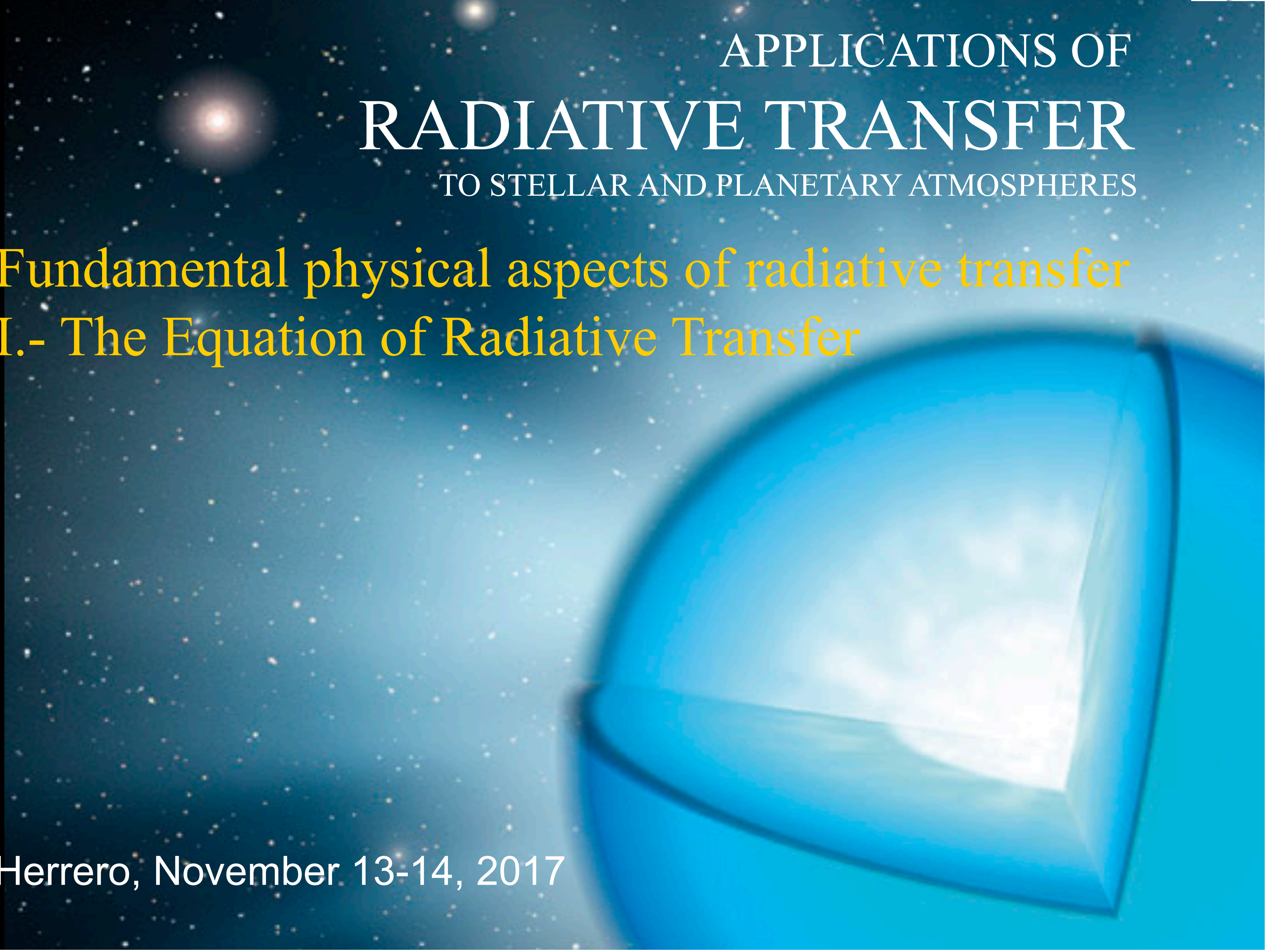

XXIX IAC Winter School of Astrophysics

APPLICATIONS OF

RADIATIVE TRANSFER

TO STELLAR AND PLANETARY ATMOSPHERES

Fundamental physical aspects of radiative transfer

I.- The Equation of Radiative Transfer

Artemio Herrero, November 13-14, 2017

Bibliography

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I. Hubeny & D. Mihalas (2014): Theory of stellar atmospheres

D. Mihalas (1978): Stellar Atmospheres

R.J. Rutten: Radiative transfer in stellar atmospheres, web lectures notes

Others:

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Unsöld (1955): Physik der Sternatmosphären

K.R. Lang (2006): Astrophysical Formulae

Warning: Not much time for lectures. Fell free to ask questions during the WS

Outline

I. The Radiative Transfer Equation

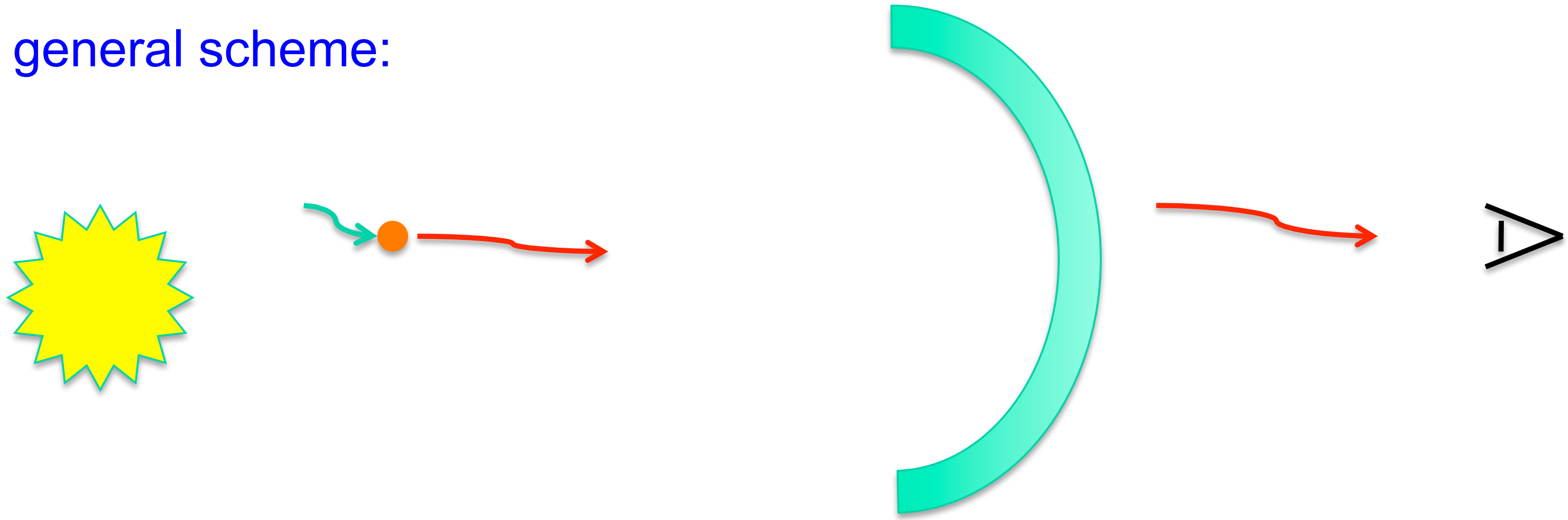
- Introduction
- Optical depth and source function
- Formal Solution of the RTE
- The bottom of the atmosphere: diffusion approximation
- The surface of the atmosphere: Eddington-Barbier approximation

Intro

The energy emitted by the stars has properties that we can register

- A smooth variation with the wavelength (that we call “the continuum”)
- Spectral features that vary rapidly with wavelength (“spectral lines”) and that reflect the physical conditions in the stellar (or planetary) atmosphere

A general scheme:



Energy is generated in the stellar interior by nuclear reactions mainly in form of photons

Radiation interacts with matter and the properties of both of them change during interaction

When radiation reaches the outermost layers it suffers a last interaction with matter and scapes

The observed spectrum carries information on the physical conditions found by radiation in its way out

Introduction

We will assume the macroscopic view presented by Prof. L. Crivellari

- Equation of radiative transfer

$$\frac{\partial}{\partial t}\left(\frac{I_\nu}{c}\right)+\vec{\nabla}\left(\frac{I_\nu}{c}c\vec{n}\right)=\eta_\nu-\chi_\nu I_\nu$$

- Specific Intensity

$$I\left(\vec{n},\nu,t,\vec{r}\right)=\lim_{\Delta A,\Delta\Omega,\Delta t,\Delta\nu\to 0}\frac{\Delta E_\nu}{\Delta A\cos\theta\Delta\Omega\Delta\nu\Delta t}=\frac{dE_\nu}{dA\cos\theta d\Omega d\nu dt}$$

  - Distance independent

- Emission and absorption coefficients

$$\eta_\nu=\lim_{\Delta V,\Delta\Omega,\Delta t,\Delta\nu\to 0}\frac{\Delta E_\nu}{\Delta V\Delta\Omega\Delta\nu\Delta t}$$

$$\frac{\delta I_\nu}{I_\nu}=-\chi_\nu\delta s$$

- Note that the inverse of the absorption coefficient is the mean free path, and that both coefficients may be anisotropic

Introduction

Let's consider some particular cases of the RTE

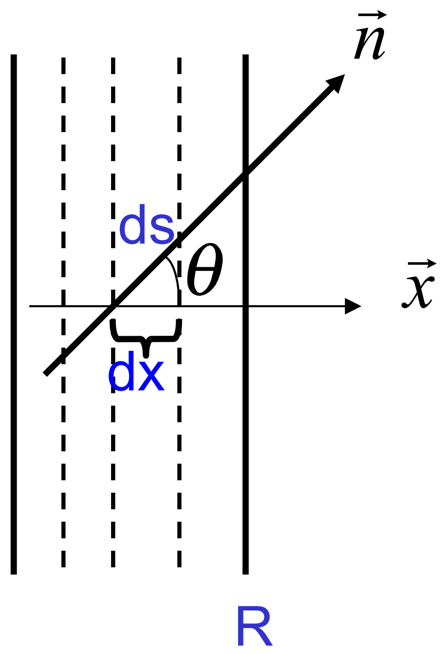
1. Stationary atmosphere

$$\frac{\partial I_\nu}{\partial t} = 0$$

$$\vec{\nabla}\left(I_\nu \vec{n}\right) = \eta_\nu - \chi_\nu I_\nu = \frac{\partial I_\nu}{\partial x}\vec{n}_x \cdot \vec{n} + \frac{\partial I_\nu}{\partial y}\vec{n}_y \cdot \vec{n} + \frac{\partial I_\nu}{\partial z}\vec{n}_z \cdot \vec{n}$$

2. Plane-parallel geometry

The atmosphere is composed of parallel planes. The angle beteen the light ray propagation direction and the normal to the surface is constant



$$\frac{\partial I_\nu}{\partial z} = \frac{\partial I_\nu}{\partial y} = 0 \text{ y } \left(\vec{n}_x \cdot \vec{n}\right) = \cos\theta = \mu$$

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mu \frac{\partial I_\nu}{\partial x} = \eta_\nu - \chi_\nu I_\nu$$

3. Plane-parallel and stationary atmosphere

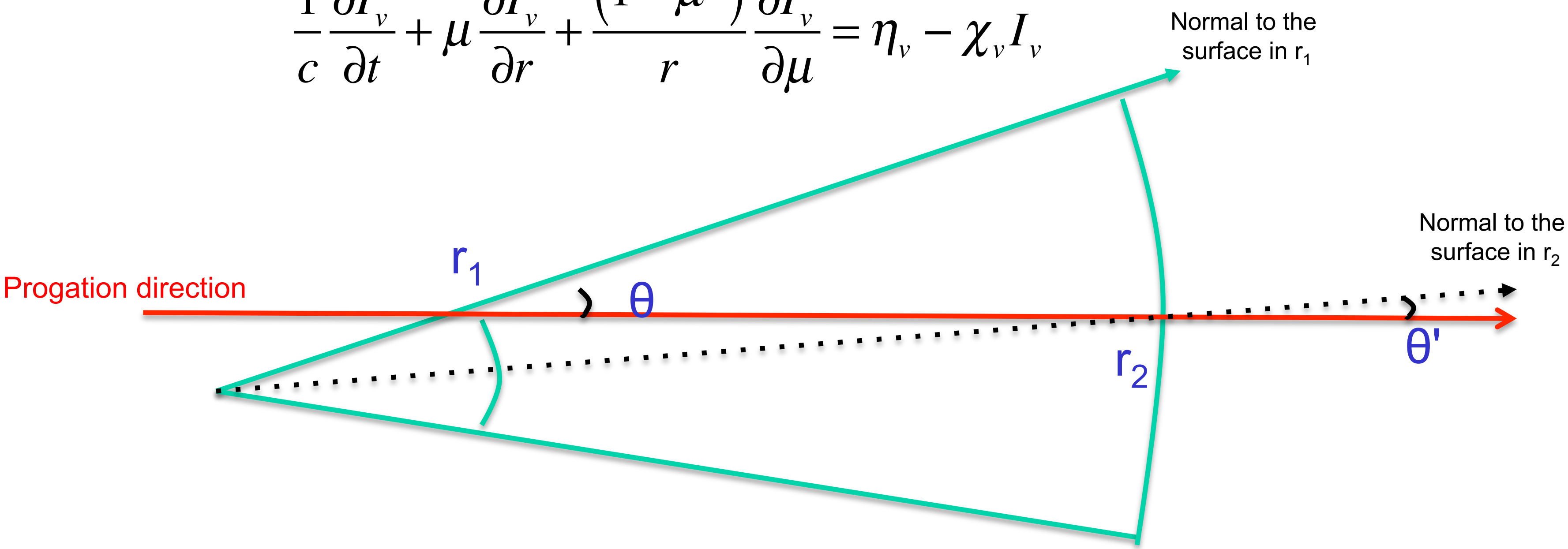
$$\mu \frac{dI_\nu}{dx} = \eta_\nu - \chi_\nu I_\nu$$

Introduction

4. Spherical geometry

Now we are interested in the radial and angular coordinates (we assume azimutal simmetry

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t}+\mu\frac{\partial I_\nu}{\partial r}+\frac{\left(1-\mu^2\right)}{r}\frac{\partial I_\nu}{\partial \mu}=\eta_\nu-\chi_\nu I_\nu$$



In spherical geometry the angle between the light ray propagation direction and the normal to surface changes when the ray propagates

Introduction

Intensity moments in plane parallel geometry (Eddington formulation)

- Mean intensity

$$J_\nu(\vec{r},t) = \frac{1}{4\pi}\oint I_\nu(\vec{n},\vec{r},t)\,d\Omega \Rightarrow J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu^0 \,d\mu$$

Related to energy density $u = \frac{4\pi}{c}J$

we speak of emergent intensity (I^+) when $\mu>0$ and of incident intensity (I^-) when $\mu<0$

- Radiative flux

$$\vec{\mathfrak{F}}_\nu(\vec{r},t) = \frac{1}{4\pi}\oint I_\nu(\vec{n},\vec{r},t)\vec{n}\,d\Omega \Rightarrow H_\nu = \frac{\mathfrak{F}_\nu}{4\pi} = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu \,d\mu$$

H_ν is the Eddington flux. Sometimes, also the astrophysical flux is used: $F_\nu = \frac{\mathfrak{F}_\nu}{\pi} = \frac{H_\nu}{4}$

Related to the total amount of energy crossing a surface

$$\mathfrak{F}_\nu^+ = \pi B_\nu \qquad \mathfrak{F}^+ = \sigma T_{\rm eff}^4$$

$$L = 4\pi R^2 \sigma T_{\rm eff}^4$$

- Second moment

$$\underline{\underline{T}}_\nu(\vec{r},t) = \frac{1}{c}\oint I_\nu(\vec{n},\vec{r},t)(\vec{n}\vec{n})\,d\Omega \Rightarrow K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu^2 \,d\mu$$

Related to radiation pressure

$$P_R = \frac{1}{3}u \qquad K = \frac{1}{3}J$$

Introduction

Some flux properties(1):

$$\mathfrak{F}_\nu = \mathfrak{F}_\nu^+ - \mathfrak{F}_\nu^- = 2\pi \int_0^{+1} I_\nu \mu d\mu - 2\pi \int_{-1}^{0} I_\nu \mu d\mu \Rightarrow \mathfrak{F}_\nu^+ = 2\pi \int_0^{+1} I_\nu \mu d\mu$$

assuming an isotropic field

$$\mathfrak{F}_\nu^+ = 2\pi I_\nu \int_0^{+1} \mu\, d\mu = 2\pi I_\nu \left[\mu^2 / 2 \right]_0^{+1} = \pi I_\nu = \{\text{if } I_\nu = B_\nu\} = \pi B_\nu$$

$$\mathfrak{F}_\nu = \mathfrak{F}_\nu^+ - \mathfrak{F}_\nu^- = 0$$

The total amount of energy flux scaping the star will be

$$\mathfrak{F}^+ = \int_0^\infty \mathfrak{F}_\nu^+ d\nu = \pi \int_0^\infty I_\nu^+ d\nu = \{ \text{ assuming again } I_\nu = B_\nu \} = \pi B = \sigma T^4$$

from where we can define the effective temperature of the star as

$$\mathfrak{F}^+ = \sigma T_{\text{eff}}^4$$

and the total amount of energy leaving the star per unit time will be

$$L = \oint \mathfrak{F}^+ dS = 4\pi R_*^2 \sigma T_{\text{eff}}^4$$

Introduction

Some flux properties(2):

By energy conservation the flux reaching an external observer at Earth will decay with the square of the distance

$$4\pi R_*^2 \mathfrak{F}^+ = 4\pi d^2 f_{obs}$$

Thus

$$\mathfrak{F}^+ = \sigma T_{\rm eff}^4 = \left(\frac{d}{R}\right)^2 f_{obs}$$

which can be used to determine any of the magnitudes, assuming we know the others (and that we know the changes suffered by radiation in their travel from the star to the Earth)

In plane parallel geometry, the net flux is conserved.

In spherical geometry, it decreases with r^2

Optical Depth

From the expression of the absorption coefficient we have

$$\frac{dI_\nu}{I_\nu} = -\chi_\nu ds$$

the solution of this equation is

$$I_\nu(s) = I_\nu(0)e^{-\int_0^s \chi_\nu ds}$$

We define the optical depth as

$$\tau_\nu = \int_0^s \chi_\nu ds$$

$$I_\nu(s) = I_\nu(0)e^{-\tau_\nu}$$

We see that for $\tau_\nu > 1$ the emergent intensity decays rapidly.

An observer will see mostly (but not only!) radiation coming from $\tau_\nu \le 1$

Optical depth

An important property is that photons will travel a mean optical distance $\Delta\tau_\nu \sim 1$

The mean value of a variable x is: $\langle x \rangle = \dfrac{\int f(x)\, x\, dx}{\int f(x)\, dx}$

Thus $\langle \tau_\nu \rangle = \dfrac{\int_0^\infty e^{-\tau_\nu} \tau_\nu\, d\tau_\nu}{\int_0^\infty e^{-\tau_\nu}\, d\tau_\nu}$ (for simplicity, now we don't write the subindex ν)

The first integral is:

$$\int_0^\infty e^{-\tau} \tau\, d\tau = \left\{ \begin{array}{c} u = \tau \rightarrow du = d\tau \\ d\upsilon = e^{-\tau} d\tau \rightarrow \upsilon = -e^{-\tau} \end{array} \right\} = \left[-\tau e^{-\tau} \right]_0^\infty - \int_0^\infty \left(-e^{-\tau} \right) d\tau =$$

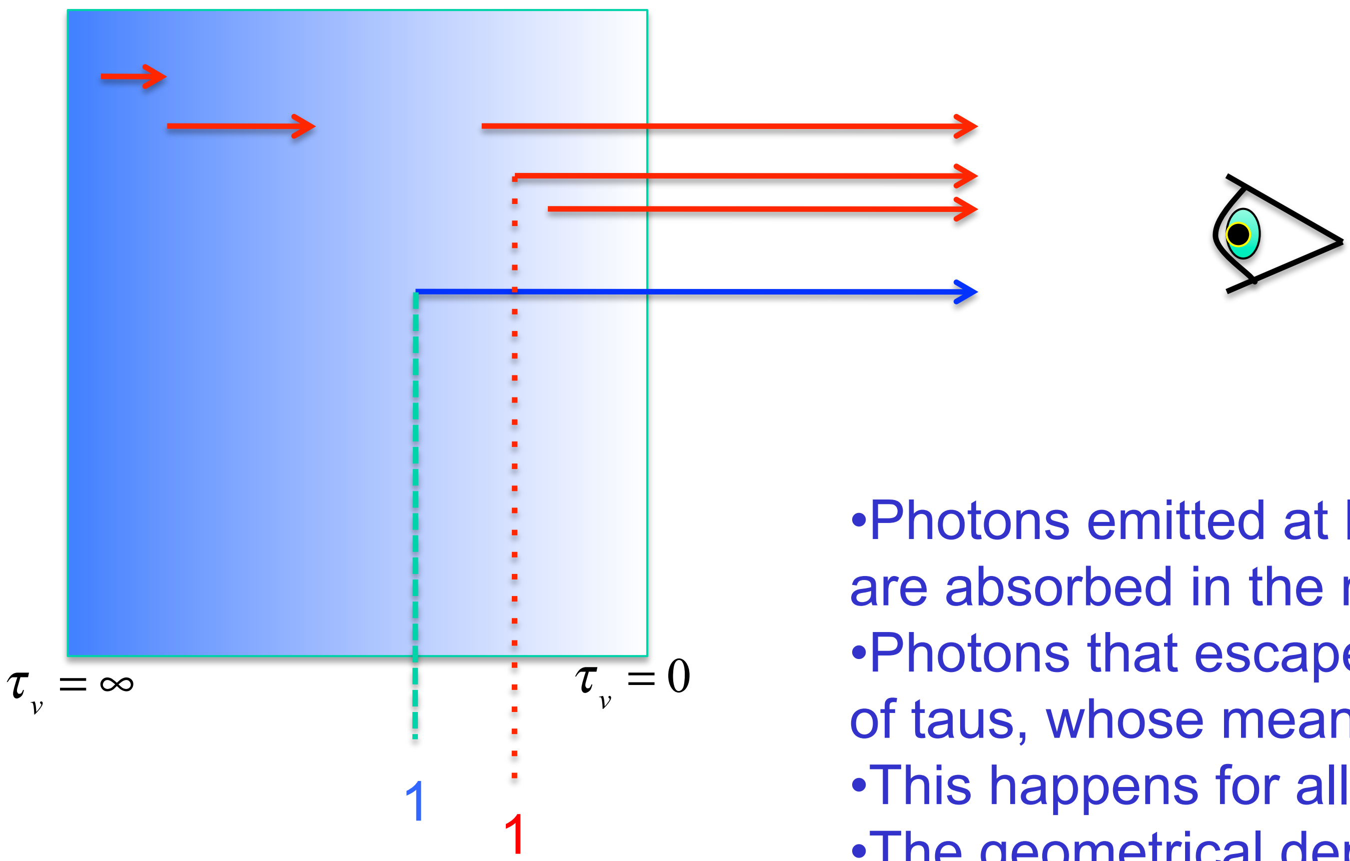
$$= \left[\tau e^{-\tau} \right]_\infty^0 + \int_0^\infty e^{-\tau}\, d\tau = 0 + \left[-e^{-\tau} \right]_0^\infty = \left[e^{-\tau} \right]_\infty^0 = 1$$

and the second, $\int_0^\infty e^{-\tau}\, d\tau = 1$

Thus $\langle \tau_\nu \rangle = 1$

Optical depth

If we have a medium (like a stellar atmosphere of semi-infinite optical depth) photons will come from $\tau_\nu = 1$



- Photons emitted at larger optical depth are absorbed in the medium
- Photons that escape come from a range of taus, whose mean value is 1.
- This happens for all frequencies
- The geometrical depth from which photons come will depend on frequency :

$$d\tau_\nu = \kappa_\nu ds$$

Source function

The optical depth is a convenient variable to study radiative transfer phenomena

- We reformulate the radiative transfer equation. For a given direction

$$\mu \frac{dI_\nu}{dx} = \eta_\nu - \chi_\nu I_\nu$$

with $d\tau_\nu = -\chi_\nu dx$ we have $\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \frac{\eta_\nu}{\chi_\nu}$

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

where S_ν is the so-called source function, that can be interpreted as the energy emitted along a photon mean free path

- The problem of knowing the emergent intensity is solved if we know $S_\nu(\tau_\nu)$

Source function – some cases

Thermodynamic Equilibrium

- We know that in TE (Kirchhoff law)

$$\frac{\eta_\nu}{\chi_\nu} = S_\nu = B_\nu(T)$$

- In TE the source function is the Planck function and is completely linked to T

Local Thermodynamic Equilibrium

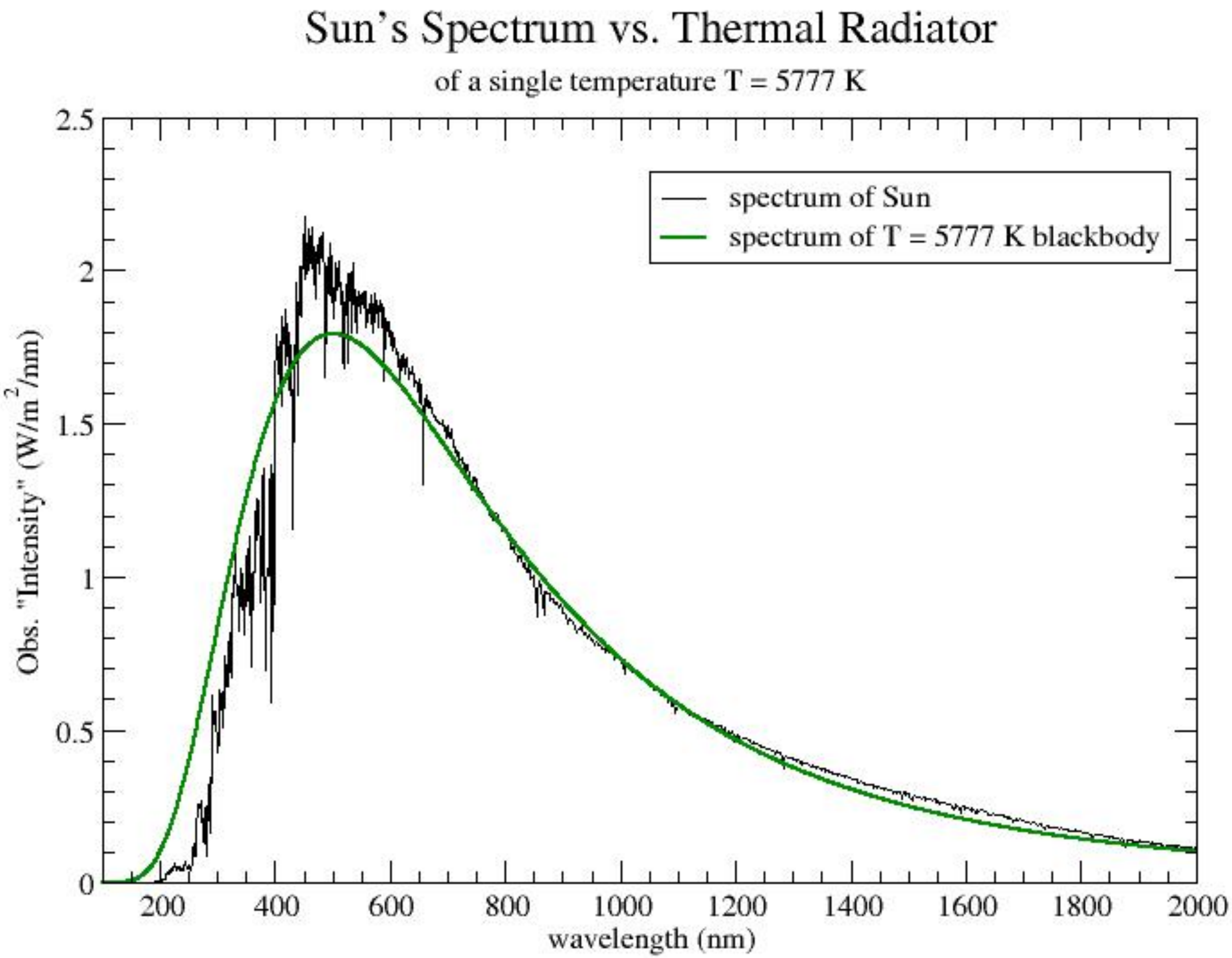
- We know that (in general) T decreases outwards in the stellar atmosphere
- Let's assume that we can set the *local* source function to the Planck function at the *local* temperature

$$\frac{\eta_\nu}{\chi_\nu} = S_\nu = B_\nu(T(\tau_\nu)) \text{ at any point in the atmosphere}$$

$\Rightarrow$ complete coupling of the source function to temperature

- If we know the temperature structure, the RTE can be solved

Source function – some cases

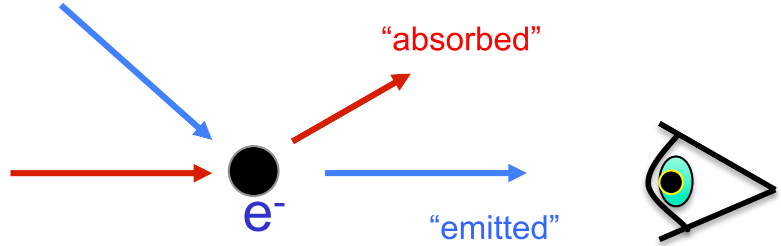


Provided by Rob Jeffries in https://physics.stackexchange.com/questions/235157/

Source function – some cases

Coherent scattering

- We have a photon scattered by a particle (usually a free electron). What will be the source function?



Be ε_ν and σ_ν the emission and absorption coefficients due to coherent scattering. The energy "emitted" and "absorbed" in all directions will be

$$E_\nu^e = \oint \varepsilon_\nu d\Omega \qquad E_\nu^a = \oint \sigma_\nu I_\nu d\Omega$$

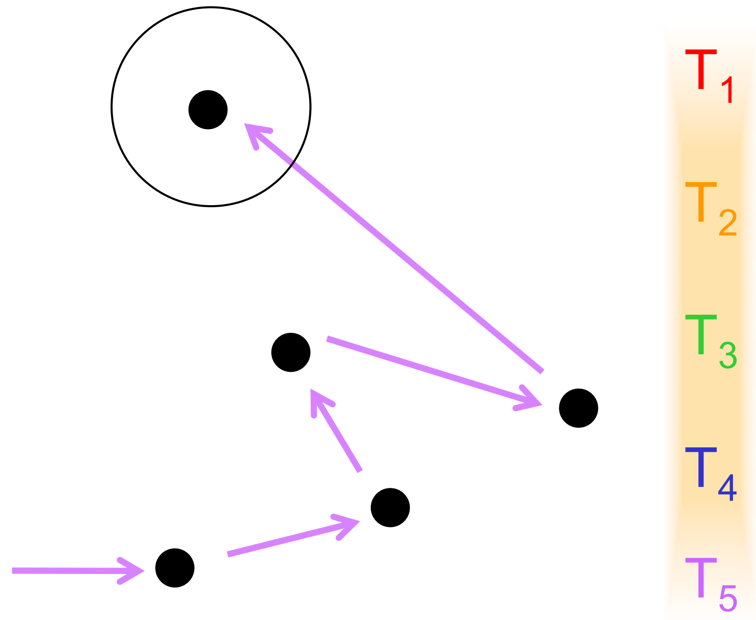
As photons have simply been scattered, we have

$$E_\nu^e = E_\nu^a \Rightarrow \oint \varepsilon_\nu d\Omega = \oint \sigma_\nu I_\nu d\Omega$$

Assuming isotropic coefficients:

$$\frac{\varepsilon_\nu}{\sigma_\nu} = \frac{\oint I_\nu d\Omega}{\oint d\Omega} = \frac{1}{4\pi}\oint I_\nu d\Omega = J_\nu = S_\nu$$

The radiation field at T_1 has characteristic of conditions at T_5



Scattering tends to decouple the source function from the local conditions

Source function – some cases

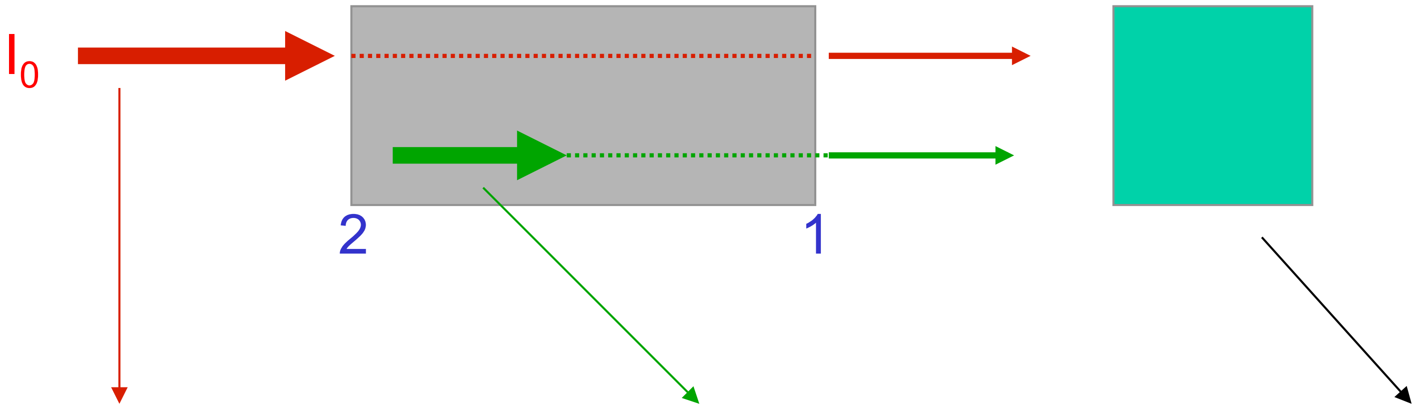
If the scattering is non-coherent or non-isotropic it will couple different frequencies and directions:

$$E_\nu^{em} = \oint_\Omega \varepsilon_\nu(n')\, d\Omega$$

$$E_\nu^{abs} = \int_0^\infty d\nu' \oint_\Omega \sigma(\nu,\nu';n,n') I_{\nu'}(n')\, d\Omega$$

Formal solution of the transfer equation

Let's have a simple view of what happens. Assume we are in a layer between two points, 1 and 2, with $\tau_2 > \tau_1$



Incident intensity, I_0 , that will be attenuated in our layer.

The emergent intensity will be $I_0 e^{-\tau}$, being τ the optical depth between 1 and 2

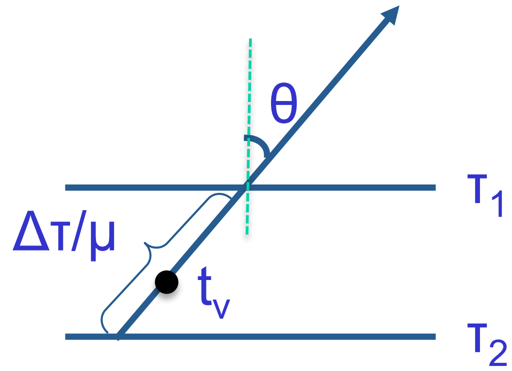
Energy emitted within our layer, attenuated within the same layer. From each point c escapes $S_c e^{-\tau_c}$, being τ_c the optical depth between la profundidad óptica entre 1 and c· We have to integrate to all points in the layer.

Both components constitute the incident intensity for the next layer

$$I_e = I_0 e^{-\tau} + \int_1^2 S_c e^{-\tau_c} d\tau$$

Formal solution of the transfer equation

Assume a layer with an outer point τ_1 and an inner point $\tau_2 > \tau_1$, so that radiation travels from τ_2 to τ_1 forming an angle θ with the normal to the surface



- The intensity emerging in τ_1 will be

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu)e^{-(t_\nu-\tau_1)/\mu}\frac{dt_\nu}{\mu}$$

- In a semi-infinite atmosphere with τ_1=0 and τ_2=∞

$$I_\nu(\tau_\nu = 0,\mu) = \int_0^\infty S_\nu(t_\nu)e^{-t_\nu/\mu}\frac{dt_\nu}{\mu}$$

In differential form we had a first-order linear differential equation

$$\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

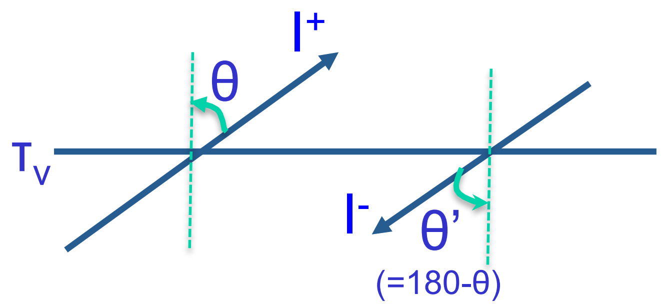
With S known, the RTE can be solved either in integral or differential form.

The Schwarzschild-Milne equations

In an intermediate point in the atmosphere (τ_ν) we will have the emergent intensity, I^+ (from $\tau'_\nu > \tau_\nu$) and incident itensity, I^- (from $\tau'_\nu < \tau_\nu$)

$$I_\nu^+(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} S(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu} \frac{dt_\nu}{\mu} \quad ; \quad \mu > 0$$

$$I_\nu^-(\tau_\nu,\mu) = \int_0^{\tau_\nu} S(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu} \frac{dt_\nu}{|\mu|} \quad ; \quad \mu < 0$$



for intensities coming from the bottom and the surface of the atmosphere.

In the first integral, $t_\nu \geq \tau_\nu$, whereas in the second $t_\nu \leq \tau_\nu$

The Schwarzschild-Milne equations

Integrating over angle we obtain the intensity moments

$$\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\mu^n d\mu = \int_0^{+1}\mu^n\, d\mu \int_{\tau_\nu}^{\infty} S(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu}\frac{dt_\nu}{\mu} + \int_{-1}^{0}\mu^n\, d\mu \int_0^{\tau_\nu} S(t_\nu) e^{-(\tau_\nu-t_\nu)/-\mu}\frac{dt_\nu}{-\mu} =$$

$$= \left\{ \begin{array}{c} \text{in the first term: } 1/\mu=\omega; \mu=0\rightarrow\omega=\infty; d\mu=-d\omega/\omega^2 \\ \text{and in the second one: } \quad -1/\mu=\omega; \mu=0\rightarrow\omega=\infty; d\mu=d\omega/\omega^2 \end{array} \right\} =$$

$$= \int_{\tau_\nu}^{\infty} S(t_\nu)dt_\nu \int_1^{\infty}\frac{e^{-(t_\nu-\tau_\nu)\omega}}{\omega^{n+1}}d\omega + (-1)^n\int_0^{\tau_\nu} S(t_\nu)dt_\nu \int_{+1}^{\infty}\frac{e^{-(\tau_\nu-t_\nu)\omega}}{\omega^{n+1}}d\omega =$$

$$= \int_{\tau_\nu}^{\infty} S(t_\nu)E_{n+1}(t_\nu-\tau_\nu)dt_\nu + (-1)^n\int_0^{\tau_\nu} S(t_\nu)E_{n+1}(\tau_\nu-t_\nu)dt_\nu$$

where the exponential integrals are defined as

$$\mathrm{E}_n(x) \equiv \int_1^{\infty}\frac{e^{-x\omega}}{\omega^n}d\omega = \int_0^1 e^{-x/\mu}\mu^{n-1}\frac{d\mu}{\mu}$$

that assymptotically behave as ($x \gg 1$):

$$E_n(x) = \frac{e^{-x}}{x}\left[1-\frac{n}{x}+\frac{n(n+1)}{x^2}+\dots\right] \approx \frac{e^{-x}}{x}$$

The Schwarzschild-Milne equations

This way we obtain the Schwarzschild-Milne equations:

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\,d\mu =$$

$$= \frac{1}{2}\int_{\tau_\nu}^{\infty} S(t_\nu)\,E_1(t_\nu - \tau_\nu)\,dt_\nu + \frac{1}{2}\int_{0}^{\tau_\nu} S(t_\nu)\,E_1(\tau_\nu - t_\nu)\,dt_\nu =$$

$$= \frac{1}{2}\int_{0}^{\infty} S(t_\nu)\,E_1(|t_\nu - \tau_\nu|)\,dt_\nu$$

Source function

Weight given to the source function of each point

→ Depending on the behaviour of S it can be J>S or J<S at a given point (particularly the surface)

$$\pi F_\nu(\tau_\nu) = 2\pi\int_{\tau_\nu}^{\infty} S(t_\nu)\,E_2(t_\nu - \tau_\nu)\,dt_\nu - 2\pi\int_{0}^{\tau_\nu} S(t_\nu)\,E_2(\tau_\nu - t_\nu)\,dt_\nu$$

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_{0}^{\infty} S(t_\nu)\,E_3(|t_\nu - \tau_\nu|)\,dt_\nu$$

The Schwarzschild-Milne equations

This can be written in operator form. Defining the so-called Lambda Operator

$$\Lambda_\tau\left[f(t)\right] \equiv \frac{1}{2}\int_0^\infty f(t)\, E_1(|t-\tau|)\,dt$$

we get

$$\Lambda_{\tau_\nu}\left[S_\nu(t_\nu)\right] = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t_\nu-\tau_\nu|)\,dt_\nu = J_\nu(\tau_\nu)$$

At the bottom: the diffusion approximation

Assume that the source function can be expanded as:

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!} \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu}$$

Substituting in the emergent and incident instensity expressions:

$$I_\nu^+(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} S_\nu(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu/\mu = \int_{\tau_\nu}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!} \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu} \right) e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu/\mu$$

$$I_\nu^-(\tau_\nu,\mu) = \int_{0}^{\tau_\nu} S_\nu(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu/|\mu| = \int_{0}^{\tau_\nu} \left(\sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!} \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu} \right) e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu/|\mu|$$

For the emergent intensity we obtain:

$$I_\nu^+(\tau_\nu,\mu) = \sum_{n=0}^{\infty} \mu^n \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu}$$

for $I_\nu^-(\tau_\nu,\mu)$ however we obtain a more complicated expression (with μ<0):

$$I_\nu^-(\tau_\nu,\mu) = \sum_{n=0}^{\infty} \mu^n \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu} \left[1 - \frac{e^{-(\tau_\nu/|\mu|)}}{n!} \left\{ (\tau_\nu/|\mu|)^n + n(\tau_\nu/|\mu|)^{n-1} + \ldots + n! \right\} \right]$$

where for τ_ν >> the expression $\left[1 - \frac{e^{-(\tau_\nu/|\mu|)}}{n!} \left\{ (\tau_\nu/|\mu|)^n + n(\tau_\nu/|\mu|)^{n-1} + \ldots + n! \right\} \right]$ tends to 1.

At the bottom: the diffusion approximation

then

$$I_\nu(\tau_\nu,\mu) = I_\nu^+(\tau_\nu,\mu) + I_\nu^-(\tau_\nu,\mu) = \sum_{n=0}^{\infty} \mu^n \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu}$$

$$I_\nu(\tau_\nu,\mu) = S_\nu(\tau_\nu) + \mu \left[\frac{dS_\nu(t_\nu)}{dt_\nu} \right]_{\tau_\nu} + \mu^2 \left[\frac{d^2 S_\nu(t_\nu)}{dt_\nu^2} \right]_{\tau_\nu} + \mu^3 \left[\frac{d^3 S_\nu(t_\nu)}{dt_\nu^3} \right]_{\tau_\nu} + \dots$$

for $-1 \leq \mu \leq +1$ when $\tau_\nu >> 1$ (and for $\mu \geq 0$ at any τ_ν)

Now we can calculate the mean intensity:

$$J_\nu(\tau_\nu) = \frac{1}{2} \int_{-1}^{+1} \sum_{n=0}^{\infty} \mu^n \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu} d\mu = \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{d^n S_\nu(t_\nu)}{dt_\nu^n} \right]_{\tau_\nu} \int_{-1}^{+1} \mu^n \, d\mu = \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[\frac{d^{(2k)} S_\nu(t_\nu)}{dt_\nu^{(2k)}} \right]_{\tau_\nu}$$

if we only consider the first terms:

$$J_\nu(\tau_\nu) = S_\nu(\tau_\nu) + \frac{1}{3} \left[\frac{d^2 S_\nu(t_\nu)}{dt_\nu^2} \right]_{\tau_\nu} + \dots$$

At the bottom: the diffusion approximation

so that, at enough depth in the atmosphere and retaining only the first terms:

$$I_\nu(\tau_\nu,\mu) = S_\nu(\tau_\nu) + \mu\left[\frac{dS_\nu(t_\nu)}{dt_\nu}\right]_{\tau_\nu} + \ldots \Rightarrow I_\nu(\tau_\nu,\mu) \approx S_\nu(\tau_\nu) + \mu\left[\frac{dS_\nu(t_\nu)}{dt_\nu}\right]_{\tau_\nu}$$

$$J_\nu(\tau_\nu) = S_\nu(\tau_\nu) + \frac{1}{3}\left[\frac{d^2S_\nu(t_\nu)}{dt_\nu^2}\right]_{\tau_\nu} + \ldots \Rightarrow J_\nu(\tau_\nu) \approx S_\nu(\tau_\nu)$$

$$F_\nu(\tau_\nu) = \frac{4}{3}\left[\frac{dS_\nu(t_\nu)}{dt_\nu}\right]_{\tau_\nu} + \frac{4}{5}\left[\frac{d^3S_\nu(t_\nu)}{dt_\nu^3}\right]_{\tau_\nu} + \ldots \Rightarrow F_\nu(\tau_\nu) \approx \frac{4}{3}\left[\frac{dS_\nu(t_\nu)}{dt_\nu}\right]_{\tau_\nu} \quad \text{(astrophysical flux)}$$

$$K_\nu(\tau_\nu) = \frac{1}{3}S_\nu(\tau_\nu) + \frac{1}{5}\left[\frac{d^2S_\nu(t_\nu)}{dt_\nu^2}\right]_{\tau_\nu} + \ldots \Rightarrow K_\nu(\tau_\nu) \approx \frac{1}{3}S_\nu(\tau_\nu)$$

At the bottom: the diffusion approximation

if we assume LTE, then $S_\nu(\tau_\nu) = B_\nu(\tau_\nu)$

$$I_\nu(\tau_\nu,\mu) \approx B_\nu(\tau_\nu) + \mu\left[\frac{dB_\nu(t_\nu)}{dt_\nu}\right]_{\tau_\nu} \qquad J_\nu(\tau_\nu) \approx B_\nu(\tau_\nu)$$

$$F_\nu(\tau_\nu) \approx \frac{4}{3}\left[\frac{dB_\nu(t_\nu)}{dt_\nu}\right]_{\tau_\nu} \qquad K_\nu(\tau_\nu) \approx \frac{1}{3}B_\nu(\tau_\nu)$$

where the flux expression has the form of a diffusion process: the flux transported is equal to the product of a diffusion coefficient times the spatial gradient of a physical magnitude

The first and third moment of the intensity have the same relation than in TE:

$$\frac{K_\nu(\tau_\nu)}{J_\nu(\tau_\nu)} = \frac{1}{3}$$

At the bottom of the atmosphere we recover a nearly isotropic field and conditions close to TE, provided that the optical depth is sufficiently large (radiation is trapped)

The K/J= f= 1/3 ratio is known as Eddington factor. It can be generalized for zones where the difussion approximation is not valid. We talk then of variable Eddington factors, f(τ)

At the surface: the Eddington-Barbier approx.

Assume now that the source function has the simple form

$$S_\nu(\tau_\nu) = S_{0,\nu} + S_{1,\nu}\tau_\nu$$

then the emergent intensity from a semi-infinite atmosphere will be

$$I_\nu(0,\mu) = \int_0^\infty (S_0 + S_1\tau_\nu) e^{-\tau_\nu/\mu} \, d\tau_\nu / \mu = S_0 + S_1\mu = S_\nu(\tau_\nu = \mu)$$

Eddington-Barbier Relationship for the specific intensity:
the emergent intensity is characteristic of the value of the source function at optical depth unity along the line of view

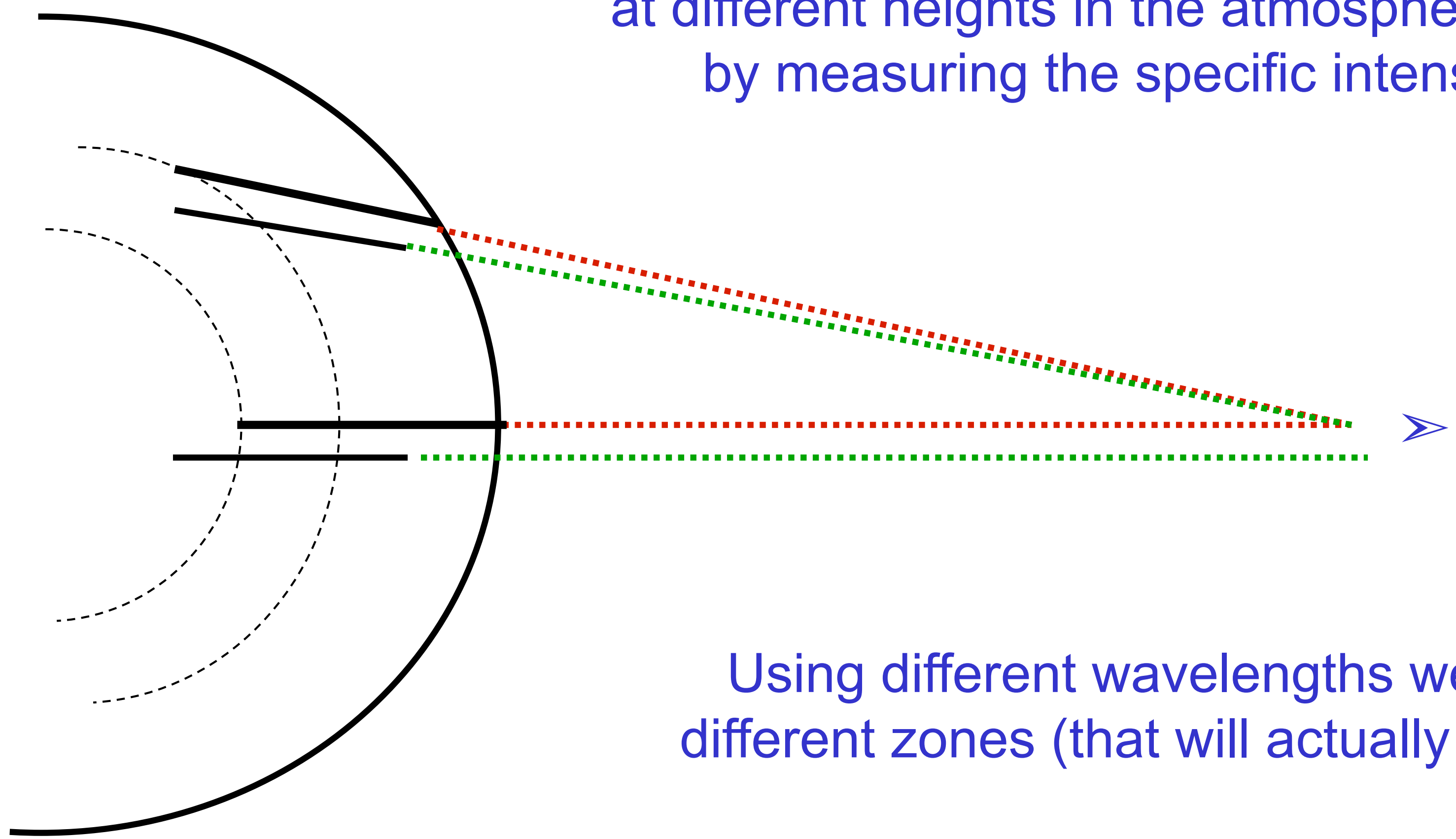
For most stars we have no specific intensities, but fluxes. In that case

$$H_\nu^+(0,\nu) = \frac{1}{4}\left(S_{0,\nu} + \frac{2}{3}S_{1,\nu}\right) = \frac{1}{4}S_\nu\left(\tau_\nu = \frac{2}{3}\right)$$

Eddington-Barbier Relationship for the flux:
The stellar flux is characteristic of the value of the source function at optical depth 2/3 along the line of view

At the surface: the Eddington-Barbier approx.

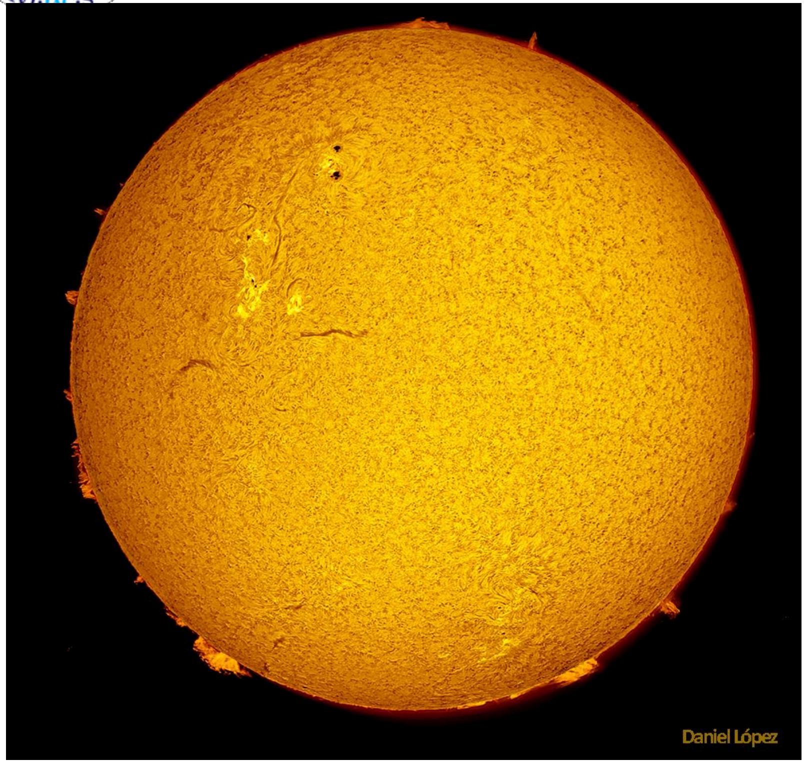
This allows us to know the source function at different heights in the atmosphere just by measuring the specific intensity



Using different wavelengths we map different zones (that will actually overlap)

But we need spatial resolution

At the surface: the limb darkening



The Eddington-Barbier relationship predicts that for $S_\nu = A + B\tau_\nu$

$$I_\nu(0,\mu) = S_\nu(\tau_\nu = \mu) = A + B\mu$$

and thus

$$I_\nu(0,\mu)/I_\nu(0,1) = \frac{A + B\mu}{A + B} \equiv f(\theta)$$

Actually, a typical limb-darkening law adopts the form

$$I_\nu(0,\mu) = I_\nu(0,1)\left(1 - \varepsilon + \varepsilon\cos\theta\right)$$

where ε varies with wavelength and stellar temperature (for the Sun in the visible, ε=0.6; see Gray Fig. 17.6)

Note that the form of the source function implies that it increases inwards, i.e., the temperature increases inwards assuming a connection between source function and temperature (like in LTE, but not limited to LTE)

At the surface: intuitive line formation

we see radiation coming from $\tau_\nu \sim 1 \Rightarrow$ $\begin{array}{l} \tau_\nu^c \sim 1 = \kappa_\nu^c s_c \\ \tau_\nu^L \sim 1 = \kappa_\nu^L s_L \end{array}$

Because spectral lines (bound-bound transitions) are more optically thick than continuum (bound-free transitions),

$\kappa_\nu^L \gg \kappa_\nu^c \Rightarrow s_\nu^L \ll s_\nu^c$

therefore the line forms (radiation escapes) in higher layers than the continuum

Note the role of T stratification in line formation