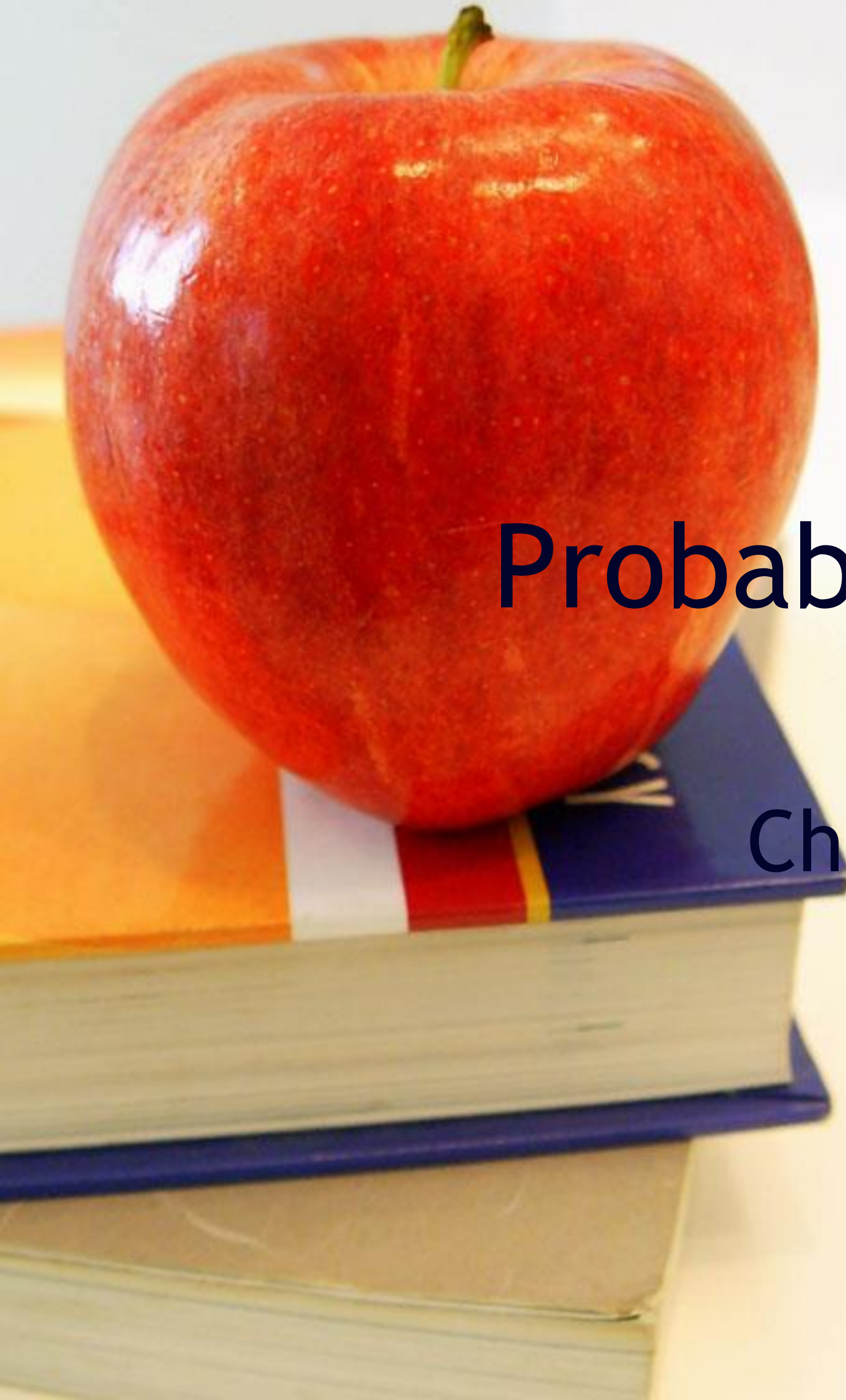


Probability Rules!

Chapter 14

Objectives:

1. General Addition Rule
2. Conditional probability
3. General Multiplication Rule
4. Independence
5. Tree diagram

The General Addition Rule

- When two events **A** and **B** are disjoint, we can use the addition rule for disjoint events (mutually exclusive) from Chapter 14:

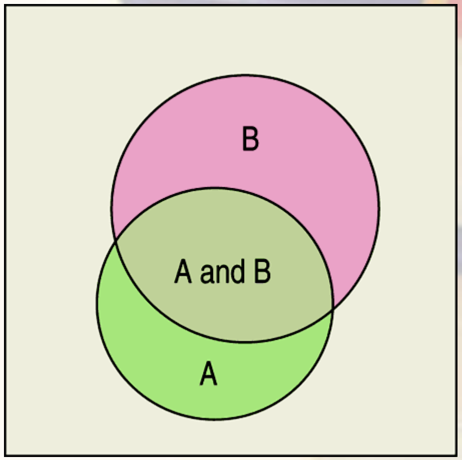
$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B})$$

- However, when our events are not disjoint, this earlier addition rule will double count the probability of *both* **A** and **B** occurring. Thus, we need the General Addition Rule.
- Let’s look at a picture…

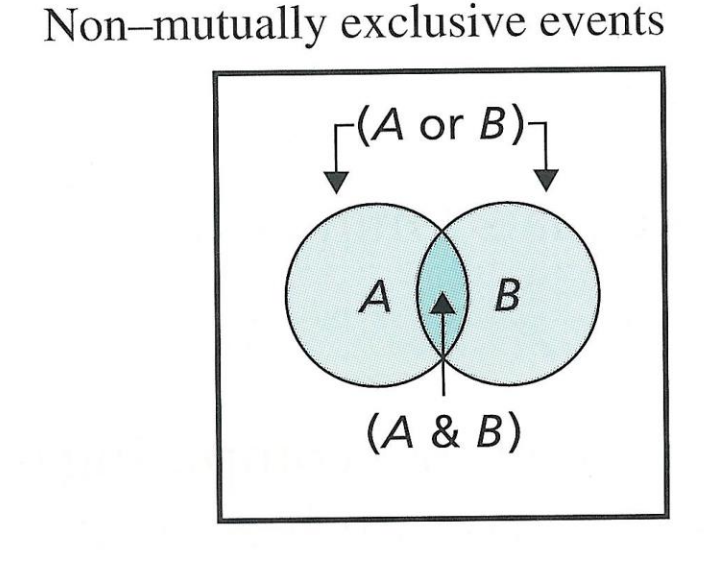
The General Addition Rule

- General Addition Rule:
  - For any two events **A** and **B**,

    $P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \cap \mathbf{B})$
- The following Venn diagram shows a situation in which we would use the general addition rule:

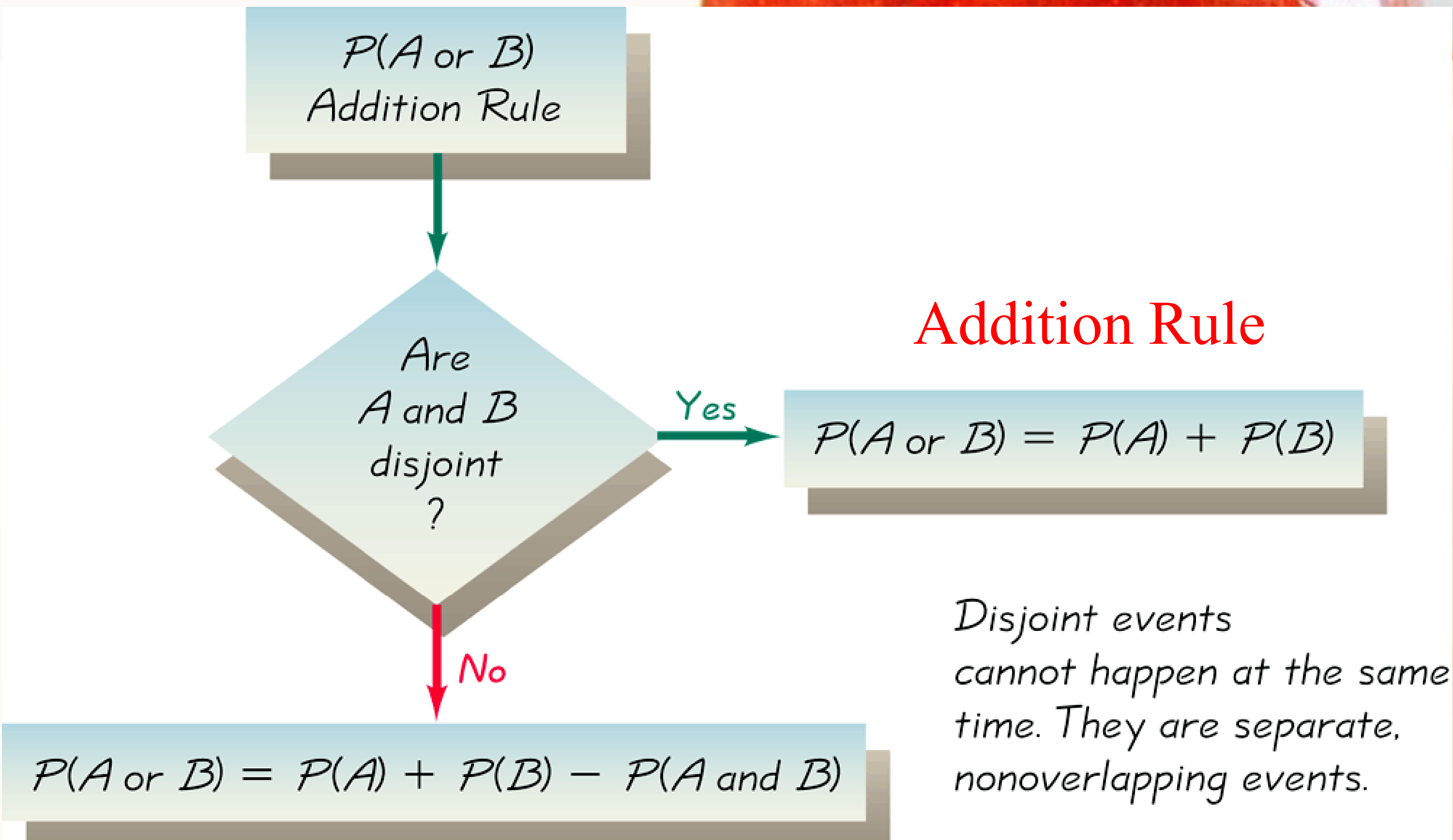


The General Addition Rule



- For two non-mutually exclusive events A and B, the probability that one or the other (or both) occurs is the sum of the probabilities of the two events minus the probability that both occur.
- P(A or B) = P(A) + P(B) – P(A and B)

Applying the Addition Rule



Addition Rule

Disjoint events cannot happen at the same time. They are separate, nonoverlapping events.

General Addition Rule

Addition Rule-Example

- A single card is drawn from a deck of cards. Find the probability that the card is a king or a queen.

Addition Rule-Example

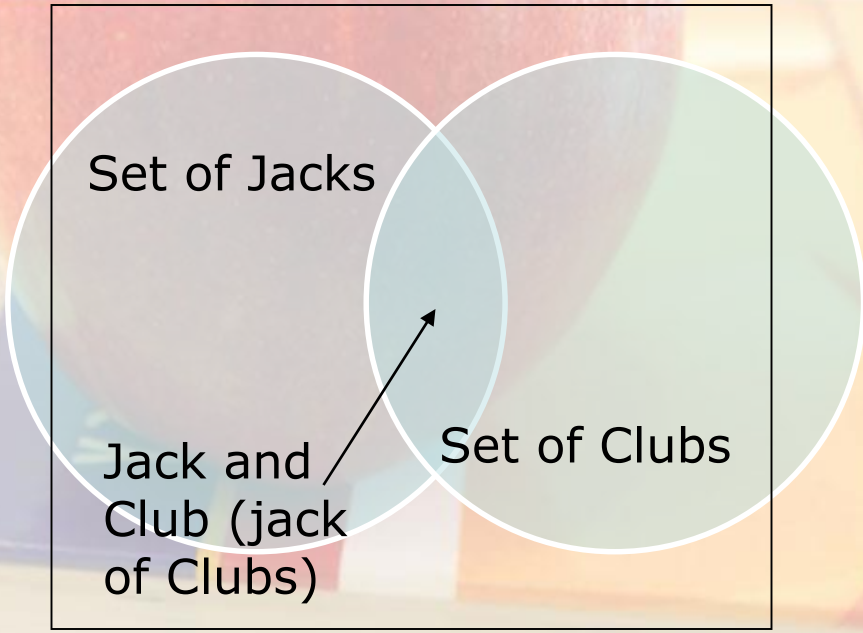
- A single card is drawn from a deck of cards. Find the probability that the card is a king or a queen.
- The events *King* and *Queen* are **disjoint**. They cannot occur at the same time. So the probability of *King and Queen* is zero.

$$P(K \cup Q) = p(K) + P(Q)$$

P(KUQ) = 4/52 + 4/52 = 8/52 = 2/13

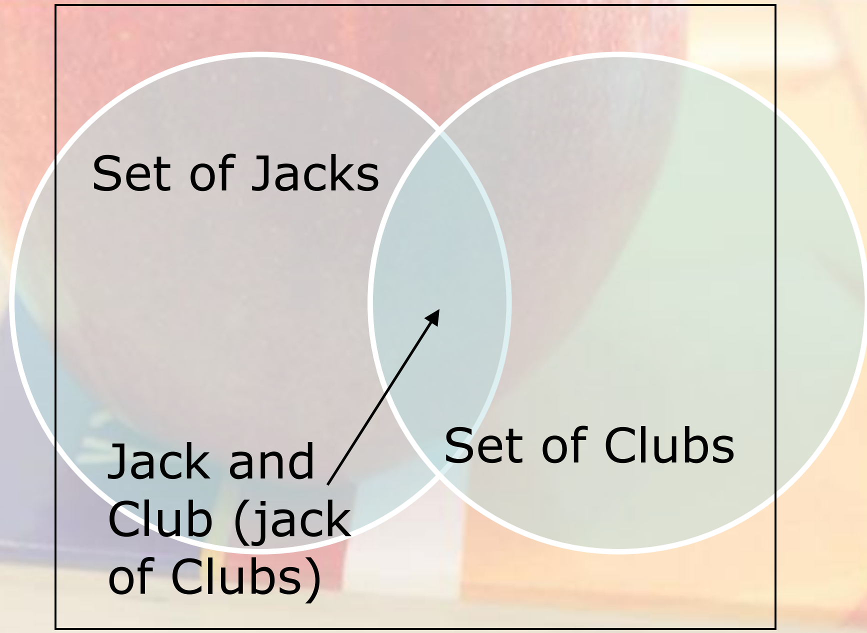
General Addition Rule-Example

- A single card is drawn from a deck of cards. Find the probability that the card is a jack or club.



Example of Addition Rule

- A single card is drawn from a deck of cards. Find the probability that the card is a jack or club.



$$P(J \text{ or } C) = P(J) + P(C) - P(J \text{ and } C)$$

$$P(J \cup C) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Addition Rule – Example

- When tossing a die once, find the probability of rolling a 5 **or** an even number.
    1. Compound or event – addition rule.
    2. Disjoint events, a 5 and an even number (2,4,6) cannot occur at the same time.
    3. Therefore use: P(A or B) = P(A) + P(B).

Addition Rule – Example

4. The probability is given by:

$$P(5 \cup \text{even}) = P(5) + P(\text{even})$$

$$P(5 \text{ or even}) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$$

Probability of rolling a 5

Probability of rolling an even number

General Addition Rule-Example

- When tossing a die once, find the probability of rolling a 5 **or** a number greater than 3.

  1. Compound or event – addition rule.
  2. Not disjoint events, a 5 and a number greater than 3 (4,5,6) can occur at the same time (the number 5).
  3. Therefore use:

     P(A or B) = P(A) + P(B) – P(A and B).

General Addition Rule-Example

4. There are three numbers greater than 3 on a die and one of them is the 5. We cannot count the 5 twice.
5. The probability is given by:

$$P(5 \cup >3) = P(5) + P(>3) - P(5 \cap >3)$$

$$P(5 \text{ or greater than } 3) = \frac{1}{6} + \left(\frac{3}{6} - \frac{1}{6}\right) = \frac{3}{6} = \frac{1}{2}$$

Probability of rolling a 5

Probability of rolling a number greater than 3

Probability of rolling the same 5

It Depends…

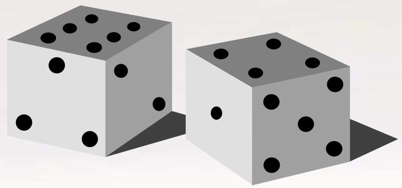
- Back in Chapter 3, we looked at contingency tables and talked about conditional distributions.
- When we want the probability of an event from a *conditional* distribution, we write $P(\mathbf{B}|\mathbf{A})$ and pronounce it "the probability of **B** *given* **A**."
- A probability that takes into account a given condition is called a conditional probability.

It Depends…

- To find the probability of the event **B** *given* the event **A**, we restrict our attention to the outcomes in **A**. We then find the fraction of *those* outcomes **B** that also occurred.

$$P(\mathbf{B}|\mathbf{A})=\frac{P(\mathbf{A}\cap\mathbf{B})}{P(\mathbf{A})}$$

- Note: P(**A**) cannot equal 0, since we know that **A** has occurred.



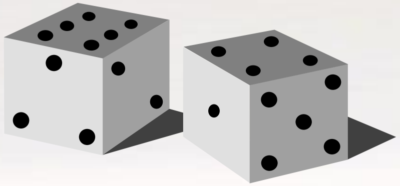
Conditional Probability

Conditional probability is the probability of an event occurring, given that another event has already occurred.

Conditional probability restricts the sample space.

The conditional probability of event B occurring, given that event A has occurred, is denoted by P(B|A) and is read as "probability of B, given A."

We use conditional probability when two events occurring in sequence are not independent. In other words, the fact that the first event (event A) has occurred affects the probability that the second event (event B) will occur.

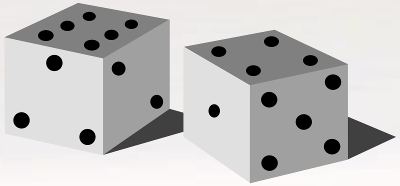


Conditional Probability

Formula for Conditional Probability

$$P(A|B)=\frac{P(A\cap B)}{P(B)} \quad or \quad P(B|A)=\frac{P(B\cap A)}{P(A)}$$

Better off to use your brain and work out conditional probabilities from looking at the sample space, otherwise use the formula.



Conditional Probability

e.g. There are 2 red and 3 blue counters in a bag and, without looking, we take out one counter and do not replace it.

The probability of a 2^{nd} counter taken from the bag being red depends on whether the 1^{st} was red or blue.

Conditional probability problems can be solved by considering the individual possibilities or by using a table, a Venn diagram, a tree diagram or a formula.

Harder problems are best solved by using a formula together with a tree diagram.

The General Multiplication Rule

- When two events **A** and **B** are independent, we can use the multiplication rule for independent events from Chapter 14:

$$P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B})$$

- However, when our events are not independent, this earlier multiplication rule does not work. Thus, we need the General Multiplication Rule.

The General Multiplication Rule

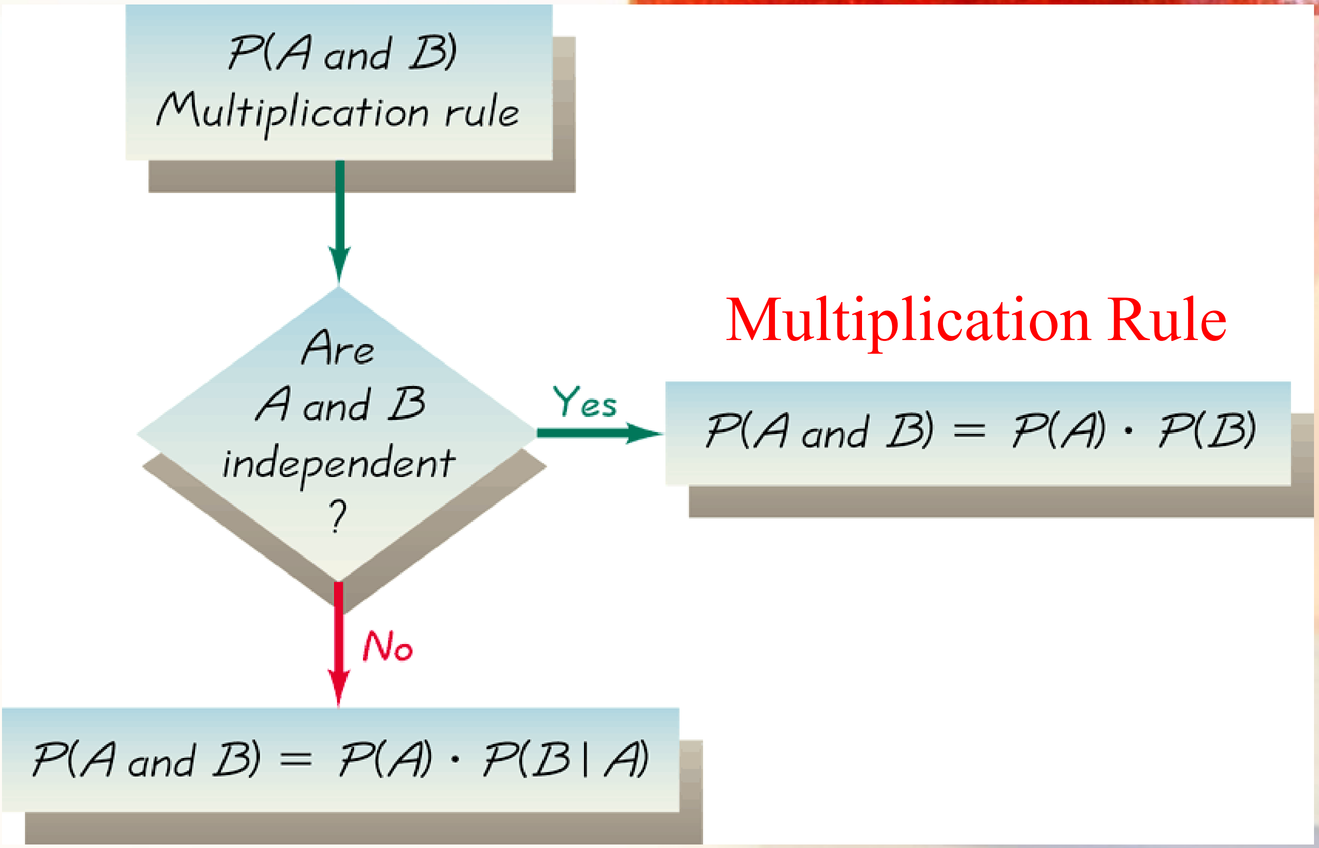
- We encountered the general multiplication rule in the form of conditional probability.
- Rearranging the equation in the definition for conditional probability, we get the General Multiplication Rule:
  - For any two events **A** and **B**,

$$P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B} | \mathbf{A})$$

or

$$P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{B}) \times P(\mathbf{A} | \mathbf{B})$$

Applying the Multiplication Rule



Multiplication Rule

General Multiplication Rule
(conditional probability)

Independence

- Independence of two events means that the outcome of one event does not influence the probability of the other.
- With our new notation for conditional probabilities, we can now formalize this definition:
  - Events **A** and **B** are independent whenever $P(\mathbf{B}|\mathbf{A}) = P(\mathbf{B})$. (Equivalently, events **A** and **B** are independent whenever $P(\mathbf{A}|\mathbf{B}) = P(\mathbf{A})$.)

Independence

- Examples:
  - If you roll two dice and obtain a sum of seven, the result of that roll has no effect on the next roll, so the two rolls are ***independent***.
  - But if you draw an ace from a deck of cards P(ace) = 4/52 and without replacing it draw a second card, the probability that it is an ace is P(ace) = 3/51. These events are ***not independent***.

Independence

- Determining independence
  - Definition: Events A and B are independent if the probability of event B occurring was not influenced by the occurrence of event A.
  - Thumb Rule: If two events are “physically independent” then they will also be statistically independent.
  - Never assume that two events are independent unless you are absolutely certain that they are independent.

Independence

- Determining independence
  - Make sure you understand the difference between mutually exclusive events and independent events.
    - Mutually Exclusive means that the events A and B have nothing in common and so there is no intersection, i.e., P(A and B) = 0.
    - Independent means that the outcome of event A will not influence the outcome of event B.

Independent ≠ Disjoint (Mutually Exclusive)

- Disjoint events *cannot* be independent! Well, why not?
  - Since we know that disjoint events have no outcomes in common, knowing that one occurred means the other didn't.
  - Thus, the probability of the second occurring changed based on our knowledge that the first occurred.
  - It follows, then, that the two events are *not* independent.
- A common error is to treat disjoint events as if they were independent, and apply the Multiplication Rule for independent events—don't make that mistake.

Depending on Independence

- It’s much easier to think about independent events than to deal with conditional probabilities.
  - It seems that most people’s natural intuition for probabilities breaks down when it comes to conditional probabilities.
- Don’t fall into this trap: whenever you see probabilities multiplied together, stop and ask whether you think they are really independent.

General Multiplication Rule – Conditional Probability

- General Multiplication Rule: $P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B}|\mathbf{A})$
  - Normally trying to find the conditional probability P(B|A), not the P(A and B).
- Conditional Probability:
  - P(B|A) = P(A and B)/P(A)
  - In words

    "the ***conditional*** probability ***equals*** the probability of the ***and*** of the events ***divided*** of the probability of the ***given*** event".

Reversing the Conditioning

- Reversing the conditioning of two events is rarely intuitive.
- Suppose we want to know $P(\mathbf{A}|\mathbf{B})$, and we know only $P(\mathbf{A})$, $P(\mathbf{B})$, and $P(\mathbf{B}|\mathbf{A})$.
- We also know $P(\mathbf{A} \cap \mathbf{B})$, since

$$P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B}|\mathbf{A})$$

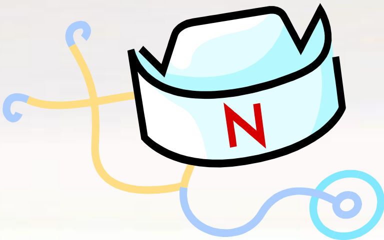
- From this information, we can find $P(\mathbf{A}|\mathbf{B})$:

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})}$$

Bayes's Rule

- When we reverse the probability from the conditional probability that you're originally given, you are actually using Bayes's Rule.

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^C)P(B^C)}$$



Example-Experimental Probability Multiplication Rule & Conditional Probability

- There are two majors of a particular college: Nursing and Engineering. The number of students enrolled in each program is given in the table on the next slide. The row total gives the total number of each category and the number in the bottom-right cell gives the total number of students. A single student is selected at random from this college. Assuming that each student is equally likely to be chosen, find :

Example (multiplication rule)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- 1. *P*(Nursing)
- 2. *P*(Grad Student)
- 3. *P*(Nursing and Grad student)
- 4. *P*(Engineering and Grad Student)

Example (conditional prob.)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- Given that an undergraduate student is selected at random, what is the probability that this student is a nurse?
- Restricting our attention to the column representing undergrads, we find that of the 90 undergrad students, 53 are nursing majors. Therefore, $P(N|U)=53/90$

Example (conditional prob.)

	Undergrads	Grads	Total
Nursing	53	47	100
Engineering	37	13	50
Total	90	60	150

- Given that an engineering student is selected, find the probability that the student is an undergraduate student.
- Restricting the sample space to the 50 engineering students, 37 of the 50 are undergrads.
- Therefore, $P(U|E) = 37/50 = 0.74$.

Drawing Without Replacement

- Sampling without replacement means that once one individual is drawn it doesn't go back into the pool.
  - We often sample without replacement, which doesn't matter too much when we are dealing with a large population.
  - However, when drawing from a small population, we need to take note and adjust probabilities accordingly.
- Drawing without replacement is just another instance of working with conditional probabilities.

Example - Conditional Probability

- Two cards are drawn without replacement from an ordinary deck of cards . Find the probability that two clubs are drawn in succession.

Since we assume that the first card drawn is a club, there are 12 remaining clubs and 51 total remaining cards.

$$P(C_1 \cap C_2) = p(C_1) \cdot p(C_2 | C_1)$$

$$= \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17}$$

Examples of Independence

1. Two cards are drawn in succession **with replacement** from a standard deck of cards. What is the probability that two kings are drawn?

$$P(K_1 \cap K_2) = p(K_1) \cdot p(K_2)$$

$$= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$$

2. Two marbles are drawn **with replacement** from a bag containing 7 blue and 3 red marbles. What is the probability of getting a blue on the first draw and a red on the second draw?

$$p(B \cap R) = p(B) \cdot p(R)$$

$$= \frac{7}{10} \cdot \frac{3}{10} = \frac{21}{100} = 0.21$$

Example: Conditional Probability - Dependent Events

- Two events are **dependent** when the outcome of one event affects the outcome of the second event.
- **Example:** Draw two cards in succession **without replacement** from a standard deck. Find the probability of a king on the first draw and a king on the second draw.
- **Answer:** $$P(K_1 \cap K_2) = p(K_1) \cdot p(K_2 | K_1)$$

$$= \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

Calculating Independence/Dependence

- Use the definition of independence and the multiplication rule.
- A and B are independent if and only if $P(A \cap B) = P(A)\,P(B)$.
- If A and B are independent events, then
  $P(A|B) = P(A)$ and $P(B|A) = P(B)$.
- If $P(A|B) = P(A)$ or $P(B|A) = P(B)$, then A and B are independent.

Calculating Independence/Dependence

Are smoking and lung disease dependent?

	Smoker	Non-smoker
Has Lung Disease	0.12	0.03
No Lung Disease	0.19	0.66

Calculating Independence/Dependence

Are smoking and lung disease dependent?

Step 1. Find the probability of lung disease.

$P(L)$ = 0.15 (row total)

Step 2. Find the probability of being a smoker

$P(S)$ = 0.31 (column total)

Step 3. Check

$P(L \cap S) = 0.12 \neq P(L) \bullet P(S) = 0.0465$

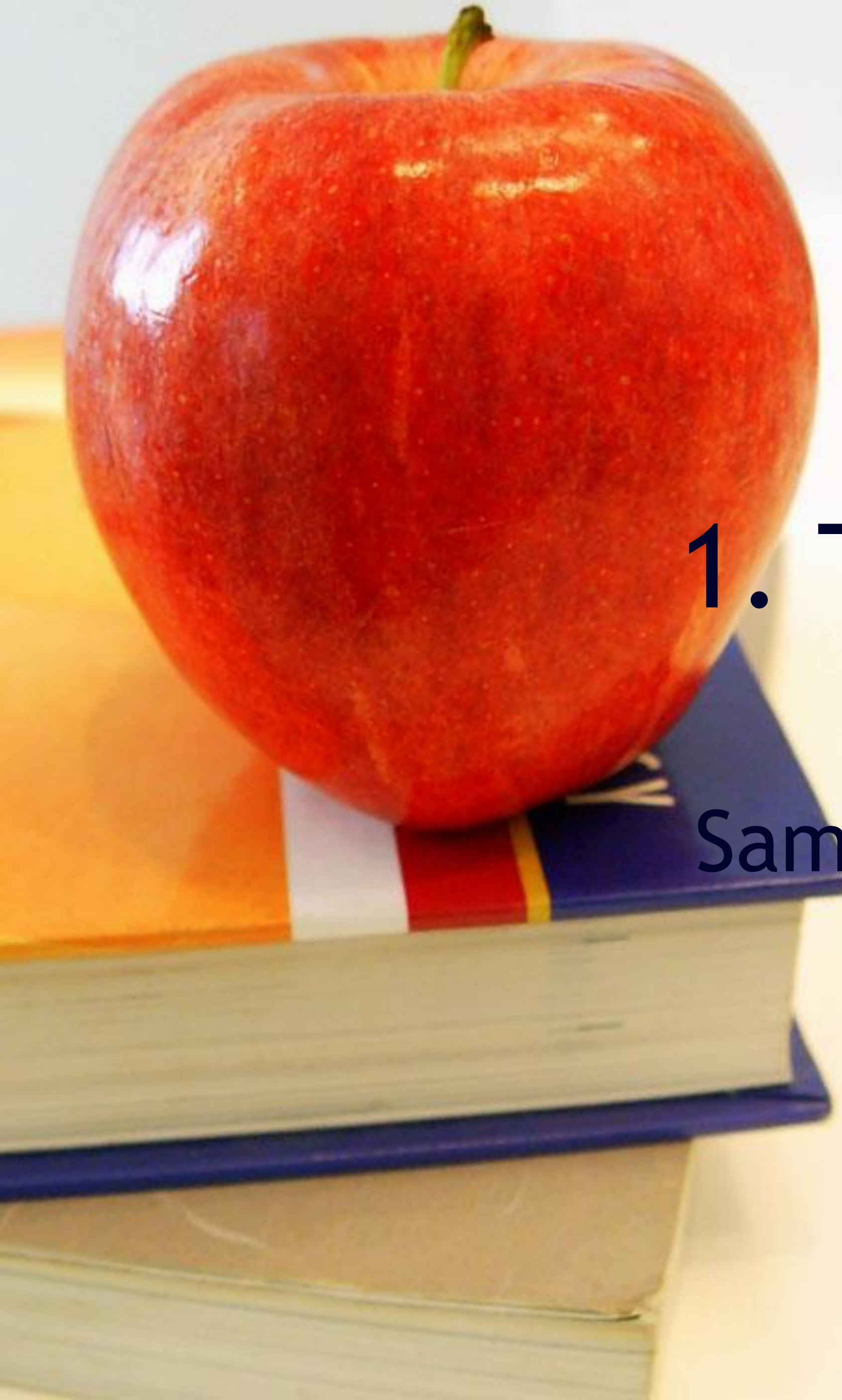
L and S are **dependent.**

	Smoker	Non-smoker
Has Lung Disease	0.12	0.03
No Lung Disease	0.19	0.66

Problem Solving Strategies in Probability

Visualizing
Sample Space & Probabilities

- 1. List – Sample Space
- 2. Venn Diagram
- 3. Lattice Diagram
- 4. Tree Diagram
- 5. Geometric Probability



1. The List

Sample Space

List Sample Space-Example

- List the sample for flipping a coin and rolling a die.
- {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}
- Find the probability of getting a 6.
- $P(6) = 2/12 = 1/6$
- Find the probability of getting heads or a 3.
- $P(H \cup 3) = 7/12$
- Find the probability of getting heads and an even number.
- $P(H \cap \text{even\#}) = 3/12$
- Given the coin is tails, what is the probability of getting a 2?
- $P(2|T) = 1/6$

List - Contingency Table

Experiment: Toss 2 Coins. Note Faces.

	2nd Coin		
1st Coin	Head	Tail	Total
Head	HH	HT	HH, HT
Tail	TH	TT	TH, TT
Total	HH, TH	HT, TT	S

Simple Event (Head on 1st Coin)

Outcome (Count, Total % Shown Usually)

Sample Space

S = {HH, HT, TH, TT}

Example

The following table gives data on the type of car, grouped by petrol consumption, owned by 100 people.

	Low	Medium	High	Total
Male	**12**	**33**	**7**	
Female	**23**	**21**	**4**	
				100

One person is selected at random.

L is the event "the person owns a low rated car"

Example

The following table gives data on the type of car, grouped by petrol consumption, owned by 100 people.

	Low	**Medium**	**High**	**Total**
Male	**12**	**33**	**7**	
Female	**23**	**21**	**4**	
				100

One person is selected at random.

L is the event "the person owns a low rated car"

F is the event "a female is chosen".

Example

The following table gives data on the type of car, grouped by petrol consumption, owned by 100 people.

	Low	Medium	High	Total
Male	12	33	7	
Female	23	21	4	
				100

One person is selected at random.

L is the event "the person owns a low rated car"

F is the event "a female is chosen".

Example

The following table gives data on the type of car, grouped by petrol consumption, owned by 100 people.

	Low	Medium	High	Total
Male	12	33	7	
Female	23	21	4	
				100

One person is selected at random.

L is the event "the person owns a low rated car"

F is the event "a female is chosen".

Find (i) $P(L)$ (ii) $P(F \cap L)$ (iii) $P(F \mid L)$

There is no need for a Venn diagram or a formula to solve this type of problem.

We just need to be careful which row or column we look at.

Example

Solution:

	Low	Medium	High	Total
Male	12	33	7	
Female	23	21	4	
	35			100

Find (i) $P(\mathrm{L})$ (ii) $P(\mathrm{F} \cap \mathrm{L})$ (iii) $P(\mathrm{F} \mid \mathrm{L})$

(Best to leave the answers as fractions)

$$\text{(i) } P(\mathrm{L}) = \frac{\cancel{35}^{\,7}}{\cancel{100}_{\,20}} = \frac{7}{20}$$

Example

Solution:

	Low	Medium	High	Total
Male	12	33	7	
Female	23	21	4	
				100

Find (i) $P(\text{L})$ (ii) $P(\text{F} \cap \text{L})$ (iii) $P(\text{F} \mid \text{L})$

$$\text{(i) } P(\text{L}) = \frac{\cancel{35}^{\,7}}{\cancel{100}_{\,20}} = \frac{7}{20}$$

$$\text{(ii) } P(\text{F} \cap \text{L}) = \frac{23}{100}$$

The probability of selecting a female with a low rated car.

Example

Solution:

	Low	Medium	High	Total
Male	12	33	7	
Female	23	21	4	
	35			100

Find (i) $P(L)$ (ii) $P(F \cap L)$ (iii) $P(F \mid L)$

(i) $P(L) = \dfrac{\cancel{35}^{\,7}}{\cancel{100}_{\,20}} = \dfrac{7}{20}$

(ii) $P(F \cap L) = \dfrac{23}{100}$

(iii) $P(F \mid L) = \dfrac{23}{35}$

The sample space is restricted from 100 to 35.

We must be careful with the denominator in (iii). Here we are given the car is low rated. We want the total of that column.

Example

Solution:

	Low	Medium	High	Total
Male	12	33	7	
Female	23	21	4	
				100

Find (i) $P(L)$ (ii) $P(F \cap L)$ (iii) $P(F \mid L)$

(i) $P(L) = \dfrac{\cancel{35}^{\,7}}{\cancel{100}_{\,20}} = \dfrac{7}{20}$

(ii) $P(F \cap L) = \dfrac{23}{100}$

(iii) $P(F \mid L) = \dfrac{23}{35}$

Notice that

$$P(L) \times P(F \mid L) = \frac{{}^{1}\cancel{7}}{20} \times \frac{23}{\cancel{35}_{5}} = \frac{23}{100}$$

$$= P(F \cap L)$$

So, $P(F \cap L) = P(F|L) \times P(L)$

List of Probabilities

Six candidates running for an elective office.

Mary (CA)	.25
John (OR)	.15
Abe (NY)	.20
Sue (PA)	.20
Buddy (GA)	.15
Carl (MA)	.05

Sample Space

List of Probabilities

Mary (CA)	.25
John (OR)	.15
Abe (NY)	.20
Sue (PA)	.20
Buddy (GA)	.15
Carl (MA)	.05

E = event "winner is from the northeast"

P(E) =

Mary (CA)	.25
John (OR)	.15
Abe (NY)	.20
Sue (PA)	.20
Buddy (GA)	.15
Carl (MA)	.05

$A = \{\text{John}, \text{Sue}\}$

$B = \{\text{Buddy}, \text{Mary}\}$

Mary (CA)	.25
John (OR)	.15
Abe (NY)	.20
Sue (PA)	.20
Buddy (GA)	.15
Carl (MA)	.05

A = {Abe, Sue, Buddy, Carl}

B = {Buddy, Mary}

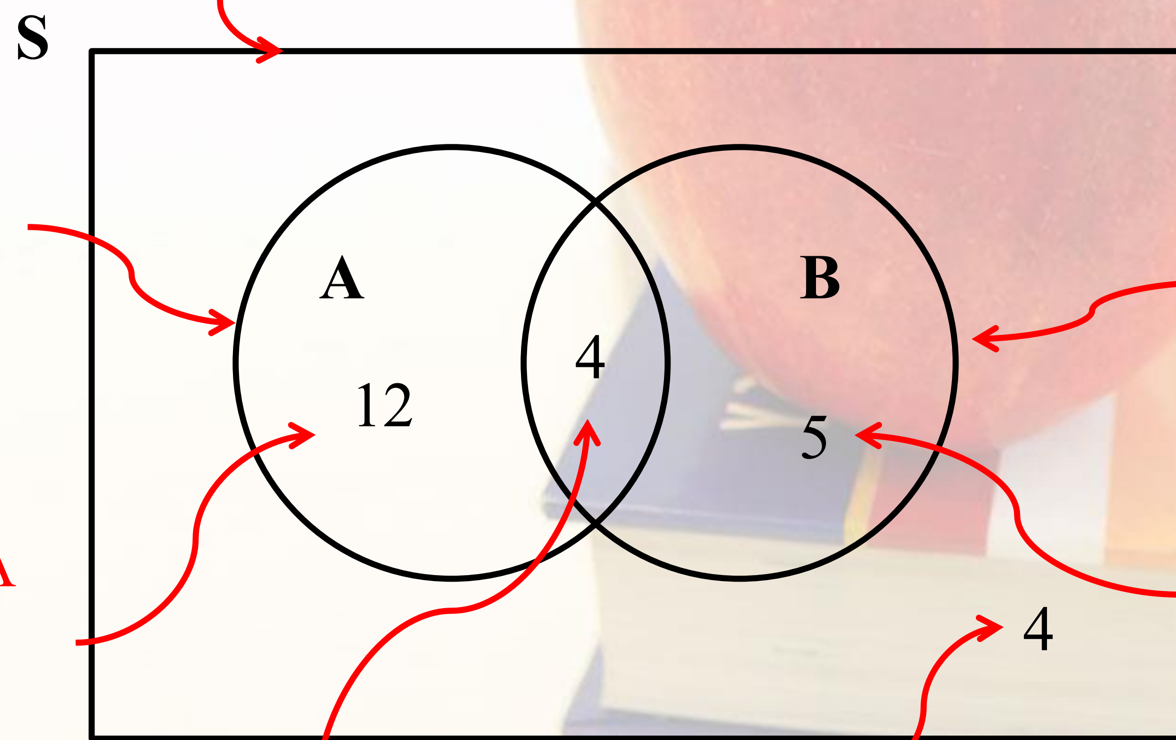
Mary (CA)	.25
John (OR)	.15
Abe (NY)	.20
Sue (PA)	.20
Buddy (GA)	.15
Carl (MA)	.05

A = {Abe, Sue, Buddy, Carl}

2. Venn Diagram

Venn Diagram

Sample space total # outcomes (25)



outcomes A (16)

outcomes B (9)

outcomes A only (12)

outcomes B only (5)

outcomes A and B (4)

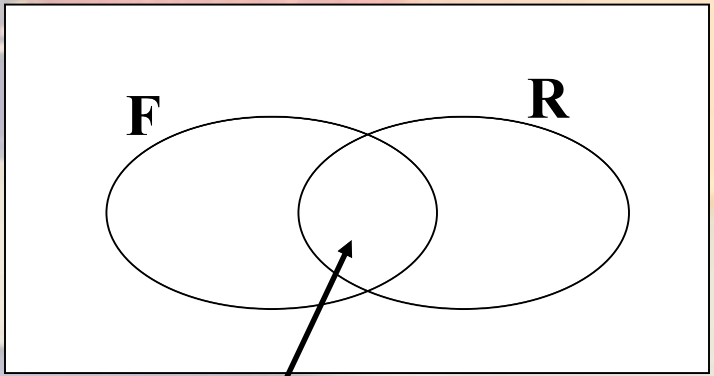
outcomes sample space only (4)

Venn Diagram

I have 2 packets of seeds. One contains 20 seeds and although they look the same, 8 will give red flowers and 12 blue. The 2nd packet has 25 seeds of which 15 will be red and 10 blue.

Draw a Venn diagram and use it to illustrate the conditional probability formula.

Solution: Let R be the event " Red flower " and
F be the event " First packet "



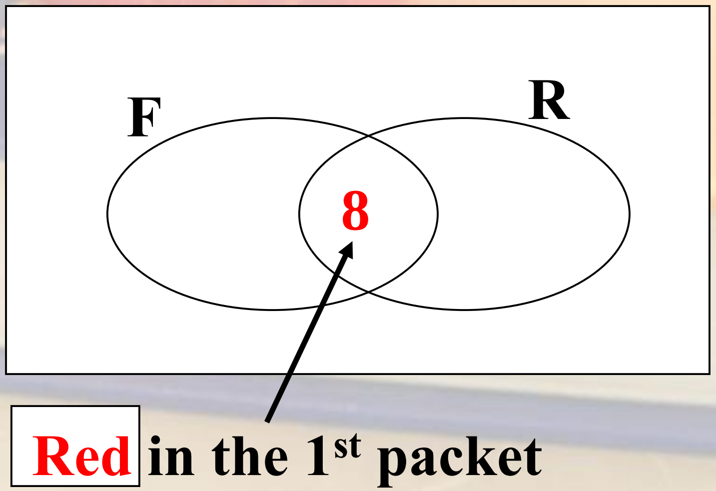
Red in the 1st packet

Venn Diagram

I have 2 packets of seeds. One contains 20 seeds and although they look the same, 8 will give red flowers and 12 blue. The 2nd packet has 25 seeds of which 15 will be red and 10 blue.

Draw a Venn diagram and use it to illustrate the conditional probability formula.

Solution: **Let R be the event " Red flower " and**
F be the event " First packet "

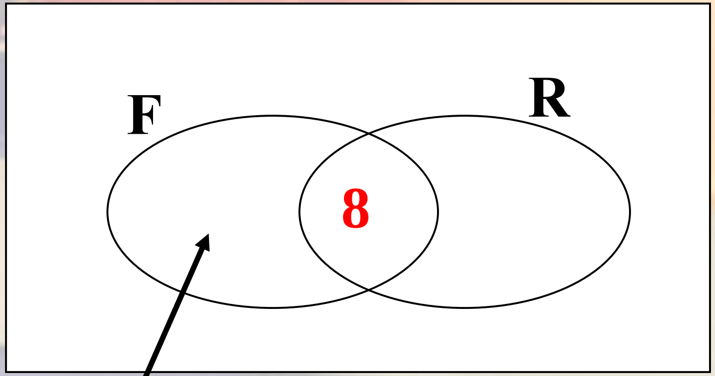


Venn Diagram

I have 2 packets of seeds. One contains 20 seeds and although they look the same, 8 will give red flowers and 12 blue. The 2nd packet has 25 seeds of which 15 will be red and 10 blue.

Draw a Venn diagram and use it to illustrate the conditional probability formula.

Solution: Let R be the event " Red flower " and
F be the event " First packet "



Blue in the 1st packet

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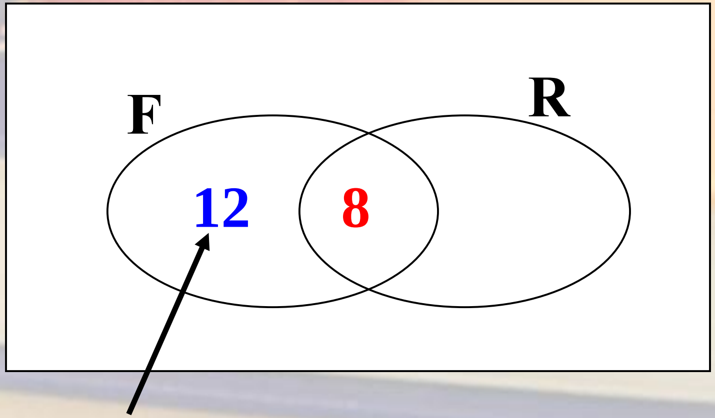

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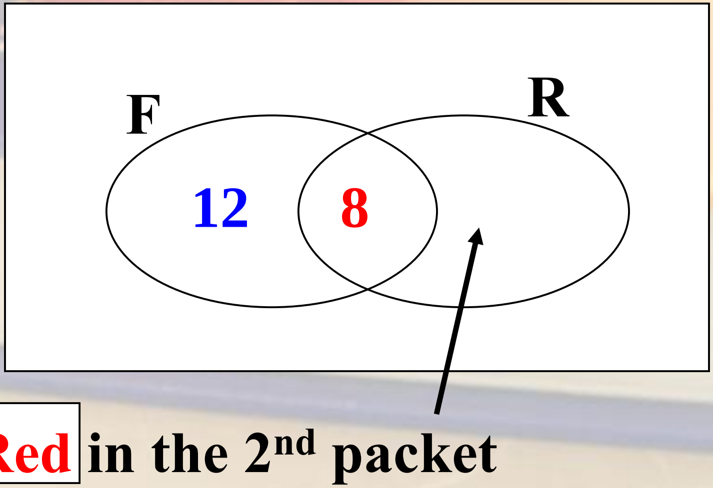

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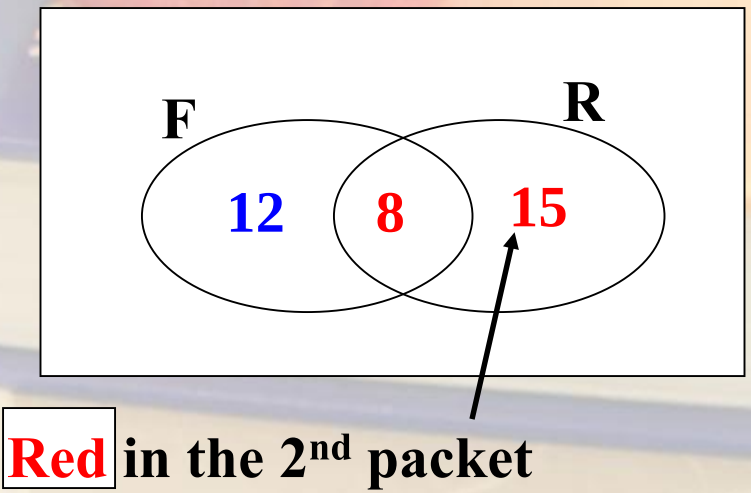

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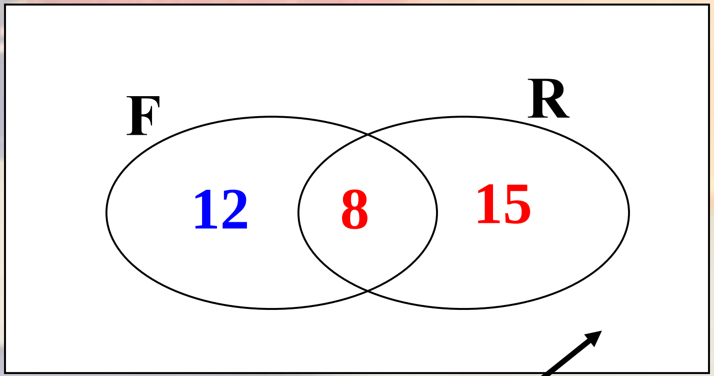

Blue in the 2^{nd} packet

Venn Diagram

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Draw a Venn diagram and use it to illustrate the conditional probability formula.

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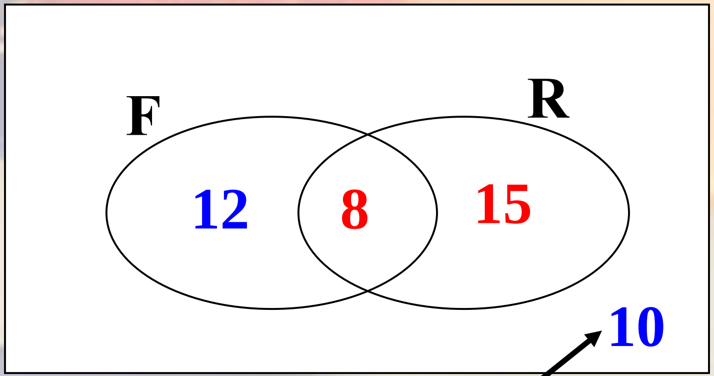

Blue in the 2nd packet

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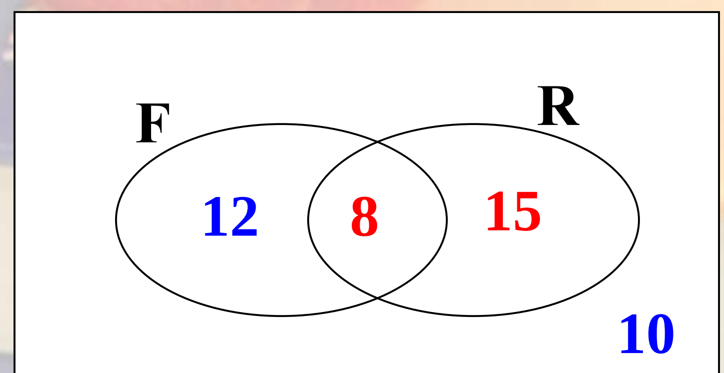

Total: 20 + 25

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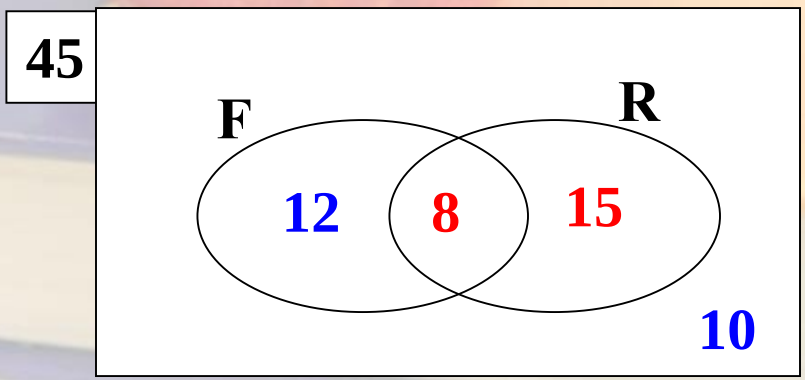

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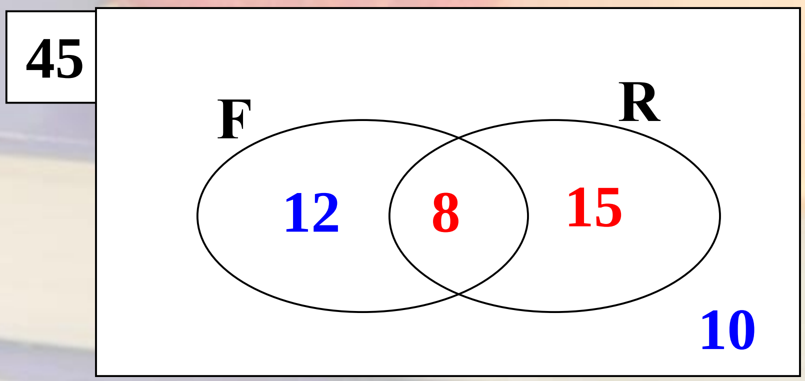

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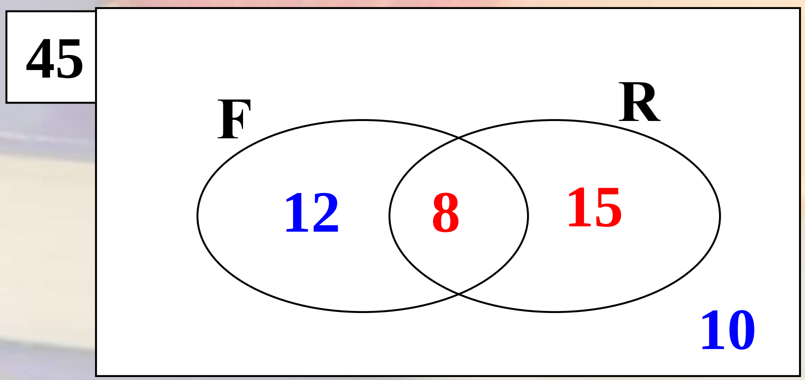
$P(R \cap F) =$



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Solution: Let R be the event " Red flower " and
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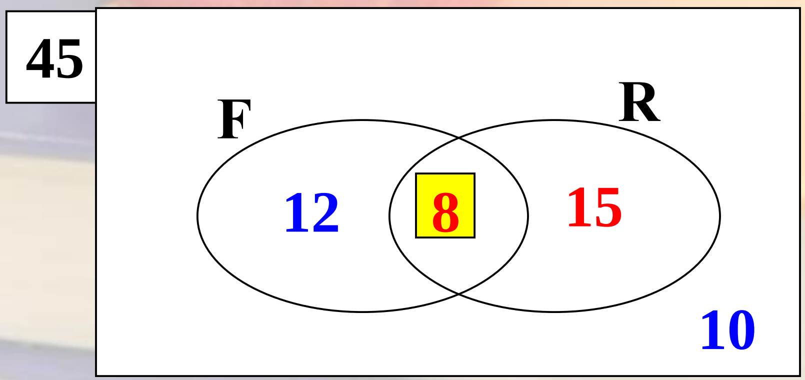
$$P(\mathrm{R} \cap \mathrm{F}) = \frac{8}{\ }$$



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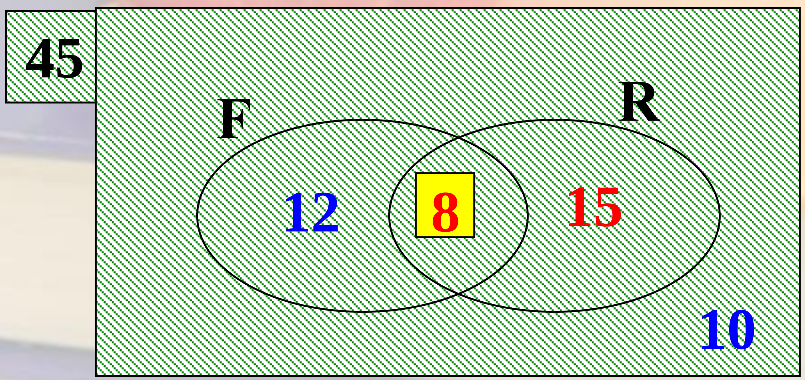
Draw a Venn diagram and use it to illustrate the conditional probability formula.

Solution: Let R be the event " Red flower " and F be the event " First packet "

$$P(R \cap F) = \frac{8}{45}$$



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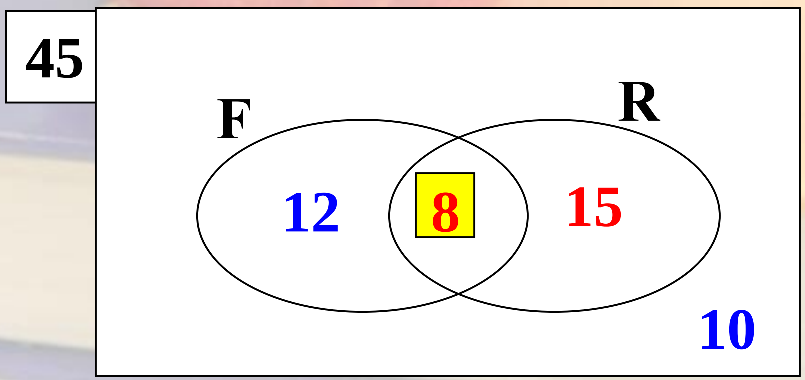

$$P(R|F) = \frac{8}{\quad}$$

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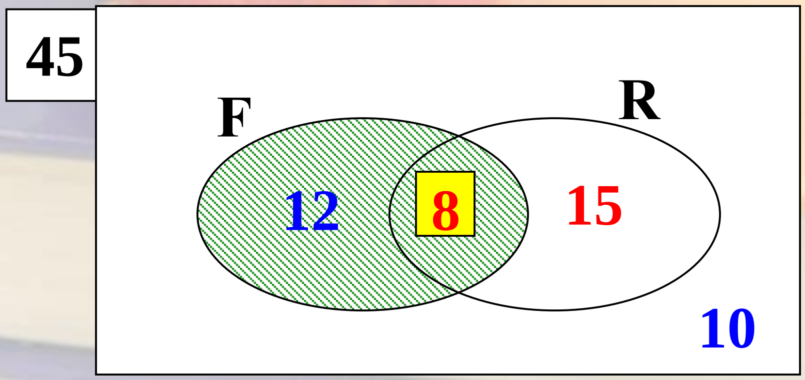
$$P(R \cap F) = \frac{8}{45}$$

$$P(R|F) = \frac{8}{20} \qquad P(F) =$$



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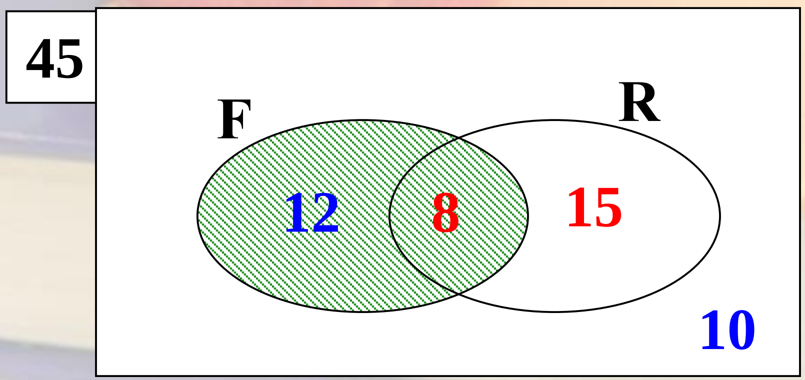

$$P(R|F) = \frac{8}{20} \qquad P(F) = \frac{20}{}$$

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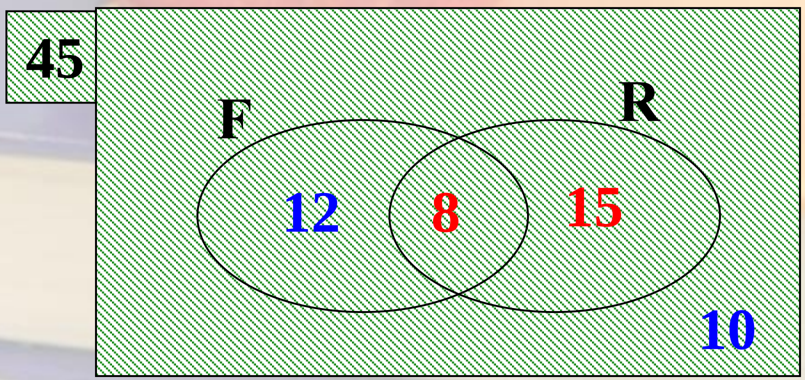

$$P(R \cap F) = \frac{8}{45}$$

$$P(R|F) = \frac{8}{20} \qquad P(F) = \frac{20}{45}$$

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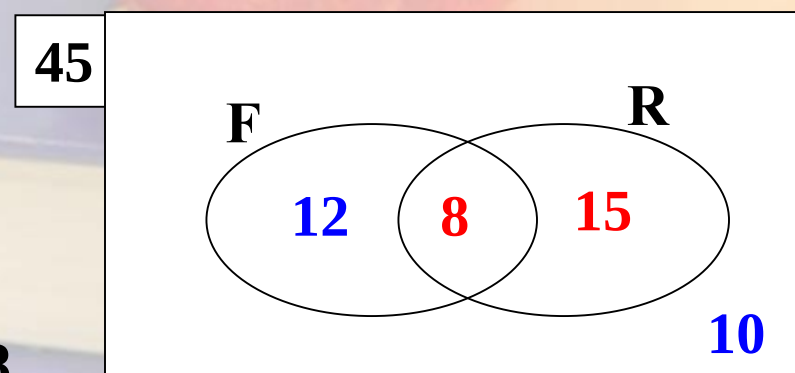


$$P(R \cap F) = \frac{8}{45}$$

$$P(R|F) = \frac{8}{20} \qquad P(F) = \frac{20}{45}$$

$$\Rightarrow P(R|F) \times P(F) = \frac{8}{{}_{1}\cancel{20}} \times \frac{{}^{1}\cancel{20}}{45} = \frac{8}{45}$$

So, $\boxed{P(R \cap F) = P(R|F) \times P(F)}$

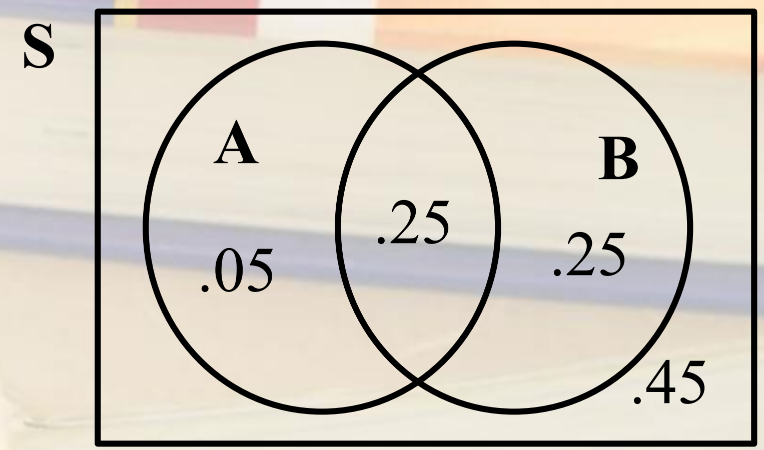
Example

- A and B are two events such that P(A)=.3, P(B)=.5 and the P(A or B)=.55. Find the probabilities of the following events:
  1. P(A and B)
  2. P(B’)
  3. $P(A' \cap B)$
  4. P(A|B)
  5. P(B’|A)

Solution

A and B are two events such that P(A)=.3, P(B)=.5 and the P(A or B)=.55.

1. $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$
   - $.55 = .3 + .5 - P(A \cap B)$
   - $P(A \cap B) = .25$
   - The Venn Diagram is:

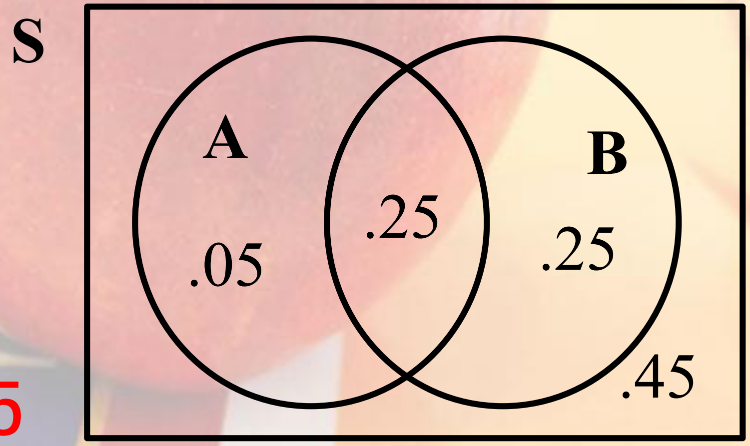


Solution

2. $P(B')$?
   - $P(B') = 1 - P(B)$
   - $P(B') = 1 - .5$
   - $P(B') = .5$
   - Or can use the venn diagram

Solution

3. P(A'∩B)?
   - Use the venn diagram
   - The region when A' intersects B, is B only. Therefore, P(A'∩B) = .25

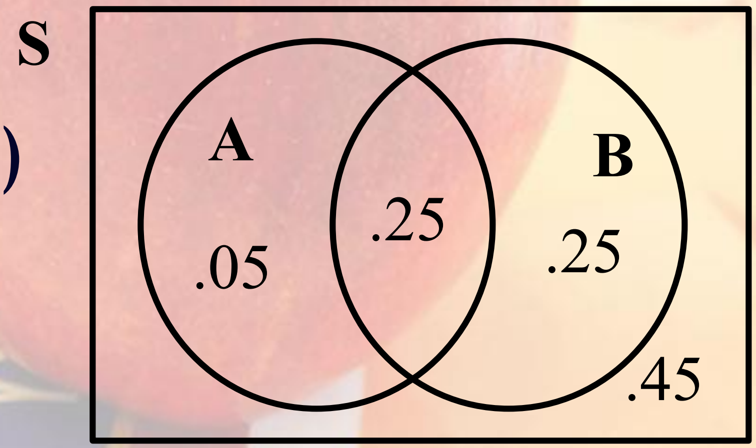


Solution

4. P(A|B)?
   - $P(A|B) = P(A \cap B)/P(B)$
   - $P(A|B) = .25/.5$
   - $P(A|B) = .5$

Solution

5. P(B’|A)?
   - P(B’|A) = P(B’∩A)/P(A)
   - P(B’|A) = .05/.3
   - P(B’|A) = .167



3. Lattice Diagram

Lattice Diagram

- Used for two dice

One die

The other die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The sample space – total # outcomes 36

Problem

- Roll two dice and observe the sum. Define the events as follows:
  - A = {the value is even}
  - B = {the value is odd}
  - C = {the value is less than 6}
  - D = {the value is greater than 6}
- Find the following probabilities
  1. P(A)
  2. P(D)
  3. P(B or C)
  4. P(sum=4|C)
  5. P(B|D)

Solution

1. P(A) = P(the value is even)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Even outcomes 18
- Sample space - total outcomes 36
- P(A) = 18/36 = 1/2

Solution

2. P(D) = P(the value is greater than 6)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Value greater than 6 = 21
- Sample space – total outcomes = 36
- P(D) = 21/36

Solution

3. P(B or C) = P(value odd or is less than 6)

	1	2	3	4	5	6
1	2	~~3~~	4	~~5~~	6	7
2	~~3~~	4	~~5~~	6	7	8
3	4	~~5~~	6	7	8	9
4	~~5~~	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Odd = 18
- Less than 6 = 4 (don’t count twice)
- P(B or C) = (18 + 4)/36 = 22/36 =11/18

Solution

4. P(sum=4|C) = P(sum=4|value less than 6)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Given: value less than 6 = 10 (sample space)
- (Sum=4) = 3
- P(sum=4|C) = 3/10

Solution

5. P(B|D) = P(odd|value greater than 6)

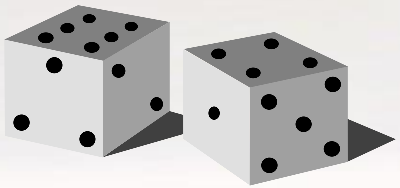
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Given: greater than 6 = 21 (sample space)
- Value odd = 12
- P(B|D) = 12/21 = 4/7

4. Tree diagram

Tree Diagrams

- A tree diagram helps us think through conditional probabilities by showing sequences of events as paths that look like branches of a tree.
- Making a tree diagram for situations with conditional probabilities is consistent with our "make a picture" mantra.



Probability Tree Diagrams

The probability of a complex event can be found using a probability tree diagram.

1. Draw the appropriate tree diagram.

2. Assign probabilities to each branch.
(Each section sums to 1.)

3. Multiply the probabilities along individual branches to find the probability of the outcome at the end of each branch.

4. Add the probabilities of the relevant outcomes, depending on the event.

Tree Diagram

In November, the probability of a man getting to work on time if there is fog on I-95 is $\frac{2}{5}$.

If the visibility is good, the probability is $\frac{9}{10}$.

The probability of fog at the time he travels is $\frac{3}{20}$.

(a) Calculate the probability of him arriving on time.

(b) Calculate the probability that there was fog given that he arrives on time.

There are lots of clues in the question to tell us we are dealing with conditional probability.

In November, the probability of a man getting to work on time if there is fog on the M6 is $\frac{2}{5}$.

If the visibility is good, the probability is $\frac{9}{10}$.

The probability of fog at the time he travels is $\frac{3}{20}$.

(a) Calculate the probability of him arriving on time.

(b) Calculate the probability that there was fog given that he arrives on time.

There are lots of clues in the question to tell us we are dealing with conditional probability.

Solution: Let T be the event " getting to work on time "

Let F be the event " fog on the M6 "

Can you write down the <u>notation</u> for the probabilities that we want to find in (a) and (b)?

(a) Calculate the probability of him arriving on time. $P(T)$

(b) Calculate the probability that there was fog given that he arrives on time. $P(F \mid T)$

Can you also write down the notation for the three probabilities given in the question?

" the probability of a man getting to work on time if there is fog is $\frac{2}{5}$ "

$$P(T \mid F) = \frac{2}{5}$$

Not foggy

" If the visibility is good, the probability is $\frac{9}{10}$ ".

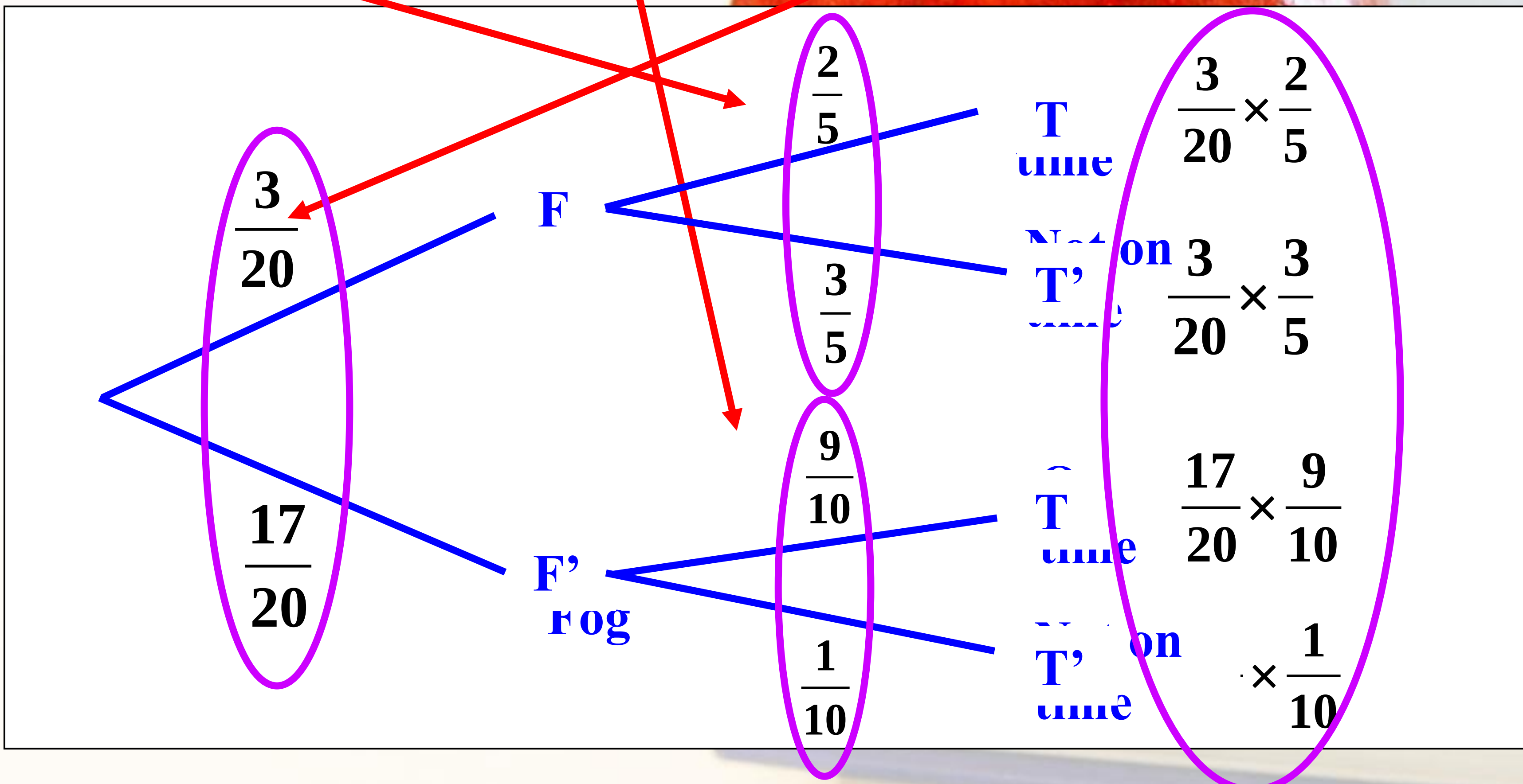
$$P(T \mid F') = \frac{9}{10}$$

" The probability of fog at the time he travels is $\frac{3}{20}$ ".

$$P(F) = \frac{3}{20}$$

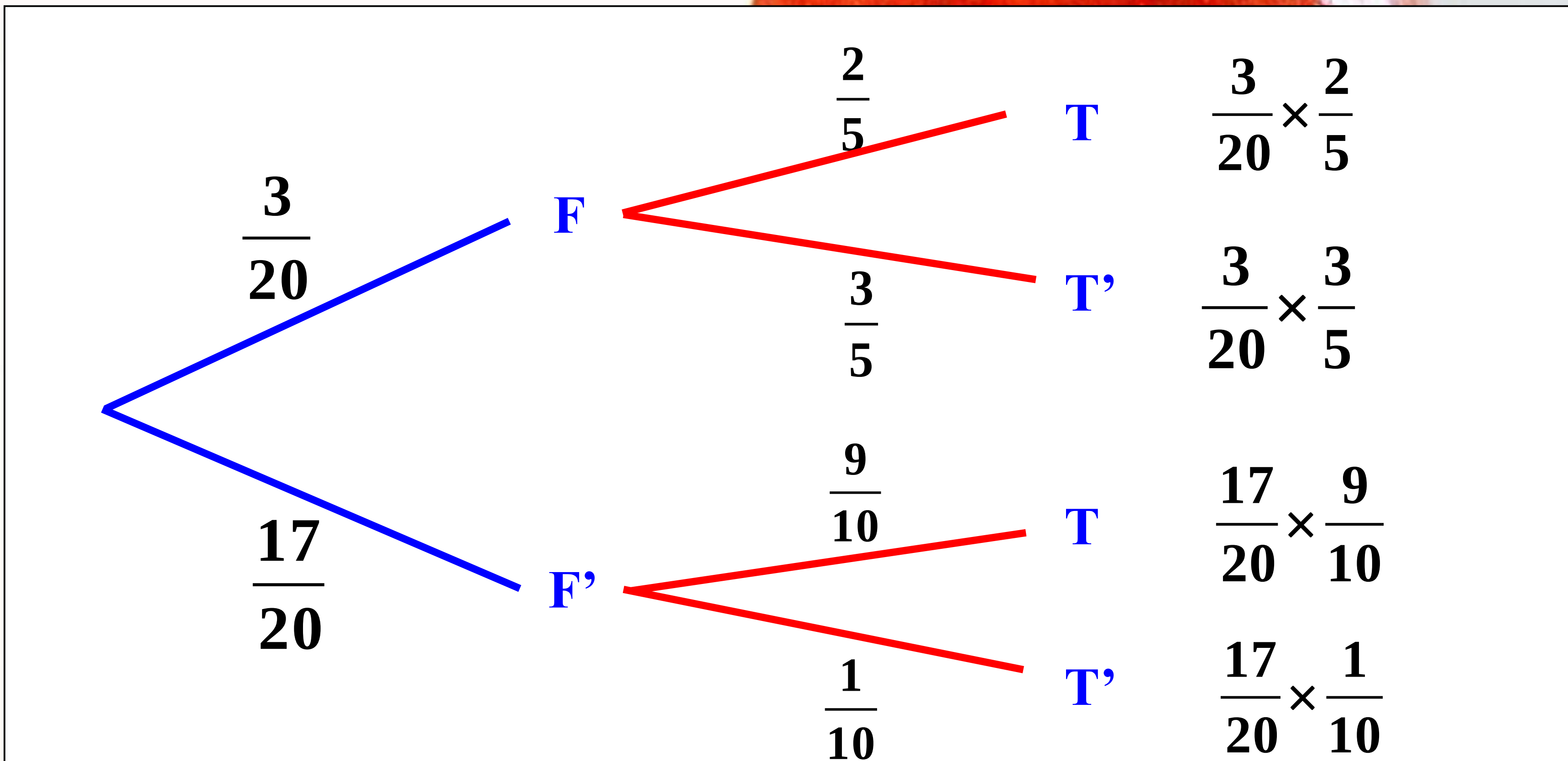
This is a much harder problem so we draw a tree diagram.

$P(T \mid F) = \dfrac{2}{5}$ $\quad P(T \mid F') = \dfrac{9}{10}$ $\quad P(F) = \dfrac{3}{20}$



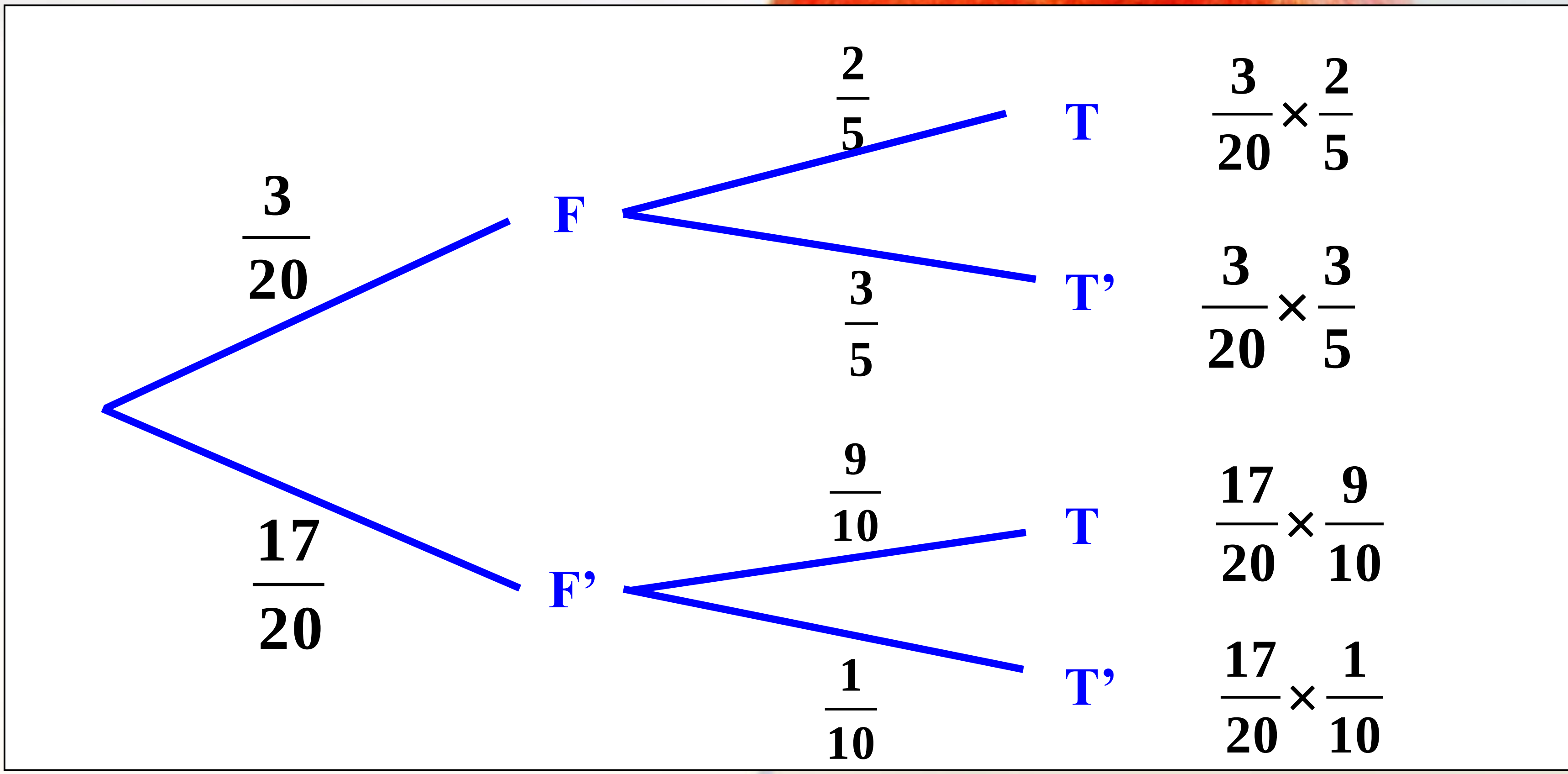
Each section sums to 1

$P(\mathrm{T} \mid \mathrm{F}) = \frac{2}{5}$ $P(\mathrm{T} \mid \mathrm{F'}) = \frac{9}{10}$ $P(\mathrm{F}) = \frac{3}{20}$



Because we only reach the 2nd set of branches after the 1st set has occurred, the 2nd set must represent conditional probabilities.

(a) Calculate the probability of him arriving on time.



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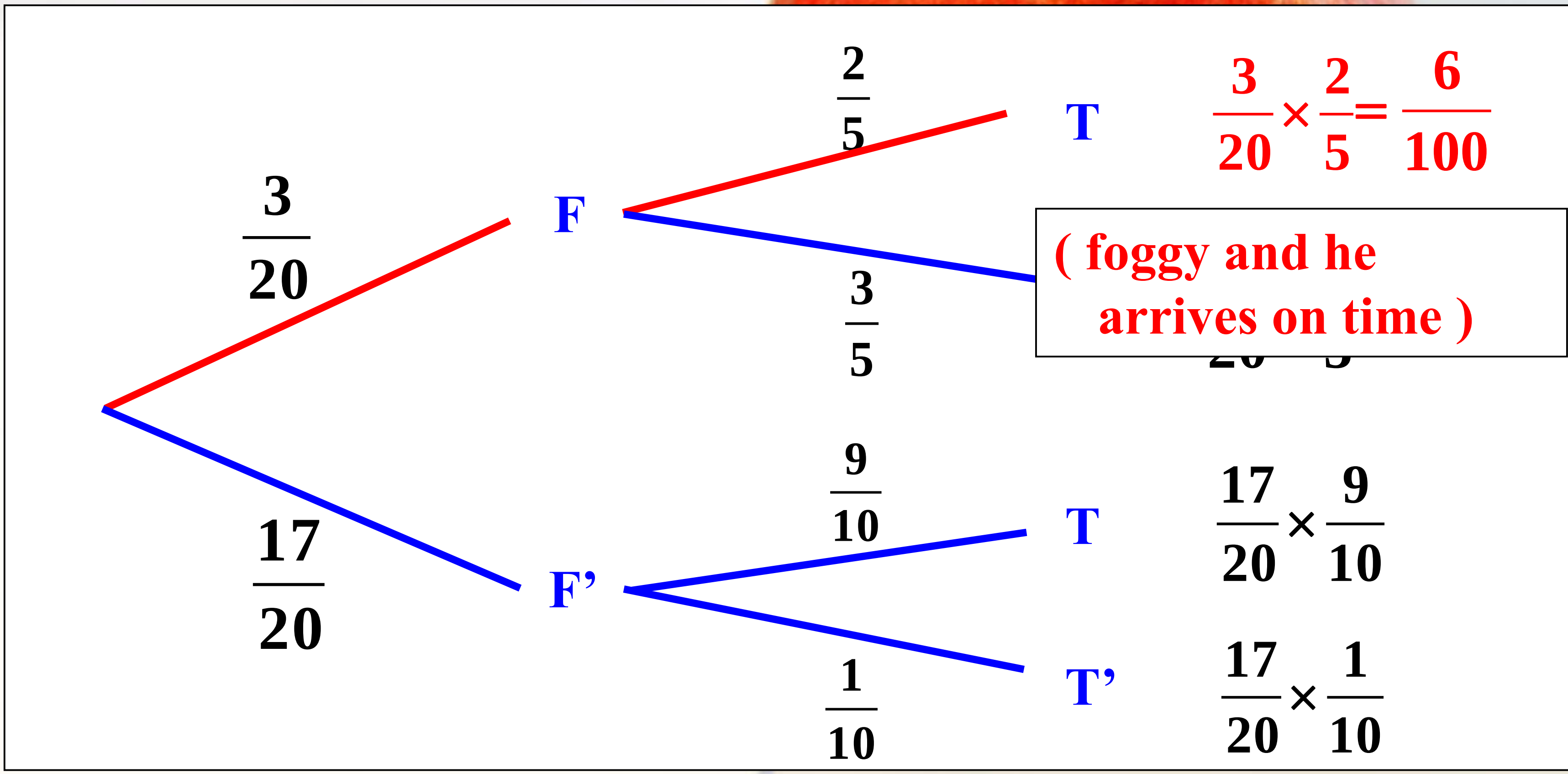

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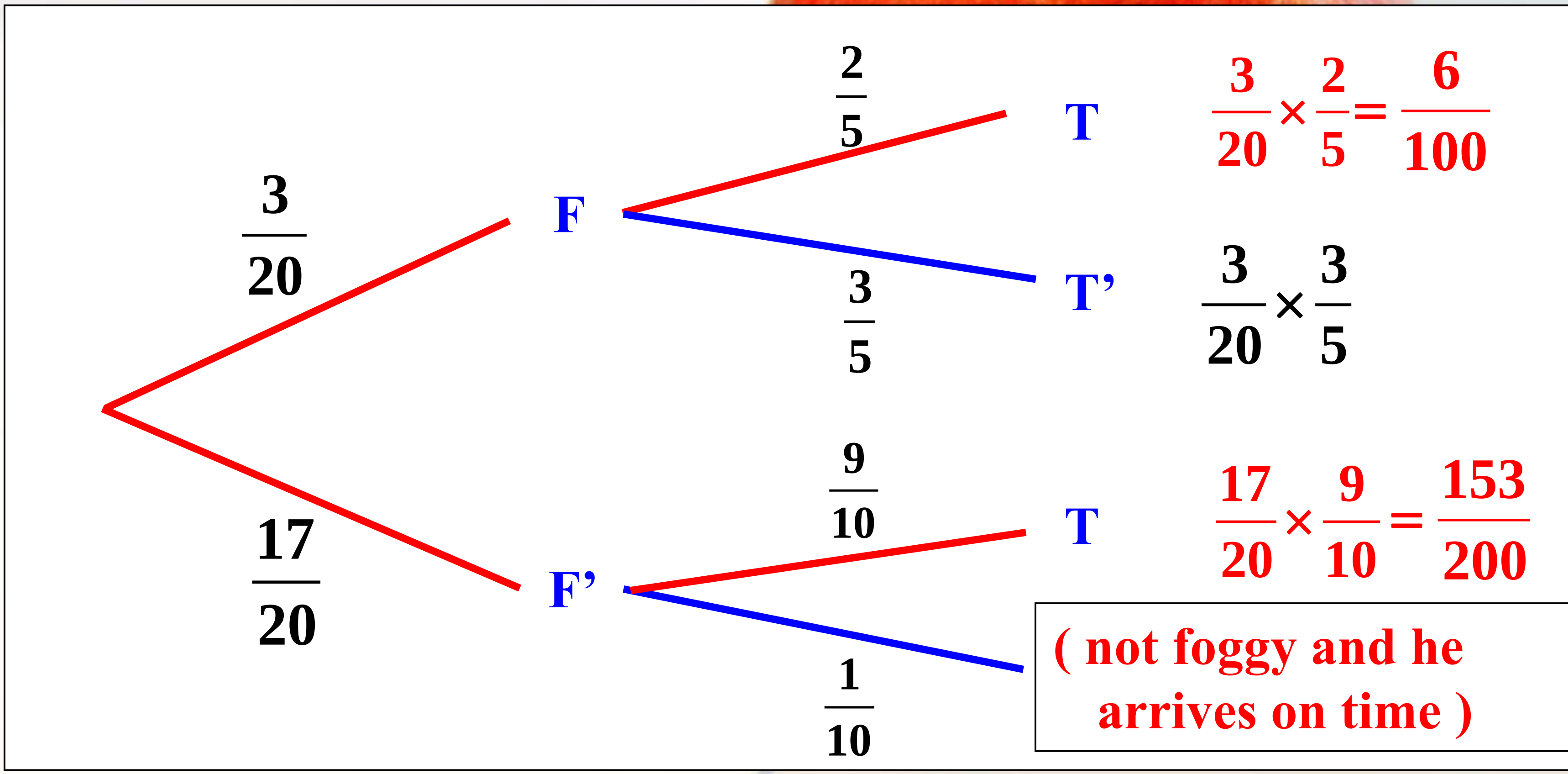
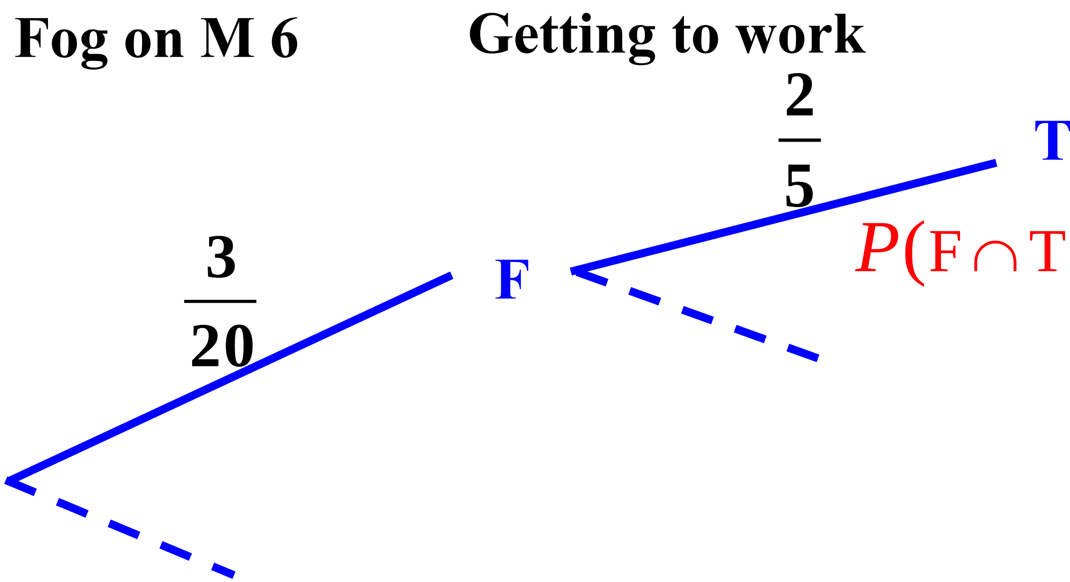


$$P(\mathrm{T}) = P(\mathrm{F} \cap \mathrm{T}) + P(\mathrm{F'} \cap \mathrm{T}) = \frac{6}{100} + \frac{153}{200} = \frac{\cancel{165}\;33}{\cancel{200}\;40} = \frac{33}{40}$$

(b) Calculate the probability that there was fog given that he arrives on time.

We need $P(\mathsf{F}|\mathsf{T})$

$$P(\mathrm{F}|\mathrm{T}) = \frac{P(\mathrm{F}\cap\mathrm{T})}{P(\mathrm{T})}$$



$$P(\mathrm{F}\cap\mathrm{T}) = \frac{3}{20}\times\frac{2}{5} = \frac{6}{100}$$

$$P(\mathrm{F}|\mathrm{T}) = \frac{P(\mathrm{F}\cap\mathrm{T})}{P(\mathrm{T})}$$

From part (a), $P(\mathsf{T}) = \dfrac{33}{40}$

$$\Rightarrow \quad P(\mathsf{F}|\mathsf{T}) = \frac{6}{100} \div \frac{33}{40} = \frac{\cancel{6}^{\,2}}{\cancel{100}_{\,5}} \times \frac{\cancel{40}^{\,2}}{\cancel{33}_{\,11}} \quad \Rightarrow \quad P(\mathsf{F}|\mathsf{T}) = \frac{4}{55}$$

Probability Trees

- In the preceding slide we saw an example of a **probability tree**. The procedure for constructing a probability tree is as follows:
  - Draw a tree diagram corresponding to all combined outcomes of the sequence of experiments.
  - Assign a probability to each tree branch.
  - The probability of the occurrence of a combined outcome that corresponds to a path through the tree is the product of all branch probabilities on the path.

Another example of a probability tree is given on the next slide.

The probability of a maximum temperature of 28° or more on the 1st day of Wimbledon (tennis competition!) has been estimated as $\frac{3}{8}$. The probability of a particular Aussie player winning on the 1st day if it is below 28° is estimated to be $\frac{3}{4}$ but otherwise only $\frac{1}{2}$.

Draw a tree diagram and use it to help solve the following:

(i) the probability of the player winning,

(ii) the probability that, if the player has won, it was at least 28°.

Solution: Let T be the event " temperature 28° or more "

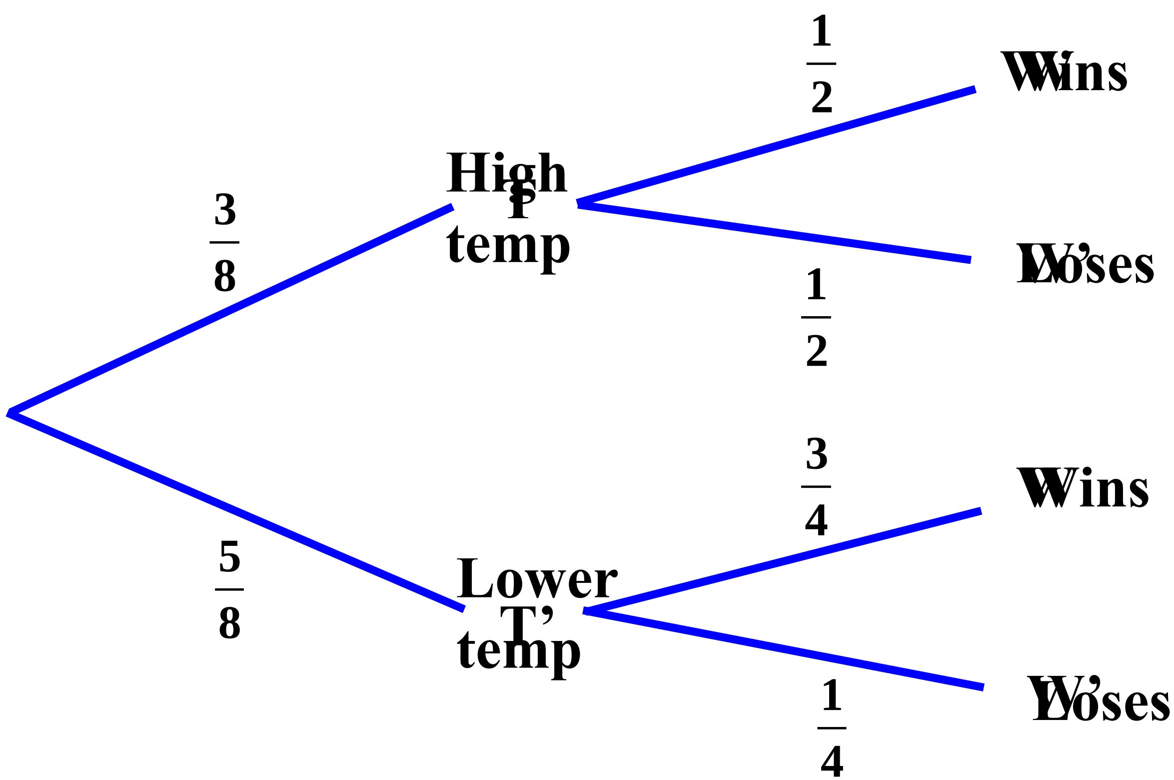
Let W be the event " player wins "

Then, $P(T)=\frac{3}{8}$ $\quad P(W|T')=\frac{3}{4}$ $\quad P(W|T)=\frac{1}{2}$

Let T be the event " temperature 28° or more "

Let W be the event " player wins "

Then, $P(T)=\dfrac{3}{8}$ $\quad P(W|T')=\dfrac{3}{4}$ $\quad P(W|T)=\dfrac{1}{2}$



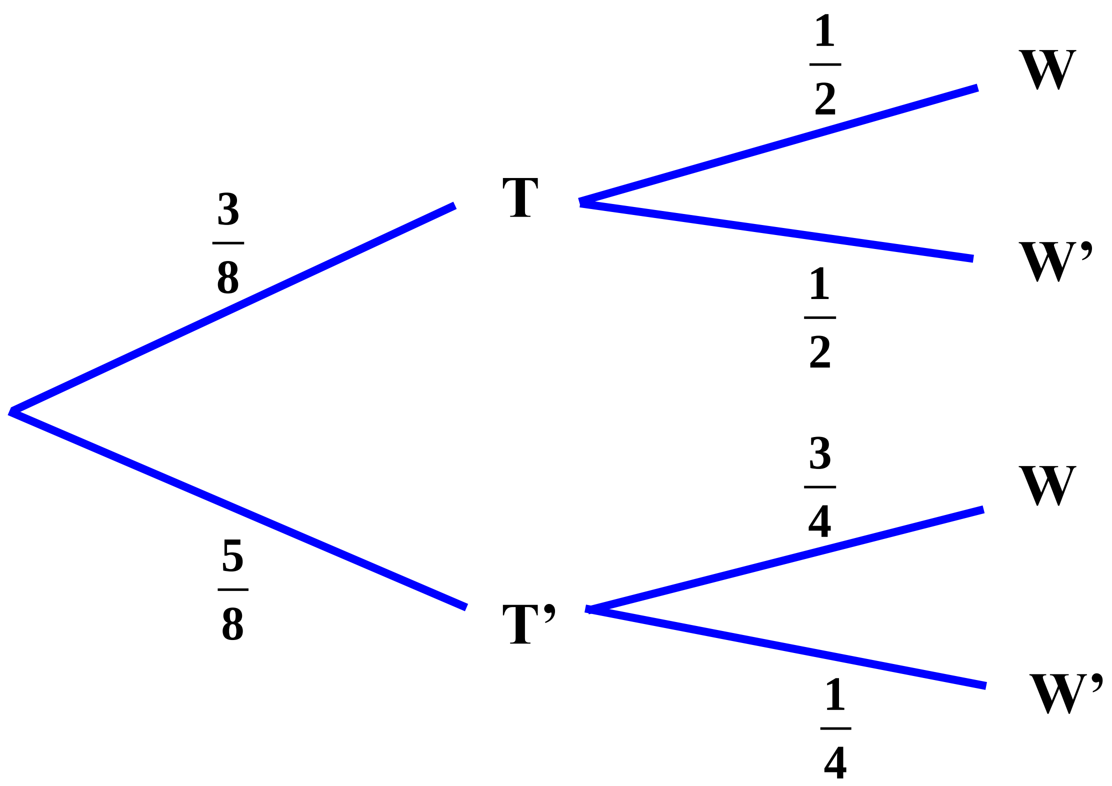
$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

$$\frac{5}{8}\times\frac{3}{4}=\frac{15}{32}$$

$$\frac{5}{8}\times\frac{1}{4}=\frac{5}{32}$$

Sum = 1



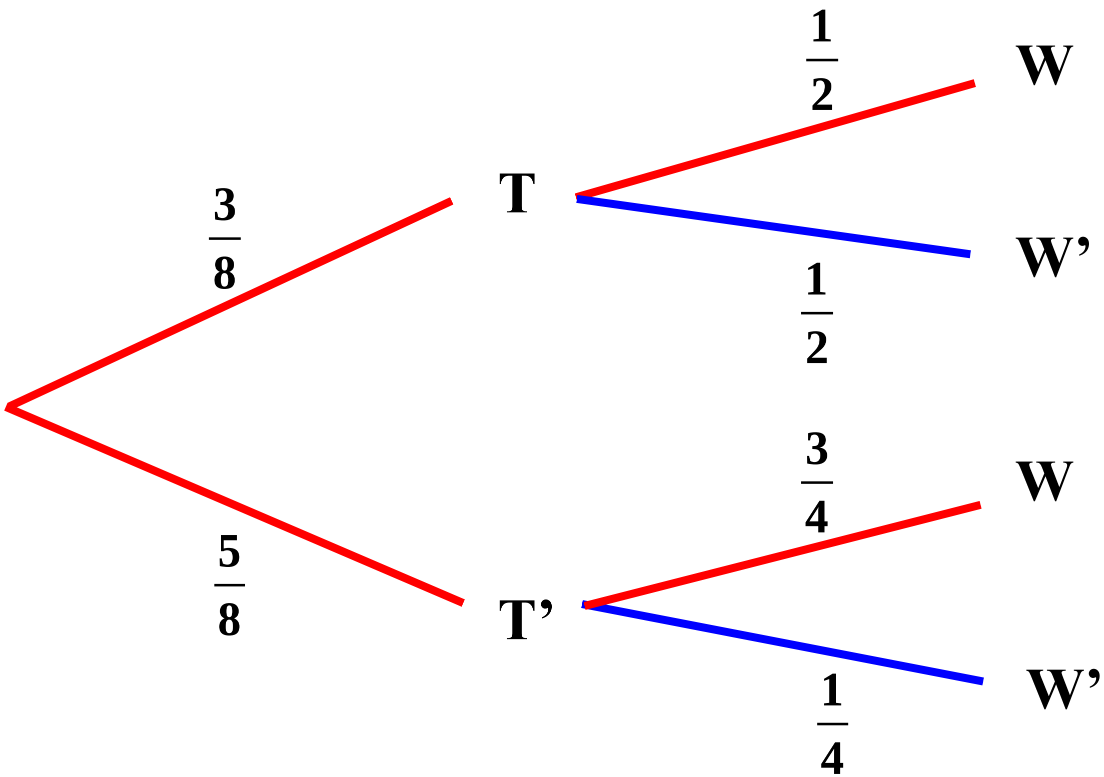
$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

$$\frac{5}{8}\times\frac{3}{4}=\frac{15}{32}$$

$$\frac{5}{8}\times\frac{1}{4}=\frac{5}{32}$$

(i) $P(\mathrm{W}) = P(\mathrm{T} \cap \mathrm{W}) + P(\mathrm{T'} \cap \mathrm{W})$



$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

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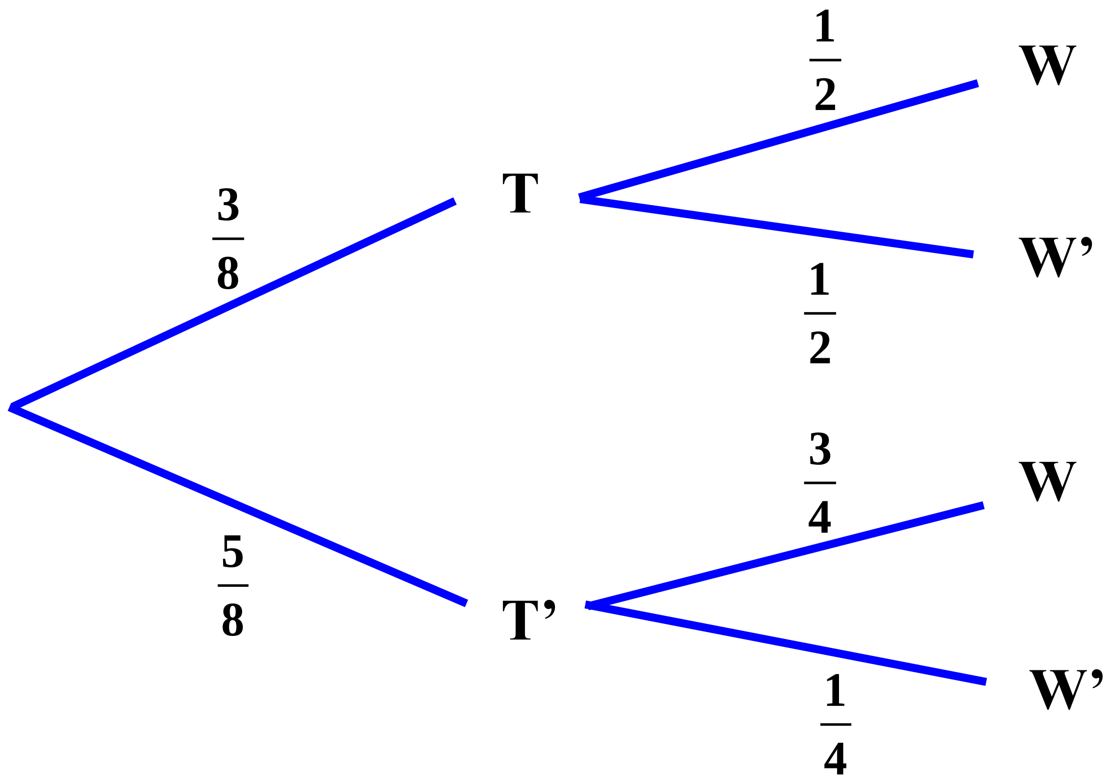
$$\frac{5}{8}\times\frac{3}{4}=\frac{15}{32}$$

$$\frac{5}{8}\times\frac{1}{4}=\frac{5}{32}$$

(i) $$P(W)=P(T\cap W)+P(T'\cap W)=\frac{3}{16}+\frac{15}{32}=\frac{6+15}{32}=\frac{21}{32}$$



$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

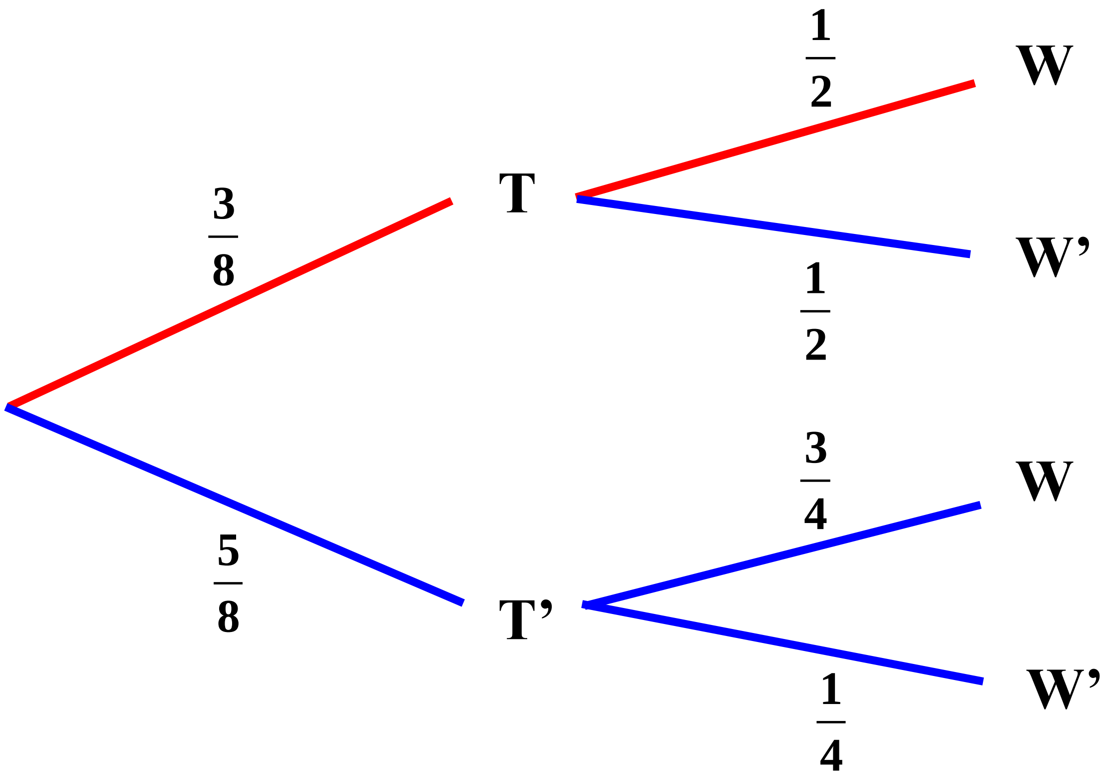
$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

$$\frac{5}{8}\times\frac{3}{4}=\frac{15}{32}$$

$$\frac{5}{8}\times\frac{1}{4}=\frac{5}{32}$$

$$P(W)=\frac{21}{32}$$

(ii) $$P(\mathrm{T}\,|\,\mathrm{W})=\frac{P(\mathrm{T}\cap\mathrm{W})}{P(\mathrm{W})}$$



$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

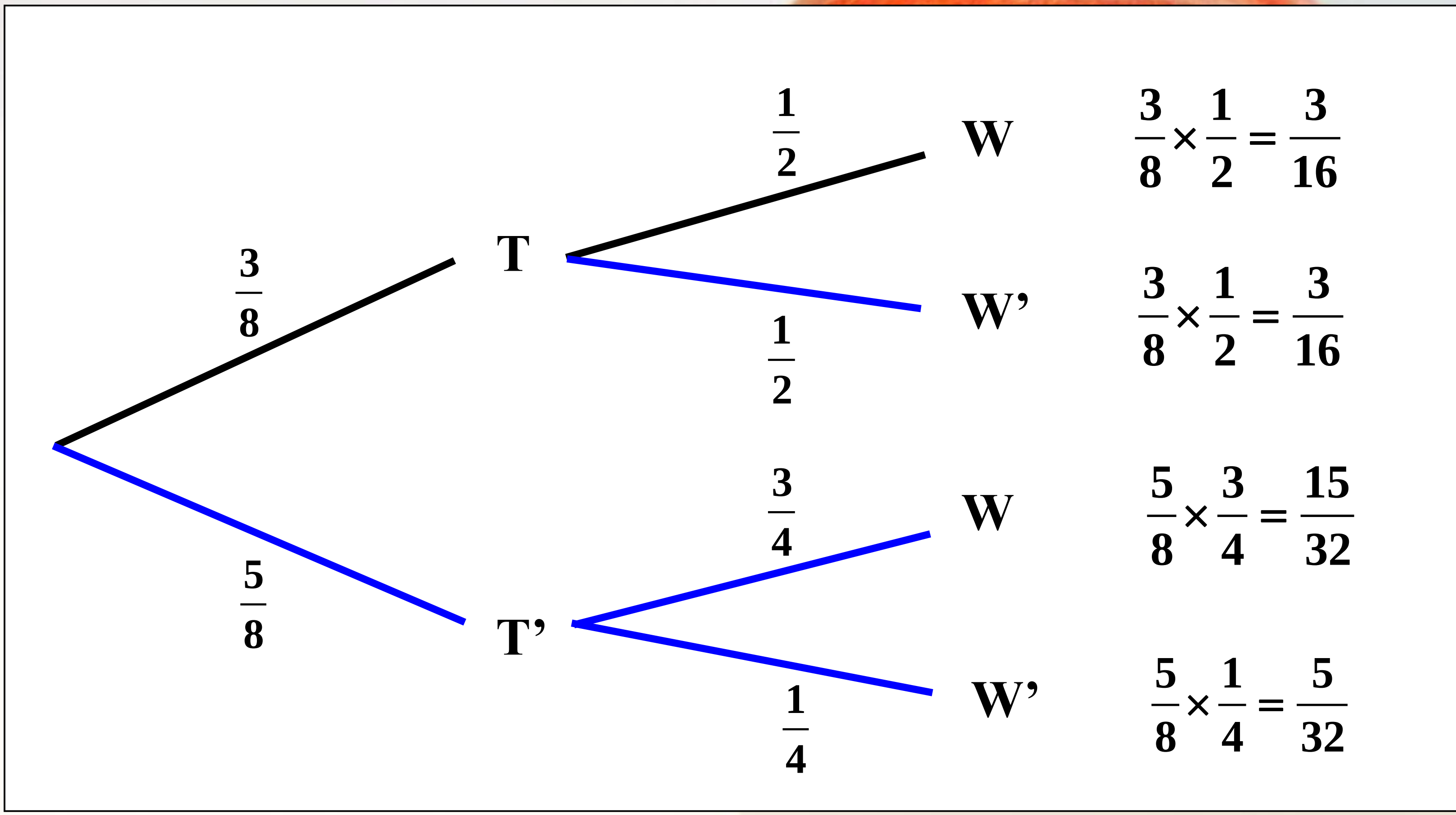
$$\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$$

$$\frac{5}{8}\times\frac{3}{4}=\frac{15}{32}$$

$$\frac{5}{8}\times\frac{1}{4}=\frac{5}{32}$$

$$P(W)=\frac{21}{32}$$

(ii) $$P(T\,|\,W)=\frac{P(T\cap W)}{P(W)} \Rightarrow P(T\,|\,W)=\frac{3}{16}\div\frac{21}{32}$$



$$P(W)=\frac{21}{32}$$

(ii) $$P(T\,|\,W)=\frac{P(T\cap W)}{P(W)} \Rightarrow P(T\,|\,W)=\frac{3}{16}\div\frac{21}{32}=\frac{\cancel{3}^{1}}{\cancel{16}_{1}}\times\frac{\cancel{32}^{2}}{\cancel{21}_{7}}=\frac{2}{7}$$

Your Turn:

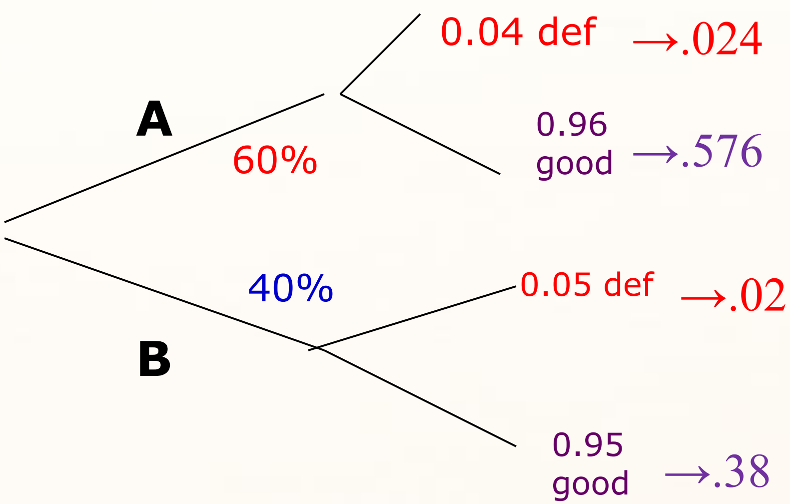
Two machines are in operation. Machine A produces 60% of the items, whereas machine B produces the remaining 40%. Machine A produces 4% defective items whereas machine B produces 5% defective items. An item is chosen at random.

What is the probability that it is defective?

Solution

Two machines are in operation. Machine A produces 60% of the items, whereas machine B produces the remaining 40%. Machine A produces 4% defective items whereas machine B produces 5% defective items. An item is chosen at random.

What is the probability that it is defective?



$$P(\text{defective}) = P(A \cap D) + P(B \cap D)$$

$$= P(A)P(D \mid A) + P(B)P(D \mid B)$$

$$= 0.6(0.04) + 0.4(0.05) = 0.044$$

Your Turn: Conditional Probability

- The disease X has a 1% prevalence in the population. There is a test for X, and
  - If you are sick, the test is positive in 90% of cases.
  - If you are not sick, the test is positive in 10% of cases.
- You have a positive test: What is the probability that you are sick?

1^{st} Event
Disease Status

2^{nd} Event
Test results

SampleSpace

.01 → Disease
- .9 → Positive Test → **Has Disease & Tests Positive .009**
- .1 → Negative Test → **Has Disease & Tests Negative .001**

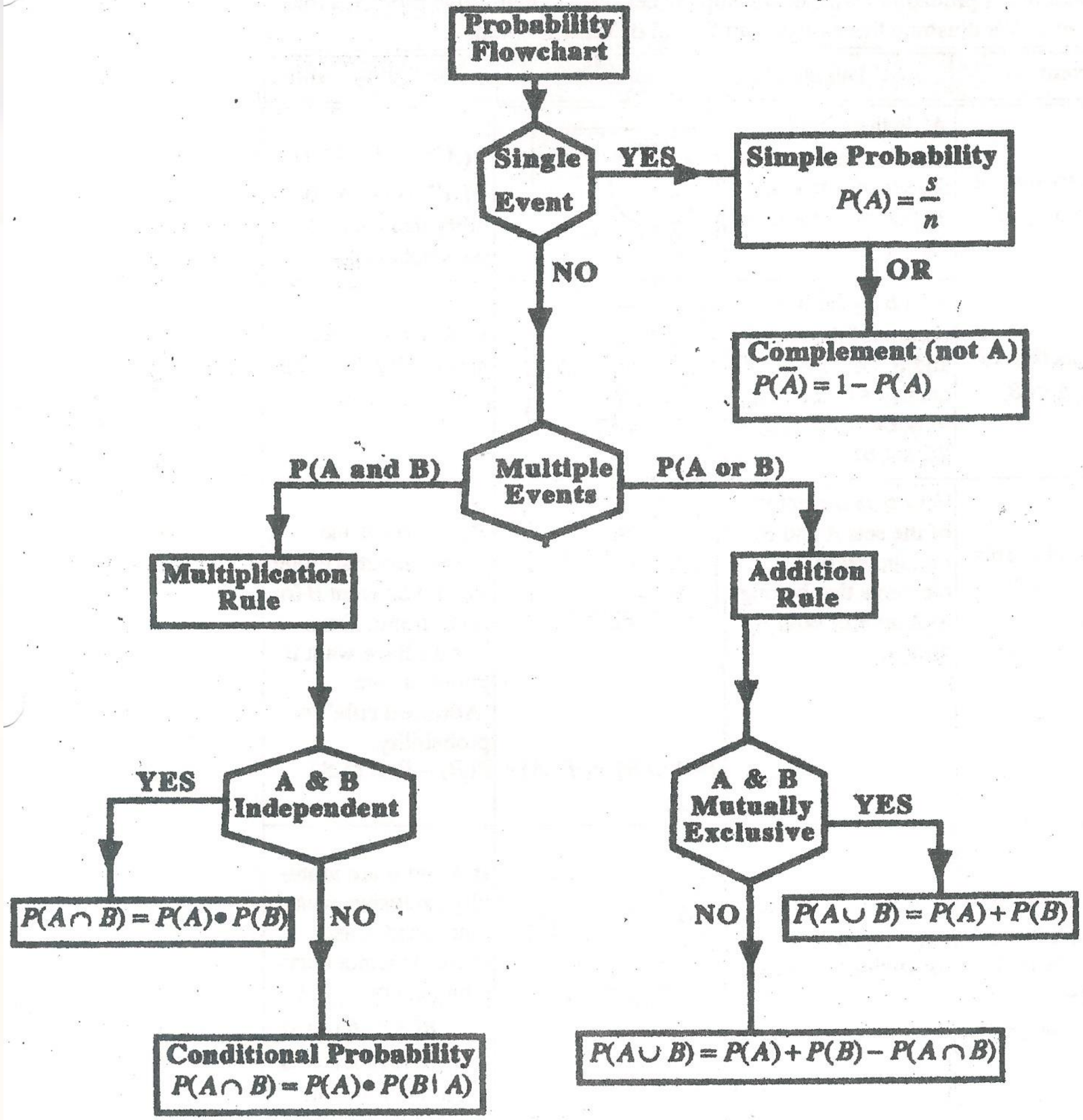
.99 → No Disease
- .1 → Positive Test → **No Disease & Tests Positive .099**
- .9 → Negative Test → **No Disease & Tests Negative .891**

Solution

- P(Disease | Positive test)
- $P(D \mid P) = P(D \cap P)/P(P)$
- $\qquad = .009/(.009 + .099)$
- $\qquad = .009/.108 = .0833$

Review

Probability



What Can Go Wrong?

- Don't use a simple probability rule where a general rule is appropriate:
  - Don't assume that two events are independent or disjoint without checking that they are.
- Don't find probabilities for samples drawn without replacement as if they had been drawn with replacement.
- Don't reverse conditioning naively.
- Don't confuse "disjoint" with "independent."

What have we learned?

- The probability rules from Chapter 14 only work in special cases—when events are disjoint or independent.
- We now know the General Addition Rule and General Multiplication Rule.
- We also know about conditional probabilities and that reversing the conditioning can give surprising results.

What have we learned?

- Venn diagrams, tables, and tree diagrams help organize our thinking about probabilities.
- We now know more about independence—a sound understanding of independence will be important throughout the rest of this course.

Assignment

- Pg. 361 – 365: #1 – 9 odd, 15, 21, 33, 41
- Read Ch-16, pg. 366 - 382