4$ and has the following properties:

- The graph of f has odd symmetry around the point $(2, 0)$.
- On the interval $[0, 2]$, the graph of f is symmetric with respect to the line $t = 1$.
- $\int_0^1 f(t)\,dt = \dfrac{4}{3}$.



Graph of $y = f(t)$

Special Focus: The Fundamental Theorem of Calculus

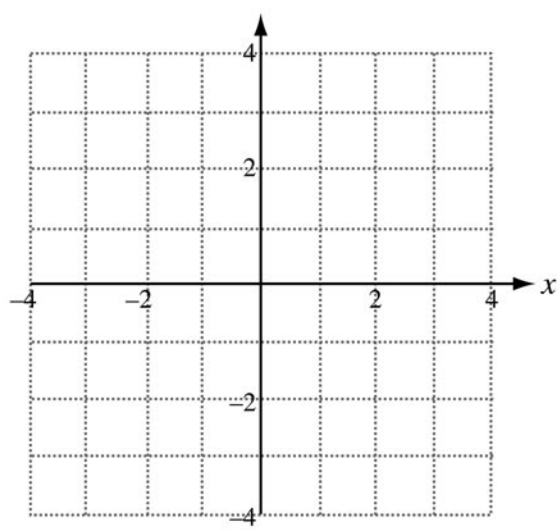
1. Let $F(x) = \int_0^x f(t)\,dt$.

a. Complete the following table of values.

x	0	1	2	3	4
F(x)					

b. Sketch your best estimate of the graph of *F* on the grid below.

$y = F(x)$

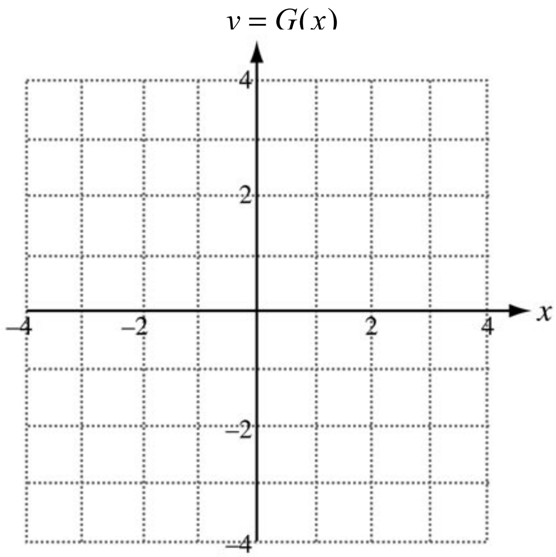


2. Let $G(x) = \int_2^x f(t)\,dt$.

a. Complete the following table of values.

x	0	1	2	3	4
$G(x)$					

b. Sketch the graph of G on the grid below.

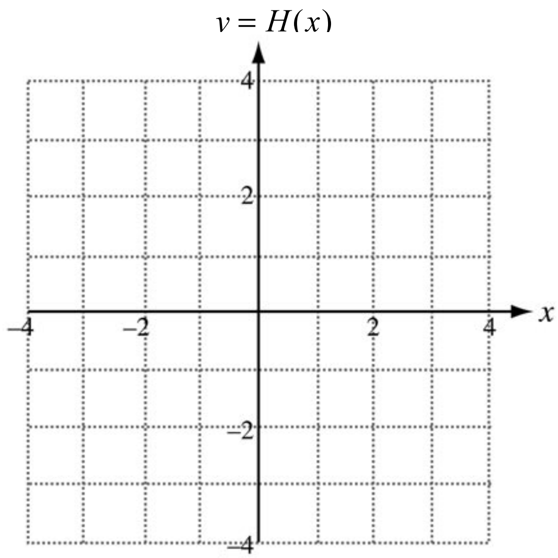


3. Let $H(x) = \int_4^x f(t)\,dt$.

a. Complete the following table of values.

x	0	1	2	3	4
H(x)					

b. Sketch the graph of *H* on the grid below.



4. Complete the following table.

	$F(x)$	$G(x)$	$H(x)$
The maximum value of the function occurs at what *x*-value(s)?			
The minimum value of the function occurs at what *x*-value(s)?			
The function increases on what interval(s)?			
The function decreases on what interval(s)?			

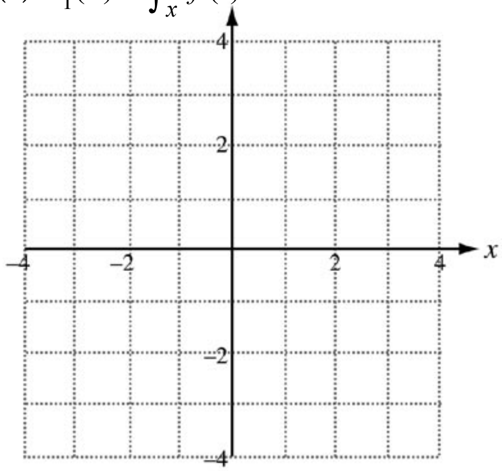
5. Although the tables in questions 1, 2, and 3 asked only for the three functions to be evaluated at integer values of x, those functions were all continuous on the domain of $0 \le x \le 4$. Refer back to the answers you gave for function F in the table above, and explain why you believe each of these answers is correct when one considers F on its entire domain. Write your arguments in the table below. Your explanations should not rely on the graphs you sketched.

	Justification of the answers above for $F(x)$
The maximum value of the function occurs at what x-value(s)?	
The minimum value of the function occurs at what x-value(s)?	
The function increases on what interval(s)?	
The function decreases on what interval(s)?	

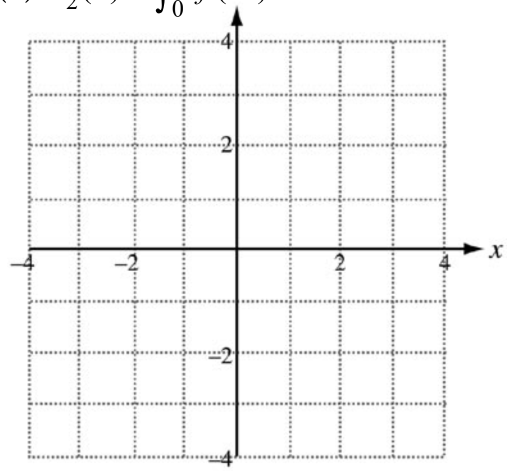
6. What conjectures would you make about the family of functions of the form $W(x) = \int_k^x f(t)\,dt$ for $0 \le k \le 4$, where f is the graph given at the beginning of this worksheet?

7. Extend your understanding by sketching each of the following functions.

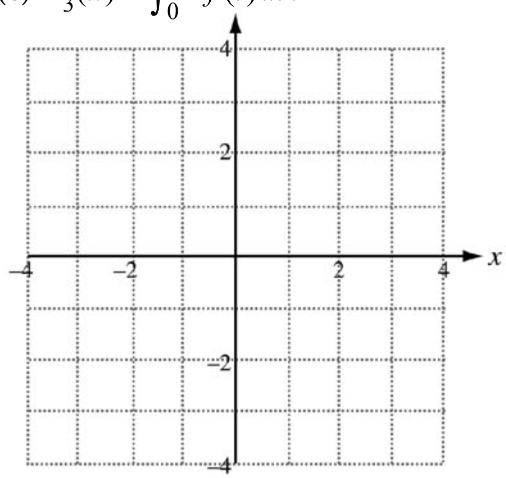
(a) $F_1(x) = \int_x^0 f(t)\,dt$.



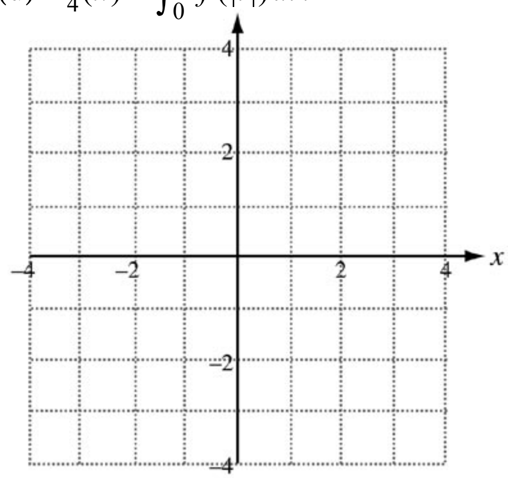
(b) $F_2(x) = \int_0^x f(-t)\,dt$.



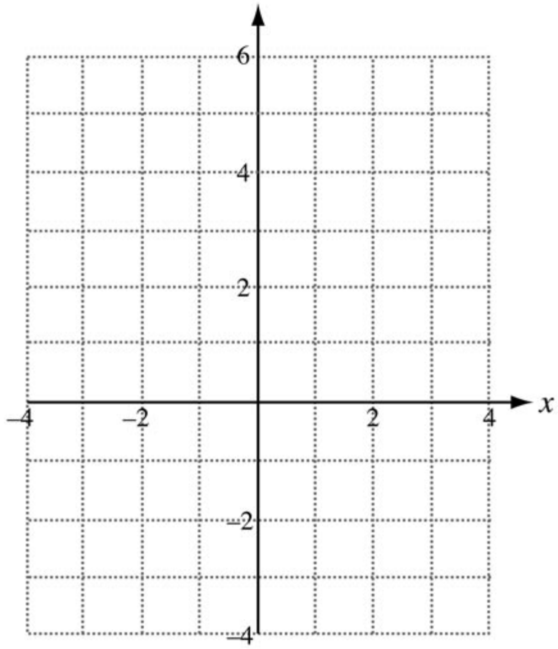
(c) $F_3(x) = \int_0^{2x} f(t)\,dt$.



(d) $F_4(x) = \int_0^x f(|t|)\,dt$.



(e) $F_5(x) = \int_0^x \left| f(t) \right| dt.$



The Integral Function—Teacher Notes

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Prerequisite Knowledge

This worksheet is intended to encourage exploration of some elementary properties of definite integrals and to extend student understanding of functions defined by integrals. For students to make reasonable progress with this worksheet, however, some prerequisite knowledge is required:

- $\int_a^b f(x)\,dx$ represents area when $f(x) \geq 0$ on the interval $[a,b]$, and $\int_a^b f(x)\,dx$ takes on a negative value that is the additive inverse of area when $f(x) \leq 0$ on the interval $[a,b]$.

Or

- $\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(b) - s(a)$ yields displacement when $v(t)$ represents velocity and when $s(t)$ represents position. In fact, $\int_a^b v(t)\,dt \geq 0$ when on the interval $[a,b]$, and $\int_a^b v(t)\,dt \leq 0$ when $v(t) \leq 0$ on the interval $[a,b]$. (Note that this is the approach Ray Cannon uses in the preceding article.)

Or

- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$ when a continuous function f on $[a,b]$ is partitioned into equal subintervals. When $f(x) > 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ will yield a positive sum, and when $f(x) < 0$ on $[a,b]$, then $\int_a^b f(x)\,dx$ will yield a negative sum.

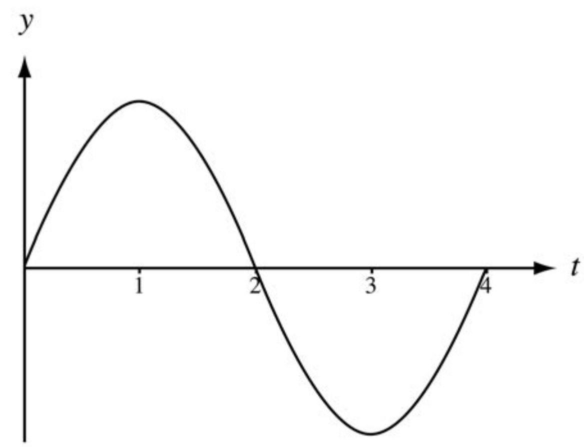
- Students need to know or discover that $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$. Many teachers may choose to see if students can wrestle with this successfully on their own when answering question 2 of this worksheet. Others will choose to introduce the concept before students tackle this worksheet. In the notes for question 2, suggestions are made for how this property might be developed either by insightful students or through a teacher-led class discussion.

Teachers should also be sure that students understand the three given properties of the function *f*. If students do not understand the symmetry properties, it will be time well spent to clarify what the properties tell us about the graph of *f*. Specifically, in order to be successful in answering the questions in this worksheet, students also need to understand that

$$\int_0^1 f(t)\,dt = \int_1^2 f(t)\,dt = \left| \int_2^3 f(t)\,dt \right| = \left| \int_3^4 f(t)\,dt \right| = \frac{4}{3}\,.$$

Once this prerequisite knowledge is in place, students should be allowed to work individually or in groups; they can explore and discover additional properties on their own.

As students begin to answer the questions, teachers should be especially alert to students who try to generate a rule for *f*. It is not the intent of the worksheet to have students find antiderivatives; it is expected that students will be applying their conceptual understanding to answer the questions rather than applying an algorithmic integration or evaluation process.



Graph of $f(t)$

Question 1

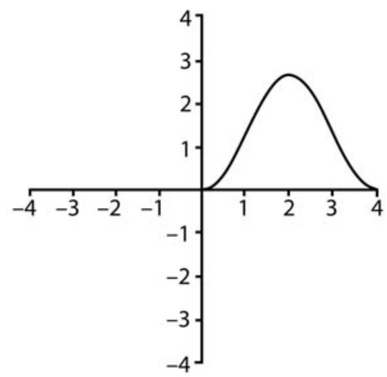
To answer question 1 correctly, students must understand that $\int_2^3 f(t)\,dt = \int_3^4 f(t)\,dt = -\frac{4}{3}$. This level of understanding was indicated as a prerequisite above (i.e., $\int_a^b f(x)\,dx$ takes on a negative value that is the additive inverse of area when $f(x) \leq 0$ on the interval $[a, b]$), but teachers should be alert for students who have not yet achieved this level of understanding. If a student needs additional help with this concept, teachers have several options. Among them are:

- If the class has investigated a velocity context such as the one Ray Cannon introduced, f could be identified as a velocity function, and students could rethink why these integrals have a negative value. For example: During the first 2 seconds, a car moves away from a point of reference, thus increasing its distance from that point. In these last 2 seconds, the car moves back toward the point of reference, and therefore its distance from that reference point is decreasing.
- If the class has investigated Riemann sums, the students could partition the interval $[0, 4]$ into eight subintervals and sketch a left- or right-hand Riemann sum. It should then be clear that the products of the form $f(x) \times \Delta x$ are positive on $[0, 2]$ and negative on $[2, 4]$.

(a) Given that $F(x) = \int_0^x f(t)\,dt$,

x	0	1	2	3	4
$F(x)$	0	4/3	8/3	4/3	0

(b) Teachers should only expect the five specific points from the above table to be graphed carefully. Celebrate graphs that show appropriate increasing and decreasing tendencies. A graph of F based on knowing the rule for f is provided below.



The graph of F on $[0,4]$

Question 2

Many students will have trouble evaluating G at $x=0$ and $x=1$. Since $G(0) = \int_2^0 f(t)\,dt$ and $G(1) = \int_2^1 f(t)\,dt$, both integrals have an upper limit that is smaller than the lower limit. Some students will believe that there is no such expression and will leave those entries in the table blank. Usually these students believe that the domain of the function defined by the integral has the lower limit as its minimum domain value. Others may merely ignore the difficulty and write the answers that would result if the limits were reversed. In any case, this question usually generates considerable debate among students. If students have trouble resolving this issue, teachers might return to the velocity context introduced in "From Riemann Sums to Net Change" and think about running time backward, as if playing a video in reverse.

We know that $\int_a^b h'(t)\,dt = h(b) - h(a)$. Therefore, if we interchange the upper and lower limits, it would also be true that $\int_b^a h'(t)\,dt = h(a) - h(b)$. Since

$$\int_a^b h'(t)\,dt = h(b) - h(a) = -(h(a) - h(b)) = -\int_b^a h'(t)\,dt,$$

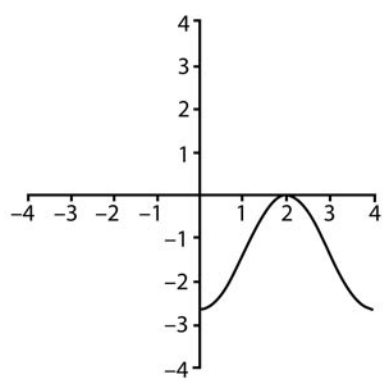
we can generalize to $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$. Applying this property will allow students to recognize that $G(0) = \int_2^0 f(t)\,dt = -\int_0^2 f(t)\,dt = -\frac{8}{3}$.

$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$ is one of the important properties of integrals that students should understand and be able to apply. Subsequent questions in this worksheet give students practice with this property so that its application will become a natural response.

(a) Given that $G(x) = \int_2^x f(t)\,dt$,

x	0	1	2	3	4
$G(x)$	–8/3	–4/3	0	–4/3	–8/3

(b) Once again, this graph of G is based upon knowledge of the rule for f. If students have plotted the five points correctly and have identified the appropriate intervals of increase and decrease, their graph of a continuous function on the interval [0,4] should be accepted.



The graph of G on $[0,4]$

Special Focus: The Fundamental Theorem of Calculus

Question 3

From class discussion of the properties required for questions 1 and 2, students have learned the properties needed to correctly answer this question. However, many students will feel even more uncomfortable evaluating this function. Students are now confronted with an integral expression that is not only of the form $\int_a^x f(t)\,dt$ where $x \le a$ for all domain values but also one for which the integrand is negative over some subintervals. That is, they must put everything together and understand the following:

If $b < a$ and if $f(t) < 0$ for all t such that $b < t < a$, then

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt = -\left(\text{negative number}\right).$$

Specifically, in this question, students must understand that

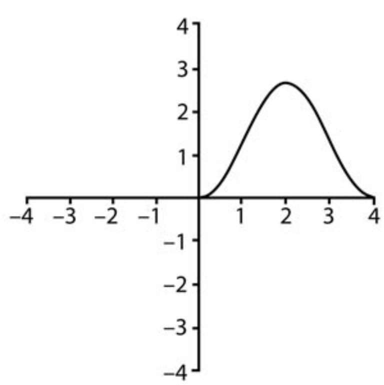
$$H(3) = \int_4^3 f(t)\,dt = -\int_3^4 f(t)\,dt = -\left(-\frac{4}{3}\right) = \frac{4}{3}.$$

Extra time, patience, and reflection invested at this point in the development of the properties of integrals will reap huge rewards for both students and teacher down the road.

(a) Given $H(x) = \int_4^x f(t)\,dt$,

x	0	1	2	3	4
$H(x)$	0	4/3	8/3	4/3	0

(b) The graph of H is sketched below.



The graph of H on $[0,4]$

Some interesting class discussion should arise when students discover that this graph is the same as that of F. If they understand the integral properties, then they might think to reexpress H as follows:

$$H(x)=\int_4^x f(t)\,dt=\int_4^0 f(t)\,dt+\int_0^x f(t)\,dt.$$

Since $\int_4^0 f(t)\,dt=-\int_0^4 f(t)\,dt$, and since $\int_0^4 f(t)\,dt=0$ as discovered in question 1,

$$H(x)=\int_4^x f(t)\,dt=\int_0^x f(t)\,dt=F(x).$$

At this point, the above argument may be a real stretch for most students, and teachers should not be disappointed if students need help making this important connection.

Suggested Extension or Homework Assignment

Before going on, or as a homework assignment, students might also explore $P(x)=\int_1^x f(t)\,dt$ and $R(x)=\int_3^x f(t)\,dt$ in tables and in graphs. These two functions provide evidence to help students create more informed conjectures about the effect of the lower limit on functions defined as $\int_k^x f(t)\,dt$.

Question 4

	$F(x)$	$G(x)$	$H(x)$
The maximum value of the function occurs at what x-value(s)?	$x=2$	$x=2$	$x=2$
The minimum value of the function occurs at what x-value(s)?	$x=0$ and $x=4$	$x=0$ and $x=4$	$x=0$ and $x=4$
The function increases on what interval(s)?	$[0,2]$	$[0,2]$	$[0,2]$
The function decreases on what interval(s)?	$[2,4]$	$[2,4]$	$[2,4]$

Students who have been successful in correctly completing the tables of questions 1, 2, and 3 and in plotting points on the grids will probably have little difficulty in completing this table. Whether they are doing more than "connecting the dots" on the graphs will become apparent as they try to defend some of their answers for question 4 in the next question.

Question 5

During class discussion or when walking around the room as students attempt to explain their reasoning, teachers may discover that some students have correct answers to question 4 alone because they have considered only integer values of x and thus are not worrying about what happens between the integers. Teachers should help students understand, for example, that just because the greatest table value occurred at $x=2$ doesn't explain why the maximum value for F on $[0, 4]$ must occur at $x=2$.

Students are only asked to support their answers for function F since they can provide reasonable arguments to support their answers based upon the properties they have learned previously or that they discovered as they worked through this exploration. They have not been asked to support their answers for functions G and H because it is not assumed that they have learned the antiderivative part of the Fundamental Theorem, which says that $\frac{d}{dx}\int_k^x f(t)\,dt = f(x)$. If the class has already studied the antiderivative part of the Fundamental Theorem, they should be asked to support their answers for G and H as well. Teachers of classes that have not yet considered the antiderivative part of the Fundamental Theorem may want to return to this part of the worksheet after doing so.

The maximum value of the function *F* occurs at what *x*-value(s)?

An example of a student's analysis using velocity:

> The maximum value of 8/3 occurs at $x=2$. I thought of f as a velocity function. The object moved forward (positive velocity) for 2 seconds then moved backward (negative velocity) the same distance for the final 2 seconds. This means the maximum displacement occurred when the direction of the object changed from moving forward to moving backward.

An example of a student's analysis using Riemann sums:

> I pictured a Riemann sum with tiny Δt values. For F, I was adding positive value products from $t=0$ to $t=2$, but then I started adding on negative value products from $t=2$ to $t=4$. This means that the greatest sum would have occurred at $t=2$.

The minimum value of the function *F* occurs at what x-value(s)?
The minimum value for F is 0, and it occurs at both $x=0$ and at $x=4$.

An example of a student's analysis using velocity:

> I thought of f as a velocity function. The object moved forward (positive velocity) for 2 seconds then moved backward (negative velocity) the same distance for the final 2 seconds. The original displacement is 0 because the object hasn't yet moved anywhere. At the end of 4 seconds, the object would be back where it started because it moved the same distance forward as backward. This means that once again, the displacement would be 0. For all other values of x between 0 and 4, the displacement would be positive.

An example of a student's analysis using Riemann sums:

> I pictured a Riemann sum with tiny Δt values. For F, I was adding positive value products from $t=0$ to $t=2$, but then I started adding on negative value products from $t=2$ to $t=4$. When $x=0$, I am evaluating $\int_0^0 f(t)\,dt = 0$. When $x=4$, I have added positive products to get larger values until I reach the sum of 8/3 for $x=2$. Then I begin to subtract values because the products are negative. From $x=2$ to $x=4$, I have subtracted off 8/3 because of the symmetry of the graph. This means that I have gotten back to a net sum of 0. Since I never subtracted off more than I had added on, my answer was never negative.

Function *F* increases on what interval(s)? Decreases on what interval(s)?
An example of a student's analysis using velocity:

> If I think of f as a velocity function, an object is moving forward (velocity positive) between $t=0$ and $t=2$. This means that the function describing displacement of the moving object increases on this interval. Another way to say this is that if the object is moving along a horizontal line, it is moving further and further to the right for these two seconds. However, between $t=2$ and $t=4$, the object is moving in reverse because the velocity is negative. So, the displacement function is decreasing on $[2,4]$; between $t=2$ and $t=4$ the object is moving left on the horizontal line I am imagining.

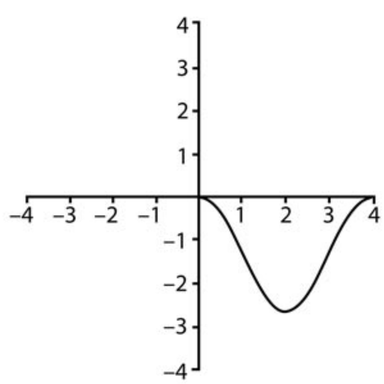
An example of a student's analysis using Riemann sums:

> I pictured a Riemann sum with tiny Δt values. For F, I was adding positive value products from $t=0$ to $t=2$, but then I started adding on negative value products from $t=2$ to $t=4$. This means that F is increasing on the interval $[0,2]$, and F is decreasing on the interval $[2,4]$.

Question 6
Without the aid of the antiderivative part of the Fundamental Theorem, it is only hoped that students will notice that the graphs differ (if at all) in their vertical placement on the grid; that is, they are vertical displacements of one another. One or more students may even propose that the functions have the same slope values; that is, $F'(x) = G'(x) = H'(x)$. If this happens, the student has provided the perfect segue into a more formal discussion of the antiderivative part of the Fundamental Theorem. Here again, teachers may want to return to this worksheet to help their students see and understand the conceptual groundwork being laid in these six questions.

Question 7
This question allows students to extend their understanding of the Fundamental Theorem, but it also requires that they understand composite functions. The correct graph and a brief commentary for each follows.

(a) Since $F_1(x) = \int_x^0 f(t)\,dt = -\int_0^x f(t)\,dt = -F(x)$, students should be able to determine the graph of F_1 shown below.

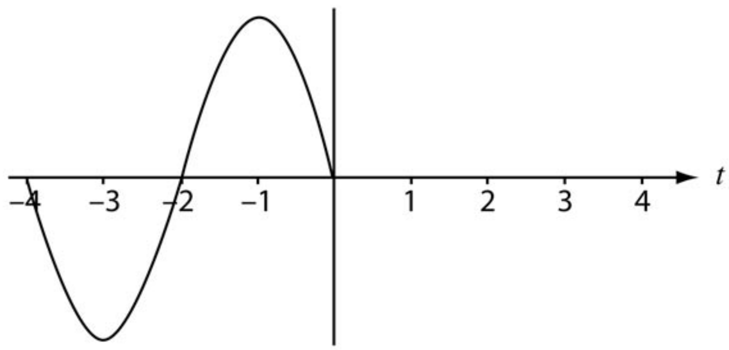


The graph of F_1 on $[0,4]$

(b) Since $y = f(-t)$ reflects the graph of $y = f(t)$ across the y-axis,

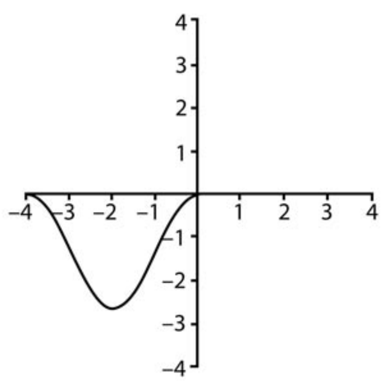
$$F_2(x) = \int_0^x f(-t)\,dt = -\int_x^0 f(-t)\,dt.$$

The graph of $y = f(-t)$ is shown below.



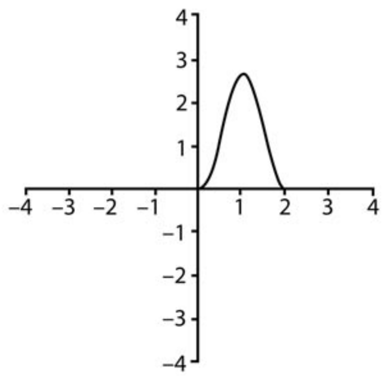
The graph of $f(-t)$ on $[-4,0]$

So, $F_2(-4) = \int_0^{-4} f(-t)\,dt = -\int_{-4}^0 f(-t)\,dt = 0$ due to the compensating loss and gain. Checking other values gives the graph of F_2.



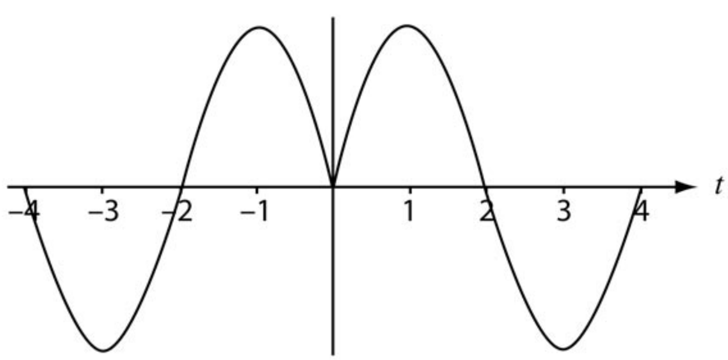
The graph of F_2 on $[-4,0]$

(c) This question will be one of the most challenging for students because the composite function is created by the upper limit. The domain of *f* is $0 \le x \le 4$. Since *f* is continuous on this interval, F_3 accepts all *x* for which $0 \le 2x \le 4$. This implies that the domain for F_3 is $0 \le x \le 2$, and the graph that results is a horizontal compression of the graph of *F*. It might help for students to build a table to investigate the values $F_3(0)$, $F_3(1/2)$, $F_3(1)$, $F_3(3/2)$, and $F_3(2)$.



The graph of F_3 on $[0,2]$

(d) Students must next consider $y = f(|t|)$ and must realize that it is an even function and that its domain requires $0 \le |t| \le 4$ or, in other words, $-4 \le t \le 4$. The graph of $y = f(|t|)$ is given below.



The graph of $f(|t|)$ on $[-4,-4]$

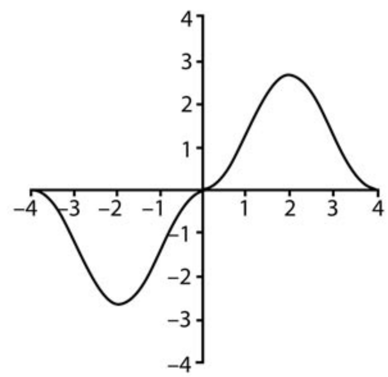
Here again, students may find it helpful to build a table of values to help them analyze function F_4. Alternatively, teachers might choose to point out that

$$F_4(x) = \begin{cases} F_2(x) & \text{for } x < 0 \\ F(x) & \text{for } x \geq 0 \end{cases}$$

since

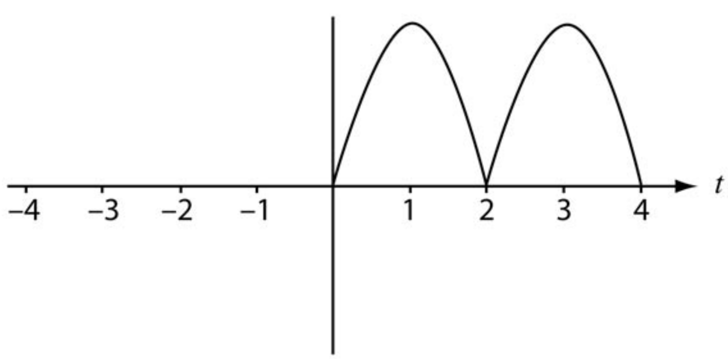
$$f(|t|) = \begin{cases} f(-t) & \text{for } t < 0 \\ f(t) & \text{for } t \geq 0 \end{cases}.$$

Hence, the graph of F_4 is just the graph of F_2 combined with the graph of F. The graph of F_4 appears below.



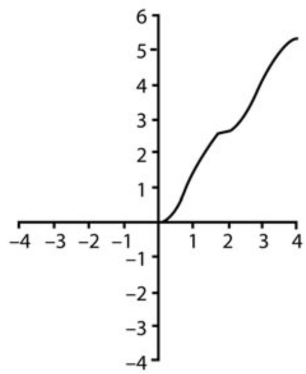
The graph of $f(|t|)$ on $[-4,4]$

(e) Finally, students should consider $F_5(x) = \int_0^x \left|f(t)\right| dt$. They should first recognize that the graph of $y = \left|f(t)\right|$ is



The graph of $\left|f(t)\right|$ on $[0,4]$

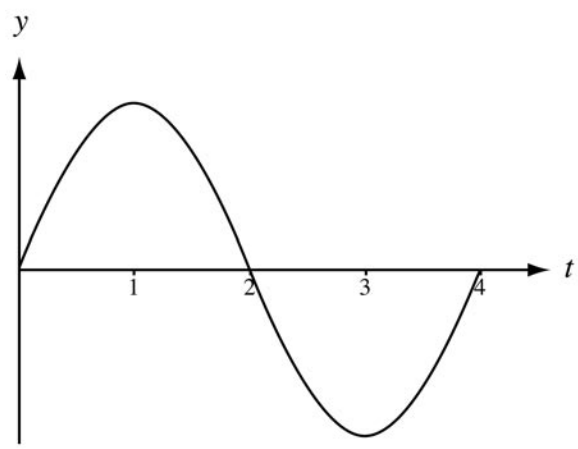
Students will find it helpful to build a table of values to help them analyze function F_5. The graph of F_5 appears below.



The graph of F_5 on $[0,4]$

A Return to the Worksheet

Once the antiderivative part of the Fundamental Theorem has been introduced, the class can return to this worksheet to probe further.



The graph of f on $[0,4]$

Questions 4 and 5—Round Two

Aha! By the antiderivative part of the Fundamental Theorem, if $F(x) = \int_0^x f(t)\,dt$, then $F'(x) = f(x)$. Likewise, if $G(x) = \int_2^x f(t)\,dt$, then $G'(x) = f(x)$. Finally, if $H(x) = \int_4^x f(t)\,dt$, then $H'(x) = f(x)$. No wonder the three graphs had the same behavior. Their slope functions are the same, and therefore they differ at most by a constant.

The maximum value of the functions *F*, *G*, and *H* occurs at what *x*-value(s)?

All three will have a local maximum value at $x=2$ because that is where the common slope function, $f(x)$ changes sign from positive to negative. This is the only critical point on the interval $[0,4]$; the absolute maximum value must also occur at $x=2$.

The minimum value of the functions *F*, *G*, and *H* occurs at what *x*-value(s)?

For each, the only critical value occurs at $x=2$, where a local maximum occurs. Therefore, the minimum must occur at an endpoint. From our previous work we know that functions *F* and *H* have a value of 0 at both endpoints and that function *G* has a value of –8/3 at both endpoints. Therefore, the absolute minimum value for each function will occur at $x=0$ and at $x=4$.

Functions *F*, *G*, and *H* increase on what interval(s)? Decrease on what interval(s)?

The common slope function is positive on $(0,2)$ and negative on $(2,4)$. Thus each of the functions *F*, *G*, and *H* increases on $[0,2]$ and decreases on $[2,4]$.

Added Question

Teachers can now ask a question about points of inflections that would not have been accessible to students earlier.

The graphs of *F*, *G*, and *H* have points of inflection at what *x*-value(s)?

We know that $F'(x) = f(x)$, $G'(x) = f(x)$, and $H'(x) = f(x)$. Therefore,

$$F''(x) = G''(x) = H''(x) = f'(x)\,.$$

The slope of *f* changes sign at $x=1$ and at $x=3$, and so the respective second derivatives change sign at these values. This is why inflection points for all three graphs occur at $x=1$ and at $x=3$.

Question 6—Round Two

From the discussion above, it is now very clear that the three functions *F*, *G*, and ***H*** differ at most by a constant and that their graphs must be (at most) vertical displacements of each other. We can even use the properties to determine those constants:

$$F(x) = \int_0^x f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^x f(t)\,dt = \frac{8}{3} + G(x)$$

and

$$F(x) = \int_0^x f(t)\,dt = \int_0^4 f(t)\,dt + \int_4^x f(t)\,dt = 0 + H(x) = H(x)\,.$$

This supports our earlier conjectures that $F(x) = H(x)$ and that $G(x) = F(x) - \frac{8}{3}$.

The above class activity also gives teachers the perfect opportunity to discuss the answer to the question many students pose when applying the antiderivative part of the Fundamental Theorem: "What happened to the constant?" This activity should help them truly understand that the constant lower limit *does* affect the vertical *placement* of the graph, but does *not* affect the ***shape*** of the graph.

Stage Three: Assessing and Improving Student Understanding and Confidence

Introduction

Goals of Stage Three

- Understand the derivatives of functions defined by integrals
- Understand the domains of functions defined by integrals
- Understand the substitution method for definite integrals
- Use the calculator to explore the FTC from numerical and graphical points of view
- Use the FTC to solve a wide variety of AP problems

Articles in Stage Three

- "Functions Defined by Integrals" by Ray Cannon
- "Exploring the FTC from Numerical and Graphical Points of View" by Mark Howell
- "Using the Fundamental Theorem of Calculus in a Variety of AP Questions" by Larry Riddle

Teachers need to select a wide variety of problems to help students develop their skills, process, and conceptual levels of understanding. This is the stage when students and teachers may discover misunderstandings. If that happens, we encourage you to reread previous chapters and select the activities that will reinforce and correct the misunderstood concepts.

The articles in this stage all give excellent examples of concepts that need some special attention in a first-year calculus class. Each chapter focuses on a special topic and how it is connected to the FTC. Teachers can choose their favorite activities and circle back to this section whenever the class schedule permits. Many teachers like to introduce AP free-response questions early in the course, even if they have not yet discussed every topic in a given problem. This allows them to return to the problem later in the year to reinforce the new topic and review previous topics.

Functions Defined by Integrals

Ray Cannon
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While students understand the part of the Fundamental Theorem that allows them to evaluate a definite integral using an antiderivative of the integrand, they often have difficulty dealing with the concept of a function defined by an integral. We give some exercises here that rely on the evaluation part of the FTC to help understand the antiderivative part of the FTC.

Example 1
Define $g(x) = \int_0^x \sin(t)dt$. What is $g'(x)$?

Solution: Since $-\cos(t)$ is an antiderivative for $\sin(t)$, we have

$$g(x) = \int_0^x \sin(t)dt = -\cos(t)\Big|_0^x = -\cos(x) + 1.$$

Thus $g'(x) = \sin(x)$.

Other examples in which the students can carry out the antidifferentiation may help them see that in fact the equation $g(x) = \int_a^x f(t)dt$ defines a function. It is also helpful to change the lower limit of integration while considering the same integrand to see that varying the lower limit changes the new function by a constant and so does not affect the derivative.

Now consider the case in which the students cannot antidifferentiate the integrand. Note that the following discussion assumes that the integrand does in fact have an antiderivative. The fact that every continuous function does have an antiderivative is one form of the FTC that is proven in Section IV of these materials. What we are hoping to accomplish here is to further students' *understanding* of the FTC statements.

Example 2
Let $g(x) = \int_1^x \sqrt{1+t^3}\,dt$. Then what is $g'(x)$?

Solution: We don't know a formula for an antiderivative of $\sqrt{1+t^3}$, but let's assume that $h(t)$ is one. Then $g(x) = h(t)\Big|_1^x = h(x) - h(1)$. Thus $g'(x) = h'(x) = \sqrt{1+x^3}$. Note the derivative of $h(1)$ is 0, since $h(1)$ is a number.

We repeat the pattern of these two examples to see how the chain rule works with functions defined by integrals.

Example 3
Let $g(x) = \int_0^{3x^2} \sin(t)\,dt$. What is $g'(x)$?

Solution: As in example 1, we have

$$g(x) = \int_0^{3x^2} \sin(t)dt = -\cos(t)\Big|_0^{3x^2} = -\cos(3x^2) + 1.$$

Now compute $g'(x) = \sin(3x^2) \cdot 6x$.

This example demonstrates that

> The derivative of $g(x)$ is the value of the integrand at the upper limit of integration times the derivative of the upper limit of integration.

Note that this procedure also works in example 1 and example 2, where the upper limit of integration is simply *x*.

Example 4
Let $g(x) = \int_1^{\sin(x)} \sqrt{1+t^3}\,dt$. What is $g'(x)$?

Solution: Again using $h(t)$ as we did in example 2,

$$g(x) = \int_1^{\sin(x)} \sqrt{1+t^3}\,dt = h(t)\Big|_1^{\sin(x)} = h(\sin(x)) - h(1).$$

Thus $g'(x) = h'\big(\sin(x)\big) \cdot \cos(x) = \sqrt{1+(\sin(x))^3} \cdot \cos(x)$. Note the careful use of the chain rule in the process of evaluating h'.

Students should be given many exercises like these four examples, and teachers may need to supplement the students' textbook to emphasize these ideas.

Domain Questions

Where are these rules valid? The rules are only valid on *intervals* on which f is continuous and which contain both limits of integration. Thus if $g(x)=\int_1^x \frac{1}{t^2-4}\,dt$, the domain of g is only the interval $-2 < x < 2$; it is **not** the set $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$. The domain of g is different than the domain of the integrand. Thus $g'(x)=\frac{1}{x^2-4}$ only for $-2 < x < 2$.

Here varying the lower limit of integration can change the domain of the function; for example, the domain of $h(x)=\int_5^x \frac{1}{t^2-4}\,dt$ is the set $(2,\infty)$, and $h'(x)=\frac{1}{x^2-4}$ only for $x > 2$.

Integration by Substitution

The technique of using a generic antiderivative, as we did in example 2 and example 4 with $h(t)$, can also be useful when dealing with integration by substitution. In terms of antidifferentiation, if $h' = f$, then $\int f(g(t))\cdot g'(t)dt = h(g(t))$. This last equation is verified simply by differentiating $h(g(t))$ to see that it is an antiderivative of $f(g(t))\cdot g'(t)$. When we consider definite integrals, however, we must note that a definite integral is a number and remember to change the bounds when doing integration by substitution.

Example of a Common Error

To evaluate $\int_0^3 \frac{2t}{1+t^2}\,dt$, let $u = 1+t^2$, so that

$$\int_0^3 \frac{2t}{1+t^2}\,dt = \int_0^3 \frac{1}{u}\,du = \ln|u|\Big|_0^3 = \ln(1+t^2)\Big|_0^3 = \ln(10)-\ln(1) = \ln(10).$$

The error here is that although the answer is correct, the second and third items, namely $\int_0^3 \frac{1}{u}\,du$ and $\ln|u|\Big|_0^3$, are meaningless. One correct way to evaluate the definite integral follows:

$$\int_{t=0}^{t=3} \frac{2t}{1+t^2}\,dt = \int_{u=1}^{u=10} \frac{1}{u}\,du = \ln|u|\Big|_1^{10} = \ln(10).$$

Here the variable of integration is shown not only by the differential, but also in the bounds on the integrand, which give the interval over which the definite integral is being computed.

Substitution for Definite Integrals

$$\int_a^b f(g(t))\cdot g'(t)\,dt = \int_{g(a)}^{g(b)} f(u)\,du.$$

Note that with $u=g(t)$, then $t=a$ corresponds to $u=g(a)$, and $t=b$ corresponds to $u=g(b)$. Indeed, if $h'=f$, then the derivative of $h(g(t))$ is $h'(g(t))\cdot g'(t)$, or $f(g(t))\cdot g'(t)$.

Thus the left-hand side is

$$\int_a^b f(g(t))\cdot g'(t)\,dt = h(g(t))\Big|_a^b = h(g(b)) - h(g(a)).$$

We have applied the evaluation part of the FTC to $f((g(t))\cdot g'(t)$ on the interval $[a,b]$.

Now on the right-hand side,

$$\int_{g(a)}^{g(b)} f(u)\,du = h(u)\Big|_{g(a)}^{g(b)} = h(g(b)) - h(g((a)).$$

Here we have applied the evaluation part of the FTC to $f(u)$ on the interval $[g(a),g(b)]$. Since both sides are equal to $h(g(b)) - h(g(a))$, we have

$$\int_a^b f(g(t))\cdot g'(t)\,dt = \int_{g(a)}^{g(b)} f(u)\,du.$$

Exploring the FTC from Numerical and Graphical Points of View

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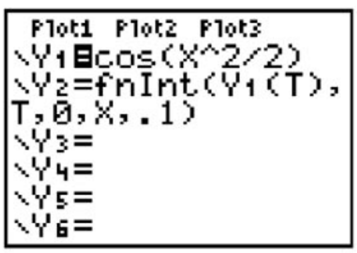
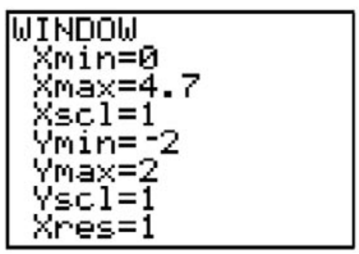
In this activity, students will explore the Fundamental Theorem of Calculus from numerical and graphical perspectives. The exploration will give students additional practice with functions of the form $F(x)=\int_0^x f(t)\,dt$. The given instructions are for students using a TI-83 calculator. A version of this activity using the TI-89 is available on AP Central.

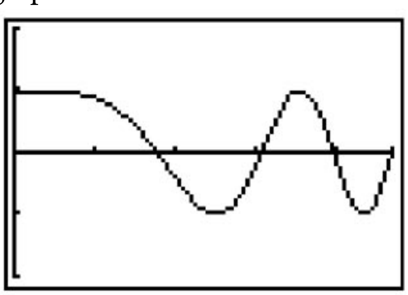
Define **Y1** and **Y2** in the **Y=** editor, and set up the viewing window as shown.
When defining **Y2**, select **9:fnInt** from the **MATH** menu, and select **1:Y1** from the **VARS-YVARS(1:Function)** menu.





Set your calculator to radian mode. Note that **Y2** is a function defined as a definite integral of **Y1**. That is, $\mathbf{Y_2}(\mathbf{X})=\int_0^{\mathbf{X}}\cos\left(\frac{T^2}{2}\right)dT$.

Make sure that **Y1** is the only function selected for graphing, and look at the graph. Your graph should look like the one shown:



If your graph is different, check that you've entered **Y1** correctly, that your window is correct, and that you are in radian mode.

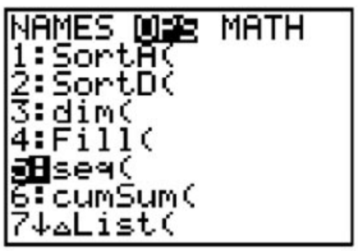
Now, you will create three lists. The first list, **L_1**, consists of **X** values from the graph screen. These will serve as inputs into **Y_1** and **Y_2**. To create **L_1**, follow these steps:

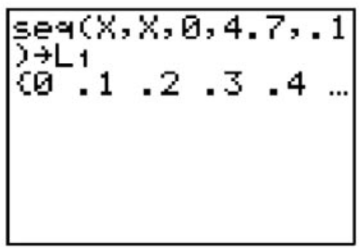
1. Select **`5:seq(`** from the **`LIST-OPS`** menu, as shown in the screen shot below.

2. Fill in the arguments as shown below.

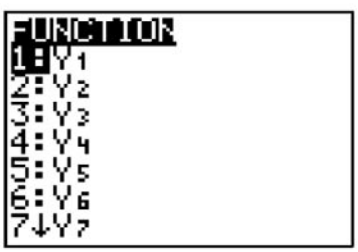
3. The **`STO>`** command is on the keypad, directly above the **`ON`** key.

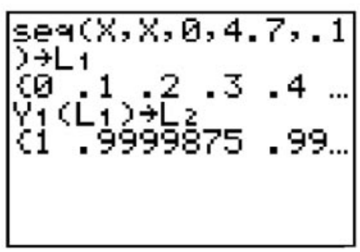
4. To enter **L_1**, type **`[2nd] 1`**.



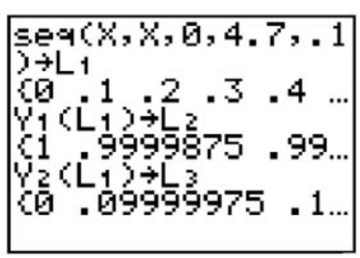


Now you will evaluate **Y_1** at each of the inputs in **L_1** and store the results in another list, **L_2**. Select **`1:Y1`** from the **`VARS-YVARS(1:Function)`** menu, and enter **L_1** as the argument to **Y_1**, as shown:





Then evaluate **Y_2** at each of the inputs in **L_1**, and store the results in list **L_3**. Select **`2:Y2`** from the **`VARS-YVARS(1:Function)`** menu, and enter **L_1** as the argument to **Y_2**, as shown:



Be prepared to wait over a minute for all the integrals to be calculated. You will examine these values in a table format and use them in calculations later on.

Special Focus: The Fundamental Theorem of Calculus

Each element of **L3** results from the evaluation of a definite integral. For example, when 0.1, the second value of **L1**, is plugged into **Y2**, the value of

$$\int_0^{0.1} \cos\left(\frac{T^2}{2}\right) dT = 0.09999975$$

results, which is the second value in **L3**.

1. Explain why the first value in **L3**, which results from evaluating **Y2(0)**, is 0.

____

Look at the graph of the integrand, **Y1(X)**. Use the graph to answer questions 2 and 3.

2. Why is the second value in **L3** greater than 0?

____

____

3. Why is the third value in **L3**, which represents $\int_0^{0.2} \cos\left(\frac{T^2}{2}\right) dT$, greater than the second value in **L3**, which represents $\int_0^{0.1} \cos\left(\frac{T^2}{2}\right) dT$?

____

____

____

Select **1:EDIT** on the ***STAT*** menu to see the table of inputs and outputs. Remember that **L1** contains the **X**-values; **L2** contains the values of the integrand, $\cos\left(\frac{T^2}{2}\right)$; and **L3** contains the values of the integral.

4. From the list of data values stored in **L3**, for what value of **X** (stored in **L1**) does **Y2**(**X**) $= \int_0^{\mathbf{X}} \cos\left(\frac{T^2}{2}\right) dT$ appear to reach its first local maximum?

____

5. Explain why the twentieth value in **L3** is smaller than the nineteenth value in **L3**. That is, why is $\int_0^{1.9} \cos\left(\frac{T^2}{2}\right) dT < \int_0^{1.8} \cos\left(\frac{T^2}{2}\right) dT$?

____

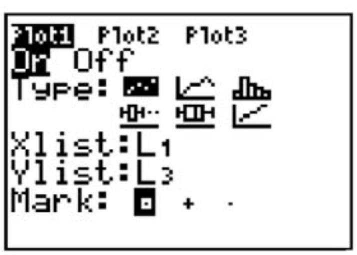
____

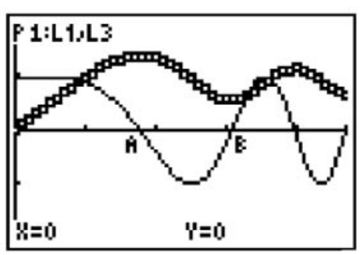
____

6. From the list of data values stored in **L3**, for what positive value of **X** does **Y2**(**X**) $= \int_0^{\mathbf{X}} \cos\left(\frac{T^2}{2}\right) dT$ appear to reach its first local minimum?

____

Select **1:Plot1** from the **STAT PLOT** menu and define **Plot1** as a scatterplot of **L3** versus **L1** as shown in the screen shot below. Select **Plot1** and **Y1** for graphing.





7. Select **2:zero** from the **CALC** menu and find the two smallest positive **X**-intercepts on the graph of **Y1**, indicated by A and B in the screen shot above.

(A)________________________ and (B) ________________________

8. Compare the answers to 7(A) and 7(B) with the answers to numbers 4 and 6 respectively. Explain why they are similar.

____

____

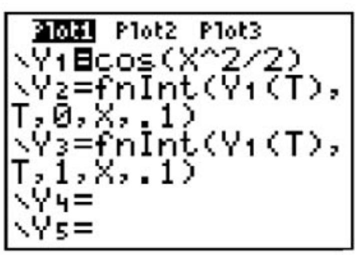
9. Find the next positive **X**-intercept of the graph of **Y1(X)** (close to **X** = 4)_____

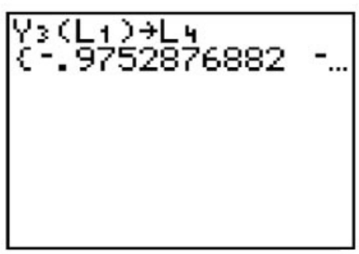
10. What happens with **Y2(X)** at the point found in number 9? ________________

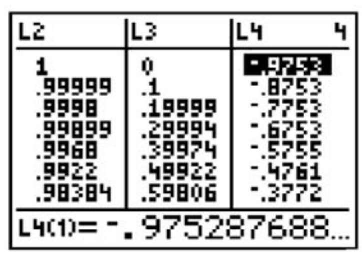
____

(Verify your answer numerically by looking at the table of values for **Y2(X)** in **L3**.)

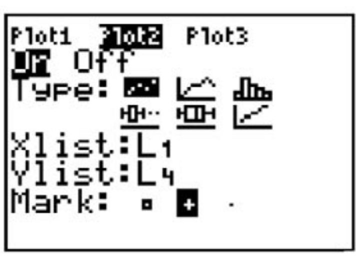
Go back to the **Y=** editor and define **Y3=fnInt(Y1(T),T,1,X,.1)**, thus changing the lower limit of integration from 0 to 1. So we have **Y3**(**X**) $= \int_1^{\mathbf{X}} \cos\left(\frac{T^2}{2}\right) dT$. Evaluate **Y3** at all the inputs in **L1**, and store the results in **L4**. See the screen shots below.

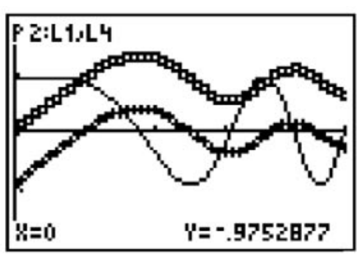






Define **Plot2** as a scatterplot of **L4** versus **L1**, using the plus symbol as the Mark. Select **Plot1**, **Plot2**, and **Y1** for graphing. Study the graph and table:





Note that the first element of **L4** represents $\texttt{Y3}(0) = \int_1^0 \cos\left(\frac{T^2}{2}\right) dT = -\int_0^1 \cos\left(\frac{T^2}{2}\right) dT$.

11. Why is the graph of **L4** versus **L1** **(Plot2)** below the graph of **L3** versus **L1** **(Plot1)**? (Remember, the only difference between the two is the lower limit of integration.)

____

____

12. It appears that **Plot1** and **Plot2** differ by a constant, i.e., that **Y3(X)-Y2(X)** is the same for all values of **X**. Find the value of this constant.

____

(You could calculate this in several different ways. If you get stuck, try expressing **Y3(X) - Y2(X)** in terms of an integral.)

13. Explain why plots 1 and 2 have the same shape.

____

____

Look at the table of values for **L3** and **L4**, and notice that the locations of the extrema are the same for both functions (as you would expect from an inspection of the scatterplots).

14. Look at the locations of the local minima and the local maximum on the graph of **Y1(X)**. At these values of **X**, describe what appears to happen with the concavity of the graphs from **Plot1** and **Plot2**.

____

____

15. Write an equation with a derivative on one side that shows how `Y1(X)` and `Y2(X)` are related.

____

16. Recall that the derivative of a function f at $x = a$ can be defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Approximate `Y2'(1)` by evaluating the expression `(Y2(1.1)−Y2(1))/0.1`.

____

Your answer should be close to the value of `Y1(1)`. Why?

____

____

17. Approximate `Y3'(1)` by evaluating the expression `(Y3(1.1)−Y3(1))/0.1`.

____

Again, your answer should be close to the value of `Y1(1)`. Why?

____

____

Teacher Notes, Answers, and Extensions

The version of the Fundamental Theorem covered by this activity says that if *f* is a continuous function on the closed interval [a,b], then the function $F(x) = \int_a^x f(t)dt$ is an antiderivative for *f*. Other ways to say this are $F'(x) = f(x)$, or $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

This is an important result in the history of the calculus, and one goal of this activity is to help students discover and appreciate its beauty. Students need to remain focused while carrying out the activity and think about each step.

Class Management

This activity takes at least one full class period (not counting the extension) to complete (45 to 50 minutes). If one class period is insufficient, students could complete the activity as a homework assignment. However, they should have at least created the lists and completed the first six questions in class. Teachers should work through the activity once before doing it with students. Depending on the calculator skills of your students, they could work independently, in groups, or with the teacher guiding the activity on an overhead display.

Calculus Prerequisites

This activity can be done any time after students have learned about a function defined by a definite integral, like $F(x) = \int_a^x g(t)dt$. Students should know that such functions will be positive when the integrand is a positive function and $x > a$, and negative when the integrand is a negative function and $x > a$. Although not essential, it is helpful if students know the addition property:

$$\int_a^b f(t)dt + \int_b^c f(t)dt = \int_a^c f(t)dt$$

Calculator Prerequisites

This activity makes extensive use of calculator functionality. Students must be able to define and graph functions and scatterplots, set up a viewing window, and find the zeros of a function on the graph screen. It introduces the calculator's integral function and shows how to create a list using the **`seq`** command. TI-83 users need to know how to enter the name of a function (retrieving **`Y1`**, for example, from the **`VARS`** menu) and the name of a list (entering **`L1`** for example, by typing **`2nd 1`**).

In the TI-83 version of the activity, an additional argument, 0.1, is supplied to the **`fnInt`** command when defining the function **`Y3`**. This argument specifies the guaranteed precision of the value returned by **`fnInt`**, and using 0.1 speeds up the plotting somewhat.

Preliminary Comments

This exploration gives students hands-on experience with functions of the form $F(x)=\int_0^x f(t)\,dt$ and emphasizes the connection between the function, *F*, and the integrand, *f*. That connection is at the heart of the Fundamental Theorem.

The hardest calculator work occurs before any questions are asked. If students get confused about what **`L3`** represents, it may be helpful to have them evaluate **`fnInt(Y1(X),X,0,0.1)`**, **`Y2(0.1)`**, and **`L3(2)`** on the **`HOME`** screen. All three of those expressions are equal.

The activity could be done using the calculator's built-in **`TABLE`** feature and by graphing **`Y2`** and **`Y3`** directly, instead of as scatterplots. However, it takes a long time to redraw the graphs or redisplay the tables for functions defined by integrals. Since students are going back and forth between the graph and the table several times in the activity, the values of the integral are stashed away in lists. This process has a side benefit. It allows you to do calculations with those values. In particular, you can approximate the derivative of **`Y2`** by forming difference quotients. A scatterplot of those approximations can then be compared with the graph of **`Y1`**. This extension appears after the answers and supports the conclusion students make at step 15.

Answers and Additional Comments

In the first six questions, students should begin to see that the function, **`Y2`**, is increasing wherever the integrand is positive and decreasing wherever the integrand is negative. This connection should resonate with students' experience with increasing/decreasing behavior of a function and the sign of its derivative.

1. The first value in **`L3`** represents $\int_0^0 \mathbf{Y1}(T)\,dT$ and so is 0. There is no area from 0 to 0.

2. The second value in **`L3`** is positive since the integrand is positive for all T in the interval [0,0.1].

3. The third value in **L3** is greater than the second since the integrand is positive for all **X** in the interval [0.1, 0.2].

4. 1.8

5. $\int_0^{1.9} \cos\left(\frac{T^2}{2}\right) dT < \int_0^{1.8} \cos\left(\frac{T^2}{2}\right) dT$ because the integrand is negative for all T in the interval [1.8,1.9]. Thus the function is decreasing.

6. 3.1

In questions 7 to 10, students should observe that the function **Y2** has a maximum where the integrand changes sign from positive to negative and a minimum where the integrand changes sign from negative to positive. Again, this connection is meant to recall students' experience with the locations of maximum and minimum points and their relationship with the sign of the derivative. Ask questions like, "Have you ever seen a situation in which one 'thing' reaches a maximum where another 'thing' changes sign from positive to negative? What are the two 'things'?"

Students make the connection explicit in question 15.

7. (A) 1.77245 and (B) 3.06998

8. At the first **X**-intercept, 1.77245, the integrand changes sign from positive to negative. Thus the function **Y2** has a local maximum there. We stop accumulating positive values of **Y1**, and start accumulating negative values at that point. Similarly, at **X** = 3.06998, the integrand changes sign from negative to positive. Thus we stop accumulating negative values and start accumulating positive values at that point, producing a local minimum on the graph of **Y2**.

9. 3.963327

10. **Y2** has a maximum there. Looking at the table of values for **L1**, **L2**, and **L3**, this occurs near **X** = 4.

Special Focus: The Fundamental Theorem of Calculus

11. Answers will vary. Focusing on **X** = 0, $\mathbf{Y_2}(0) = \int_0^0 \cos\left(\frac{T^2}{2}\right) dT$ is 0,

    since the upper and lower limits of integration are equal. However,

    $\mathbf{Y_3}(0) = \int_1^0 \cos\left(\frac{T^2}{2}\right) dT = -\int_0^1 \cos\left(\frac{T^2}{2}\right) dT < 0$ since the integral is positive. This

    puts the graph of **L4** below the graph of **L3**. Also see the answer to question 12.

12. The difference is a constant. Just calculate **Y3(0)−Y2(0)** = −0.975288.
    Students may notice that

    $$\mathbf{Y_3(X)-Y_2(X)} = \int_1^{\mathbf{X}} \cos\left(\frac{T^2}{2}\right) dT - \int_0^{\mathbf{X}} \cos\left(\frac{T^2}{2}\right) dT = -\int_0^1 \cos\left(\frac{T^2}{2}\right) dT$$

    and simply evaluate the latter integral or note that it is the negative of **L4(1)**.

The punch line is at hand in question 13! The Fundamental Theorem says that the two functions, **Y2** and **Y3**, have the same derivative, **Y1**. The derivative of a function tells you the slope on the graph of the function, and graphs that have the same slope everywhere have the same shape. In fact, **Plot1** and **Plot2** are members of the family of antiderivatives of **Y1**. Two antiderivatives of a function always differ by a constant.

13. **Plot1** and **Plot2** have the same shape since each accumulates the same amount of area from the graph of **Y1** as **X** increases. So each increases at the same rate where the integrand is positive, and each decreases at the same rate where the integrand is negative.

If you have addressed the connection between points of inflection and extrema of the first derivative (in addition to the more customary definition as points where the second derivative changes sign), then question 14 should give students yet another indicator that **Y1** is indeed the derivative of **Y2**.

14. The graphs of **Plot1** and **Plot2** both appear to have a change of concavity at the points where the graph of **Y1** has a local maximum or a local minimum. The points of inflection occur at **X** = 2.506629, **X** = 3.544908, and **X** = 4.341608. These are the extrema on the graph of **Y1**.

15. $\frac{d}{d\mathbf{X}}\mathbf{Y_2(X)} = \mathbf{Y_1(X)}$

Questions 16 and 17 lead students to approximate the derivative of Y_2 by forming difference quotients. The answers are not exactly the same as the output from Y_1. To get better agreement, you could have students form a difference quotient with a smaller step size as follows:

Evaluate $\dfrac{Y_2(1.0001)-Y_2(1)}{0.0001}$, and compare it to $Y_1(1)$. They agree to four digits to the right of the decimal!

16. $(Y_2(1.1)-Y_2(1))/0.1 = 0.851261$. $Y_1(1) = 0.877583$.

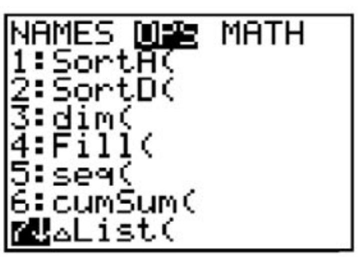
17. $(Y_3(1.1)-Y_3(1))/0.1 = 0.851261$. $Y_1(1) = 0.877583$.

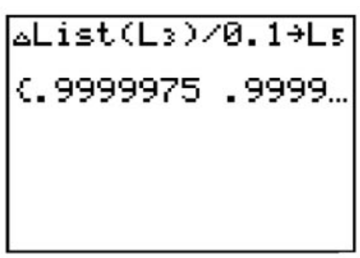
Notice that the difference quotients are identical. Y_2 and Y_3 are changing by exactly the same amount, an amount dictated by the magnitude of Y_1.

Extension

After students have seen that $\dfrac{d}{dX}Y_2(X) = Y_1(X)$, it is natural to gather more evidence in support of this conclusion. Because we have assembled data of inputs and outputs for Y_2, and we have a formula for Y_1, it is a relatively simple matter to form difference quotients for Y_2 and compare them to outputs from Y_1. This was done for a single value of X in question 16. Here's one way to do it for all the values of Y_2 that we've calculated.

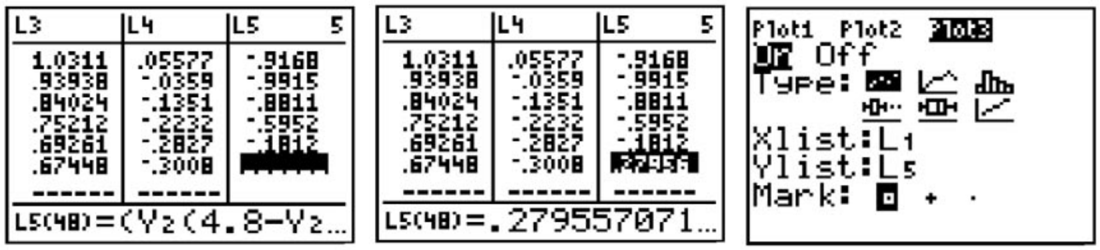
Form the difference quotients using the calculator's delta-list command. That command generates a new list, with each entry formed by subtracting adjacent pairs of the argument. Applying the delta-list command to the output list L_3, and then dividing each of the elements of that list by the differences in the input list, 0.1, gives a list of difference quotients. Select `7:ΔList(` from the `LIST OPS` menu. Store the difference quotients to L_5, so you can graph them later on. See the screen shots below.





Special Focus: The Fundamental Theorem of Calculus

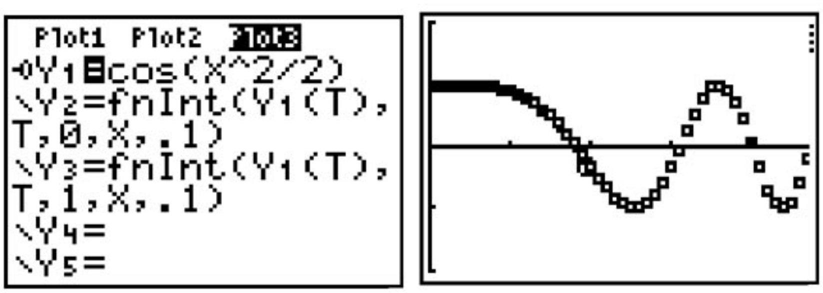
Note that there will be one entry fewer in L_5 than in L_1, L_2, L_3, or L_4. To compensate, you can enter one additional value at the end of L_5 "by hand." In the Stat Editor, position the cursor in L_5**(48)** (one cell past the last entry), press Enter, and enter the expression $(Y_2(4.8)-Y_2(4.7))/0.1$. This will even out the lists. Define **PLOT3** as L_5 versus L_1. In addition, deselect **PLOT1** and **PLOT2**. See the screen shots below.



If you fail to do this correctly, you will probably get the error **DIM MISMATCH**, telling you the lengths of the lists you are trying to graph don't match. If this happens, fix the lists so they have the same length.

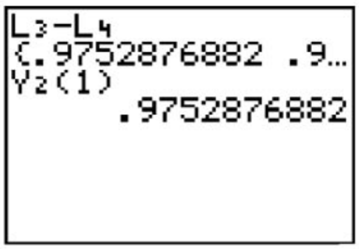
Note: It may be necessary to execute the **SetUpEditor** command in order to see L_5. On the **HOME** screen, select **5:SetUpEditor** from the **STAT** menu and press **ENTER**.

Finally, change the style of the graph of Y_1 from the **Y=** editor by positioning the cursor to the left of Y_1 and repeatedly pressing the **ENTER** key until the style shown resembles a circle with a little tail. That way, you'll be able to see the graph of Y_1 running overtop of the scatterplot. See the screen shots below.



Cool! What you just did was calculate 47 difference quotients for the function Y_2. Each difference quotient value in the list L_5 approximates the derivative of Y_2. You then graphed the integrand on top of the scatterplot of those 47 difference quotients and saw the graphs coincide. This provides more evidence in support of the Fundamental Theorem of Calculus.

You could repeat this extension, using the outputs from the function defined in L_4 instead of L_3. That is, form the difference quotients for L_4 this time and store to L_6. The results should be identical, since L_4 and L_3 differ by a constant. In fact, you could calculate just what that constant is in two different ways: by subtracting the lists or by evaluating $Y_2(1)$ (which is, of course, an integral). See the screen shot below.



Special Focus: The Fundamental Theorem of Calculus

Using the Fundamental Theorem of Calculus in a Variety of AP Questions

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Most students have no problem using the Fundamental Theorem of Calculus to evaluate a definite integral when they are able to find an antiderivative. Indeed, for many of them this is just an algorithm, not a "theorem." In the examples below, therefore, we use the Fundamental Theorem to solve other types of problems.

We use the version of the Fundamental Theorem in the form

$$\int_a^b f'(t)\,dt = f(b) - f(a),$$

or, in words, that **the definite integral of a rate of change gives the total change**. An equivalent expression is

$$f(b) = f(a) + \int_a^b f'(t)\,dt,$$

which allows us to compute the value of a function given an initial value and the rate of change. These two ideas appear frequently in AP Calculus questions. Compute position given velocity, or compute velocity given acceleration, or find the change in the amount of water in a tank given the rate at which the water has leaked from the tank, or find the temperature of an object given the rate at which the object is cooling or heating.

The second version of the FTC says that

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t).$$

In both versions, we assume all necessary differentiability and continuity conditions.

2000 AB4

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?

(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

Here we are given the rate at which water is pumped into a tank and the rate at which water leaks from the tank. How do we know these are rates of change? The word "rate" in the first two sentences is certainly a big clue. But the essential clue is the units: gallons per minute. These are the units of a rate of change of volume with respect to time, and so the Fundamental Theorem of Calculus says that the definite integral of each rate will give the corresponding change in volume. Part (a) asks for the change in the amount of water over the first 3 minutes due to the leak, and the FTC says that this change is given by $\int_0^3 \sqrt{t+1}\, dt$. The *evaluation* of this definite integral in the noncalculator section of the exam is another application of the Fundamental Theorem. For part (c) we need to use both rates of change due to the pump and the leak, as well as the initial value $A(0) = 30$. We know that $A'(t)$ = rate in − rate out = $8 - \sqrt{t+1}$, and again the FTC allows us to write

$$A(t) = A(0) + \int_0^t A'(x)\, dx = 30 + \int_0^t (8 - \sqrt{x+1})\, dx = 30 + 8t - \int_0^t \sqrt{x+1}\, dx.$$

Note the need to change the (dummy) variable for the integrand of the definite integral. The last part of the answer is a perfectly good expression for $A(t)$ and would have received full credit.

A common mistake to avoid in these types of questions is confusing $\sqrt{t+1}$ with the amount of water that leaked out, rather than the rate at which water leaked out. Students must also be reminded to consider the initial amount of 30 gallons at time $t = 0$ and how to use that value in the context of the FTC.

2002 AB2/BC2

The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour, and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time $t = 9$ there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round answer to the nearest whole number.

(c) Let $H(t) = \int_9^t \left(E(x) - L(x)\right) dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

This is again a Fundamental Theorem of Calculus question. How can we tell? We are given information about rates of change (rate at which people enter a park and rate at which people leave the park), and we are asked about the change in the number of people in the park. Notice the units of $E(t)$ and $L(t)$ are people per hour. Thus the definite integral of $E(t)$ will give the change in the number of people who enter the park, so in part (a) we need to evaluate $\int_9^{17} E(t)\, dt$ numerically with the calculator. We again need to use the FTC in part (c) to explain the meaning of $H(17)$ and to compute $H'(17)$. The second version of the FTC tells us immediately that

$$H'(t) = \frac{d}{dt}\int_9^t (E(x) - L(x))\, dx = E(t) - L(t),$$

and so $H'(17) = E(17) - L(17)$. Moreover, once we recognize that $E(t) - L(t)$ represents the rate of change of the number of people in the park, the FTC also allows us to conclude that $H(t)$ is the total change in the number of people in the park between

9 a.m. and time t. Since nobody was in the park when it opened, $H(t)$ therefore gives the number of people in the park at time t.

Another question similar in spirit to 2000 AB4 and 2002 AB2/BC2 is 2003 (Form B) AB2. In this question, the student is given the rate at which heating oil is pumped into a tank and the rate at which heating oil is removed from the tank. Questions are asked about how many gallons of heating oil are pumped into the tank over a given time interval and how many gallons are in the tank at a particular time. Both questions can be answered using the FTC. Also see 2002 (Form B) AB2/BC2, 2004 AB1/BC1, and 2004 (Form B) AB2.

2002 AB3

An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(d) What is the position of the object at time $t = 4$?

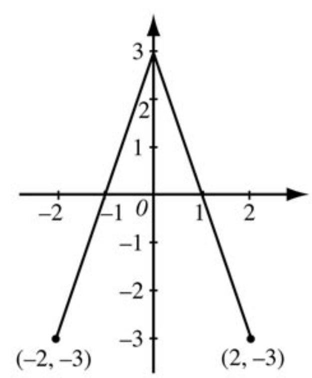
Here is a straightforward and common application of the FTC where we are given an initial position and the velocity function (rate of change of position). Because the question asks only for the position at a specific time and not the position function, a general antiderivative is not needed. Rather, the FTC can be used directly to find that

$$x(4) = x(0) + \int_0^4 x'(t)\,dt = x(0) + \int_0^4 v(t)\,dt = 2 + \int_0^4 \sin\left(\frac{\pi}{3}t\right)dt = 3.432,$$

where the evaluation is done on the calculator. This is a common type of question on the AP Exam.

2002 AB4/BC4

The graph of the function f shown to the right consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.



Graph of f

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.

Special Focus: The Fundamental Theorem of Calculus

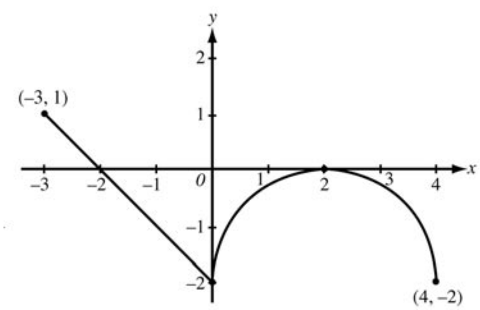
In this question we are given a function defined by a definite integral. One difficulty for students in this type of question is computing $g(-1) = \int_0^{-1} f(t)\,dt$ because the upper limit is less than the lower limit, and the integrand is described by a graph, not a symbolic function. It is probably best to have the students use the properties of definite integrals to switch the order of the limits to get the integral in "standard" form, then evaluate the new definite integral using the geometry of the triangle:

$$g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt = -\left(\frac{1}{2}\cdot 1\cdot 3\right) = -\frac{3}{2}.$$

The key to answering parts (b) and (c) is using the FTC to recognize that $g'(x) = f(x)$. Therefore information about the behavior of the function *g* can be read from the graph of *f*. For example, the function *g* is increasing on the interval $-1 < x < 1$ because $g'(x) = f(x) > 0$ on this interval, and the graph of *g* is concave down on $0 < x < 2$ because $g'(x) = f(x)$ is decreasing on this interval.

For similar questions, see 1995 AB6, 1995 BC6, 1997 AB5/BC5, 1999 AB5, 2002 (Form B) AB4/BC4, 2003 (Form B) AB5/BC5, and 2004 AB5.

2003 AB4/BC4



Graph of f'

Let *f* be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of *f*, consists of one line segment and a semicircle, as shown above.

(d)Find *f*(–3) and *f*(4). Show the work that leads to your answers.

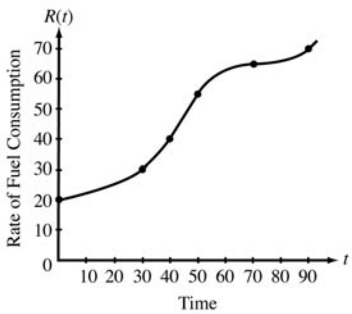
Here the Fundamental Theorem can be used to compute values for *f* since we are given an initial value and a rate of change via the graph of the derivative f'. Hence

$$f(0) - f(-3) = \int_{-3}^{0} f'(x)\,dx = \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}, \text{ and so } f(-3) = f(0) + \frac{3}{2} = \frac{9}{2};$$

$$f(4) - f(0) = \int_0^4 f'(x)\,dx = -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi,$$

and so $f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$.

2003 AB3



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

(d)For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in [the] answer.

The function $R(t)$ is the rate of change of the amount of fuel with units of gallons per minute. Therefore the FTC tells us that the definite integral of this rate of change is the total change in the amount of fuel, or more specifically in this particular question, the total amount of fuel in gallons consumed for the first b minutes.

A similar interpretation question was asked in 1999 AB3/BC3 and in 2004 (Form B) AB3/BC3.

1976 AB6

(a) Given $5x^3 + 40 = \int_c^x f(t)\,dt$
   (i) Find $f(x)$.
   (ii) Find the value of c.

(b) If $F(x) = \int_x^3 \sqrt{1+t^{16}}\,dt$, find $F'(x)$.

These two questions also make use of the Fundamental Theorem. The information in part (a) says that $5x^3+40$ is an antiderivative of $f(x)$. One can differentiate both sides and use the FTC to get $15x^2 = f(x)$. The value of c can be determined by letting $x = c$ in the given equation since the value of the definite integral will be 0. This gives $5c^3+40 = \int_c^c f(t)\,dt = 0$, and so $c = -2$.

Before the FTC can be applied to the function in part (b), it is necessary to switch the order of the limits:

$$F(x) = \int_x^3 \sqrt{1+t^{16}}\,dt = -\int_3^x \sqrt{1+t^{16}}\,dt, \text{ and so } F'(x) = -\sqrt{1+x^{16}}.$$

2003 Multiple-Choice AB23

$$\frac{d}{dx}\left(\int_0^{x^2} \sin\left(t^3\right)dt\right) =$$

(A) $-\cos\left(x^6\right)$ (B) $\sin\left(x^3\right)$ (C) $\sin\left(x^6\right)$ (D) $2x\sin\left(x^3\right)$ (E) $2x\sin\left(x^6\right)$

The evaluation of this derivative requires the use of the chain rule in addition to the FTC. These typically are difficult questions for students on the AP Exam unless they have had lots of previous practice with such situations. Indeed, only 29 percent of the AB students correctly solved this question on the 2003 exam. To get students to recognize this as a chain rule problem, first have them write the definite integral as a composition. Let $F(x) = \int_0^{x^2} \sin\left(t^3\right)dt$, and let $G(x) = \int_0^{x} \sin\left(t^3\right)dt$. Then students should recognize that $F(x) = G(x^2)$. Now use the chain rule to get $F'(x) = G'(x^2)\cdot 2x$ and the FTC to get $G'(x) = \sin(x^3)$. Putting these two results together yields $F'(x) = \sin\left((x^2)^3\right)\cdot 2x = 2x\sin\left(x^6\right)$. The correct answer is option (E).

Multiple-Choice Questions on the Fundamental Theorem of Calculus

1. 1969 BC12

If $F(x) = \int_0^x e^{-t^2}\,dt$, then $F'(x) =$

(A) $2xe^{-x^2}$ (B) $-2xe^{-x^2}$ (C) $\dfrac{e^{-x^2+1}}{-x^2+1} - e$ (D) $e^{-x^2} - 1$ (E) e^{-x^2}

2. 1969 BC22

If $f(x) = \int_0^x \dfrac{1}{\sqrt{t^3+2}}\,dt$, which of the following is FALSE?

(A) $f(0) = 0$
(B) f is continuous at x for all $x \ge 0$
(C) $f(1) > 0$
(D) $f'(1) = \dfrac{1}{\sqrt{3}}$
(E) $f(-1) > 0$

3. 1973 AB20

If F and f are continuous functions such that $F'(x) = f(x)$ for all x, then $\int_a^b f(x)\,dx$ is

(A) $F'(a) - F'(b)$
(B) $F'(b) - F'(a)$
(C) $F(a) - F(b)$
(D) $F(b) - F(a)$
(E) none of the above

4. 1973 BC45

Suppose $g'(x) < 0$ for all $x \ge 0$ and $F(x) = \int_0^x t\,g'(t)\,dt$ for all $x \ge 0$. Which of the following statements is FALSE?

(A) F takes on negative values.
(B) F is continuous for all $x > 0$.
(C) $F(x) = x\,g(x) - \int_0^x g(t)\,dt$
(D) $F'(x)$ exists for all $x > 0$.
(E) F is an increasing function.

Special Focus: The Fundamental Theorem of Calculus

5. 1985 AB42

$$\frac{d}{dx}\int_2^x \sqrt{1+t^2}\,dt =$$

(A) $\dfrac{x}{\sqrt{1+x^2}}$ (B) $\sqrt{1+x^2}-5$ (C) $\sqrt{1+x^2}$ (D) $\dfrac{x}{\sqrt{1+x^2}}-\dfrac{1}{\sqrt{5}}$

(E) $\dfrac{1}{2\sqrt{1+x^2}}-\dfrac{1}{2\sqrt{5}}$

6. 1988 AB13

If the function f has a continuous derivative on $[0,c]$, then $\int_0^c f'(x)\,dx =$

(A) $f(c)-f(0)$ (B) $|f(c)-f(0)|$ (C) $f(c)$ (D) $f(x)+c$

(E) $f''(c)-f''(0)$

7. 1988 AB25

For all $x>1$, if $f(x)=\int_1^x \frac{1}{t}\,dt$, then $f'(x)=$

(A) 1 (B) $\dfrac{1}{x}$ (C) $\ln x - 1$ (D) $\ln x$ (E) e^x

8. 1988 BC14

If $F(x)=\int_1^{x^2}\sqrt{1+t^3}\,dt$, then $F'(x)=$

(A) $2x\sqrt{1+x^6}$ (B) $2x\sqrt{1+x^3}$ (C) $\sqrt{1+x^6}$ (D) $\sqrt{1+x^3}$

(E) $\displaystyle\int_1^{x^2}\frac{3t^2}{2\sqrt{1+t^3}}\,dt$

9. 1993 AB41

$\dfrac{d}{dx}\displaystyle\int_0^x \cos(2\pi u)\,du$ is

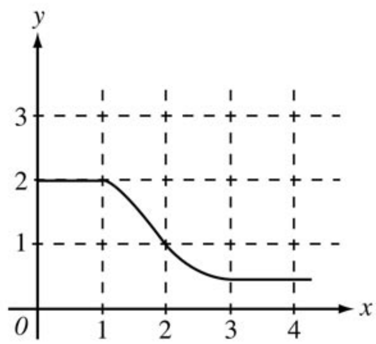
(A) 0 (B) $\dfrac{1}{2\pi}\sin x$ (C) $\dfrac{1}{2\pi}\cos(2\pi x)$ (D) $\cos(2\pi x)$ (E) $2\pi\cos(2\pi x)$

10. 1993 BC41

Let $f(x)=\int_{-2}^{x^2-3x} e^{t^2}\,dt$. At what value of x is $f(x)$ a minimum?

(A) For no value of x (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

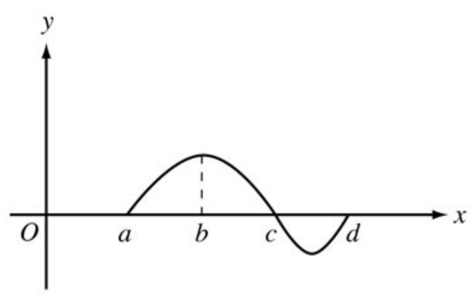
11. 1997 AB78



The graph of f is shown in the figure above. If $\int_1^3 f(x)\,dx = 2.3$ and $F'(x)=f(x)$, then $F(3)-F(0)=$

(A) 0.3 (B) 1.3 (C) 3.3 (D) 4.3 (E) 5.3

12. 1997 BC22



The graph of f is shown in the figure above. If $g(x)=\int_a^x f(t)\,dt$, for what value of x does $g(x)$ have a maximum?

(A) a (B) b (C) c (D) d

(E) It cannot be determined from the information given.

Special Focus: The Fundamental Theorem of Calculus

13. 1997 BC88

Let $f(x)=\int_0^{x^2}\sin t\,dt$. At how many points in the closed interval $\left[0,\sqrt{\pi}\right]$ does the instantaneous rate of change of f equal the average rate of change of f on that interval?

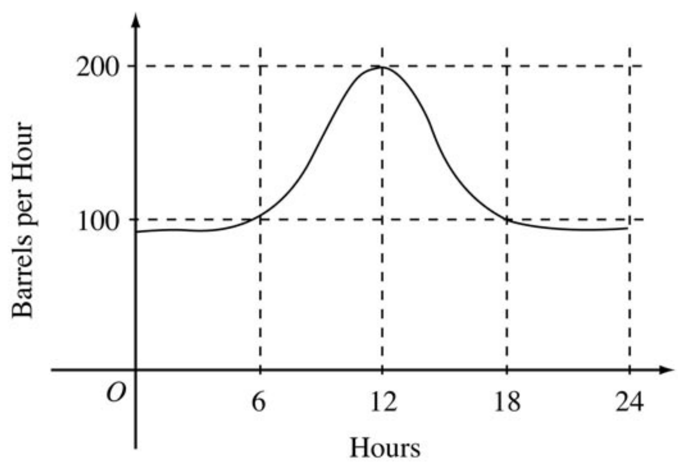
(A) Zero
(B) One
(C) Two
(D) Three
(E) Four

14. 1997 BC89

If f is the antiderivative of $\dfrac{x^2}{1+x^5}$ such that $f(1)=0$, then $f(4)=$

(A) -0.012 (B) 0 (C) 0.016 (D) 0.376 (E) 0.629

15. 1998 AB9



The flow of oil, in barrels per hour, through a pipeline on July 9 is given by the graph shown above. Of the following, which best approximates the total number of barrels of oil that passed through the pipeline that day?

(A) 500 (B) 600 (C) 2,400 (D) 3,000 (E) 4,800

16. 1998 AB11

If f is a linear function and $0 < a < b$, then $\int_a^b f''(x)\,dx =$

(A) 0 (B) 1 (C) $\frac{ab}{2}$ (D) $b - a$ (E) $\frac{b^2 - a^2}{2}$

17. 1998 AB15

If $F(x) = \int_0^x \sqrt{t^3 + 1}\,dt$, then $F'(2) =$

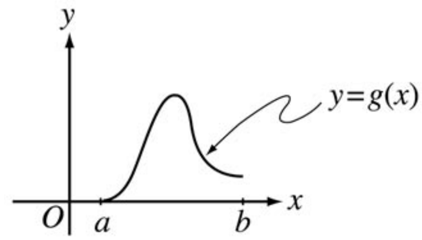
(A) –3 (B) –2 (C) 2 (D) 3 (E) 18

18. 1998 AB88

Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(1) = 0$ then $F(9) =$

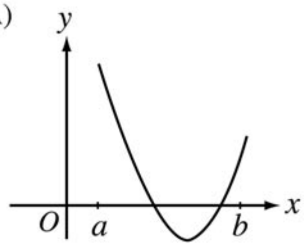
(A) 0.048 (B) 0.144 (C) 5.827 (D) 23.308 (E) 1,640.250

19. 1998 BC88

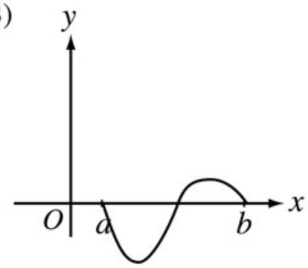


Let $g(x)=\int_a^x f(t)\,dt$, where $a \le x \le b$. The figure above shows the graph of g on $[a,b]$. Which of the following could be the graph of f on $[a,b]$?

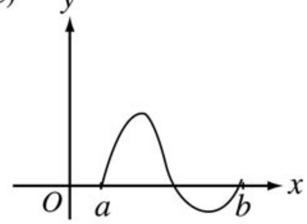
(A)



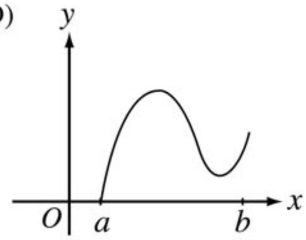
(B)



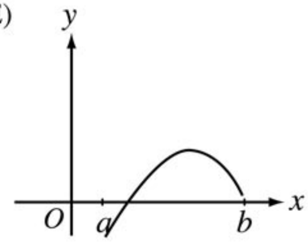
(C)



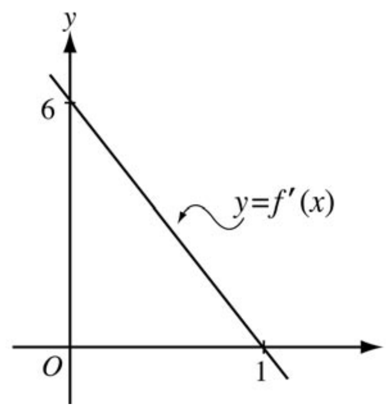
(D)



(E)



20. 2003 AB22



The graph of f', the derivative of f, is the line shown in the figure above. If $f(0) = 5$, then $f(1) =$

(A) 0 (B) 3 (C) 6 (D) 8 (E) 11

21. 2003 AB82/BC82

The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for $0 \le t \le 8$. Which of the following expressions gives the change in altitude of the balloon during the time the altitude is decreasing?

(A) $\int_{1.572}^{3.514} r(t)\,dt$

(B) $\int_{0}^{8} r(t)\,dt$

(C) $\int_{0}^{2.667} r(t)\,dt$

(D) $\int_{1.572}^{3.514} r'(t)\,dt$

(E) $\int_{0}^{2.667} r'(t)\,dt$

Special Focus: The Fundamental Theorem of Calculus

22. 2003 AB84

A pizza, heated to a temperature of 350 degrees Fahrenheit (°F) is taken out of an oven and placed in a 75°F room at time $t = 0$ minutes. The temperature of the pizza is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. To the nearest degree, what is the temperature of the pizza at time $t = 5$ minutes?

(A) 112°F (B) 119°F (C) 147°F (D) 238°F (E) 335°F

23. 2003 AB91

A particle moves along the x-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln\left(1 + 2^t\right)$. If the velocity of the particle is 2 at time $t = 1$ then the velocity of the particle at time $t = 2$ is

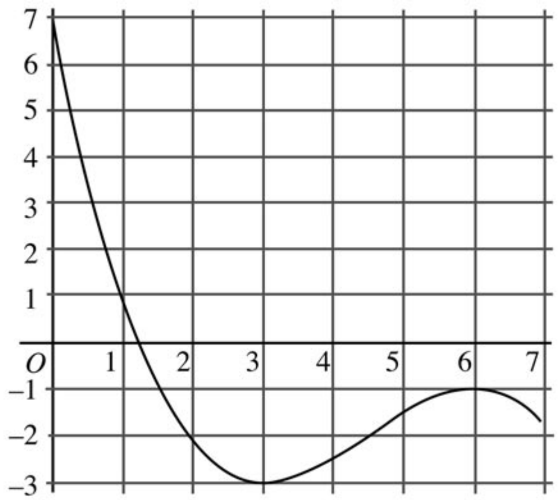
(A) 0.462 (B) 1.609 (C) 2.555 (D) 2.886 (E) 3.346

24. 2003 AB92

Let g be the function given by $g(x) = \int_0^x \sin\left(t^2\right) dt$ for $-1 \le x \le 3$. On which of the following intervals is g decreasing?

(A) $-1 \le x \le 0$
(B) $0 \le x \le 1.772$
(C) $1.253 \le x \le 2.171$
(D) $1.772 \le x \le 2.507$
(E) $2.802 \le x \le 3$

25. 2003 BC18



Graph of f

The graph of the function f shown in the figure above has horizontal tangents at $x = 3$ and $x = 6$. If $g(x) = \int_0^{2x} f(t)\,dt$, what is the value of $g'(3)$?

(A) 0 (B) –1 (C) –2 (D) –3 (E) –6

26. 2003 BC27

$$\frac{d}{dx}\left(\int_0^{x^3} \ln\left(t^2+1\right)dt\right) =$$

(A) $\dfrac{2x^3}{x^6+1}$ (B) $\dfrac{3x^2}{x^6+1}$ (C) $\ln\left(x^6+1\right)$ (D) $2x^3\ln\left(x^6+1\right)$

(E) $3x^2\ln\left(x^6+1\right)$

27. 2003 BC80

Insects destroyed a crop at the rate of $\dfrac{100e^{-0.1t}}{2-e^{-3t}}$ tons per day, where time t is measured in days. To the nearest ton, how many tons did the insects destroy during the time interval $7 \le t \le 14$?

(A) 125 (B) 100 (C) 88 (D) 50 (E) 12

28. 2003 BC87

A particle moves along the x-axis so that at any time $t \ge 0$, its velocity is given by $v(t) = \cos\left(2 - t^2\right)$. The position of the particle is 3 at time $t = 0$. What is the position of the particle when its velocity is first equal to 0?

(A) 0.411 (B) 1.310 (C) 2.816 (D) 3.091 (E) 3.411

Multiple-Choice Question Solutions

1. (1969 BC12)

(E) By the Fundamental Theorem of Calculus, if $F(x) = \int_0^x e^{-t^2}\,dt$, then $F'(x) = e^{-x^2}$.

2. (1969 BC22)

(E) Since $f(x) = \int_0^x \frac{1}{\sqrt{t^3+2}}\,dt$, $f(-1) = \int_0^{-1} \frac{1}{\sqrt{t^3+2}}\,dt = -\int_{-1}^0 \frac{1}{\sqrt{t^3+2}}\,dt < 0$ because the integrand is positive on the interval $[-1,0]$. Since $f(-1) < 0$, E is false. A is true because $f(0) = \int_0^0 \frac{1}{\sqrt{t^3+2}}\,dt = 0$. B is true because f is differentiable for all $x \ge 0$. C is true because the integrand is positive on the interval $[0,1]$. D is true because $f'(x) = \frac{1}{\sqrt{x^3+2}}$ by the FTC.

3. (1973 AB20)

(D) By the Fundamental Theorem of Calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$ since $F'(x) = f(x)$.

4. (1973 BC45)

(E) $F'(x) = xg'(x)$ with $x \geq 0$, and $g'(x) < 0 \Rightarrow F'(x) < 0 \Rightarrow F$ is not increasing. Hence E is false. A is true because $F(1) = \int_0^1 tg'(t)\,dt < 0$ since the integrand is negative on the interval (0, 1). B is true because by the FTC, the function *F* is differentiable for all $x > 0$. C is true using integration by parts. D is true because $F'(x) = xg'(x)$ for all $x > 0$.

5. (1985 AB42)

(C) This is a direct application of the Fundamental Theorem of Calculus:
$f'(x) = \sqrt{1+x^2}$.

6. (1988 AB13)

(A) By the Fundamental Theorem of Calculus, $\int_0^c f'(x)\,dx = f(c) - f(0)$.

7. (1988 AB25)

(B) Use the Fundamental Theorem of Calculus: $f'(x) = \dfrac{1}{x}$.

8. (1988 BC14)

(A) Use the Fundamental Theorem of Calculus and the chain rule:

$$F'(x) = \sqrt{1+\left(x^2\right)^3} \cdot \frac{d\left(x^2\right)}{dx} = 2x\sqrt{1+x^6}\,.$$

9. (1993 AB41)

(D) This is a direct application of the Fundamental Theorem.

10. (1993 BC41)

(C) Use the Fundamental Theorem and the chain rule: $f'(x) = \left(2x-3\right)e^{\left(x^2-3x\right)^2}$.
Therefore $f' < 0$ for $x < \dfrac{3}{2}$ and $f' > 0$ for $x > \dfrac{3}{2}$, so *f* has its absolute minimum at $x = \dfrac{3}{2}$.

11. (1997 AB78)

(D) $F(3) - F(0) = \int_0^3 F'(x)dx = \int_0^3 f(x)dx = \int_0^1 f(x)dx + \int_1^3 f(x)dx = 2 + 2.3 = 4.3.$

12. (1997 BC22)

(C) By the Fundamental Theorem, $g'(x) = f(x)$. The only critical value of g on (a, d) is at $x = c$, where g' changes from positive to negative. Thus the absolute maximum for g occurs at $x = c$.

13. (1997 BC88)

(C) By the FTC and the chain rule, $f'(x) = 2x\sin(x^2)$. For the average rate of change of f we need to determine $f(0)$ and $f(\sqrt{\pi})$. We have $f(0) = 0$ and $f(\sqrt{\pi}) = \int_0^{\pi} \sin t\,dt = 2$. The average rate of change of f on the interval is therefore $\dfrac{2}{\sqrt{\pi}}$. See how many points of intersection there are for the graphs of $y = 2x\sin(x^2)$ and $y = \dfrac{2}{\sqrt{\pi}}$ on the interval $\left[0, \sqrt{\pi}\right]$. There are two.

14. (1997 BC89)

(D) Both statements below follow from the Fundamental Theorem of Calculus:

$$f(x) = \int_1^x \frac{t^2}{1+t^5}\,dt;\ f(4) = \int_1^4 \frac{t^2}{1+t^5}\,dt = 0.376, \text{ or}$$

$$f(4) = f(1) + \int_1^4 \frac{x^2}{1+x^5}\,dx = 0.376.$$

15. (1998 AB9)

(D) Let $r(t)$ be the rate of oil flow as given by the graph, where t is measured in hours. The total number of barrels is given by $\int_0^{24} r(t)dt$. This can be approximated by counting the squares below the curve and above the horizontal axis. There are approximately five squares with area 600 barrels. Thus the total is about 3,000 barrels.

16. (1998 AB11)

(A) $\int_a^b f''(x)\,dx = f'(b) - f'(a) = 0$ since the derivative is constant for a linear function. (Alternatively, $f''(x) = 0$ for a linear function, so the value of the definite integral is 0.)

17. (1998 AB15)

(D) By the Fundamental Theorem of Calculus, $F'(x) = \sqrt{x^3 + 1}$, hence $F'(2) = \sqrt{2^3 + 1} = 3$.

18. (1998 AB88)

(C) $F(9) = F(1) + \int_1^9 F'(t)\,dt = F(1) + \int_1^9 \frac{(\ln t)^3}{t}\,dt = 5.827$.

19. (1998 BC88)

(C) From the given information, $g'(x) = f(x)$. We want a graph for *f* that represents the slope of the graph of *g* in the figure. The slope of *g* is 0 at *a* and *b*. Also the slope goes from positive to negative. This is true only for the graph of *f* in figure (C).

20. (2003 AB22)

(D) By the Fundamental Theorem of Calculus,

$$f(1) = f(0) + \int_0^1 f'(x)dx = 5 + \frac{1}{2} \cdot 1 \cdot 6 = 8.$$

21. (2003 AB82/BC82)

(A) Graph $r(t)$ on the interval $0 \le t \le 8$ and solve for where $r(t) = 0$. This occurs where $t = 1.572$ and $t = 3.514$, and $r(t) < 0$ between these two values. So the altitude is decreasing for $1.572 \le t \le 3.514$. By the Fundamental Theorem of Calculus, the change in altitude over this interval is the definite integral of the rate of change of the altitude over this interval, so the change in altitude is given by the definite integral in (A).

22. (2003 AB84)

(A) Let $T(t)$ be the temperature at time *t*. Then $T'(t) = -110e^{-0.4t}$. By the Fundamental Theorem of Calculus,

$$T(5) = T(0) + \int_0^5 T'(t)\,dt = 350 + \int_0^5 -110e^{-0.4t}dt = 112.217,$$

or 112 to the nearest degree.

Special Focus: The Fundamental Theorem of Calculus

23. (2003 AB91)

(E) By the Fundamental Theorem of Calculus,

$v(2) = v(1) + \int_1^2 v'(t)\,dt = 2 + \int_1^2 \ln(1+2^t)\,dt = 3.346$ using numerical integration on the calculator.

24. (2003 AB92)

(D) By the Fundamental Theorem of Calculus, $g'(x) = \sin(x^2)$. Graph this function on the interval $-1 \le x \le 3$ and look for where the derivative is negative. This happens a bit before $x = 2$ and at approximately $x = 2.5$. Thus the answer is (D). One can check that the endpoints of the interval in (D) are indeed where $g'(x) = 0$.

25. (2003 BC18)

(C) By the Fundamental Theorem of Calculus and the chain rule, $g'(x) = f(2x) \cdot 2$. Therefore $g'(3) = 2f(6) = 2(-1) = -2$.

26. (2003 BC27)

(E) By the Fundamental Theorem of Calculus and the chain rule,

$$\frac{d}{dx}\left(\int_0^{x^3} \ln(t^2+1)\,dt\right) = \ln\left((x^3)^2+1\right) \cdot \frac{d}{dx}(x^3) = 3x^2 \ln(x^6+1).$$

27. (2003 BC80)

(A) By the Fundamental Theorem of Calculus, the total change in the amount of crops is the definite integral of the rate of change in the amount of crops.

Hence the amount destroyed is $\int_7^{14} \frac{100e^{-0.1t}}{2-e^{-3t}}\,dt = 124.994$, or 125 to the nearest ton.

28. (2003 BC87)

(C) The velocity is first equal to 0 when $2 - t^2 = \frac{\pi}{2}$ or $t = \sqrt{2 - \frac{\pi}{2}} = 0.655136$.

By the Fundamental Theorem of Calculus,

$$s(0.655136) = s(0) + \int_0^{0.655136} v(t)\,dt = 3 + \int_0^{0.655136} \cos(2-t^2)\,dt = 2.816.$$

Stage Four: Confirming Conjectures with a Formal Proof

Introduction

Goals of Stage Four

- Determine one or more suitable proofs to discuss with your students
- Celebrate an important mathematical (and cultural) achievement with your students

Article in Stage Four

- "Proving the Fundamental Theorem of Calculus" by Lisa Townsley

Teachers need to choose whether or not to prove the Fundamental Theorem in their classes. Furthermore, those who do choose to prove the theorem have many options to consider. This chapter suggests some strategies and choices for those who wish to share a formal or informal proof with their students. Some teachers may prefer to give a proof in class that is different from the one in their text. This might illustrate an alternative approach or a teacher's favorite method of proof.

Special Focus: The Fundamental Theorem of Calculus

Proving the Fundamental Theorem of Calculus

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For the purposes of this paper, we refer to the Fundamental Theorem by its parts:

- Antiderivative part of the FTC:
  If f is continuous on an open interval containing a, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Evaluation part of the FTC:
  If f is continuous on $[a,b]$, and F is any antiderivative of f,
  then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Students sometimes employ circular reasoning when trying to understand the Fundamental Theorem. They see the antiderivative part of the FTC as saying, "Well, you start with a function, take an antiderivative and evaluate at x, then differentiate with respect to x. Naturally, you end up back at the original function!" Here they have absorbed the evaluation part of the theorem as a process and think of the antiderivative part in terms of that process. Of course, you need the very theorem you are proving to know that there is indeed an antiderivative to employ, but the student usually misses the circular nature of this reasoning.

There are several choices on how to prove the two parts of the Fundamental Theorem. An informal survey of 11 frequently used texts reveals two basic proofs for each part of the theorem. Many proofs of the Fundamental Theorem invoke the Mean Value Theorem at some point. In this paper, we offer a proof of the antiderivative part of the FTC with the evaluation part of the FTC as its consequence and also an alternative proof of the evaluation part of the FTC.

In order to prove the antiderivative part of the FTC, it is worthwhile to spend some class time on understanding what the theorem is saying. A discussion of all the "letters" present in the expression $F(x) = \int_a^x f(t)\,dt$ will help students to understand $F(x)$ as a function. Some questions to ask are:

- What is the input variable for $F(x)$?
- What is the domain of the function F?
- What is a picture of a computation of $F(x)$ for a specific f, a, and x?
- Where is t in this picture?
- As x varies, how does the picture change?
- Why is f required to be continuous?

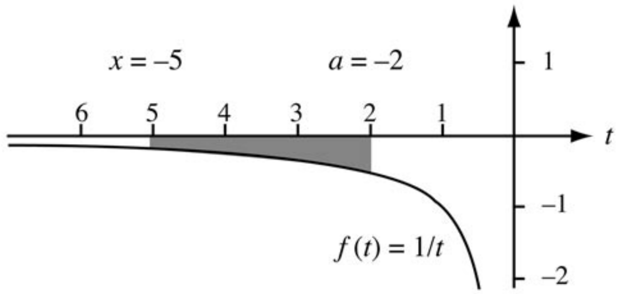

Figure 1. $F(x) = \int_{-2}^{x} \frac{dt}{t}$. Here we illustrate the case $x = -5$ with $F(-5) = \ln(5/2)$.

As an illustration of these questions, consider Figure 1 above. That the input variable of the function *F* is located in the upper limit of integration is a stumbling block for students, so the graph can help students see that the variable *x* is sliding along the *t*-axis, and there is a definite domain (in this example, the domain of *x* is $(-\infty, 0)$). As *x* varies in this example, the values of $F(x)$ are 0 (at $x = -2$), negative (for $-2 < x < 0$), and positive (for $x < -2$). Requiring *f* to be continuous helps students to recall the definition of the definite integral as a limit of Riemann sums, which are guaranteed to converge for continuous functions.

A computer algebra system or a graphing calculator allows one to graph the function $F(x)$ and then ask what properties of $F(x)$ are apparent. The students should be able to determine from the graph that the function $F(x)$ seems to be differentiable.

We can prove the antiderivative part of the Fundamental Theorem from first principles using the definition of the derivative. This is a nice approach, since the properties of the definite integral are used to simplify the result below. For convenience, assume $f(x) > 0$ on the open interval containing *a*. First notice that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt \; - \; \int_a^{x} f(t)\,dt}{h} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}.$$

To complete the proof, let $h > 0$ and bound the area representing $\int_x^{x+h} f(t)dt$ using the minimum and maximum values of the function *f* on the domain $[x, x+h]$, arriving at the inequality

$$h \cdot \left(\min_{t\in[x,x+h]} f(t) \right) \le \int_x^{x+h} f(t)dt \le h \cdot \left(\max_{t\in[x,x+h]} f(t) \right).$$

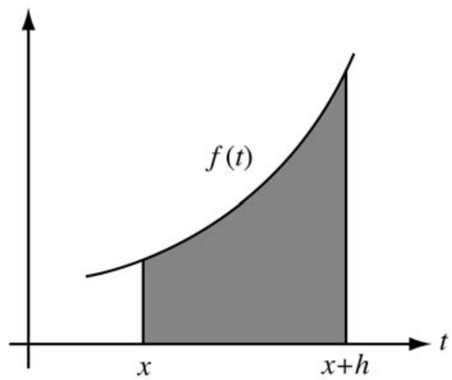

Figure 2. As $h \to 0^+$, we see $x+h \to x$, $\min\limits_{t\in[x,x+h]} f(t) \to f(x)$, and $\max\limits_{t\in[x,x+h]} f(t) \to f(x)$.

Then

$$\left(\min_{t\in[x,x+h]} f(t)\right) \le \frac{\int_x^{x+h} f(t)dt}{h} \le \left(\max_{t\in[x,x+h]} f(t)\right).$$

The limits of the left and right sides of the inequality are both equal to $f(x)$ as $h \to 0^+$ because *f* is continuous (see Figure 2), and so the Squeeze Theorem implies that

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\, dt}{h} = f(x).$$

For the limit as $h \to 0^-$, we can write

$$\lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = \lim_{k\to 0^+} \frac{F(x)-F(x-k)}{k} = \lim_{k\to 0^+} \frac{\int_{x-k}^{x} f(t)dt}{k},$$

where you can replace *h* by $-k$ with $k>0$. Using the same idea as before, but with *k* instead of *h*, we have

$$k \cdot \left(\min_{t\in[x-k,x]} f(t)\right) \le \int_{x-k}^{x} f(t)dt \le k \cdot \left(\max_{t\in[x-k,x]} f(t)\right).$$

Hence

$$\left(\min_{t\in[x-k,x]} f(t)\right) \le \frac{\int_{x-k}^{x} f(t)dt}{k} \le \left(\max_{t\in[x-k,x]} f(t)\right).$$

Now take the limit and use the Squeeze Theorem as before to obtain

$$\lim_{k\to 0^+} \frac{F(x)-F(x-k)}{k} = \lim_{k\to 0^+} \frac{\int_{x-k}^{x} f(t)\,dt}{k} = f(x).$$

Putting the two results together shows that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x).$$

The proof of the evaluation part of the Fundamental Theorem follows directly from the antiderivative part, although it is more symbolic than geometric, which can make it less satisfying to students. We begin with the function $F(x)=\int_a^x f(t)\,dt$, which we just proved is an antiderivative for *f*, and notice that we have $F(b)-F(a)=\int_a^b f(t)dt-\int_a^a f(t)dt=\int_a^b f(t)dt$. Now we want to show that this relationship is also true for any other antiderivative $G(x)$ of *f* (so $G'(x)=f(x)$). By the antiderivative part of the theorem, we know $F'(x)=f(x)=G'(x)$. Both F and G are continuous on $[a,b]$ since they are differentiable, and so $F(x)=G(x)+C$ by a direct corollary of the Mean Value Theorem (functions with the same derivative differ by a constant). Now since $F(b)-F(a)=\int_a^b f(t)dt$, and also

$$F(b)-F(a)=\big(G(b)+C\big)-\big(G(a)+C\big)=G(b)-G(a),$$

we finally arrive at $\int_a^b f(t)dt=G(b)-G(a)$ as desired for any antiderivative of *f*.

Stewart's *Calculus: Concepts and Contexts*[1] is an example of one text that proves the evaluation part of the FTC before a formal discussion of the Fundamental Theorem. Since students easily grasp the procedural nature of this theorem, this approach is certainly valid. The proof of the antiderivative part of the Fundamental Theorem follows later in the text. In this case, the proof of the evaluation part starts by building a Riemann sum from $F(b) - F(a)$ (recall in this exposition that F represents some unknown antiderivative of *f* and hence is continuous since it is differentiable). The proof then requires the clever addition of 0, many times. Whereas in the study of infinite

[1] James Stewart, *Calculus: Concepts and Contexts*, 3rd ed., Pacific Grove, California:Brooks/Cole, 2005.

series we sometimes find collapsing sums, here we have a difference of just two terms that expands instead. If we have a partition

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

of $[a,b]$ into n subintervals of width Δx, then

$$F(b) - F(a) = F(x_n) - F(x_0) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \ldots + F(x_1) - F(x_0).$$

Use the Mean Value Theorem on each pair to replace the difference $F(x_i) - F(x_{i-1})$ with $F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x$, where c_i is in the interval (x_{i-1}, x_i). Summing these and applying the limit as $n \to \infty$, which is equivalent to $\Delta x \to 0$, yields the definite integral form:

$$F(b) - F(a) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx.$$

A nice feature of proving the evaluation part of the Fundamental Theorem first is that the students' confusion about the dummy variable *t* versus the independent variable *x* is postponed until you approach the antiderivative part. The students may encounter difficulty, however, in interpreting an *expanding* sum and understanding the introduction of an index *i* and an indeterminate upper limit *n* of the sum.

Once you have completed the proof of the evaluation part of the Fundamental Theorem, you have several options on how to handle the antiderivative part of the theorem. One approach is discussed in the article "Functions Defined by Integrals." A formal proof can also be given using the definition of the derivative as explained above.

Here are some questions to ask in class after the completion of the proofs of the two parts of the Fundamental Theorem.

- At what point(s) in the proofs did we need the fact that *f* is continuous? (We need *f* continuous in the evaluation of limits when using the Squeeze Theorem.)
- At what point(s) in the proof did we use the Mean Value Theorem? (For most proofs of the evaluation part of the FTC, you use the Mean ValueTheorem on $F(x_i) - F(x_{i-1})$ in the Riemann sum, or you use the corollary about functions with the same derivative to deduce that two

antiderivatives differ only by a constant. A proof of the antiderivative part can also be given that uses the Mean Value Theorem for Definite Integrals. See Larson's *Calculus of a Single Variable*[2].)

- Looking at each part of the theorem, can you explain why we say differentiation and integration are inverse processes?

  ($\frac{d}{dx}\int_a^x f(t)dt = f(x)$, so differentiation after integration yields the original function. In reverse, we have $\int_a^b F'(x)dx = F(b) - F(a)$, so by integrating the derivative F' we get back to the function F (up to a constant).)
- Can we solve the differential equation $\frac{dF}{dx} = f$ for every continuous function f? What about the initial value?

  (Yes, the solution is $F(x) = F(a) + \int_a^x f(t)dt$, where $F(a)$ is the given initial condition.)
- How is the concept of *limit*, the foundation of both the integral and differential calculus, used in these proofs?
  (When we prove the antiderivative part, we use a limit as $h \to 0$ to compute the derivative using difference quotients, and for the evaluation part via the Mean Value Theorem, we compute the limit of a Riemann sum as $n \to \infty$.)

Whichever proofs you choose, the consideration of $\int_a^x f(t)\, dt$ as a *function* can provide opportunities for rich discussion. Identifying the domain of a particular function $\int_a^x f(t)\, dt$, understanding that the Fundamental Theorem says that continuous functions are in fact derivatives, and realizing that the theorem shows that we can find solutions to initial value differential equations even when a closed form of the antiderivative is not available—each of these requires a level of understanding that goes beyond the rote. Helping students understand the proofs you choose can contribute to the goal of helping them reach the level of understanding and appreciation we hope for them to achieve.

[2] Ron Larson, et al., *Calculus of a Single Variable: Early Transcendental Functions*, 3rd ed., Boston: Houghton Mifflin, 2003.

Stage Five: Historical Background and More Advanced Examples

Introduction

Goals of Stage Five
- Provide some historical background for the FTC
- Provide additional examples of using the FTC later in the course

Articles in Stage Five
- "Why Do We Name the Integral for Someone Who Lived in the Mid-Nineteenth Century?" by David Bressoud
- "Examples to Reinforce Concepts That Are Connected to the Fundamental Theorem of Calculus" by Steve Kokoska

David Bressoud's article provides some important background information about the historical importance of the FTC. Even experienced teachers may gain new insights about the relationship between familiar theorems and the definitions that mathematicians use.

Steve Kokoska presents a variety of examples that show how the FTC occurs in subsequent calculus, physics, probability, and statistics classes. These examples may be used as the basis for individual or group projects.

Why Do We Name the Integral for Someone Who Lived in the Mid-Nineteenth Century?

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When I ask my students to tell me what the Fundamental Theorem of Calculus (FTC) says, I usually get a response very similar to the definition given in the Wikipedia as of spring 2005 (http://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus): "The two central operations of calculus, differentiation and integration, are inverses of each other." In one sense, this is right on the mark. In another, it is very misleading.

The actual statement of the FTC comes in two parts that are true for "nice" functions:

Evaluation part of the FTC:
If *f* is continuous on $[a,b]$, and F is any antiderivative of *f*, then $\int_a^b f(x)\, dx = F(b) - F(a)$.

Antiderivative part of the FTC:
If f is continuous on an open interval containing a, then $\dfrac{d}{dx}\int_a^x f(t)\, dt = f(x)$.

The evaluation part says that if we know that f is the derivative of *F*, then integrating *f* gets us back to *F* (up to a constant). The antiderivative part says that if we integrate *f* and then differentiate the resulting function, we get back to the function *f* with which we started.

The problem with this understanding of the FTC is that most students think of integration as the inverse of differentiation: integration means reversing differentiation (up to the addition of a constant). Too often, the FTC is remembered as the definition of integration.

Of course, no modern calculus text actually defines integration this way. Most texts are careful to define the definite integral as the limit of Riemann sums:

> Given a function *f* and an interval $[a,b]$ on which *f* is bounded, the definite integral $\int_a^b f(x)\,dx$ is the limit of the Riemann sums, $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$, where $a = x_0 < x_1 < \cdots < x_n = b$, each c_k can be any

value in the interval $\left[x_{k-1}, x_k\right]$, and the limit is obtained by restricting the length of the longest subinterval. In other words, we can force $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$ to be as close as we wish to the value designated as $\int_a^b f(x)\,dx$ just by restricting the size of $x_k - x_{k-1}$ for all k.

What the FTC really says is that if you define the definite integral as the limit of Riemann sums, then it has the very desirable property of being the inverse operation to differentiation.

The power of calculus comes from the fact that two operations that appear to be doing very different things, reversing differentiation and taking limits of Riemann sums, are, in fact, linked. The FTC is this link. This is what students need to know about the FTC.

It was not always the case that textbooks defined integration as a limit of Riemann sums. Until the 1820s, mathematicians defined integration as the inverse operation of differentiation. This continued in textbooks until the mid-twentieth century. The most widely used English language calculus textbook from 1900 until well into the 1950s was Granville's *Elements of the Differential and Integral Calculus*. It defines integration as "having given the differential of a function, to find the function itself." In other words, integration is the inverse of differentiation. The definite integral is then defined as "the difference of the values of $\int y\,dx$ for $x = a$ and $x = b$." Though he never labels it as such, Granville does make the connection that constitutes the FTC. He explains how the definite integral can be used to evaluate the limit of a summation of the form $\sum f(x_i)\Delta x$ as the length of the subintervals (Δx) approaches zero. The link is still there. All that differs is the choice of which interpretation of the integral should be used as the definition.

The problem with defining the integral as the inverse of differentiation is that there are many functions for which we want to be able to evaluate a definite integral but for which there is no explicit formula for an antiderivative. If integration is antidifferentiation, then there are too many functions for which we do not have an easy answer to the question, What do we mean by an antiderivative?

In the 1820s, Cauchy saw that he needed a definition of integration that was independent of the idea of differentiation. For example, even though there is no closed form for the antiderivative of e^{-x^2}, we should be able to assign a value to

$$\int_0^1 e^{-x^2}\,dx.$$

Cauchy became the first to insist that the definite integral be defined as a limit of what would later become known as Riemann sums. Specifically, Cauchy used left-hand sums:

$$\int_a^b f(x)\,dx = \lim_{|x_k - x_{k-1}| \to 0} \sum_k f(x_{k-1})(x_k - x_{k-1}).$$

Mathematicians recognized the usefulness of this definition, but it was slow to appear in the popular textbooks.

The shift in textbooks to defining integration in terms of Riemann sums began with Richard Courant's *Differential and Integral Calculus*, first published in English in 1934. This is also the first time that the term Fundamental Theorem of Calculus was used in its modern sense. Courant realized that if the integral is defined as a limit of Riemann sums, then the fact that integration is the inverse operation to differentiation is something that requires proof and elevation to the level of a fundamental theorem. When George Thomas wrote the first edition of his influential calculus text in the late 1940s, he chose Courant's approach to the integral. Calculus texts since then have followed his example.

Mathematically, the definition chosen by Cauchy, Courant, and Thomas makes sense. Pedagogically, it carries a potential trap. Students at this stage in their mathematical development usually ignore formal definitions in favor of a working understanding in terms of the examples they have seen. Traditional texts move quickly from the definition of the integral to extensive practice in how to use antidifferentiation to accomplish integration. Students internalize integration as the inverse of differentiation, and so the FTC lacks meaning.

There are two practices that can be used to help counter this. The first is to spend time on the implications of the integral as a limit of Riemann sums before exhibiting the power of the FTC. This means familiarizing students with a variety of problems that can be formulated as limits of Riemann sums: areas and volumes, population counts derived from density functions, distances from velocities, and velocities from accelerations. Twenty years ago, the complexity of calculating these limits without the aid of the FTC

made this approach very difficult, but today we have calculators. A calculator that is not using a computer algebra system is using the Riemann sum definition to approximate the definite integral. Students can learn the power of the definite integral as a model for a wide class of problems, problems for which the simplest case requires multiplying two quantities (distance = velocity × change in time, mass = density × volume), but for which the first quantity varies as a function of a variable whose change is measured by the second quantity (velocity depends on time, density depends on location).
The second practice is to emphasize what is truly important about the FTC. It informs us that two very different ways of understanding the integral are, in fact, equivalent (or almost so). This is the true power of calculus. We can take a problem modeled by a definite integral that reflects the nature of the problem as a limit of a Riemann sum, and we can find an elegant solution to this problem using antidifferentiation. Students need to be fully aware that the power runs both ways. Given a problem that seems to call for an antiderivative:

$$\text{Solve for } f: \quad \frac{df}{dx} = e^{-x^2}, \quad f(0) = 0,$$

the Riemann sum definition can provide a solution:

$$f(x) = \int_0^x e^{-t^2}\, dt.$$

Students must realize that this is a meaningful solution.

So Where Does Riemann Come into This?

Riemann's definition of the integral was published in 1867 in a paper on Fourier series. Riemann wanted to determine when a function could be integrated. Cauchy had shown in the 1820s that every continuous function could be integrated. It is not too hard to see that there are also discontinuous functions that can be integrated, and the study of Fourier series had created situations where scientists needed to integrate discontinuous functions.

A simple example of an integrable discontinuous function is the signum function, $\operatorname{sgn}(x)$ which is 1 when x is positive, −1 when x is negative, and 0 when $x = 0$. We can integrate this function from −1 to 1 and get 0 or from −2 to 5 and get 3. Even though this function is discontinuous at $x = 0$, its definite integral is well defined over any interval. In fact, with just a little thought about what it means to integrate this function from 0 to x when x is negative, we see that

$$\int_0^x \operatorname{sgn}(t)\, dt = |x|.$$

Riemann wanted to know how discontinuous a function could be and still be integrable. To even begin to ask this question, he needed a very clear definition of what it means to be integrable. As he showed, his definition of integration allows for very strange functions. One of his examples is a function that makes a discontinuous jump at every rational number with an even denominator (every interval contains an infinite number of these), that is continuous at all other numbers, and that is integrable.[1]

The ability to integrate discontinuous functions creates problems for the antiderivative part of the FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$ Derivatives cannot have jumps. This statement is deeper than the Intermediate Value Theorem. If a function has a derivative, then it must be continuous, but if a function is a derivative, it does not have to be continuous. Nevertheless, even discontinuous derivatives are almost continuous. If the limit from the left and the limit from the right at a exist, then they must be equal. If we let f be Riemann's very discontinuous but still integrable function, then the derivative of $\int_a^x f(t)\,dt$ cannot exist at rational numbers with even denominators.

If f is continuous at every point in $\left[a,b\right]$, then it is true that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ at every x in (a,b). The remainder of the nineteenth century would be spent figuring out what could be said about this antiderivative part of the FTC when f is not continuous at every point in $\left[a,b\right]$.

Beyond Riemann

A derivative cannot have jumps, but it does not have to be continuous. One example is the derivative of

$$F(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

As long as x is not zero, we can use the standard techniques of differentiation to find its derivative:

$$F'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\left(\frac{-1}{x^2}\right)\cos\left(\frac{1}{x}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad x \neq 0.$$

When x is 0, we have to rely on the limit definition:

$$F'(0) = \lim_{h\to 0} \frac{F(h)-F(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

This function is differentiable at all values of x, but the limits from the left and right of $F'(x)$ as x approaches 0 do not exist. The derivative $F'(x)$ oscillates forever between 1 and −1 as x approaches 0, and so F' is not continuous at $x = 0$.

If we modify this function slightly:

$$G(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then we get a function that is still differentiable at every value of x, including $x = 0$, but the derivative is

$$G'(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - 2x^{-1} \cos\left(\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This derivative is not bounded near $x = 0$. For this function, the conclusion of the evaluation part of the FTC fails. Since G' is not bounded, the Riemann integral $\int_a^b G'(x)\,dx$ does not exist if 0 is in the interval $[a,b]$, so it does not equal $G(b) - G(a)$.

You can get around this difficulty for the function G by using improper integrals, only integrating up to a value ε near 0 and then taking the limit as ε approaches 0. But in 1881, Vito Volterra showed how to build a function (call it V) that is differentiable at every x and whose derivative stays bounded, but for which the derivative cannot be integrated.[2] Although the derivative stays bounded, the limit of the Riemann sums of the derivative does not exist, and we cannot get around the problem by using improper integrals. Volterra's function is an even stranger example of a function for which the

[1] You can find this function in my book, *A Radical Approach to Real Analysis* (Mathematical Association of America, 1994), pages 254–256.

evaluation part of the FTC is not true: $\frac{d}{dx}V(x)=v(x)$, but $\int_a^b v(x)\,dx$ does not exist, even if we allow improper integrals, and so it cannot equal $V(b)-V(a)$.

These problems would eventually lead to an entirely different definition of integration, the Lebesgue integral, first proposed by Henri Lebesgue in his doctoral thesis of 1902. It takes a very different approach from Riemann integration. Instead of dividing up the x-axis (the domain of f) and choosing a value of the function at some point in each subinterval, it divides up the y-axis (the range of f) and looks at the length of the set of x's that yield values of f in that subinterval of the range. For nice functions, the limits of Riemann sums and of Lebesgue sums are the same, but Lebesgue's approach turns out to be much more useful. In particular, the first statement of the FTC always holds: If f is the derivative of F, then the Lebesgue integral of f from a to b will always exist and will equal $F(b)-F(a)$.

I find it deliciously ironic that the standard definition of integration found in all calculus texts today, Riemann's definition, was created almost two centuries after the beginnings of calculus, was developed specifically to identify strange functions that would create problems for integration, and lasted barely over 30 years before it was superceded by a more inclusive and useful definition of integration, the Lebesgue integral. Despite the irony of this situation, I am not arguing to replace Riemann's definition. Students in first-year calculus are not ready for the subtleties of Lebesgue's approach. More importantly, Riemann's definition is useful for driving home the critical observation that integrals enable us to evaluate limits of sums of products.

But it is also worth dropping a few hints to your students that the FTC is, in fact, more complicated than it may look at first glance.

[2] Volterra's function is explained in Chae's *Lebesgue Integration*, 2nd ed. (Springer-Verlag, 1995), pages 46–47.

Special Focus: The Fundamental Theorem of Calculus

Examples to Reinforce Concepts That Are Connected to the Fundamental Theorem of Calculus

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When the Fundamental Theorem of Calculus is introduced, most students have plenty of straightforward practice evaluating definite integrals and using the inverse relationship between the process of differentiation and of integration:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

The following examples focus on applications that involve the FTC and usually appear later in a calculus, physics, probability, or statistics course.

1. The Fresnel function defined by $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$ can be used to measure the amount of reflection versus refraction. Even for dark, dense material, some reflection takes place, especially at sharp angles, and $S(x)$ can also be used to determine the color reflected off metal.

   (a) Use the Fundamental Theorem of Calculus to find the derivative $S'(x)$ of the Fresnel function.

   (b) Sketch a graph of $y = S'(x)$ on $[-4, 4]$. This is a little tricky by hand because of the t^2 in the argument of the sine. Since the derivative involves a common trigonometric function, you can be pretty sure the graph oscillates. However, the graph does not have a constant period. Use technology to help construct this graph and find the pattern for the zeros of the derivative—the values of x for which the derivative crosses the x-axis.

   (c) Use the derivative, $S'(x)$, and its graph to find the find the intervals on which the graph of $y = S(x)$ is increasing and the intervals on which the graph is decreasing. Even though the function $S(x)$ is an accumulation function, its derivative still provides the usual information about the graph of $y = S(x)$. The graph and pattern of zeros in part (b) lead to an interesting pattern of intervals here.

(d) We can learn more about the shape of the graph $y = S(x)$ by considering the second derivative. Find $S''(x)$ and the intervals on which the graph of $y = S(x)$ is concave up and the intervals on which the graph is concave down. The second derivative also involves a trigonometric function and leads to a similar pattern of intervals on which the Fresnel function is concave up and concave down.

(e) Put this information (increasing, decreasing, and concavity) together to sketch a graph of $y = S(x)$. Find $S(0)$ first, and be careful as you sketch the local maximum peaks and local minimum valleys as $|x|$ increases. What do you suppose happens to the values of $S(x)$ as $|x|$ increases? Use technology to sketch $y = S(x)$ and to find some values of $S(x)$ for large values of $|x|$. (Even with a fast calculator, this may take a while.)

2. Consider the sine integral function defined by $Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt$.

The integrand, $f(t) = \frac{\sin(t)}{t}$, is not defined at $t = 0$, but the limit is 1. Define $f(0) = 1$ to make f continuous everywhere. The integrand is called the Sine Cardinal function, Cardinal Sine function, or the sinc function. As you will see in this problem, the graph of the integrand exhibits damped harmonic oscillation, that is, harmonic oscillation with diminishing amplitude. The Sine Cardinal function is used in signal processing and Fourier analysis and is related to the spherical Bessel function of the first kind.

(a) Use the Fundamental Theorem of Calculus to find $Si'(x)$ on $[-10,10]$. Use technology to sketch a graph of $y = Si'(x)$. Approximate several of the zeros of the derivative. Try to find a pattern in the zeros.

(b) For $x > 0$, find the first five local maximum values and first five local minimum values. You should be able to solve this problem analytically: find the values of x such that $Si'(x) = 0$ and use a sign chart. Check your solutions using technology. Describe the behavior of these values (the maximum and minimum values). What characteristic of the graph of $y = Si(x)$ does this suggest? Do the same thing for $x < 0$.

(c) Unfortunately, there is no *nice*, inherent pattern in the inflection points. However, we want to find at least one of these. Find the coordinates of the first inflection point to the right of the origin.

(d) Use the information in parts (a) through (c) to sketch the graph of $y = Si(x)$. Then, use technology, if available, to confirm the behavior seen in the sketch.

3. Consider the error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. The integrand is the Gauss curve, and the error function is used extensively in probability and statistics calculations.

(a) Complete the following table of values for $erf(x)$.

x	1	2	3	4	5	6
erf(x)						

x	–1	–2	–3	–4	–5	–6
erf(x)						

(b) Using the table in part (a), describe the behavior of $erf(x)$ as $|x|$ gets large What does this suggest for the values of $\lim_{x \to \infty} erf(x)$ and $\lim_{x \to -\infty} erf(x)$? What characteristics of the graph of $y = erf(x)$ does this suggest?

(c) Use the Fundamental Theorem of Calculus to find the derivative, $erf'(x)$, of the error function.

(d) Sketch a graph of the derivative of the error function and use this graph to determine the intervals on which the graph of $y = erf(x)$ is increasing and decreasing, and where the graph is concave up and concave down.

(e) Find $erf(0)$ and use the information about increasing, decreasing, and concavity to sketch a graph of $y = erf(x)$.

4. The purpose of this problem is to combine the ideas of the Fundamental Theorem of Calculus and the chain rule.

(a) Consider an arbitrary function f and the function $g(x) = x^2$. Let $h(x) = (f \circ g)(x)$ and find $h'(x)$.

(b) Consider the function H defined by $H(x) = \int_0^{x^2} (t-4)e^{-t/10}\,dt$. Although this is still an accumulation function, this is slightly different than the previous three examples because the upper bound is x^2 (a function of x), not simply x. Rewrite H as a composition function (similar to part (a)) and find $H'(x)$.

(c) Use your result in part (b) to find the absolute maximum value and the absolute minimum value of the function H (if they exist).

(d) Use technology to sketch a graph of $y = H(x)$ and confirm your results in part (c).

Additional Problems

1. Suppose f is a continuous function such that

$$\int_0^x f(t)\,dt = x^2 \sin x - \int_0^x t^2 f(t)\,dt$$

for all x. Find an explicit formula for $f(x)$ and sketch a graph of $y = f(x)$.

2. Consider the function F defined by $F(x) = \int_0^x t\left|\sin(t)\right| dt$.

(a) Compute $F'(x)$ and use this to find the intervals on which the graph of $y = f(x)$ is increasing and the intervals on which the graph is decreasing.

(b) Find the absolute maximum value and the absolute minimum value of $y = f(x)$ (if they exist).

(c) Sketch a graph of $y = f(x)$.

3. Consider the function F defined by $F(x) = \int_1^x (\sin t + \ln t)\,dt$ for $x > 0$.

(a) Approximate the value(s) of x for which the graph of $y = f(x)$ has a local minimum value or local maximum value.

(b) Sketch a graph of $y = f(x)$.

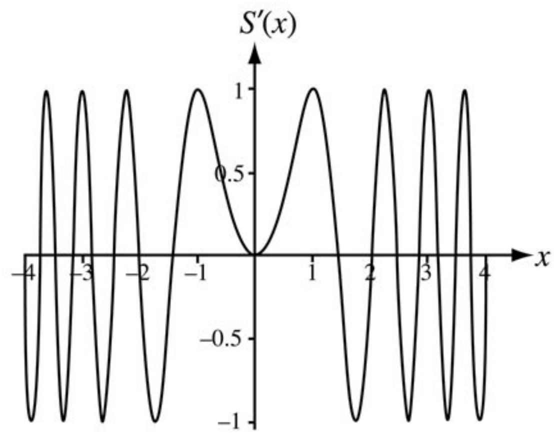
Special Focus: The Fundamental Theorem of Calculus

Solutions

1. (a) Using the FTC, the derivative of $S(x)$ is the integrand evaluated at x:

$$S'(x) = \sin\left(\frac{\pi x^2}{2}\right).$$

(b) Technology produces the following graph of $y = S''(x)$.



Notice that the graph oscillates faster and faster as $|x|$ increases. By using a calculator to find the zeros, students might recognize the numerical approximations for $\pm\sqrt{2k}$. The zeros of $S'(x)$ are $0, \pm\sqrt{2}, \pm\sqrt{4}, \pm\sqrt{6}, \ldots$. By hand, students can set $S'(x)=0$ and continue in the following manner:

$\sin\left(\frac{\pi x^2}{2}\right)=0$	(Use the expression for $S'(x)$.)
$\frac{\pi x^2}{2} = k\pi$	(Set the argument equal to $k\pi$.)
$x = \pm\sqrt{2k}$	(Solve for x.)
$x = 0, \pm\sqrt{2}, \pm\sqrt{4}, \pm\sqrt{6}, \ldots$	(Let $k = 0, 1, 2, 3, \ldots$.)

(c) Consider these zeros, the graph of $y = S'(x)$, and/or a sign chart to find the intervals on which $S(x)$ is increasing and decreasing.

Increasing: $\ldots,[-\sqrt{10},-\sqrt{8}],\ [-\sqrt{6},-\sqrt{4}],\ [-\sqrt{2},\sqrt{2}],\ [\sqrt{4},\sqrt{6}],\ [\sqrt{8},\sqrt{10}],\ldots$

Decreasing: $\ldots,[-\sqrt{8},-\sqrt{6}],\ [-\sqrt{4},-\sqrt{2}],\ [\sqrt{2},\sqrt{4}],\ [\sqrt{6},\sqrt{8}],\ldots$

(d) Find the second derivative using the chain rule: $S''(x) = \pi x \cos\left(\frac{\pi x^2}{2}\right)$.

Set $S''(x) = 0$ and solve for x: $\pi x \cos\left(\frac{\pi x^2}{2}\right) = 0$.

We conclude that $x = 0$ or $\cos\left(\frac{\pi x^2}{2}\right) = 0$.

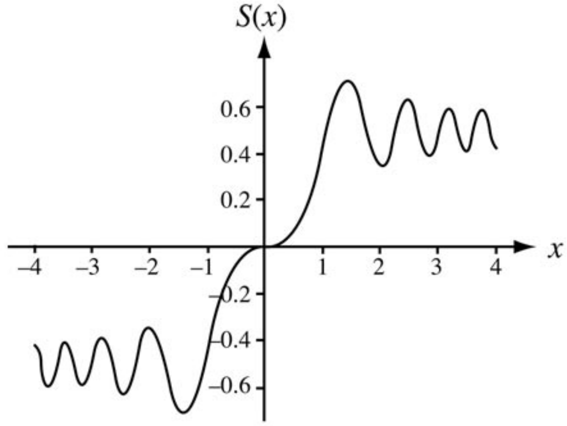
Since the cosine is zero when its argument is equal to $\pi/2 + k\pi$, for some integer k, $\frac{\pi x^2}{2} = \frac{\pi}{2} + k\pi$ for some integer k. In other words, $\frac{x^2}{2} = \frac{1}{2} + k$ for some integer k, and consequently, $x^2 = 1 + 2k$ for some integer k. This implies that $x = \pm\sqrt{1+2k}$ for some integer k. Finally, we conclude that $S''(x) = 0$ when $x = 0, \pm\sqrt{1}, \pm\sqrt{3}, \pm\sqrt{5}, \ldots$.

Consider these values and where the graph of $y = S'(x)$ is increasing/decreasing to find the intervals on which the graph of $y = S(x)$ is concave up/concave down.

Concave up: $\ldots,(-\sqrt{7},-\sqrt{5}),\ (-\sqrt{3},-\sqrt{1}),\ (0,1),\ (\sqrt{3},\sqrt{5}),\ (\sqrt{7},\sqrt{9}),\ldots$

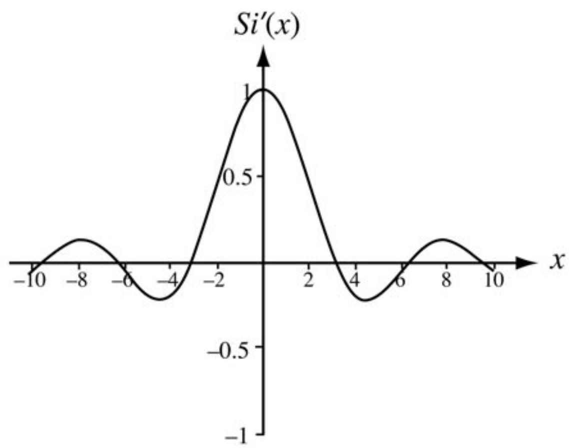
Concave down: $\ldots,(-\sqrt{9},-\sqrt{7}),\ (-\sqrt{5},-\sqrt{3}),\ (-1,0),\ (\sqrt{1},\sqrt{3}),\ (\sqrt{5},\sqrt{7}),\ldots$

(e) $S(0) = \int_0^0 \sin\left(\frac{\pi t^2}{2}\right) dt = 0$



Due to the behavior of the graph of $\sin\left(\frac{\pi x^2}{2}\right)$, the oscillations of the graph of S become flatter and flatter as $|x|$ increases. This means that the graph of $y = S(x)$ tends to *level out*. Even if you create a table of values for $S(x)$ as $|x|$ increases, it is hard to tell that as $x \to \infty$, $S(x) \to 0.5$, and as $x \to -\infty$, $S(x) \to -0.5$.

2. (a) Using the FTC, $Si'(x) = \frac{\sin(x)}{x}$. Using technology, here is a graph of $y = Si'(x)$.



Use technology to approximate some of the zeros:
$x \approx \pm 3.1416, \pm 6.2832, \pm 9.4248$.

These values look like multiples of π! By hand, we investigate when $Si'(x)=0$.
$Si'(x)=0 \Rightarrow \dfrac{\sin(x)}{x}=0 \;\Rightarrow \sin(x)=0$ and $x \neq 0 \;\Rightarrow x=\pm k\pi$ for some integer $k \neq 0$

$x=\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

(b) Consider these zeros and the graph of $y=Si'(x)$ to find local maximum and local minimum values.

For $x>0$:

Local maximum values:
$Si(\pi)=1.8594$, $Si(3\pi)=1.6748$, $Si(5\pi)=1.6340$, $Si(7\pi)=1.6161$,
$Si(9\pi)=1.6061$

Local minimum values:
$Si(2\pi)=1.4182$, $Si(4\pi)=1.4922$, $Si(6\pi)=1.5180$, $Si(8\pi)=1.5311$,
$Si(10\pi)=1.5390$

The local maximum values are getting smaller as x increases, and the local minimum values are getting bigger as x increases. Although it's hard to tell, even using technology, these values are converging, and the graph of $y=Si(x)$ has a horizontal asymptote at $y=\dfrac{\pi}{2}\approx 1.571$.

For $x<0$:

Local maximum values:
$Si(-2\pi)=-1.4182$, $Si(-4\pi)=-1.4922$, $Si(-6\pi)=-1.5180$, $Si(-8\pi)=-1.5311$,
$Si(-10\pi)=-1.5390$

Local minimum values:
$Si(-\pi)=-1.8594$, $Si(-3\pi)=-1.6748$, $Si(-5\pi)=-1.6340$, $Si(-7\pi)=-1.6161$,
$Si(-9\pi)=-1.6061$

The local maximum values are decreasing as x decreases, and the local minimum values are increasing as x decreases. Again, although it's hard to tell, even using technology, these values are converging, and the graph of $y = Si(x)$ has another horizontal asymptote at $y = -\frac{\pi}{2} \approx -1.571$.

(c) There are a couple of ways to find this inflection point.

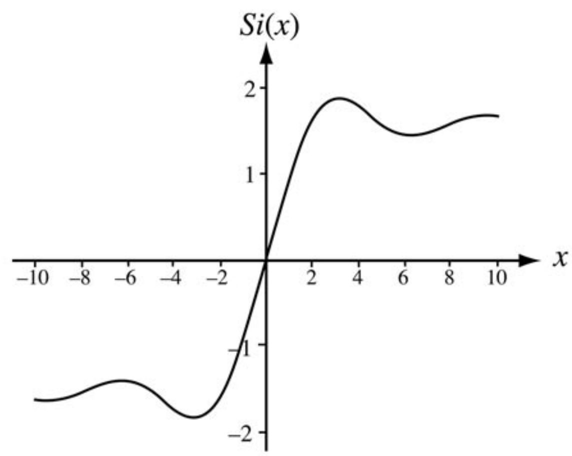
(i) The graph of $y = Si(x)$ changes from decreasing to increasing somewhere between $x = 4$ and $x = 5$, indicating an inflection point. Use technology to approximate this value.

(ii) Find $Si''(x)$, set $Si''(x) = 0$, and solve (numerically):

$$Si''(x) = \frac{x\cos(x) - \sin(x)}{x^2} = 0.$$

Either method produces the inflection point (4.49341, 1.65557).

(d) Using technology, here is a graph of $y = Si(x)$.



3. (a) Using a graphing calculator, here is the table of values.

x	1	2	3	4	5	6
$erf(x)$	0.84270	0.99532	0.99998	1.00000	1.00000	1.00000

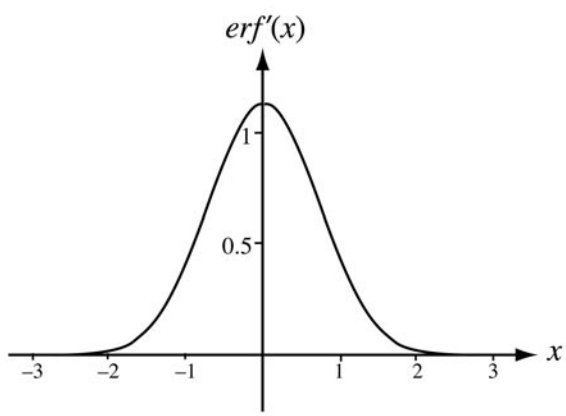
x	–1	–2	–3	–4	–5	–6
$erf(x)$	–0.84270	–0.99532	–0.99998	–1.00000	–1.00000	–1.00000

(b) The table in part (a) suggests as $x \to \infty$, $erf(x) \to 1$, and as $x \to -\infty$, $erf(x) \to -1$. Therefore, technology and the table suggest $\lim_{x\to\infty} erf(x) = 1$ and $\lim_{x\to-\infty} erf(x) = -1$.

These limits, if true, indicate the graph of $y = erf(x)$ has two horizontal asymptotes: $y = 1$ and $y = -1$.

(c) Using the FTC, $erf'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$.

(d) Using technology, here is a graph of $y = erf'(x)$:



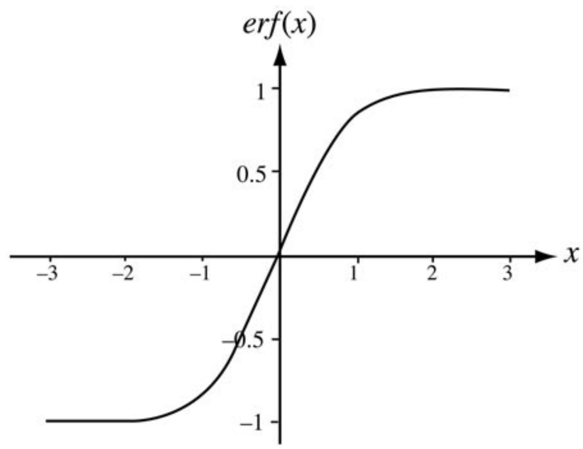
Since $erf'(x) > 0$ for all x, the graph of $y = erf(x)$ is always increasing.

The graph of $y = erf'(x)$ is increasing for $x < 0$ and decreasing for $x > 0$. Therefore, there is a change in concavity (an inflection point) at $x = 0$.

The graph of $y = erf(x)$ is concave up on the interval $(-\infty, 0)$ and concave down on the interval $(0, \infty)$.

(e) $erf(0) = \dfrac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2}\, dt = 0$.

Using technology, here is a graph of $y = erf(x)$. Notice that this graph confirms the results obtained in parts (a) through (d).



4. (a) $h(x)$ is defined as the composition of two functions. Although we do not have a specific rule for the function *f*, we can still simplify the derivative of *h*:

$h(x) = (f \circ g)(x) = f(g(x)) = f(x^2)$.

We use the chain rule to obtain that $h'(x) = \dfrac{d}{dx} f(x^2) = f'(x^2)(2x)$.

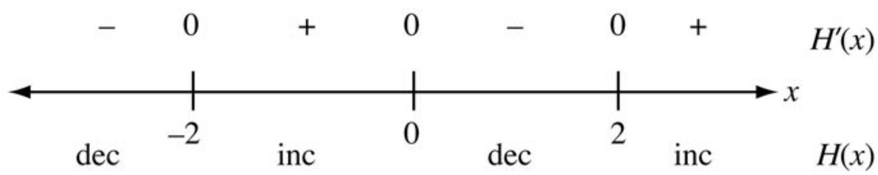
(b) Let $G(x) = x^2$ and $F(x) = \int_0^x (t-4)e^{-t/10}\, dt$.

Then $(F \circ G)(x) = F(G(x)) = F(x^2) = \int_0^{x^2} (t-4)e^{-t/10}\, dt = H(x)$.

As in part (a), $H'(x) = (2x)F'(x^2) = (2x)(x^2 - 4)e^{-x^2/10}$.

(c) $H'(x) = 0 \Leftrightarrow (2x)(x^2-4)e^{-x^2/10} = 0 \Leftrightarrow 2x = 0$ or $x^2 - 4 = 0$ or $e^{-x^2/10} = 0$.
Thus we conclude that $H'(x) = 0 \Leftrightarrow x = 0$ or $x = \pm 2$ (since $e^{-x^2/10}$ is never zero).

Since there are no x-values where the derivative does not exist, the only critical points are $x = 2, 0, -2$. The domain of H is all real numbers, and H is continuous everywhere (its derivative exists everywhere). The Extreme Value Theorem does not apply because the domain is not a closed interval. We are not guaranteed the existence of any absolute extrema.
Consider a sign chart:

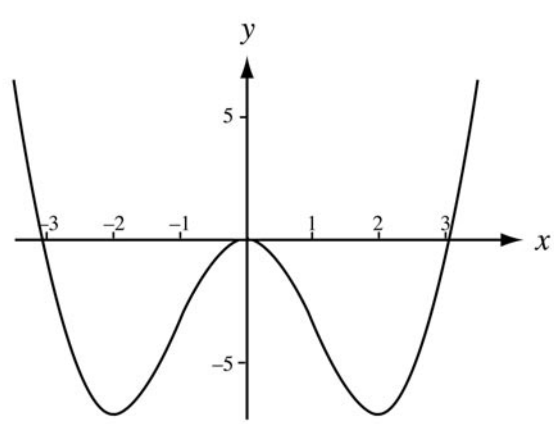


H has a local minimum at $x = -2$ and at $x = 2$ because $H'(x)$ changes from negative to positive at $x = -2$ and at $x = 2$. H has a local maximum at $x = 0$ because $H'(x)$ changes from positive to negative at $x = 0$.

H is increasing on $[-2,0] \cup [2,\infty)$.
H is decreasing on $(-\infty,-2] \cup [0,2]$.

$H(-2) = H(2) = 60 - 100e^{-2/5} = -7.032$, $H(0)=0$, and $H(-4)=H(4)>0$.
The absolute minimum value of H is $H(-2) = H(2) = 60 - 100e^{-2/5} = -7.032$.
H has no absolute maximum value.

(d) Using technology, here is a graph of $y = H(x)$:



Special Focus: The Fundamental Theorem of Calculus

Solutions to the Additional Problems

1. Take the derivative with respect to x of both sides of the given equation:

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right)=\frac{d}{dx}\left(x^2\sin x-\int_0^x t^2 f(t)\,dt\right).$$

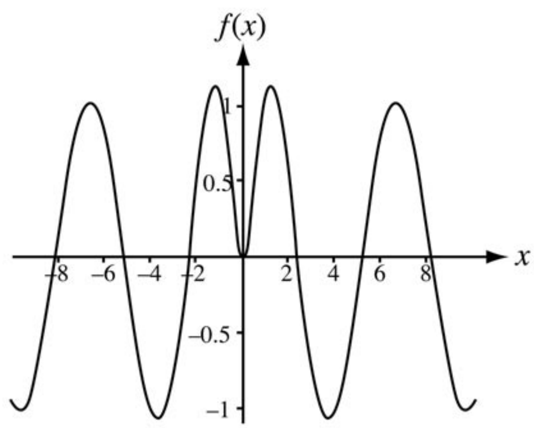
Use the antiderivative part of the FTC to evaluate each side of this equation:

$$f(x)=x^2\cos(x)+2x\sin(x)-x^2 f(x).$$

Solve this equation for $f(x)$:

$$f(x)\left(1+x^2\right)=x^2\cos(x)+2x\sin(x),$$

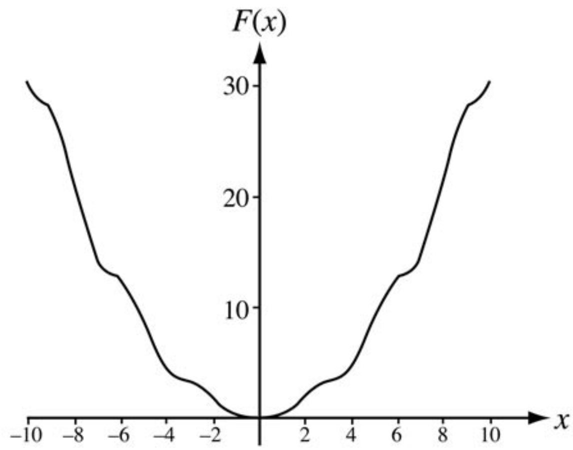
$$f(x)=\frac{x^2\cos(x)+2x\sin(x)}{1+x^2}.$$



2. (a) $F'(x)=x|\sin x|$.
F is increasing on $[0,\infty)$. F is decreasing on $(-\infty,0]$.

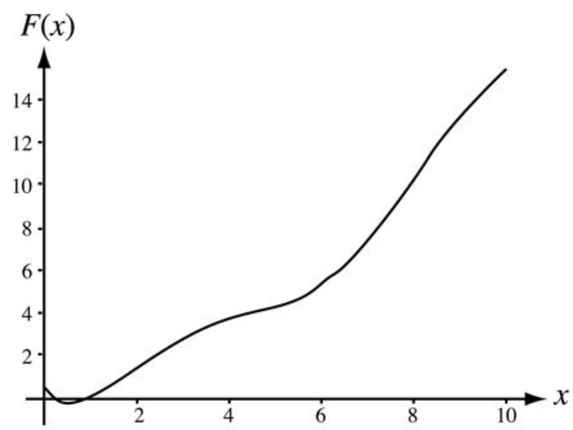
(b) Absolute minimum value is $F(0)=0$. There is no absolute maximum value.

(c)



3. (a) The local minimum value at $x = 0.57871$. There are no local maximum values.

(b)



Special Focus: The Fundamental Theorem of Calculus

Contributors

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About the Editor

Caren Diefenderfer joined the Hollins University faculty in 1977. Her two terms as chair of the mathematics and statistics department at Hollins were preceded by a term as chair of the division of natural and mathematical sciences. Caren became the Chief Reader for AP Calculus in 2004. Caren also has a long relationship with the Mathematical Association of America (MAA). She was the secretary of the MD-DC-VA section, currently serves on the Coordinating Council on Competitions, is the chair of the newly formed special interest group in quantitative literacy (SIGMAA QL), and has been a resource faculty member for the summer MAA PREP workshops on quantitative reasoning.

Benita Albert has taught AP Calculus BC for over 36 years at Oak Ridge High School. She has been an AP Exam Reader, Table Leader, and member of the AP Calculus Development Committee as well as a consultant for the College Board in the southeast. Recently, her committee work with the College Board has concentrated on Pre-AP educational services. Benita received one of the 2004 Edyth May Sliffe Awards for her efforts in educating several years' worth of students who scored well on the American Mathematics Competition (AMC).

David Bressoud is the DeWitt Wallace Professor of Mathematics at Macalester College. He was chair of the AP Calculus Development Committee from 2002 to 2005 and currently serves as the chair of the MAA's Committee on the Undergraduate Program. David's research interests lie in number theory and combinatorics with occasional forays into analysis (special functions, modular forms) and algebra. He enjoys the history of mathematics and has drawn on it in his books, which include: *A Course in Computational Number Theory*, *A Radical Approach to Real Analysis*, *Second Year Calculus: From Celestial Mechanics to Special Relativity*, and *Factorization and Primality Testing*.

Ray Cannon is a professor of mathematics at Baylor University. He started at the AP Exam Reading as a Reader in 1978 and served as the Chief Reader from 1992 to 1995. Ray continues to serve as an AP Calculus consultant for the College Board and leads AP institutes around the country. His major areas of research are analysis and the AP Calculus program.

Mark Howell is a teacher of AP Calculus and AP Computer Science at Gonzaga College High School. Mark has served the AP community as a workshop leader, AP Exam Reader, Table Leader, and Question Leader. He was also a member of the AP Calculus Development Committee from 1996 to 1999. Mark won the Siemens Award for Advanced Placement in 1999, the Tandy Technology Scholars Award the same year, and a state-level Presidential Teacher Award in 1993.

Stephen Kokoska is a professor of mathematics at Bloomsburg University. He is a consultant for the College Board and has been the BC Exam Leader at the AP Calculus Reading. Steve's research interests include the statistical analysis of cancer chemoprevention experiments. This work involves the development of mathematical models and innovative statistical methods to evaluate cancer-inhibiting or enhancing substances.

Steven Olson teaches mathematics at both Hingham High School and Northeastern University. Steve has been an AP Exam Reader since 1980 and has taught AP Calculus since 1970. He was a member of the AP Calculus Development Committee from 1993 to 1996 and continues to lead AP institutes around the country for the College Board. Steve is a state-level winner of the Presidential Award and a winner of the Siemens Award, and he was the New England Outstanding AP Teacher of the Year in 2001.

Larry Riddle is a professor of mathematics at Agnes Scott College. He served as the Chief Reader for the AP Calculus Exam from 2000 to 2003. Larry and his math students have created a Web site on women mathematicians, which has been featured on NPR's *51 Percent* and in numerous articles. His interest in the use of technology in teaching mathematics has led him to develop a software program called IFS Construction Kit for teaching about the fractals associated with iterated function systems.

J. T. Sutcliffe, who holds the Founders Master Teaching Chair at St. Mark's School of Texas, was an AP Calculus Reader and Table Leader from 1977 to 1988 and an Exam Leader from 1989 to 1992. She served on the AP Calculus Development Committee from 1979 to 1982. A recipient of the Presidential Award for Mathematics Teaching and Siemens and Tandy Technology Scholars Awards, she has served as a College Board consultant and led many AP institutes since 1980.

Lisa Townsley joined the faculty at Benedictine University in 1987. Lisa teaches a variety of mathematics courses, including calculus, linear and abstract algebra, and discrete mathematics. Her interests lie in writing about group cohomology, working with student researchers, and effectively employing technology in classroom discovery. Lisa has been a Question Leader at the AP Calculus Reading and is a frequent contributor to AP Central.

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