Applications of Double Integrals: Center of Mass and Surface Area

1. A flat plate ("lamina") is described by the region $\mathcal{R}$ bounded by $y = 0$, $x = 1$, and $y = 2x$. The density of the plate at the point (x, y) is given by the function $f(x, y)$.

   (a) Write double integrals giving the first moment of the plate about the x-axis and the first moment of the plate about the y-axis. (You need not convert to iterated integrals.)

   (b) The center of mass of the plate is defined to be the point $(\overline{x}, \overline{y})$ where

   $$\overline{x} = \frac{\text{first moment of plate about } y\text{-axis}}{\text{mass of plate}} \qquad \text{and} \qquad \overline{y} = \frac{\text{first moment of plate about } x\text{-axis}}{\text{mass of plate}}.$$

   Write expressions for $\overline{x}$ and $\overline{y}$ in terms of iterated integrals.

2. In this problem, we will look at the portion of the paraboloid $z = x^2 + y^2 + 1$ with $z < 10$. Let's call this surface $\mathcal{S}$.

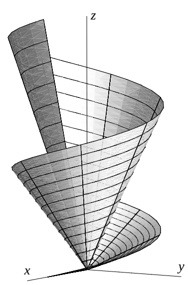
   (a) Parameterize the surface $\mathcal{S}$.[1] Describe any restrictions on the parameters.

   (b) Find the surface area of $\mathcal{S}$.

[1] Remember that this basically means we want to describe the surface using two variables — those are the parameters. Although we may use cylindrical or spherical coordinates to come up with a parameterization, our final parameterization should always describe the surface in Cartesian coordinates.

3. In each part, write a double integral that expresses the surface area of the given surface $\mathcal{S}$. Sketch the region of integration of your double integral. (You do not need to convert the double integral to an iterated integral or evaluate it.)

   (a) $\mathcal{S}$ is parameterized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, uv\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.



   (b) $\mathcal{S}$ is the part of the surface from (a) under the plane $z = 20$.

4. Find the surface area of the following surfaces.

   (a) $\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $x^2 + y^2 = 1$.

   (b) $\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $y^2 + z^2 = 1$.

   (c) $\mathcal{S}$ is a sphere of radius 1.

Applications of Double Integrals: Center of Mass and Surface Area

1. A flat plate ("lamina") is described by the region $\mathcal{R}$ bounded by $y = 0$, $x = 1$, and $y = 2x$. The density of the plate at the point (x, y) is given by the function $f(x, y)$.

   (a) *Write double integrals giving the first moment of the plate about the x-axis and the first moment of the plate about the y-axis. (You need not convert to iterated integrals.)*

   Solution. The first moment about the x-axis is $\iint_{\mathcal{R}} y f(x, y)\ dA$, and the first moment about the y-axis is $\iint_{\mathcal{R}} x f(x, y)\ dA$.

   (b) *The center of mass of the plate is defined to be the point $(\overline{x}, \overline{y})$ where*

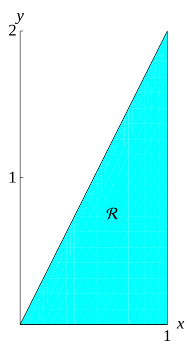
   $$\overline{x} = \frac{\text{first moment of plate about } y\text{-axis}}{\text{mass of plate}} \qquad \text{and} \qquad \overline{y} = \frac{\text{first moment of plate about } x\text{-axis}}{\text{mass of plate}}.$$

   Write expressions for $\overline{x}$ and $\overline{y}$ in terms of iterated integrals.

   Solution. We know that the mass of the plate is obtained by integrating the density, so the mass is equal to $\iint_{\mathcal{R}} f(x, y)\ dA$. So, in terms of double integrals,

   $$\overline{x} = \frac{\iint_{\mathcal{R}} x f(x, y)\ dA}{\iint_{\mathcal{R}} f(x, y)\ dA} \qquad \text{and} \qquad \overline{y} = \frac{\iint_{\mathcal{R}} y f(x, y)\ dA}{\iint_{\mathcal{R}} f(x, y)\ dA}.$$

   Since we are asked to write this in terms of iterated integrals, we need to actually look at the region $\mathcal{R}$. It looks like this:

   Let's slice vertically. Then, we are slicing the interval $[0, 1]$ on the x-axis, so the outer integral will be $\int_0^1$ something dx. Each slice has its bottom end on $y = 0$ and its top end on $y = 2x$, so the inner integral has y going from 0 to $2x$. This is true for all of the integrals we have, so

   $$\boxed{\overline{x} = \frac{\int_0^1 \int_0^{2x} x f(x, y)\ dy\ dx}{\int_0^1 \int_0^{2x} f(x, y)\ dy\ dx} \qquad \text{and} \qquad \overline{y} = \frac{\int_0^1 \int_0^{2x} y f(x, y)\ dy\ dx}{\int_0^1 \int_0^{2x} f(x, y)\ dy\ dx}}$$

2. In this problem, we will look at the portion of the paraboloid $z = x^2 + y^2 + 1$ with $z < 10$. Let's call this surface $\mathcal{S}$.

   (a) *Parameterize the surface $\mathcal{S}$. Describe any restrictions on the parameters.*

   Solution. This is the same problem as #1 on the worksheet "Parametric Surfaces". There, we came up with three possible parameterizations:

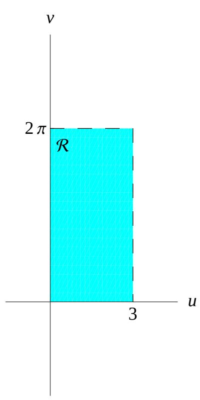
   i. $\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1 \rangle$ with $u^2 + v^2 < 9$.

   ii. $\vec{r}(u, v) = \langle u \cos v, u \sin v, u^2 + 1 \rangle$ with $0 \le u < 3$, $0 \le v < 2\pi$.

   iii. $\vec{r}(u, v) = \langle \sqrt{u-1} \cos v, \sqrt{u-1} \sin v, u \rangle$ with $1 \le u < 10$, $0 \le v < 2\pi$.

   (b) *Find the surface area of $\mathcal{S}$.*

   Solution. We can do this using any of the parameterizations from (a). Let's use the second, $\vec{r}(u, v) = \langle u \cos v, u \sin v, u^2 + 1 \rangle$ with $0 \le u < 3$, $0 \le v < 2\pi$. The region $\mathcal{R}$ in the uv-plane described by the restrictions $0 \le u < 3$, $0 \le v < 2\pi$ is a rectangle:



   We know that the surface area is given by the double integral $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v| \, dA$. Let's first calculate $|\vec{r}_u \times \vec{r}_v|$:

$$
\begin{aligned}
\vec{r}_u &= \langle \cos v, \sin v, 2u \rangle \\
\vec{r}_v &= \langle -u \sin v, u \cos v, 0 \rangle \\
\vec{r}_u \times \vec{r}_v &= \langle -2u^2 \cos v, -2u^2 \sin v, u \rangle \\
|\vec{r}_u \times \vec{r}_v| &= \sqrt{4u^4 + u^2} \\
&= \sqrt{u^2(4u^2 + 1)} \\
&= |u| \sqrt{4u^2 + 1} \\
&= u\sqrt{4u^2 + 1} \text{ since } u \ge 0
\end{aligned}
$$

   So, the double integral expressing the surface area is $\boxed{\iint_{\mathcal{R}} u\sqrt{4u^2 + 1} \, dA}$.

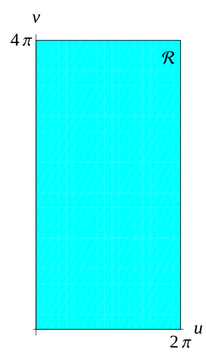
   As always, we evaluate double integrals by converting them to iterated integrals. In this case, our region of integration is a rectangle, so it makes sense to do this in Cartesian coordinates (as opposed to polar coordinates). The double integral becomes the iterated integral

$$\begin{aligned}\int_0^3 \int_0^{2\pi} u\sqrt{4u^2+1}\, dv\, du &= \int_0^3 2\pi u\sqrt{4u^2+1}\, du \\ &= \frac{\pi}{6}(4u^2+1)^{3/2}\Big|_{u=0}^{u=3} \\ &= \boxed{\frac{\pi}{6}\left(37^{3/2}-1\right)}\end{aligned}$$

3. In each part, write a double integral that expresses the surface area of the given surface $\mathcal{S}$. Sketch the region of integration of your double integral. (You do not need to convert the double integral to an iterated integral or evaluate it.)

(a) *$\mathcal{S}$ is parameterized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, uv\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.*

Solution. Since we are given a parameterization of $\mathcal{S}$, we can just write down the double integral: it is $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\, dA$, where $\mathcal{R}$ is the region in the uv-plane which describes the possible (u,v). The restrictions $0 \le u \le 2\pi$, $0 \le v \le 4\pi$ define a rectangle in the uv-plane:



Let's compute the integrand $|\vec{r}_u \times \vec{r}_v|$:

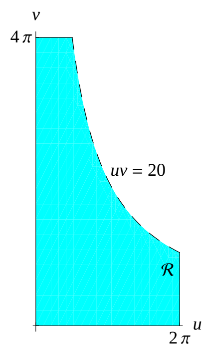
$$\begin{aligned}\vec{r}_u &= \langle \cos v, \sin v, v\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, u\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle u\sin v - uv\cos v, -u\cos v - uv\sin v, u\rangle \\ |\vec{r}_u \times \vec{r}_v| &= u\sqrt{2+v^2} \text{ since } u \ge 0\end{aligned}$$

So, a double integral which gives the surface area of $\mathcal{S}$ is $\iint_{\mathcal{R}} u\sqrt{2+v^2}\, dA$, where $\mathcal{R}$ is the region shown (the rectangle $0 \le u \le 2\pi$, $0 \le v \le 4\pi$).

(b) *$\mathcal{S}$ is the part of the surface from (a) under the plane $z = 20$.*

Solution. We can use the same parameterization as in (a), so the integrand $|\vec{r}_u \times \vec{r}_v|$ for the double integral will not change. What *will* change is the region of integration: the restriction $z < 20$ imposes restrictions on u and v.

In our parameterization $\vec{r}(u,v) = \langle u\cos v, u\sin v, uv\rangle$, $z = uv$, so the restriction $z < 20$ means $uv < 20$. So, the region of integration $\mathcal{R}$ now consists of points (u,v) satisfying $0 \le u \le 2\pi$, $0 \le v \le 4\pi$, $uv < 20$. The region looks like this:



So, a double integral expressing the surface area is $\iint_{\mathcal{R}} u\sqrt{2+v^2}\ dA$, where $\mathcal{R}$ is the region shown.

4. Find the surface area of the following surfaces.

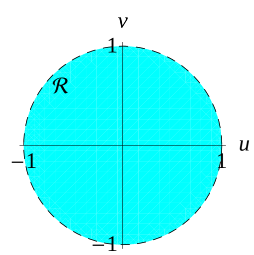
(a) *$\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $x^2 + y^2 = 1$.*

Solution. First, we need to parameterize the surface $\mathcal{S}$.

Since our surface is part of a plane, let's first parameterize the plane. Then, since we only want the part of the plane inside the cylinder $x^2 + y^2 = 1$, we'll use the inequality $x^2 + y^2 < 1$ to figure out restrictions on our parameters.

Remember that parameterizing a surface amounts to describing each point (x, y, z) on the surface using just two variables. In this case, we can easily write z in terms of x and y: $z = 12 - 3x + 3y$, so let's use x and y as our parameters. To avoid confusion, we'll rename them u and v, so our parameterization is $x = u$, $y = v$, $z = 12 - 3u + 3v$. We can write this as a parametric vector function $\vec{r}(u, v) = \langle u, v, 12 - 3u + 3v\rangle$.

Since we want $x^2 + y^2 < 1$ and $x = u$, $y = v$, the restriction on parameters that we have is $u^2 + v^2 < 1$. This describes a unit disk in the uv-plane, which we'll call $\mathcal{R}$:



We know that the double integral expressing the surface area is $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\ dA$, so let's calculate the integrand:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, -3\rangle \\ \vec{r}_v &= \langle 0, 1, 3\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 3, -3, 1\rangle \\ |\vec{r}_u \times \vec{r}_v| &= \sqrt{19} \end{aligned}$$

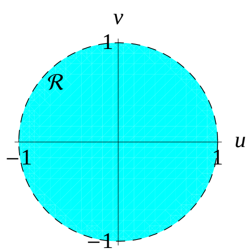
So, the double integral expressing the surface area is $\boxed{\displaystyle\iint_{\mathcal{R}} \sqrt{19}\ dA}$.

Since the region $\mathcal{R}$ is a disk, we could do this integral in polar coordinates. However, since the integrand is just a constant, there's an even easier way: we can pull the constant out to get $\iint_{\mathcal{R}} \sqrt{19}\ dA = \sqrt{19}\iint_{\mathcal{R}} 1\ dA$, and we know that $\iint_{\mathcal{R}} 1\ dA$ is the area of $\mathcal{R}$. In this case, $\mathcal{R}$ is a unit disk, so its area is π. Therefore, the value of the double integral is $\boxed{\sqrt{19}\pi}$.

(b) *$\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $y^2 + z^2 = 1$.*

Solution. Since we are talking about the same plane as in (a), we might think to parameterize the surface the same way. However, then the part of the plane lying inside the cylinder is described by $v^2 + (12 - 3u + 3v)^2 < 1$, which is a hard region to describe in the uv-plane. So, let's try a different parameterization.

The plane is described by $3x - 3y + z = 12$, which means we can easily describe any of the three variables x, y, and z in terms of the other two. Since we are restricting ourselves to points where $y^2 + z^2 < 1$, let's use $y = u$ and $z = v$; then, the region in the uv-plane is easy to describe: it's just the disk $u^2 + v^2 < 1$. Since $3x - 3y + z = 12$, $x = y - \frac{z}{3} + 4 = u - \frac{v}{3} + 4$. So, we have the parameterization $\vec{r}(u, v) = \left\langle u - \frac{v}{3} + 4, u, v\right\rangle$ with $u^2 + v^2 < 1$. If we let $\mathcal{R}$ denote the disk $u^2 + v^2 < 1$, then a double integral giving the surface area is $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\ dA$. Here's a picture of the region:



Let's write out the integrand:

$$\begin{aligned}
\vec{r}_u &= \langle 1, 1, 0\rangle \\
\vec{r}_v &= \left\langle -\frac{1}{3}, 0, 1\right\rangle \\
\vec{r}_u \times \vec{r}_v &= \left\langle 1, -1, \frac{1}{3}\right\rangle \\
|\vec{r}_u \times \vec{r}_v| &= \frac{\sqrt{19}}{3}
\end{aligned}$$

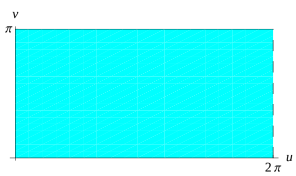
So, the double integral we want to compute is $\iint_{\mathcal{R}} \frac{\sqrt{19}}{3}\ dA = \frac{\sqrt{19}}{3}\iint_{\mathcal{R}} 1\ dA$, which is $\frac{\sqrt{19}}{3}$ times the area of $\mathcal{R}$. Since $\mathcal{R}$ is a disk of radius 1, its area is π. So, the surface area of $\mathcal{S}$ is $\boxed{\frac{\sqrt{19}}{3}\pi}$.

(c) *$\mathcal{S}$ is a sphere of radius 1.*

Solution. We can position our sphere anywhere we want, so let's put it with its center at the origin. Then, the sphere can be described very simply in spherical coordinates as the surface $\rho = 1$, so it makes sense to use θ and ϕ from spherical coordinates as our parameters. The point $(1, \theta, \phi)$ in spherical coordinates is $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. If we write $u = \theta$ and $v = \phi$, this gives us the parameterization $\vec{r}(u,v) = \langle \sin v\cos u, \sin v\sin u, \cos v\rangle$, and $0 \le u < 2\pi$, $0 \le v \le \pi$. So, if $\mathcal{R}$ is the rectangle $0 \le u < 2\pi$, $0 \le v \le \pi$, then the surface area is the double integral $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\ dA$. Let's compute the integrand:

$$\begin{aligned}
\vec{r}_u &= \langle -\sin v\sin u, \sin v\cos u, 0\rangle \\
\vec{r}_v &= \langle \cos v\cos u, \cos v\sin u, -\sin v\rangle \\
\vec{r}_u \times \vec{r}_v &= \langle -\sin^2 v\cos u, -\sin^2 v\sin u, -\sin v\cos v\rangle \\
|\vec{r}_u \times \vec{r}_v| &= \sqrt{\sin^4 v\cos^2 u + \sin^4 v\sin^2 u + \sin^2 v\cos^2 v} \\
&= \sqrt{\sin^4 v + \sin^2 v\cos^2 v} \\
&= \sqrt{\sin^2 u(\sin^2 v + \cos^2 v)} \\
&= \sqrt{\sin^2 v} \\
&= |\sin v|
\end{aligned}$$

Since $0 \le v \le \pi$, $\sin v \ge 0$, so $|\sin v| = \sin v$. Therefore, an integral giving the surface area of the sphere is $\boxed{\iint_{\mathcal{R}} \sin v\ dA \text{ where } \mathcal{R} \text{ is the rectangle } 0 \le u < 2\pi,\ 0 \le v \le \pi}$. Here is a sketch of the region:



To compute, we need to convert this to an iterated integral. Since the region is a rectangle, it's easiest to do this in Cartesian coordinates, and we get

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} \sin v\ dv\ du &= \int_0^{2\pi} \left(-\cos v\Big|_{v=0}^{v=\pi}\right)\ du \\
&= \int_0^{2\pi} 2\ du \\
&= \boxed{4\pi}
\end{aligned}$$