Applications of Sheaf Cohomology and Exact Sequences on Network Codings

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Abstract—**Sheaf cohomology is a mathematical tool for collating local algebraic data into global structures. The purpose of this paper is to apply sheaf theory into network coding problems. After the definition of sheaves, we define so called network coding sheaves for a general multi source network coding scenario, and consider various forms of sheaf cohomologies. The main theorem states that 0-th network coding sheaf cohomology is equivalent to information flows for the network coding. Then, this theorem is applied to several practical problems in network codings such as maxflow bounds, global extendability, network robustness, and data merging, by using some of the standard exact sequences of homological algebra.**

I. Introduction

This paper introduces new tools for the analysis of data flows over networks, especially focusing on network codings. The problem of network coding is one of a host of problems in data analysis and management that require an understanding of local-to-global transitions. A new set of tools we present in this paper is based on sheaf theory.

Sheaf theory was invented in the mid 1940s as a branch of algebraic topology to deal with the collation of local data on topological spaces. Through the success in the theory of functions of several complex variables and algebraic geometry, this theory is now indispensable in modern mathematics. However, instead of its generality dealing with local-to-global transitions, applications to other areas in science or engineering have not been well established so far except for logic and semantics in computer science with the notion of Topos (e.g., [2][8][9]).

On the other hand, there is a powerful mathematical tool in sheaf theory, so called sheaf cohomology, which can treat local-to-global transition in algebraic data level. One of the important messages in this paper is to show usefulness of sheaf cohomology for applications to analysis of data on a network. This viewpoint provides us with a lot of analytical tools from homological algebra such as exact sequences for various forms of sheaf cohomology. Moreover, not only on a network, sheaf cohomological tools are easily extendable on a higher dimensional base space, such as simplicial complexes or product of them, which might be useful in situations with spacial expanse like wireless settings or with time dependence (product with time axis $\mathbb{R}$). As a first step to this direction, we study applications of sheaf cohomology to some basic problems in network codings by means of some fundamental exact sequences in this paper.

This paper is organized as follows. In Section II, we explain fundamental sheaf theory which will be required to understand this paper. In particular, sheaf cohomology and its basic operations will be discussed in detail. It should be noted that our tools presented here are computable by elementary module calculus (or by linear algebra for data in a field), although sheaf theory itself is highly abstract subject in mathematics. An important notion introduced in this section is a *network coding sheaf* (NC sheaf for short), which gives a relationship between sheaf theory and network coding problems. Especially, information theoretical meaning of NC sheaf cohomology plays important roles for applications. In Section III, NC sheaf cohomology is applied into some practical problems (maxflow bound, global extendability, network robustness, and data merging). All of the techniques used in this section are standard long exact sequences in homological algebra. In Section IV, we explain some future directions of this work. Throughout the paper, we refer to [1][4][6] for general discussions on sheaf theory.

II. Sheaf Formulation of Network Codings

A. Definition of Sheaves

Let X be a topological space (e.g., network) and $\mathcal{R}$ be a commutative ring.

Definition 1. (Presheaf). A presheaf F on X consists of the following data:
- an $\mathcal{R}$-module $F(U)$ for each open subset $U \subset X$.
- an $\mathcal{R}$-linear map $\rho_{VU} : F(U) \to F(V)$ for each pair $V \subset U \subset X$.

These data satisfy the following conditions:

$$\rho_{UU} = \mathrm{id}_U,\ \rho_{WV} \circ \rho_{VU} = \rho_{WU} \text{ for } W \subset V \subset U,$$

where id_U is the identity map on $F(U)$.

An element $\sigma \in F(U)$ is called a section on $F(U)$, and an $\mathcal{R}$-linear map ρ_{VU} is called a restriction map. We often write $\sigma|_V$ instead of $\rho_{VU}(\sigma)$, and call it the restriction of σ to V.

Definition 2. (Sheaf). A presheaf F on X is called a sheaf if it satisfies the following two conditions:

1) For any open set $U \subset X$, any open covering $U = \cup_{i\in I} U_i$, and any section $\sigma \in F(U)$, $\sigma|_{U_i} = 0$ for all $i \in I$ implies $\sigma = 0$.
2) For any open set $U \subset X$, any open covering $U = \cup_{i\in I} U_i$, any family $\sigma_i \in F(U_i)$ satisfying $\sigma_i|_{U_i\cap U_j} = \sigma_j|_{U_i\cap U_j}$ for all pairs (i,j), there exists $\sigma \in F(U)$ such that $\sigma|_{U_i} = \sigma_i$ for all $i \in I$.

Remark 3.
1) Each $\mathcal{R}$-module $F(U)$ is regarded as local data storage for applications.
2) From the conditions in Definition 2, a sheaf F allows one to glue a set of local data together into global data uniquely.

B. Network Coding Sheaves

Let us at first explain the problem setting of network codings (e.g.,[7][10]) on which we construct sheaves. Let k be an $\mathcal{R}$-module, or simply a (finite) field. Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph(not necessarily acyclic), where $\mathcal{V}$ and $\mathcal{E}$ are finite sets of nodes and edges, respectively. A directed edge $e \in \mathcal{E}$ from $v \in \mathcal{V}$ to $w \in \mathcal{V}$ is denoted by $e = |vw|$ ($\mathrm{Head}(e) := w$, $\mathrm{Tail}(e) := v$). We assume that there exists a subset $\mathcal{S} = \{s_1, \cdots, s_\alpha\} \subset \mathcal{V}$ of nodes called sources which transmit elements in $k^{n_{s_i}}, n_{s_i} \in \mathbb{N}$, as information for each $s_i \in \mathcal{S}$. We often identify a graph G with a topological space by the geometrical representation with the usual Euclidean topology.

We also assume that there exists a subset $\mathcal{R} = \{r_1, \cdots, r_\beta\} \subset \mathcal{V}$ of nodes called receivers. Each receiver requires information from some sources and this assignment is determined by $S : R \to 2^{\mathcal{S}}$ in the sense that a receiver r_i requires all transmitted information from $S(r_i) \in 2^{\mathcal{S}}$.

Let $\mathrm{cap} : \mathcal{E} \to \mathbb{N}$ be a capacity function which assigns for each edge $e \in \mathcal{E}$ its edge capacity $\mathrm{cap}(e)$. The set of the incoming (outgoing, resp.) edges in the sense of edge directions at a node $v \in \mathcal{V}$ is denoted by $\mathrm{In}(v)$ ($\mathrm{Out}(v)$, resp.). A local coding map ϕ_{wv} determines a data assignment of the incoming data at v into an outgoing edge e with $\mathrm{Head}(e) = w$ given by

$$\phi_{wv} : k^{n_v} \oplus k^{l_v} \to k^{\mathrm{cap}(e)},\ \text{where } l_v = \sum_{e\in \mathrm{In}(v)} \mathrm{cap}(e),$$

where it is assumed that $n_v = 0$ for $v \in \mathcal{V}\backslash\mathcal{S}$. Especially, a local coding map $\phi_{s_i r_j}$ from a receiver r_j to a source $s_i \in S(r_j)$ corresponds to the decoding map. Let us denote the set of all local coding maps by $\Phi = \{\phi_{wv}\}$.

In order to express decodable information flows on a network as a network coding sheaf cohomology, we extend the graph $G = (\mathcal{V}, \mathcal{E})$ to $X = (\mathcal{V}, \tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}}$ is given by adding edges $e = |r_j s_i|$ in $\mathcal{E}$ from each receiver r_j to all of its requesting sources $s_i \in S(r_j)$ with $\mathrm{cap}(e = |r_j s_i|) = n_{s_i}$. For removing ambiguity, we denote the set of incoming edges at $v \in \mathcal{V}$ in $\mathcal{E}$ or $\tilde{\mathcal{E}}$ by $\mathrm{In}(v; \mathcal{E})$ or $\mathrm{In}(v; \tilde{\mathcal{E}})$, respectively. $\mathrm{Out}(v; \mathcal{E})$ and $\mathrm{Out}(v; \tilde{\mathcal{E}})$ are similarly defined.

This extension enables one to compare decoded information at each receiver r_j with transmitted information from $s_i \in S(r_j)$ as the glueing condition of the network coding sheaf on the added edge $e = |r_j s_i|$.

Let us equip X with the usual Euclidean topology. For the definition of a network coding sheaf F (NC sheaf for short), we at first assign sections for some special open sets.

Definition 4. (Local Sections).

1) For a connected open set U contained in an edge $e \in \tilde{\mathcal{E}}$, $F(U) := k^{\mathrm{cap}(e)}$.
2) For a connected open set U which only contains one node $v \in \mathcal{V}$, $F(U) := k^{n_v} \oplus k^{l_v}$, where $l_v = \sum_{e\in \mathrm{In}(v;E)} \mathrm{cap}(e)$.

Definition 5. (Local Restriction Maps).

1) For connected open sets $V \subset U \subset e$ for some edge e, $\rho_{VU} := \mathrm{id} : F(U) \to F(V)$.
2) For connected open sets $V \subset U$, where U contains only one node v and V is located in $e \in \mathrm{In}(v; \tilde{\mathcal{E}})$, $\rho_{VU} : F(U) \to F(V)$ is given by the projection map induced by the product structure in Definition 4 2).
3) For connected open sets $W \subset U$, where U contains only one node v and W is located in $e \in \mathrm{Out}(v; \tilde{\mathcal{E}})$, $\rho_{WU} := \phi_{wv} : F(U) \to F(W)$, where $w = \mathrm{Head}(e)$.

From these local definitions of sections and restriction maps, the network coding sheaf is defined by the *sheafification*. It is a process to construct $F(U)$ for arbitrary open set $U \subset X$ and the restriction map $\rho_{VU} : F(U) \to F(V)$ by using the glueing condition in Definition 2. More precisely, it is explained in the following way.

Definition 6. (NC Sheaf). For an open set $U \subset X$, $F(U)$ is defined by the set of all equivalent classes $\sigma = [(\sigma_i, U_i)_{i\in I}]$, where a representative $(\sigma_i, U_i)_{i\in I}$ with a covering $U = \cup_{i\in I} U_i$ is given by a family of sections $\sigma_i \in F(U_i)$ satisfying $\sigma_i|_{U_i\cap U_j} = \sigma_j|_{U_i\cap U_j}$, and the equivalent relation $\sim$ is defined by

$$(\sigma_i, U_i)_{i\in I} \sim (\tau_j, V_j)_{j\in J} \Leftrightarrow \sigma_i|_{U_i\cap V_j} = \tau_j|_{U_i\cap V_j} \text{ for } i \in I, j \in J.$$

The restriction map $\rho_{VU} : F(U) \to F(V)$ is induced by local restriction maps on a representative (of course, it is independent of the choice of a representative). The sheaf F obtained by the sheafification process is called the network coding sheaf of (X, Φ).

C. Sheaf Cohomology

For a sheaf F, the global structure of data on X is characterized by sheaf cohomology $H^\bullet(X;F)$. We explain it via Čech cohomology. Precisely speaking, Čech cohomology is not the same as sheaf cohomology in general, however both cohomologies turn out to be equivalent for NC sheaves. Hence, we define sheaf cohomology $H^\bullet(X;F)$ in the following way.

First of all, let us take the open covering $X = (\cup_{v\in\mathcal{V}} U_v) \cup (\cup_{e\in\tilde{\mathcal{E}}} U_e)$ by using open stars U_v and U_e for each $v \in \mathcal{V}$ and $e \in \tilde{\mathcal{E}}$. Here, an open star U_v for a node $v \in \mathcal{V}$ is the maximal connected open set containing only one node v, and an open star U_e for an edge $e \in \tilde{\mathcal{E}}$ is the maximal open set contained in the edge e. Then, let us define the Čech complex $0 \to C^0(X;F) \xrightarrow{\partial} C^1(X;F) \to 0$ by

$$C^0(X;F) = \prod_{v\in\mathcal{V}} F(U_v), \tag{II.1}$$

$$C^1(X;F) = \prod_{e\in\tilde{\mathcal{E}}} F(U_e), \tag{II.2}$$

where the boundary map $\partial = (\partial_e)_{e\in\tilde{\mathcal{E}}}$ is defined for each product element $F(U_e)$ of $C^1(X;F)$ with $e = |vw|$ by

$$\begin{aligned} &\partial_e : F(U_v) \times F(U_w) \to F(U_e), \\ &\partial_e(\sigma_v, \sigma_w) = \rho_{U_e U_v}(\sigma_v) - \rho_{U_e U_w}(\sigma_w). \end{aligned} \tag{II.3}$$

Definition 7. (Sheaf Cohomology). The i-th sheaf cohomology $H^i(X;F)$ is defined by $H^i(X;F) := H^i(C^\bullet)$, i.e.,

$$H^0(X;F) := \mathrm{Ker}(\partial), \tag{II.4}$$

$$H^1(X;F) := C^1(X;F)/\mathrm{Im}(\partial). \tag{II.5}$$

For an open set $A \overset{\iota}{\hookrightarrow} X$, a sheaf F on X induces a sheaf on A called the inverse image ι^*F. It is defined by $\iota^*F(U) := F(U)$ for an open set $U \subset A$ and the restriction maps are induced by the original ones of the sheaf F. Then, by constructing an open covering for A from $X = (\cup_{v\in\mathcal{V}} U_v) \cup (\cup_{e\in\tilde{\mathcal{E}}} U_e)$, we can define the sheaf cohomology $H^\bullet(A;\iota^*F)$ on A as the cohomology of the Čech complex $C^\bullet(A;\iota^*F)$. We will often use the notations $H^\bullet(A;F) = H^\bullet(A;\iota^*F)$ and $C^\bullet(A;F) = C^\bullet(A;\iota^*F)$.

Finally, let me introduce the relative sheaf cohomology $H^\bullet(X,A;F)$ of a pair (X,A), where $A \subset X$ is an open set. Let us note that we can define a surjective chain map $p^\bullet : C^\bullet(X;F) \to C^\bullet(A;F)$. Then the chain complex for a pair (X,A) is defined by the subcomplex $C^\bullet(X,A;F) := \mathrm{Ker}(p^\bullet)$. The relative sheaf cohomology for the pair (X,A) is defined by $H^i(X,A;F) := H^i(C^\bullet(X,A;F))$.

Basic other operations on sheaves and these cohomologies (e.g., $f_*, f^*, \otimes, \mathcal{H}om(\bullet,\bullet)$, see [1][4][6]) will be useful to construct a new NC sheaf from known ones, or to investigate relationships between different NC sheaves and their cohomologies. Applications of these operations to network coding problems will be studied in [3].

D. Computation

It should be noted that computations for sheaf cohomologies only require module operations under the situation of this paper, because of the definition from Čech cohomology. Especially, from the definition of the sheaf cohomology (II.4),(II.5), all we need to do is to check the kernel and the cokernel of the boundary map $\partial : C^0(X;F) \to C^1(X;F)$. These calculations are performed by means of Smith normal forms. We refer to [5] for details of computations of Smith normal forms including fast algorithms.

E. Information Theoretical Meaning of NC Sheaf Cohomology

In this subsection, we show an information theoretical meaning of $H^0(X;F)$ for a NC sheaf F. Since $H^0(X;F)$ is defined by (II.4), let us study a meaning of the kernel of the boundary map $\partial : C^0(X;F) \to C^1(X;F)$.

First of all, let us recall the definition of information flow on a network G. An information flow ψ for a family of transmitted data $z = (z_{s_1}, \cdots, z_{s_\alpha})$, $z_{s_i} \in k^{n_{s_i}}, s_i \in \mathcal{S}$, is defined by an assignment $\psi(e) \in k^{\mathrm{cap}(e)}$ for each edge $e \in \mathcal{E}$ satisfying the *flow conditions*, some of which are expressed as follows: Under $e = |vw|$ and $e_i \in \mathrm{In}(v;\mathcal{E})$ $(i = 1, \cdots, K)$,

1) $\phi_{wv}(\psi(e_1), \cdots, \psi(e_K)) = \psi(e)$ for $v \notin \mathcal{S} \cup \mathcal{R}$,
2) $\phi_{wv}(z_{s_i}, \psi(e_1), \cdots, \psi(e_K)) = \psi(e)$ for $v = s_i \in \mathcal{S} \setminus \mathcal{R}$,
3) $\phi_{wv}(\psi(e_1), \cdots, \psi(e_K)) = z_{s_i}$ for $v = r_j \in \mathcal{R} \setminus \mathcal{S}$ and $w = s_i \in S(r_j)$,
4) $\phi_{wv}(\psi(e_1), \cdots, \psi(e_K)) = \psi(e)$ for $v = r_j \in \mathcal{R} \setminus \mathcal{S}$, $w \notin S(r_j)$,

and so on. The other cases are similarly derived by taking proper domain and target spaces of local coding maps.

We recall that the boundary map $\partial : C^0(X;F) \to C^1(X;F)$ is determined by a family of maps ∂_e $(e \in \tilde{\mathcal{E}})$ by (II.3). Hence, $\sigma = (\sigma_v)_{v\in\mathcal{V}} \in C^0(X;F) \in \mathrm{Ker}(\partial)$ if and only if $\partial_e(\sigma_v, \sigma_w) = 0$ for all $e = |vw| \in \tilde{\mathcal{E}}$. Let us also recall that the restriction map $\rho_{U_e U_v}$ from the tail node v is determined by the local coding map ϕ_{wv}, and the restriction map $\rho_{U_e U_w}$ from the head node is determined by the projection $F(U_w) \to F(U_e)$.

Let us construct an assignment $\psi(e) \in k^{\mathrm{cap}(e)}$ for each $e = |vw| \in \mathcal{E}$ by $\psi(e) = \rho_{U_e U_w}(\sigma_w)$ (i.e., projecting $\sigma_w \in F(U_w)$ into $F(U_e)$). Let $z = (z_{s_1}, \cdots, z_{s_\alpha})$, $z_{s_i} \in k^{n_{s_i}}$, be the product element in $\sigma \in C^0(X;F)$ corresponding to the transmitted data from S. Then, It can be checked by direct calculation that the kernel condition on ∂_e expressed as $\rho_{U_e U_v}(\sigma_v) = \rho_{U_e U_w}(\sigma_w)$ is equivalent to be the above flow condition for the assignment ψ on the edge e.

Therefore, we obtained the equivalent expression of an information flow for a family of transmitted data z by means of $\sigma \in \mathrm{Ker}(\partial)$, and proved:

Theorem 8. *(Information Theoretical Meaning of* $H^0(X;F)$*). For a NC sheaf* F *of* (X,Ψ)*,* $H^0(X;F)$ *is equivalent to the information flows on the network.*

This theorem makes it possible to apply homological algebraic tools for cohomologies into network coding problems. We will see some of the applications in the next section.

III. Applications

A. Relative Cohomology and Maxflow Bound

We discussed the definition of relative NC sheaf cohomology $H^\bullet(X,A;F)$ for an open set $A \subset X$. At that point, the relative chain complex $C^\bullet(X,A;F)$ is derived as a subcomplex of $C^\bullet(X;F)$ which is mapped to 0 by the surjective chain map $p^\bullet : C^\bullet(X;F) \to C^\bullet(A;F)$. Hence, we have a short exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(X,A;F) & \xrightarrow{i^0} & C^0(X;F) & \xrightarrow{p^0} & C^0(A;F) & \longrightarrow & 0 \\ & & \downarrow \partial & \circlearrowleft & \downarrow \partial & \circlearrowleft & \downarrow \partial & & \\ 0 & \longrightarrow & C^1(X,A;F) & \xrightarrow{i^1} & C^1(X;F) & \xrightarrow{p^1} & C^1(A;F) & \longrightarrow & 0 \end{array} \tag{III.1}$$

meaning that the above diagram is commutative and $\mathrm{Ker}(i^k) = 0$, $\mathrm{Im}(i^k) = \mathrm{Ker}(p^k)$, $\mathrm{Im}(p^k) = C^k(A;F)$ for $k = 0,1$. It should be remarked that the boundary maps in (III.1) are all induced by the original one $\partial : C^0(X;F) \to C^1(X;F)$.

One of the important techniques in homological algebra is to construct a long exact sequence on cohomologies from (III.1) expressed as

$$0 \to H^0(X,A;F) \xrightarrow{i^0} H^0(X;F) \xrightarrow{p^0} H^0(A;F) \xrightarrow{\delta^0} H^1(X,A;F) \xrightarrow{i^1} \cdots \tag{III.2}$$

where the maps $i^\bullet$ and $p^\bullet$ in (III.2) are induced by those in the short exact sequence (III.1). The map δ^0 is called the connecting homomorphism defined by $\delta^0(\sigma_A) := \partial(\sigma_X)$, where $p^0(\sigma_X) = \sigma_A$ and $\partial(\sigma_X) \in C^1(X;F)$ is identified with the element in $C^1(X,A;F)$ because $p^1\partial(\sigma_X) = \partial p^0(\sigma_X) = \partial\sigma_A = 0$ leads to $\partial(\sigma_X) \in C^1(X,A;F)$.

We now study the single source scenario $\alpha = 1$ for applications of relative NC sheaf cohomology. Let us take an open set A which does not include the source node $s \in \mathcal{S}$, but includes some receiver $r_j \in \mathcal{R}$. Then, the incoming edges into A define a cut C between s and r_j. Especially, it follows from the definition of $H^0(A;F)$ that $\mathrm{cap}(C) = \dim H^0(A;F)$, where $\mathrm{cap}(C)$ represents the cut capacity for C. On the other hand, the following lemma holds [3]:

Lemma 9. *Under the above condition on* A*,* $H^0(X,A;F) = 0$ *for any NC sheaf* F*.*

From this lemma, $H^0(X,A;F) = 0$ in (III.2) and it follows that $p^0 : H^0(X;F) \to H^0(A;F)$ is injective. Hence $\dim H^0(X;F) \leq \dim H^0(A;F)$ for any network coding sheaf F and any $A \subset X$ with the above condition. We note that $\dim H^0(A;F)$ does not depend on the choice of a NC sheaf F, and the minimum on the righthand side by changing A gives us the minimum cut capacity between s and r_j. On the other hand, let us recall that $\dim H^0(X;F)$ expresses the dimension of information flow on the network by Theorem 8. Therefore, by taking the maximum of $\dim H^0(X;F)$ under changing NC sheaves, we have proved the maxflow bound inequality in a homological algebraic way:

$$\mathrm{maxflow} = \max_F \dim H^0(X;F) \leq \mathrm{mincut} = \min_A \dim H^0(A;F).$$

We admit that the maxflow bound itself is an easy consequence from the information flow property of the single source situation. The reason to show this example is to see that each standard tool in homological algebra has corresponding practical applications.

B. Connecting Morphism and Global extendability

Let us again go back to the general multi-source problem setting and think the long exact sequence of a pair (X,A) (III.2) in more detail. It should be noted that this exact sequence holds for any open set $A \subset X$.

Let us suppose that we have known a local information flow σ_A on A and study a condition which allows one to globally extend σ_A into the whole network. Since the information flows on A and X are represented by elements in $H^0(A;F)$ and $H^0(X;F)$, respectively, the global extendability of a local information flow $\sigma_A \in H^0(A;F)$ asks the existence of $\sigma \in H^0(X;F)$ such that $p^0(\sigma) = \sigma_A$. On the other hand, the exactness of the sequence (III.2) at $H^0(A;F)$ is expressed as $\mathrm{Im}(p^0) = \mathrm{Ker}(\delta^0)$. Thus, we proved the following proposition

Proposition 10. *(Global Extendability). A local information flow* $\sigma_A \in H^0(A;F)$ *is globally extendable if and only if* $\delta^0(\sigma_A) = 0$*.*

C. Excision and Network Robustness

Let $A \subset G$ be an open set and $Z = X \setminus A$ be its complementary closed set. For a section $\sigma \in F(U)$, the support of σ denoted by $|\sigma|$ is defined by

$$|\sigma| := \{x \in U \mid s|_V \neq 0 \text{ for any neighborhood } V \subset U \text{ of } x\}.$$

Let us also consider a subspace of $F(U)$ for each open set $U \subset X$ given by

$$F_Z(U) := \{\sigma \in F(U) \mid |\sigma| \subset Z\}.$$

Then by replacing $F(U_v)$, $F(U_e)$ in (II.1), (II.2) with $F_Z(U_v)$, $F_Z(U_e)$, respectively, the local cohomology with support Z denoted by $H^\bullet_Z(X;F)$ is defined in the same way.

One of the useful exact sequence relating to the local cohomology $H^{\bullet}_{Z}(X;F)$ is expressed as follows (Proposition II.9.2 in [4]):

$$0 \to H^0_Z(X;F) \xrightarrow{i^0} H^0(X;F) \xrightarrow{p^0} H^0(A;F) \xrightarrow{\delta^0} H^1_Z(X;F) \xrightarrow{i^1} \cdots \tag{III.3}$$

where i^k is induced by the inclusion map $i_U : F_Z(U) \to F(U)$.

Now let us suppose that we have a network failure on A and study the robustness problem of information flows. Precisely speaking, we consider the condition under which the global information flow $\sigma \in H^0(X;F)$ persists on Z with the removal of A. From the definition of the local cohomology $H^0_Z(X;F)$, a section $\sigma_Z \in H^0_Z(X;F)$ represents an information flow on the network G such that $|\sigma_Z| \subset Z$. It means that, for any point $x \in A$, there exists an open set $U \subset A$ such that $\sigma_Z|_U = 0$. Hence, it follows that $H^0_Z(X;F)$ represents the information flows which are not affected by the removal of A. By combining this argument to the exact sequence (III.3), we have the following proposition:

Proposition 11. *(Network Robustness). Let $A \subset G$ be an open set and $Z = X \setminus A$ be the complementary closed set. Then $H^0_Z(X;F)$ represents the global information flow on the failure network $G \setminus A$. Moreover, the network coding of F is robust to this failure if and only if $p^0 = 0$.*

Remark 12. By the five lemma (e.g., [1][4][6]), we have the isomorphism $H^k(X,A;F) \simeq H^k_Z(X;F)$. Hence the above argument can be explained by using only relative cohomology. On the other hand, the long exact sequence (III.3) is one of the examples showing *excision property*. There are several versions of long exact sequences related to excision property (see, e.g., II.9 in [4]) each of which will be used to analyze local information flows like this example.

D. Mayer-Vietoris and Data Merging

In this subsection, we study a data merging problem by using a homological algebraic tool. Let U, V be open sets in X such that $X = U \cup V$. Then we are interested in relationship between local and global information flows on U, V, and X.

In homological algebra, one of the appropriate tools for this purpose is known as Mayer-Vietoris long exact sequence given by (see, e.g., II.5.10 in [4])

$$0 \to H^0(X;F) \xrightarrow{f^0} H^0(U;F) \oplus H^0(V;F) \xrightarrow{g^0} H^0(U \cap V;F) \xrightarrow{\delta^0} (\dagger)$$

$$(\dagger) \xrightarrow{\delta^0} H^1(X;F) \xrightarrow{f^1} \cdots, \tag{III.4}$$

where $f^{\bullet}$ and $g^{\bullet}$ are defined by the sum and difference of restriction maps, respectively.

Let $\sigma_U \in H^0(U;F)$ and $\sigma_V \in H^0(V;F)$ be local information flows on U and V, respectively. The question of data merging asks the existence of a global information flow $\sigma \in H^0(X;F)$ which induces σ_U and σ_V, i.e., $f^0(\sigma) = (\sigma_U, \sigma_V)$. Therefore, again by using exactness property, we have the following proposition on data merging problem

Proposition 13. *(Data Merging). Let U and V be open sets in X. Let $\sigma_U \in H^0(U;F)$ and $\sigma_V \in H^0(V;F)$ be local information flows on U and V, respectively. Then these two local information flows can be merged into a global information flow on X if and only if $g^0(\sigma_U, \sigma_V) = 0$.*

IV. Conclusion

In this paper, we introduced sheaf theoretical tools on network codings and showed that sheaf cohomologies and these long exact sequences can be applied several practical problems.

An important application of sheaf theory will be a characterization of maxflows on general multi-source network codings. One of the possibilities to attack this problem is to give another sheaf theoretic proof of the maxflow characterization on a single source scenario, which will be potentially applicable to multi-source one. As is seen in the single source case, the maxflow characterization given by the mincut can be regarded as a flow-cut duality. From this viewpoint of duality, a derived categorical formulation of network coding sheaves may provide us with some useful characterizations through Poincaré-Verdier duality and Morse theory.

It should be also mentioned that we can differently formulate network coding problems without the extension of graph $G \to X$. Even in this formulation, information flows on a network can be treated by using NC sheaf cohomologies [3].

Acknowledgment

The authors express their sincere gratitude to S. Hara and M. Robinson for valuable discussions on this paper. This work is partially supported by the AFOSR for the first author, and by JST PRESTO program for the second author.

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