Application of fractional calculus in modeling and solving the bioheat equation

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Abstract

Fractional calculus provides novel mathematical tools for modeling physical and biological processes. The bioheat equation is often used as a first order model of heat transfer in biological systems. In this paper we describe formulation of bioheat transfer in one dimension in terms of fractional order differentiation with respect to time. The solution to the resulting fractional order partial differential equation reflects the interaction of the system with the dynamics of its response to surface or volume heating. An example taken from a study involving pulsating (on-off) cooling of a peripheral tissue region during laser surgery is used to illustrate the utility of the method. In the future we hope to interpret the physical basis of fractional derivatives using *Constructal Theory*, according to which, the geometry biological structures evolve as a result of the optimization process.
Keywords: bioheat transfer, diffusion, fractional calculus, modelling, laser surgery, fractals, temperature.

1 Introduction

The present paper considers the application of fractional calculus to the analysis of problems in bioheat transfer. The methods of fractional calculus, reviewed recently by Magin [1], are developed as the basis for formulation and solution of the bioheat transfer problem in peripheral tissue regions. Investigators have studied bioheat transfer using mathematical models for more than 50 years [5-7]. In these models tissue cooling (or warming) is approximated by coupling tissue perfusion to the bulk tissue temperature through Newton's law of cooling (or heating). In addition to full body models, there are numerous models in literature

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 ISBN 1-85312-721-3

that describe heat transfer mechanisms in a single organ or a portion of the body. In this regard, an analytical model developed by Keller and Seiler examines bio-heat transport phenomena with heat generation (metabolism) occurring in the peripheral tissue regions. The Keller and Seiler [8] model was solved numerically using parallel computers to simulate all possible modes of bioheat transfer by Boregowda et al. [9].

Recently a number of investigators [10-12] have applied the bioheat transfer model to periodic diffusion problems in localized tissue regions such as that which occurs in the skin when laser heating and/or cryogen cooling is applied. Fractional calculus is ideally suited to address this kind of periodic heating or cooling, but to our knowledge has not been used in modeling bioheat transfer either at the tissue, organ or whole body level. The present study demonstrates that fractional calculus can provide a unified approach to examine periodic heat transfer in peripheral tissue regions. For example, in an experimental study conducted by Pikkula et al. [13], cryogen spray cooling is utilized to cool the skin surface during the laser skin surgery. A generalized fractional calculus approach developed by Kulish and Lage [14 -16] is adopted to model the localized periodic bioheat transfer problems similar to the one posed by Pikkula et al. [13].

The one-dimensional heat flow problem can be completely solved for well defined surface temperature or thermal flux boundary conditions by applying Laplace transforms [17,18]. The solution can also be expressed as a fractional differential equation for the semi-infinite peripheral tissue region [14,15]. Further, the fractional differential equation can be solved to compute the heat flux at the boundary for different periodic or on-off boundary conditions that closely represent the heating and cooling of skin surface during laser surgery.

The approach offered by fractional calculus models a large class of biomedical problems that involve localized pulse heating and/or cooling. One advantage of this approach is that there is no need to solve first for the temperature in the entire domain.

2 General formulation

The approach used in this study is an approximation to the physical model developed in the study by Deng and Liu [10]. The region of interest is the boundary and its vicinity, and the total thickness is assumed to be large, so that rectangular coordinates in one dimension can be used for the analysis. Note that the outermost portion, the skin, is considered to be thin so that its thickness is not explicitly incorporated into the model. The localized tissue region that is represented by this approximate physical model is shown in fig. 1.

The generalized one-dimensional bioheat transfer equation for the temperature $T(x,t)$ in the tissue developed by Pennes [2] can be written as:

$$\rho c \frac{\partial T(x,t)}{\partial t} = K \frac{\partial^2 T(x,t)}{\partial x^2} + \omega_b \rho_b c_b (T_a - T(x,t)) + Q_m + Q_r(x,t) \qquad (1)$$

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where $\rho, c,$ and K are the density, specific heat and thermal conductivity of the tissue and ρ_b, c_b the density and specific heat of the blood, ω_b is the blood perfusion, T_a is the arterial blood temperature (assumed to be constant), Q_m is the metabolic heat generation and $Q_r(x,t)$ the heat generation due to spatial heating in the medium.

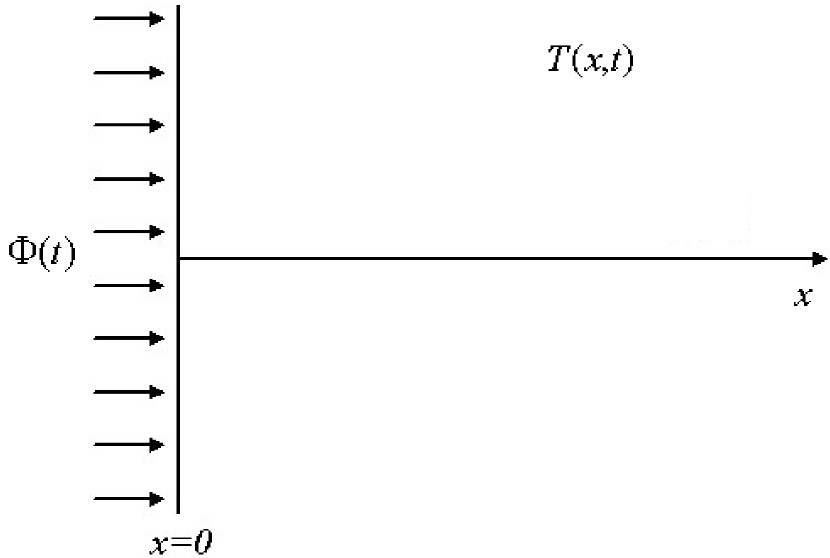

Figure 1: Assumed physical model of the localized tissue region. $T(x,t)$ represents the temperature in the tissue, while $\Phi(t)$ describes the surface thermal flux at $x=0$.

We assume that the problem has the following boundary conditions:

$T(x,0^+) = T_i(x,0)$	initial temperature distribution
$\Phi(t) = -K\dfrac{\partial T(0,t)}{\partial x}$	surface flux
$\lim\limits_{x\to\infty} T(x,t) = T_C$	constant core temperature

If we initially assume Q_r to be zero we can solve this problem following Liu et al. [12] in terms of $\widetilde{T}(x,t) = T(x,t) - T_i(x,0)$ where we have subtracted the

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initial temperature distribution $T_i(x,0)$ which is just the solution of steady state problem. Applying the Laplace transformation to eqn. (1) for the given boundary conditions we obtain for $h = \omega_b \rho_b c_b / \rho c$ and $k = K/\rho c$.

$$k\frac{\partial t(x,s)}{\partial x^2} - (s+h)\tilde{t}(x,s) = 0 \qquad \phi(s) = -K\left[\frac{\partial \tilde{t}(x,s)}{\partial x}\right]_{x=0}$$

$$\tilde{T}(x,0^+) = 0, \qquad \lim_{x\to\infty}\tilde{t}(x,s) = 0,$$

This second order ordinary differential equation has the following solution for the specified boundary conditions

$$\tilde{t}(x,s) = \frac{\sqrt{k}\phi(s)e^{-x\sqrt{(s+h)/k}}}{K\sqrt{s+h}}. \tag{2}$$

If we consider only the relationship between the flux and the temperature at the $x = 0$ boundary, then the result can be written in terms of a Laplace convolution integral as

$$\tilde{T}(0,t) = \frac{\sqrt{k}}{K}\int_0^t \Phi(t-\tau)\frac{e^{-h\tau}}{\sqrt{\pi\tau}}\,d\tau = \frac{\sqrt{k}}{K}\ \Phi(t) * \frac{e^{-ht}}{\sqrt{\pi t}}, \tag{3}$$

where we have used the Laplace transform pair $L^{-1}\left\{\frac{1}{\sqrt{s+h}}\right\} = \frac{e^{-ht}}{\sqrt{\pi t}}$.

Thus, if the surface flux is modelled by $\Phi(t) = \Phi_0 u(t)$, where $u(t)$ is the unit step function then the surface temperature will increase as

$$\tilde{T}(0,t) = \frac{\sqrt{k}}{K}\int_0^t \frac{\Phi_0 e^{-h\tau}}{\sqrt{\pi\tau}}\,d\tau = \frac{\sqrt{k}\Phi_0}{K\sqrt{h}}\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{ht}} e^{-u^2}\,du = \frac{\sqrt{k}\Phi_0}{K\sqrt{h}}\,erf\left(\sqrt{ht}\right)$$

where $u^2 = h\tau$ and the error function is defined by $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$

However, the convolution integral, eqn. (3) can also be written as

$$\tilde{T}(0,t) = \frac{\sqrt{k}}{K}\int_0^t \frac{\Phi(\tau)e^{h(t-\tau)}}{\sqrt{\pi(t-\tau)}}\,d\tau = \frac{\sqrt{k}}{K}\frac{e^{-ht}}{\sqrt{\pi}}\int_0^t \frac{\Phi(\tau)e^{h\tau}}{\sqrt{t-\tau}}\,d\tau. \tag{4}$$

which can be written in terms of the Riemann-Liouville fractional integral [19-22] defined by

$$ {}_0D_t^{-\alpha}F(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{F(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau\,, \text{ where } \Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{u}du\,. $$

Thus, eqn. (4), can be simply expressed in terms of fractional integration by

$$ \tilde{T}(0,t) = \frac{\sqrt{k}}{K}e^{-ht}\,{}_0D_t^{-1/2}\left[e^{h\tau}\Phi(\tau)\right]. $$

If we assume a step input in flux at $x=0,\ \Phi(t)=\Phi_0\,u(t)$, we can write

$$ \tilde{T}(0,t) = \frac{\sqrt{k}}{K}e^{-ht}\,{}_0D_t^{-1/2}\left[\Phi_0 e^{ht}\right], $$

which since fractional integral is a linear operator and the fractional derivative ${}_0D_t^{-1/2}\left[e^{ht}\right] = \frac{e^{ht}}{\sqrt{h}}erf\left(\sqrt{ht}\right)$ [21], gives the same result for the surface temperature as that obtained above by inversion of the Laplace transform.

In the case of a specified surface temperature at the surface $x=0$, a parallel analysis gives the surface flux in terms of the fractional semiderivative of the surface temperature, which can be written

$$ \Phi(t) = -\frac{K}{\sqrt{k}}e^{-ht}\,{}_0D_t^{1/2}\left[e^{ht}T(0,t)\right], \tag{5} $$

where the fractional derivative of order $1/2$ is defined [21] as

$$ {}_0D_t^{1/2}F(t) = \frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^t \frac{F(\tau)}{(t-\tau)^{1/2}}\,d\tau\,. $$

This result can also be obtained using Babenko's method [22,23].

Thus, for the case where $T(0,t)=T_0u(t)$, a step in surface temperature of T_0 at $x=0$, and using the semiderivative of e^{ht} [21] we obtain

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$$\Phi(t) = -\frac{KT_0}{\sqrt{k}}\left[\frac{e^{-ht}}{\sqrt{\pi t}} + \sqrt{h}\, erf\left(\sqrt{ht}\right)\right]. \tag{6}$$

A graph of this result is shown in fig. (2).

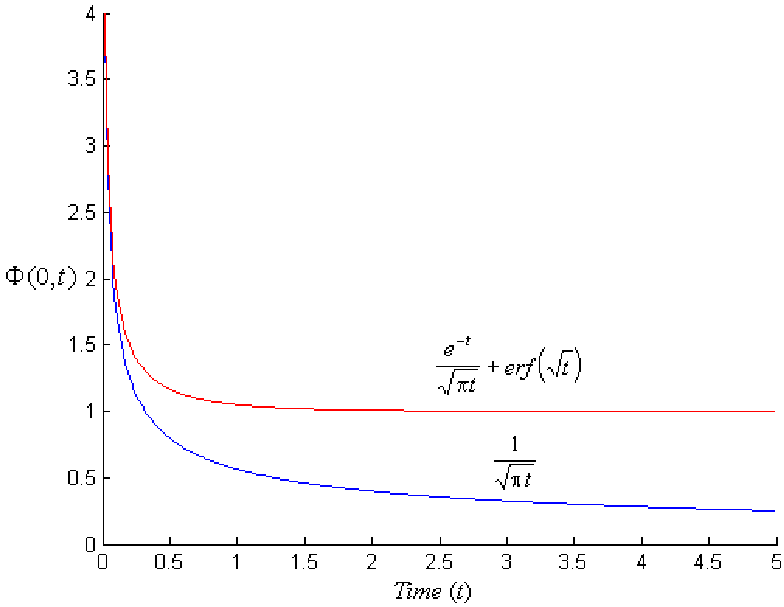

Figure 2: Graph of the flux $\Phi(0,t)$ necessary to establish a step input in temperature $T_0 u(t)$, assuming $A = K = k = 1$. Two cases are plotted: one for the bioheat equation with $h = 1$, and a second for normal diffusion without blood flow cooling, e.g., $h = 0$.

Since the relationship between flux and temperature is assumed to follow from the Fourier law for heat flux it is valid at any point in the domain, not only at the $x = 0$ surface. Therefore, for the one-dimensional problem of heating with linear surface cooling this allows us to write our fractional integral and derivative results as

$$\Phi(x,t) = -\frac{K}{\sqrt{k}} e^{-ht}\, {}_0D_t^{1/2}\left[e^{ht}\widetilde{T}(x,t)\right], \tag{7}$$

and

$$\widetilde{T}(x,t) = \frac{\sqrt{k}}{K} e^{-ht}\, {}_0D_t^{-1/2}\left[e^{ht}\Phi(x,t)\right]. \tag{8}$$

Thus, given the flux or temperature profiles at a specific location we can use this information to determine the corresponding temperature or flux. This approach could be useful in experimental situations where the half-order fractional integrals or derivatives of known functions could be used to determine the required input conditions needed for desired temperature or flux outputs [24-26]. A few examples are listed in table 1, which is adapted from Oldham and Spanier [21].

Note that, if the initial temperature distribution $T_i(x,0)$ is assumed to be uniform and constant, e.g., $T_i(x,0)=T_0$, then $\widetilde{T}(x,t)=T(x,t)-T_0$ and the flux expression, eqn. (7), becomes

$$\Phi(x,t)=-\frac{K}{\sqrt{k}}e^{-ht}{}_0D_t^{1/2}\left[e^{ht}T(x,t)\right]-\frac{KT_0}{\sqrt{k}}\left[\frac{e^{-ht}}{\sqrt{\pi t}}+\sqrt{h}\,erf\left(\sqrt{ht}\right)\right].$$

This equation simplifies for $h=0$ to

$$\Phi(x,t)=-\frac{K}{\sqrt{k}}\left[{}_0D_t^{1/2}T(x,t)-\frac{T_0}{\sqrt{\pi t}}\right]$$

which was previously derived by Kulish and Lage [14]. Kulish and Lage have recently applied fractional-diffusion theory to thermoreflectance measurements of the thermal properties of thin films under pulsed laser heating. The current bioheat model under conditions of volumetric as well as surface heating extends Kulish's results, eqn. (10) in [16], to yield

$$\Phi(x,t)=-\frac{K}{\sqrt{k}}\left[e^{-ht}{}_0D_t^{1/2}\left[e^{ht}\widetilde{T}(x,t)\right]\right]+e^{-ht}{}_0D_t^{1/2}\left[e^{ht}\widetilde{P}(x,t)\right]-K\frac{\partial P}{\partial x}$$

where $\widetilde{P}(x,t)=P(x,t)-P(x,0)$ and $P(x,t)$ represents the particular solution to the Laplace domain inhomogeneous ordinary differential equation.

In this short paper we have described a fractional calculus approach to the formulation of the bioheat equation. This method provides a simple expression for either the temperature or flux under experimental conditions often specified by laser heating and cryogen-cooling procedures. Additional studies are needed to develop a connection between the fractional order of the operators and the material structure and properties of the tissue or substate under study. Recent work by West et al. [27] and others [28-30] is directed toward establishing a stronger role for fractional calculus in describing dynamic phenomena in complex materials.

Table 1: Flux and temperature for selected input functions.

$f(t),\ t>0$	$-\frac{\sqrt{k}}{K}\Phi(x,t)=e^{-ht}\,{}_0D_t^{1/2}f(t)e^{ht}$	$\frac{K}{\sqrt{k}}T(x,t)=e^{-ht}\,{}_0D_t^{-1/2}f(t)e^{ht}$
A	$A\left[\frac{e^{-ht}}{\sqrt{\pi t}}+\sqrt{h}\,erf\left(\sqrt{ht}\right)\right]$	$A\frac{1}{\sqrt{h}}erf\left(\sqrt{ht}\right)$
Ae^{-2ht}	$A\left[\frac{e^{-ht}}{\sqrt{\pi t}}+\sqrt{h}\,e^{-ht}erf\left(\sqrt{ht}\right)\right]$	$A\frac{2e^{-ht}}{\sqrt{\pi h}}daw\left(\sqrt{ht}\right)$
$Aerf\left(\sqrt{ht}\right)$	$A\sqrt{h}$	$A\frac{1}{\sqrt{h}}\left[1-e^{-ht}\right]$
$Aerfc\left(\sqrt{ht}\right)$	$A\left[\frac{e^{-ht}}{\sqrt{\pi t}}-\sqrt{h}\,erfc\left(\sqrt{ht}\right)\right]$	$A\frac{1}{\sqrt{h}}\left[e^{-ht}-erfc\left(\sqrt{ht}\right)\right]$
$Aerfc\left(-\sqrt{ht}\right)$	$A\left[\frac{e^{-ht}}{\sqrt{\pi t}}+\sqrt{h}\,erfc\left(-\sqrt{ht}\right)\right]$	$A\frac{1}{\sqrt{h}}\left[erfc\left(-\sqrt{ht}\right)-e^{-ht}\right]$
$A/\sqrt{ht}$	$A\sqrt{h}\,\frac{\sqrt{\pi}}{2}e^{-ht/2}\left[I_1\left(\frac{ht}{2}\right)+I_0\left(\frac{ht}{2}\right)\right]$	$A\frac{\sqrt{\pi}}{\sqrt{h}}e^{-ht/2}I_0\left(\frac{ht}{2}\right)$
$Ae^{-2ht}/\sqrt{ht}$	$A\sqrt{h}\,\frac{\sqrt{\pi}}{2}e^{-3ht/2}\left[I_1\left(\frac{ht}{2}\right)-I_0\left(\frac{ht}{2}\right)\right]$	$A\frac{\sqrt{\pi}}{\sqrt{h}}e^{-3ht/2}I_0\left(\frac{ht}{2}\right)$

where $daw(x)$ is Dawson's integral defined as $daw(x)=e^{-x^2}\int_0^x e^{t^2}dt$, $I_0(x),\ I_1(x)$ are the hyperbolic Bessel functions and $erfc(x)$ is the complementary error function given as $erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt\ =\ 1-erf(x)$.

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