AP* CALCULUS RESOURCES

Applied Practice resources enable teachers to integrate both student activities and AP exam preparation into their course curriculum. For ***FREE*** downloadable activities and resources check out our Resource Library – again these materials are free to download, all you need is an account.

Explore the Online Resource Library:
https://www.appliedpractice.com/free-resources/

We are currently developing resources to support the AP Calculus classroom which align with the AP Curriculum Framework and the redesigned exam!

The new resource guides will be rolled out throughout the 2018-2019 school year. Each resource guide will include multiple choice and free response practice questions that address the topics and concepts outlined in the curriculum framework as well as an assessment made of multiple choice and free responses questions that are tied to the Learning Objective and Science Practices assessed on the actual AP exam.

Check Out our Products Page:
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Thanks,

The Applied Practice Team

L’Hosptial’s Rule

AP exam note: With the new redesign of the AP Calculus AB exam effective Fall 2016, L'Hospital's Rule is now covered on the AP Calculus AB exam.

Students are exposed to the concept of a limit early on in the AP Calculus curriculum. However, as students first study and understand limits, they are unable to implement L'Hospital's Rule due to their lack of exposure to differentiation techniques.

However, once students have learned basic differentiation, it is imperative that the concept of indeterminate forms and limits are spiraled back through the curriculum for students at various times during the year.

Indeterminate Forms

Recall from the study of limits that the forms $0/0$ and ∞/∞ are indeterminate. We were able to manipulate limit expressions to evaluate limits of this form in many cases. However, not all limits can be manipulated by the techniques from the limit chapter. In these cases, L'Hospital's Rule is especially useful.

L'Hospital's Rule

Let f and g be functions that are differentiable on an open interval (a,b) containing c, except possibly at c itself. Assume $g'(x) \neq 0$ for all x in (a,b), except possibly at c itself.

If $\lim\limits_{x\to c} f(x) = 0 \ \ and \ \ \lim\limits_{x\to c} g(x) = 0$, then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Note: This result also applies when the limits of the numerator and denominator both approach $\pm\infty$.

Important: The newly redesigned AP Calculus exams have a higher standard of communication and reasoning with theorems and definitions. To apply L'Hospital's Rule, students must first explicitly show that the conditions have been met. This requires students to examine the limits of the numerator and denominator SEPARATELY before applying L'Hospital's Rule.

Consider the two examples on the following page…

Example 1:

$$\lim_{x\to 1}\frac{e^{1-x}-x}{x^2-1}$$

<u>*Correct Method*</u>:

$$\lim_{x\to 1}(e^{1-x}-x)=0\ \ and\ \ \lim_{x\to 1}(x^2-1)=0.\ By\ L'Hospital's \Longrightarrow \lim_{x\to 1}\frac{e^{1-x}-x}{x^2-1}=\lim_{x\to 1}\frac{-e^{1-x}-1}{2x}=-1$$

<u>*Incorrect Method:*</u>

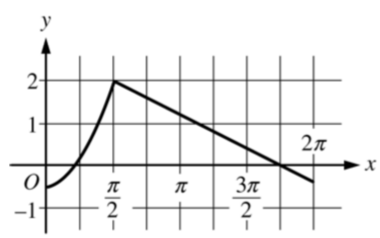
$$\lim_{x\to 1}\frac{e^{1-x}-x}{x^2-1}=\frac{0}{0}.\ By\ L'Hospital's \Longrightarrow \lim_{x\to 1}\frac{e^{1-x}-x}{x^2-1}=\lim_{x\to 1}\frac{-e^{1-x}-1}{2x}=-1$$

Students that write " $=\frac{0}{0}$ " will NOT earn full credit on free response questions requiring L'Hospital's Rule.

Example 2: (From the 2018 AP Calculus administration)

Let f be the function defined by $f(x)=e^x\cos x$.

Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right)=0$. The graph of g', the derivative of g, is shown below. Find the value of $\displaystyle\lim_{x\to\pi/2}\frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

<u>*Correct Method*</u>: (ALL was REQUIRED to earn full credit)

Since g is differentiable, then g is continuous. (Hence $\lim_{x\to\frac{\pi}{2}} g(x)=g\left(\frac{\pi}{2}\right)=0$)

$$\lim_{x\to\frac{\pi}{2}} f(x)=0,\ \lim_{x\to\frac{\pi}{2}} g(x)=0.\ \ By\ L'hospital \Longrightarrow \lim_{x\to\frac{\pi}{2}}\frac{f(x)}{g(x)}=\lim_{x\to\frac{\pi}{2}}\frac{f'(x)}{g'(x)}$$

$$=\frac{e^{\frac{\pi}{2}}\cos\left(\frac{\pi}{2}\right)-e^{\frac{\pi}{2}}\sin\left(\frac{\pi}{2}\right)}{2}=\frac{-e^{\frac{\pi}{2}}}{2}$$

Multiple Choice Practice

1. $\lim\limits_{x\to -2} \dfrac{x^3+x^2-x+2}{3x^2+5x-2}$

(A) $\frac{1}{3}$ (B) $\frac{-3}{7}$ (C) -1 (D) Does Not Exist

2. $\lim\limits_{x\to 0} \dfrac{cos(2x+\pi)+x+1}{e^{x^2}+2x-1}$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) Does Not Exist

3. $\lim\limits_{x\to 3} \dfrac{ln(x^2-x)-\ln(6)}{(x+4)(x-3)}$

(A) $\frac{1}{56}$ (B) $\frac{5}{56}$ (C) $\frac{7}{8}$ (D) $\frac{35}{8}$

4. $\lim\limits_{x\to -1} \dfrac{e^{1+x}+x}{cos\left(\frac{\pi x}{2}\right)}$

(A) $\frac{-4}{\pi}$ (B) $\frac{1}{\pi}$ (C) $\frac{4}{\pi}$ (D) -2

5. $\lim\limits_{h\to 0} \dfrac{ln(e-h)-\ln(e)}{h}$

(A) $\frac{-1}{e}$ (B) $\frac{1}{e}$ (C) -1 (D) 1

x	-7	-5	-3
$f(x)$	0	-3	0
$f'(x)$	6	-1	9

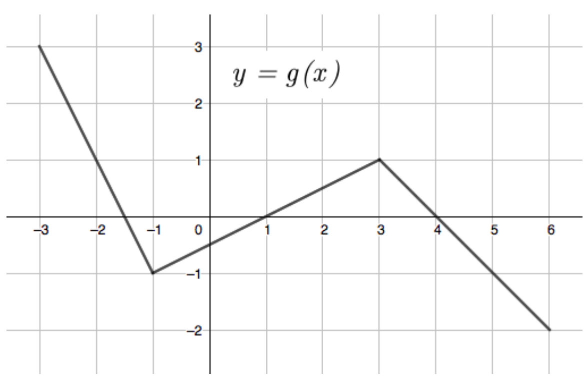
The table above gives selected values of a twice-differentiable function $f(x)$. Use the table to answer questions $6-7$.

6. $\displaystyle\lim_{x\to -3}\frac{f(2x-1)}{x^2-9}$

(A) -3 (B) -2 (C) -1 (D) Does Not Exist

7. $\displaystyle\lim_{x\to -5}\frac{f\big(f(x)\big)}{3x+15}$

(A) $\frac{1}{3}$ (B) -3 (C) 3 (D) Does Not Exist



8. The graph of $g(x)$ is above. Use the graph to find the following limit, if it exists:

$$\lim_{x\to 4}\frac{g(x-3)}{x^2-2x-8}$$

(A) $\frac{1}{12}$ (B) $-\frac{1}{6}$ (C) 3 (D) Does Not Exist

Multiple Representations – *L'Hosptial's Rule*

Activity: Multiple Representations

L'Hospital's Rule can be assessed in several ways on the AP Exam. Students should be able to recognize and use L'Hospital's Rule numerically, graphically and algebraically. Many textbooks fall short in providing student exercises that are similar in nature to what students will encounter on the AP exam.

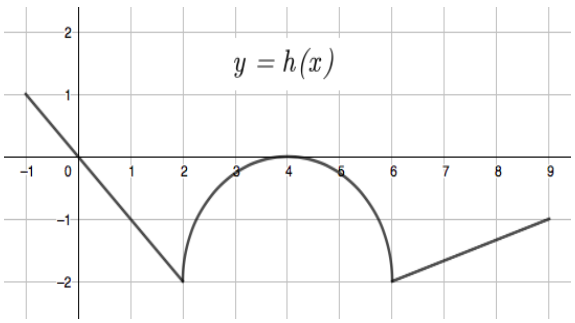
The following are two separate multiple representation activities for L'Hospital's Rule. Each page allows students to engage with a variety of potential problems on the AP exam that manifest in unique ways.

The representations/types of problems on each page are as follows:

Top Left: Graphical
Top Right: Numerical
Bottom Left: Algebraic
Bottom Right: Algebraic/Differential Equations

Students should approach these problems as they would a free response problem on the AP exam. Make sure that students explicitly check the conditions for L'hospital's Rule PRIOR to applying it. The scoring standards follow the activity.

A graph of $h(x)$ is below.



$$\lim_{x \to 4} \frac{h(x)}{x-4} =$$

Selected values of the differentiable functions f and g are given in the table below. Use the table to evaluate the following limit.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	3	0	-2

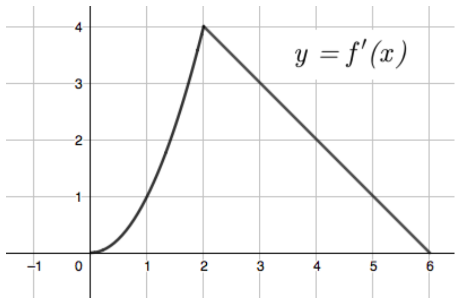
$$\lim_{x \to 2} \frac{f(x) - x^2}{g(x^2 - x)} =$$

$$\lim_{x \to 1} \frac{e^{x^2-1} - x}{\cos(\pi x) + sin\left(\frac{\pi x}{2}\right)}$$

The differential equation for the curve $y = f(x)$ is given by $\frac{dy}{dx} = \frac{2x}{y+1}$. It is known that $f(2) = 1$ and f is twice differentiable.

$$\lim_{x \to 2} \frac{f(x) + x - 3}{\ln(2x-3)}$$

$f(x)$ is a differentiable function with the graph of $f'(x)$ is below. $f(2) = 0$



$$\lim_{x \to 2} \frac{f(x)}{x^2 - 4}$$

Selected values of the differentiable functions f and g are given in the table below. Use the table to evaluate the following limit.

x	1	2	4	6
$f(x)$	3	1	0	4
$f'(x)$	4	6	-3	-1
$g(x)$	0	4	1	0
$g'(x)$	5	-2	4	9

$$\lim_{x \to 2} \frac{f(g(x))}{g(f(x))}$$

$$\lim_{x \to \infty} \frac{x^{99}}{3^x}$$

The differential equation for the curve $y = g(x)$ is given by $\frac{dy}{dx} = 2^x(y - 1)$. It is known that $g(1) = 0$ and g is twice differentiable.

$$\lim_{x \to 1} \frac{\sin(g(x))}{x^2 - x}$$

Teacher Resources and Answer Key with Rationales
– L'Hosptial's Rule

Rationales for Multiple Choice Questions:

Q1

A	*The student incorrectly plugged in -2 in the original limit*
B	*The student factored the numerator of the limit*
C	*This answer is correct*
D	*The student misunderstood the 0/0 form*

Q2

A	*The student incorrectly plugged in -2 in the original limit*
B	*The student factored the numerator of the limit*
C	*This answer is correct*
D	*The student misunderstood the 0/0 form*

Q3

A	*The student did not use the chain rule when applying L'hospital's Rule*
B	*This answer is correct*
C	*The student did not use chain rule and incorrectly handled the complex fraction*
D	*The student incorrectly handled the complex fraction*

Q4

A	*The student incorrectly evaluated the trig value after L'hospital's Rule*
B	*The student mishandled the complex fraction after evaluating*
C	*This answer is correct*
D	*The student did not use chain rule when differentiating*

Q5

A	*This answer is correct*
B	*The student did not use chain rule when differentiating the numerator*
C	*The student mishandled the natural log expression*
D	*The student mishandled the natural log expression and did not use chain rule*

Q6

A	*The student use f(-3) and f'(-3)*
B	***This answer is correct***
C	*The student did not use the chain rule*
D	*The student misunderstood the 0/0 form*

Q7

A	*The student used product rule instead of composition when differentiating*
B	***This answer is correct***
C	*The student did not use the chain rule*
D	*The student misunderstood the 0/0 form*

Q8

A	***This answer is correct***
B	*The student incorrectly differentiated g(x - 3) as g'(x)*
C	*The student mishandled the complex fraction when simplifying*
D	*The student misunderstood the 0/0 form*

Solutions:

Top Left:

$\lim_{x\to 4} h(x) = 0, \qquad \lim_{x\to 4}(x-4) = 0 \;\Longrightarrow By\ L'Hospital\text{:}\ \lim_{x\to 4}\frac{h(x)}{x-4} = \lim_{x\to 4}\frac{h'(x)}{1} = 0$

Top Right:

$\lim_{x\to 2}(f(x) - x^2) = 0, \qquad \lim_{x\to 2} g(x^2 - x) = 0 \Longrightarrow By\ L'Hospital\text{:}\ \lim_{x\to 2}\frac{f(x)-x^2}{g(x^2-x)} = \lim_{x\to 2}\frac{f'(x)-2x}{g'(x^2-x)*(2x-1)}$

$= \frac{f'(2)-4}{g'(2)*(3)} = \frac{-1}{-6} = \frac{1}{6}$

Bottom Left:

$\lim_{x\to 1}\left(e^{x^2-1} - x\right) = 0, \quad \lim_{x\to 1}\left(\cos(\pi x) + sin\left(\frac{\pi x}{2}\right)\right) = 0 \;\Longrightarrow By\ L'Hospital\text{:}$

$\lim_{x\to 1}\frac{e^{x^2-1} - x}{\cos(\pi x) + sin\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1}\frac{2xe^{x^2-1} - 1}{-\pi\sin(\pi x) + \frac{\pi}{2}cos\left(\frac{\pi x}{2}\right)} = \frac{1}{0} \Longrightarrow DNE$

Bottom Right:

Since f is differentiable, then f is continuous. Thus, $\lim_{x\to 2} f(x) = f(2)$. And since f' is differentiable, f' is continuous. Thus, $\lim_{x\to 2} f'(x) = f'(2)$.

$\lim_{x\to 2}(f(x) + x - 3) = 0, \qquad \lim_{x\to 2}(\ln(2x-3)) = 0 \;\Longrightarrow\; By\ L'Hospital\text{:}\ \lim_{x\to 2}\frac{f(x)+x-3}{\ln(2x-3)}$

$= \lim_{x\to 2}\frac{f'(x)+1}{\frac{2}{2x-3}} = \frac{f'(2)+1}{2} = \frac{\frac{4}{2}+1}{2} = \frac{3}{2}$

Top Left:

$$\lim_{x\to 2} f(x) = 0, \qquad \lim_{x\to 2}(x^2-4) = 0 \implies By\ L'Hospital:\ \lim_{x\to 2}\frac{f(x)}{x^2-4} = \lim_{x\to 2}\frac{f'(x)}{2x} = \frac{4}{4} = 1$$

Top Right:

$$\lim_{x\to 2} f(g(x)) = 0, \lim_{x\to 2} g(f(x)) = 0 \Rightarrow By\ L'Hospital:\ \lim_{x\to 2}\frac{f\big(g(x)\big)}{g\big(f(x)\big)} = \lim_{x\to 2}\frac{f'\big(g(x)\big) * g'(x)}{g'\big(f(x)\big) * f'(x)} = \frac{f'(4) * -2}{g'(1) * 6} = \frac{1}{5}$$

Bottom Left:

$$\lim_{x\to\infty}(x^{99}) = \infty \qquad \lim_{x\to\infty} 3^x = \infty \implies By\ L'Hospital\ and\ hierarchy\ of\ functions:\ \lim_{x\to\infty}\frac{x^{99}}{3^x} = 0$$

Bottom Right:

Since g is differentiable, then g is continuous. Thus, $\lim_{x\to 1} g(x) = g(1)$. And since g' is differentiable, g' is continuous. Thus, $\lim_{x\to 1} g'(x) = g'(2)$.

$$\lim_{x\to 1}(\sin(g(x))) = 0, \qquad \lim_{x\to 1}(\mathrm{x}^2 - x) = 0 \implies By\ L'Hospital:\ \lim_{x\to 1}\frac{\sin\big(g(x)\big)}{x^2-x}$$

$$= \lim_{x\to 1}\frac{\cos\big(g(x)\big) * g'(x)}{2x-1} = \frac{\cos(0) * g'(0)}{1} = g^{'}(0) = 2^1(-1) = -2$$