CHAPTER 6

Applications of the Integral

6.1 Area Between Two Curves

Preliminary Questions

1. Suppose that $f(x) \geq 0$ and $g(x) \leq 0$. True or False: the integral $\int_a^b (f(x) - g(x))\,dx$ is still equal to the area between the graphs of f and g.

2. Two airplanes take off simultaneously and travel east. Their velocities are $v_1(t)$ and $v_2(t)$. What is the physical interpretation of the area between the graphs of $v_1(t)$ and $v_2(t)$?

Exercises

1. Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$ (Figure 1).

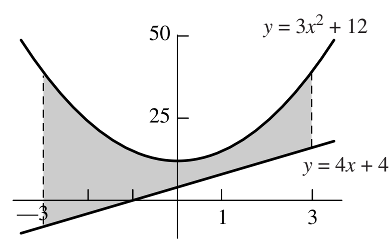


Figure 1

We have
$\int_{-3}^{3} \left(3x^2 + 12\right) - (4x + 4)\,dx = \int_{-3}^{3} \left(3x^2 - 4x + 8\right)\,dx = \left(x^3 - 2x^2 + 8x\right)\Big|_{-3}^{3} = 102.$

3. Let $f(x) = 2 - x^2$ and $g(x) = x$ as in Figure 2 (B).
(a) Determine the points of intersection of the graphs of f and g.

(b) Compute the area of the region *below* the graph of f and *above* the graph of g.

(a) $f(x) = g(x)$ gives $2 - x^2 = x$ which simplifies to $0 = x^2 + x - 2 = (x+2)(x-1)$ and thus $f(x)$ and $g(x)$ intersect at $x = -2$ and $x = 1$.

(b) We have $\int_{-2}^{1} \left(2 - x^2\right) - (x)\, dx = \left(2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_{-2}^{1} = 4.5$.

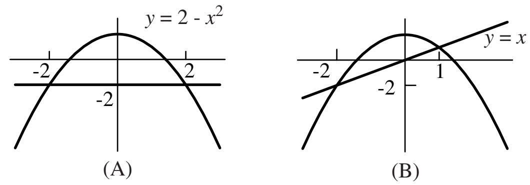
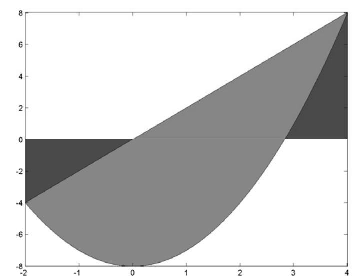


Figure 2

5. Sketch the region enclosed by the graphs of $y = 2x$ and $y = x^2 - 8$ and compute its area.

The graphs intersect at $x = 4$ and $x = -2$. We have
$\int_{-2}^{4} (2x) - \left(x^2 - 8\right)\, dx = \left(-\frac{1}{3}x^3 + x^2 + 8x\right)\Big|_{-2}^{4} = 56.$



7. Calculate the area between the graphs of $\sin x$ and $\cos x$ over the interval $[0, \pi]$ (see Figure 3).

Using the result of the previous exercise, we have
$\int_0^{\pi/4} (\cos x) - (\sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x) - (\cos x)\, dx =$
$\left(\sqrt{2} - 1\right) + (-\cos x + \sin x)\big|_{\pi/4}^{\pi} = \left(\sqrt{2} - 1\right) + 1 = \sqrt{2} = 1.414214.$

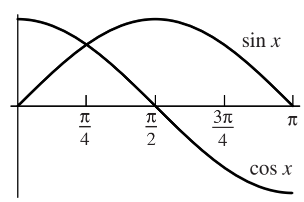


Figure 3

In Exercises 8–11, let $f(x) = 19 - 3x$ and $g(x) = 2x + 4$ as in Figure 4.

9. Find the area of the region between the lines $y = f(x)$ and $y = g(x)$ over the interval [4, 6].

We have $\int_4^6 (2x+4) - (19-3x)\,dx = \int_4^6 (5x-15)\,dx = \left(\frac{5}{2}x^2 - 15x\right)\Big|_4^6 = 20.$

11. Find the area of the region below $y = f(x)$, above $y = g(x)$, and to the right of the y-axis.

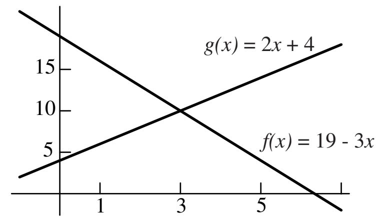


Figure 4

We have $\int_0^3 (19-3x) - (2x+4)\,dx = \int_0^3 (15-5x)\,dx = \left(15x - \frac{5}{2}x^2\right)\Big|_0^3 = 22.5.$

In Exercises 12–15, refer to the curves $y = 20 + x - x^2$ and $y = x^2 - 5x$ shown in Figure 5.

13. Which is the upper curve over the interval [6, 8]? Find the area between the curves over [6, 8].

The upper curve on the interval [1, 3] is $y = x^2 - 5x$. We have
$\int_6^8 \left(x^2 - 5x\right) - \left(20 + x - x^2\right)\,dx = \int_6^8 \left(2x^2 - 6x - 20\right)\,dx = \left(\frac{2}{3}x^3 - 3x^2 - 20x\right)\Big|_6^8 =$
$73.333333.$

15. Compute the area of the region between the two curves over [4, 8] as a sum of two integrals.

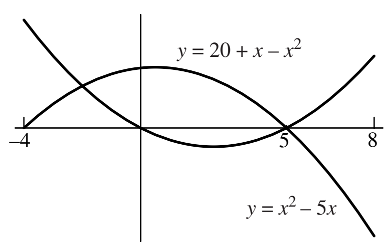


Figure 5

We have $\int_4^5 \left(20 + x - x^2\right) - \left(x^2 - 5x\right)\,dx + \int_5^8 \left(x^2 - 5x\right) - \left(20 + x - x^2\right)\,dx =$
$\int_4^5 \left(20 + 6x - 2x^2\right)\,dx + \int_5^8 \left(2x^2 - 6x - 20\right)\,dx =$
$\left(20x + 3x^2 - \frac{2}{3}x^3\right)\Big|_4^5 + \left(\frac{2}{3}x^3 - 3x^2 - 20x\right)\Big|_5^8 = 254.$

17. Calculate the shaded area in Figure 6.

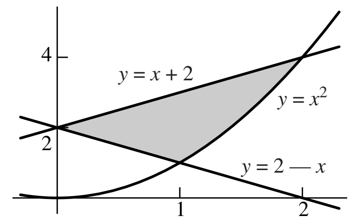


Figure 6

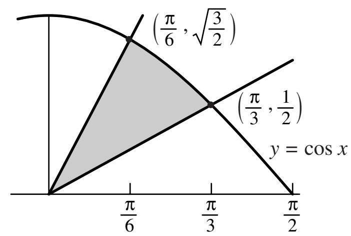


Figure 7

19. Find the area of the shaded region in Figure 8.

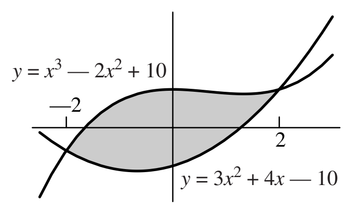


Figure 8

We have $\int_{-2}^{2} \left(x^3 - 2x^2 + 10\right) - \left(3x^2 + 4x - 10\right)\, dx = \int_{-2}^{2} \left(x^3 - 5x^2 - 4x + 20\right)\, dx = \left(\frac{1}{4}x^4 - \frac{5}{3}x^3 - 2x^2 + 20x\right)\Big|_{-2}^{2} = 53.333333.$

21. Find the area of the region enclosed by the curves $y = x^3 - 6x$ and $y = 8 - 3x^2$.

The two curves intersect at $x = -4$, $x = -1$ and $x = 2$. Thus we have
$\int_{-4}^{-1} \left(x^3 - 6x\right) - \left(8 - 3x^2\right)\, dx + \int_{-1}^{2} \left(8 - 3x^2\right) - \left(x^3 - 6x\right)\, dx =$
$\int_{-4}^{-1} \left(x^3 + 3x^2 - 6x - 8\right)\, dx + \int_{-1}^{2} \left(-x^3 - 3x^2 + 6x + 8\right)\, dx =$
$\left(\frac{1}{4}x^4 + x^3 - 3x^2 - 8x\right)\Big|_{-4}^{-1} + \left(-\frac{1}{4}x^4 - x^3 + 3x^2 + 8x\right)\Big|_{-1}^{2} = 32.5.$

23. Find the area of the shaded region in Figure 11.

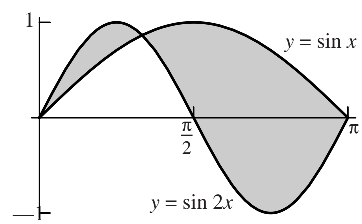


Figure 11

We have $\int_0^{\pi/3} (\sin 2x - \sin x)\, dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\, dx = \left(-\frac{1}{2}\cos 2x + \cos x\right)\Big|_0^{\pi/3} + \left(-\cos x + \frac{1}{2}\cos 2x\right)\Big|_{\pi/3}^{\pi} = 2.5.$

25. Find the area of the shaded region in Figure 13.

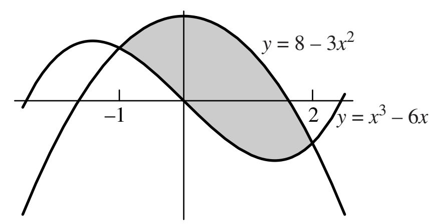


Figure 13

We have $\int_{-1}^{2} \left(8 - 3x^2\right) - \left(x^3 - 6x\right)\, dx = \int_{-1}^{2} \left(-x^3 - 3x^2 + 6x + 8\right)\, dx = \left(-\frac{1}{4}x^4 - x^3 + 3x^2 + 8x\right)\Big|_{-1}^{2} = 20.25.$

In Exercises 27–30, find the area of the shaded region by integrating along the y-axis.

27.

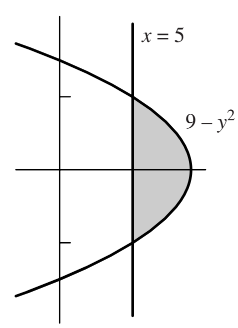


Figure 15 Figure for Problem 27.

We have $\int_{-2}^{2} \left(9 - y^2 - 5\right)\, dy = \int_{-2}^{2} \left(4 - y^2\right)\, dy = \left(4y - \frac{1}{3}y^3\right)\Big|_{-2}^{2} = 10.666667.$

29.

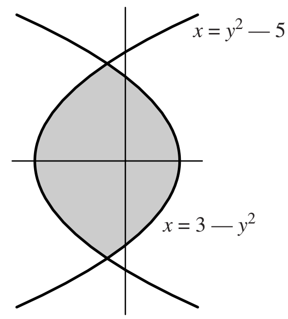


Figure 17 Figure for Problem 29.

We have $\int_{-2}^{2} \left(3 - y^2\right) - \left(y^2 - 5\right)\, dy = \int_{-2}^{2} \left(8 - 2y^2\right)\, dy = \left(8y - \frac{2}{3}y^3\right)\Big|_{-2}^{2} = 21.333333.$

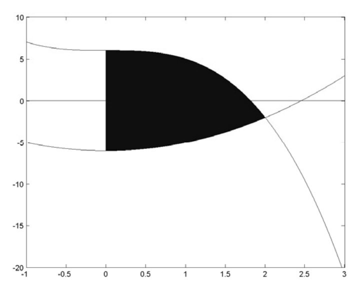
31. Find the area of the shaded region in Figure 17 by integrating with respect to the x-axis.

We have
$\int_{-5}^{-1} \left(2\sqrt{x+5}\right)\, dx + \int_{-1}^{3} \left(2\sqrt{3-x}\right)\, dx = \left(\frac{4}{3}\,(x+5)^{3/2}\right)\Big|_{-5}^{-1} - \left(\frac{4}{3}\,(3-x)^{3/2}\right)\Big|_{-1}^{3} = 0.$

In Exercises 32–48, sketch the region enclosed by the curves and compute its area.

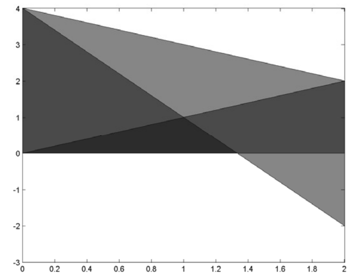
33. $y = x^2 - 6$, $y = 6 - x^3$, y-axis

We have $\int_{0}^{2} \left(6 - x^3\right) - \left(x^2 - 6\right)\, dx = \int_{0}^{2} \left(-x^3 - x^2 + 12\right)\, dx =$
$\left(-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x\right)\Big|_{0}^{2} = 13.333333.$



35. $x + y = 4$, $x - y = 0$, $y + 3x = 4$

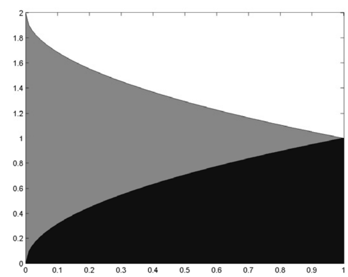
We have $\int_{0}^{1} (4 - x) - (4 - 3x)\, dx + \int_{1}^{2} (4 - x) - (x)\, dx =$
$\int_{0}^{1} (2x)\, dx + \int_{1}^{2} (4 - 2x)\, dx = \left(x^2\right)\Big|_{0}^{1} + \left(4x - x^2\right)\Big|_{1}^{2} = 2.$



Region is the large light-colored triangle.

37. $y = 2 - \sqrt{x}$, $y = \sqrt{x}$, $x = 0$

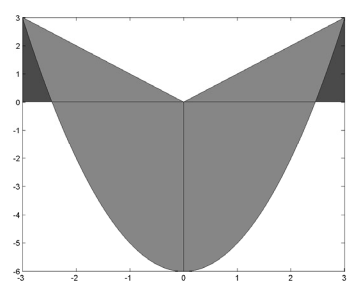
The curves intersect at $x = 1$. Thus we have
$\int_0^1 \left(2 - \sqrt{x} - \sqrt{x}\right)\, dx = \int_0^1 \left(2 - 2\sqrt{x}\right)\, dx = \left(2x - \frac{4}{3}x^{3/2}\right)\Big|_0^1 = \frac{2}{3}$.



Region is light-colored.

39. $y = |x|$, $y = x^2 - 6$

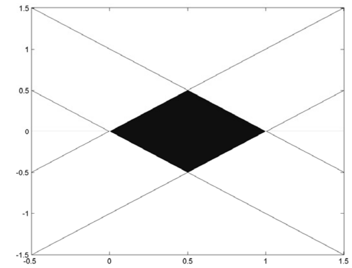
We have $\int_{-3}^0 (-x) - \left(x^2 - 6\right)\, dx + \int_0^3 (x) - \left(x^2 - 6\right)\, dx =$
$\left(-\frac{1}{2}x^2 - \frac{1}{3}x^3 + 6x\right)\Big|_{-3}^0 + \left(\frac{1}{2}x^2 - \frac{1}{3}x^3 + 6x\right)\Big|_0^3 = 27$.



Light-colored region.

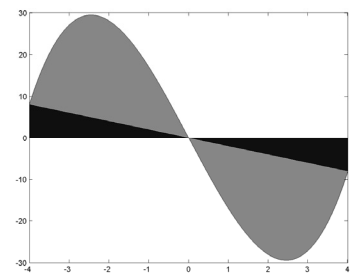
41. $x = |y|$, $x = 1 - |y|$

We have $\int_{-1/2}^0 (1 + y) - (-y)\, dy + \int_0^{1/2} (1 - y) - (y)\, dy =$
$\int_{-1/2}^0 (1 + 2y)\, dy + \int_0^{1/2} (1 - 2y)\, dy = \left(y + y^2\right)\Big|_{-1/2}^0 + \left(y - y^2\right)\Big|_0^{1/2} = 0.5$.



43. $y = x^3 - 18x,\ y + 2x = 0$

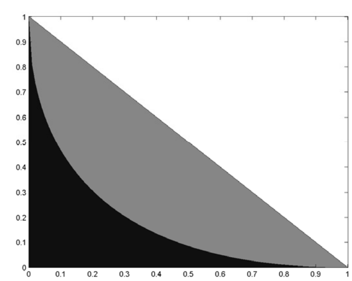
We have $\int_{-4}^{0} \left(x^3 - 18x\right) - (-2x)\ dx + \int_0^4 (-2x) - \left(x^3 - 18x\right)\ dx =$
$\int_{-4}^{0} \left(x^3 - 16x\right)\ dx + \int_0^4 \left(-x^3 + 16x\right)\ dy = \left(\frac{1}{4}x^4 - 8x^2\right)\Big|_{-4}^{0} + \left(-\frac{1}{4}x^4 + 8x^2\right)\Big|_0^4 = 128.$



Region is lighter-colored area.

45. $x + y = 1,\ x^{1/2} + y^{1/2} = 1$

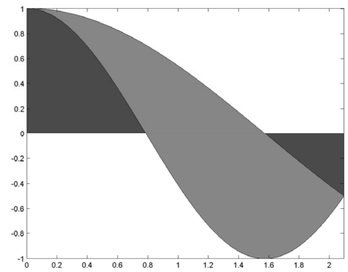
We have $\int_0^1 (1 - x) - \left(\left(1 - \sqrt{x}\right)^2\right)\ dx = \int_0^1 \left(-2x + 2\sqrt{x}\right)\ dx = \left(-x^2 + \frac{4}{3}x^{3/2}\right)\Big|_0^1 = \frac{1}{3}.$



Region is lighter-colored area.

47. $y = \cos x,\ y = \cos(2x),\ x = 0,\ x = \frac{2\pi}{3}$

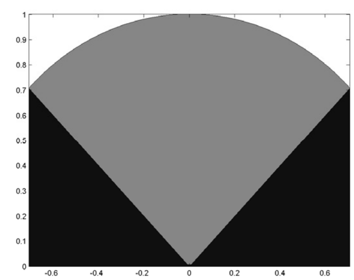
We have $\int_0^{2\pi/3} (\cos x - \cos 2x)\ dx = \left(\sin x - \frac{1}{2}\sin 2x\right)\Big|_0^{2\pi/3} = 1.299038.$



Region is lighter-colored area.

49. Sketch the region whose area is represented by the integral $\int_{-\frac{\sqrt{2}}{2}}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - |x|\right) dx$ and evaluate using geometry.

The region is a sector of a circle of radius 1 with a central angle of $\pi/4$ and is hence a quarter of a circle. The area must then be $\left(\pi r^2\right)/4 = \pi/4$.



Region is lighter-colored area.

51. Find the area enclosed by the curves $y = c - x^2$ and $y = x^2 - c$ as a function of c. Find the value of c for which this area is equal to 1.

The curves intersect at $x = \pm\sqrt{c}$. Thus we have
$\int_{-\sqrt{c}}^{\sqrt{c}} \left(c - x^2\right) - \left(x^2 - c\right)\, dx = \int_{-\sqrt{c}}^{\sqrt{c}} \left(2c - 2x^2\right)\, dx = \left(2cx - \frac{2}{3}x^3\right)\Big|_{-\sqrt{c}}^{\sqrt{c}} = \frac{8}{3}c^{3/2}$. In order for the area to equal 1, we must have $\frac{8}{3}c^{3/2} = 1$ which gives $c = \frac{9^{1/3}}{4} = 0.520021$.

53. Set up (but do not evaluate) an integral that expresses the area between the graphs of $y = (1 + x^2)^{-1}$ and $y = x^2$.

The curves intersect at $x = \pm 0.786151$ and thus the area is given by the integral
$\int_{-.786151}^{.786151} \left(\left(1 + x^2\right)^{-1}\right) - \left(x^2\right)\, dx$.

55. $\boxed{\text{CAS}}$ Find a numerical approximation to the area above $y = |x|$ and below $y = \cos x$ (find the points of intersection numerically).

The curves intersect at $x = \pm 0.739085$. Thus we have
$\int_{-.739085}^{0} (\cos x - (-x))\, dx + \int_{0}^{.739085} (\cos x - x)\, dx =$
$\left(\sin x + \frac{1}{2}x^2\right)\Big|_{-.739085}^{0} + \left(\sin x - \frac{1}{2}x^2\right)\Big|_{0}^{.739085} = 0.800977$.

57. $\boxed{\text{CAS}}$ Use a CAS to find a numerical approximation to the number c (besides 0) in $[0, \pi/2]$ where the curves $y = \sin x$ and $y = \tan^2 x$ intersect. Then find the area enclosed by the graphs over $[0, c]$.

The curves intersect at $x = 0$ and $x = 0.666239$. Thus we have $\int_0^{.666239} \left(\sin x - \tan^2 x\right)\, dx = (-\cos x - \tan x + x)|_0^{.666239} = 0.093937.$

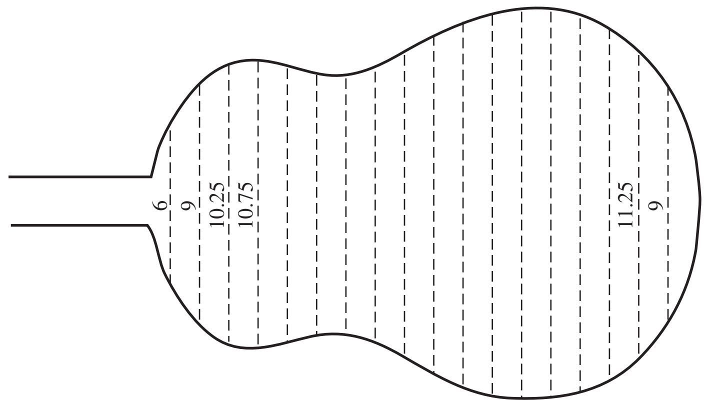


Figure 19 Guitar back.

Further Insights and Challenges

59. Find the line $y = mx$ that divides the area under the curve $y = x(1 - x)$ over $[0, 1]$ into two regions of equal area.

We have $\int_0^1 (x(1-x))\,dx = \int_0^1 \left(x - x^2\right)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{6}$. Now, let $y = mx$ and $y = x(1-x)$ intersect at $x = a$. Then $ma = a(1-a)$ which gives $m = (1-a)$. Hence $y = mx = (1-a)x$. Then $\int_0^a (x(1-x)) - ((1-a)x)\,dx = \int_0^a \left(ax - x^2\right)\,dx = \left(\frac{a}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^a = \frac{a^3}{6}$. Finally, we need $\frac{a^3}{6} = \frac{1}{2}\frac{1}{6} = \frac{1}{12}$ which gives $a = \left(\frac{1}{2}\right)^{1/3}$ and hence $m = (1-a) = 0.206299$.

6.2 Setting Up Integrals: Volumes, Density, Average Value

Preliminary Questions

1. What is the average value of $f(x)$ on $[1, 4]$ if the area between the graph of $f(x)$ and the x-axis is equal to 9?

2. What is the average value of $f(x)$ over $[0, 2]$ assuming that $d/dx(\sqrt{x^3 + 1}) = f(x)$?

Exercises

1. Consider a pyramid of height 20 whose base is a square of side 8.

(a) What is the area of the cross-section of the pyramid at a height x (where $0 \le x \le 20$)?

(b) Calculate the volume of the pyramid by integrating the cross-sectional area.

(a) Using properties of similar triangles, the area of the cross-section at height x is given by $\frac{4}{25}(20 - x)^2$.

(b) We have $\int_0^{20} \left(\frac{4}{25}(20-x)^2\right)\,dx = \left(-\frac{4}{75}(20-x)^3\right)\Big|_0^{20} = 426\frac{2}{3}$.

3. Calculate the volume of a cylinder inclined at an angle $\theta = 30°$ whose height is 10 and whose base is a circle of radius 4.

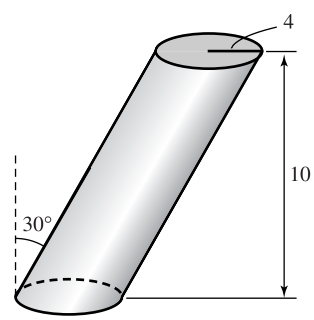


Figure 1 Cylinder inclined at an angle $\theta = 30°$.

We have $\int_0^{10} \left(\pi(4)^2\right)\,dx = (16\pi x)\big|_0^{10} = 160\pi$.

5. Calculate the volume of the ramp in Figure 3 in three ways.

(a) Integrate the area of the rectangular cross-sections perpendicular to the x-axis.

(b) Integrate the area of the triangular cross-sections perpendicular to the y-axis.

(c) Integrate the area of the rectangular cross-sections perpendicular to the z-axis.

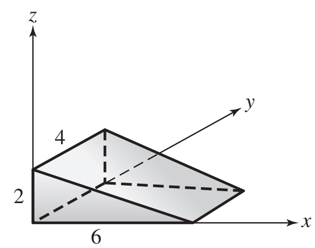


Figure 3 Ramp of length 6, width 4, and height 2.

(a) We have $\int_0^6 4\left(-\frac{1}{3}x+2\right)\,dx = \left(-\frac{2}{3}x^2+8x\right)\Big|_0^6 = 24.$

(b) We have $\int_0^4 \left(\frac{1}{2}\cdot 2\cdot 6\right)\,dy = (6y)\big|_0^4 = 24.$

(c) We have $\int_0^2 4\left(-3(z-2)\right)\,dz = \left(-6z^2+24z\right)\Big|_0^2 = 24.$

7. Derive a formula for the volume of the wedge in Figure 4 (B) in terms of the constants a, b, c.

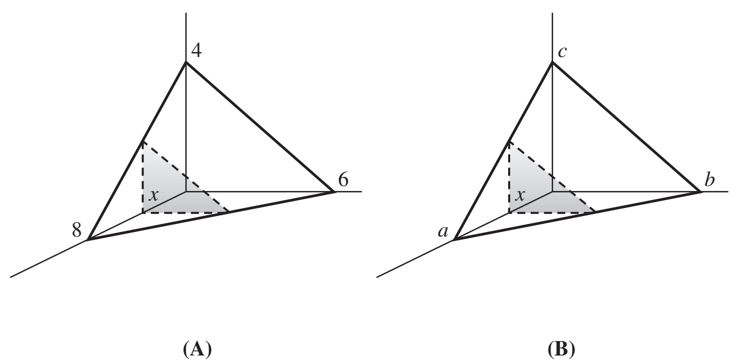


Figure 4

The line from c to a is given by the equation $(z/c)+(x/a)=1$ and the line from b to a is given by $(y/b)+(x/a)=1$. The cross-sections perpendicular to the x-axis are right triangles with height $c(1-x/a)$ and base $b(1-x/a)$ thus we have $\int_0^a \left(\frac{1}{2}bc\,(1-x/a)^2\right)\,dx = \left(-\frac{1}{6}abc(1-x/a)^3\right)\Big|_0^a = \frac{1}{6}abc.$

9. Show that the volume of a pyramid of height h whose base is an equilateral triangle of side s is equal to $\frac{\sqrt{3}}{12}hs^2$.

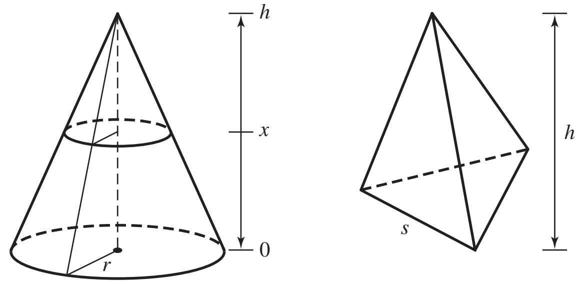


Figure 5 Right-circular cone and pyramid with triangular base.

Using similar triangles, the area of the cross-section is given by $\frac{\sqrt{3}}{4}\left(\frac{s(h-x)}{h}\right)^2$. Thus we have $\frac{s^2\sqrt{3}}{4h^2}\int_0^h \left((h-x)^2\right)\,dx = \left(-\frac{s^2\sqrt{3}}{12h^2}(h-x)^3\right)\Big|_0^h = \frac{\sqrt{3}}{12}s^2h.$

11. Find the volume of the solid B whose base is the unit circle and whose cross-sections perpendicular to the x-axis are equilateral triangles (one side of which is a chord of the circle perpendicular to the x-axis).

We have $\frac{\sqrt{3}}{4}\int_{-1}^{1}\left((2\sqrt{1-x^2})^2\right)\,dx = \sqrt{3}\int_{-1}^{1}\left(1-x^2\right)\,dx = \sqrt{3}\left(x-\frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{4\sqrt{3}}{3}.$

In Exercises 13–20, find the volume with given base and cross-sections.

13. The base is the unit circle $x^2+y^2=1$ and the cross-sections perpendicular to the x-axis are triangles whose height and base are equal.

We have $\frac{1}{2}\int_{-1}^{1}\left((2\sqrt{1-x^2})^2\right)\,dx = 2\int_{-1}^{1}\left(1-x^2\right)\,dx = \left(2x-\frac{2}{3}x^3\right)\Big|_{-1}^{1} = \frac{8}{3}.$

15. The base is the triangle enclosed by $x+y=1$, the x-axis, and the y-axis. The cross-sections perpendicular to the y-axis are equilateral triangles.

We have
$\frac{\sqrt{3}}{4}\int_0^1 (1-y)^2\,dy = \frac{\sqrt{3}}{4}\int_0^1\left(1-2y+y^2\right)\,dy = \frac{\sqrt{3}}{4}\left(y-y^2+\frac{1}{3}y^3\right)\Big|_0^1 = \frac{\sqrt{3}}{12} = 0.144338.$

17. The base is a square, one of whose sides is the interval $[0,\ell]$ along the x-axis. The cross-sections perpendicular to the x-axis are rectangles of height x^2.

We have $\int_0^l \left(lx^2\right)\,dx = \left(\frac{1}{3}lx^3\right)\Big|_0^l = \frac{1}{3}l^4.$

19. The base is the region enclosed by $y=x^2$ and $y=3$. The cross-sections perpendicular to the y-axis are rectangles of height y^3.

We have $\int_0^9 \left(y^3(3-\sqrt{y})\right)\,dy = \int_0^9\left(3y^3-y^{7/2}\right)\,dy = \left(\frac{3}{4}y^4-\frac{2}{9}y^{9/2}\right)\Big|_0^9 = 546.75.$

21. A frustum of a pyramid is a pyramid with the top cut off (Figure 8). Suppose that the height of the frustum is h, the side of the bottom is a, and the side of the top is b.

(a) Show that if the frustum were continued to a full pyramid, it would have height $ha/(a-b)$.

(b) Show that the cross-section at height x is a square of side $(1/h)(a(h-x)+bx)$.

(c) Use integration to show that the volume of the frustum is $(1/3)h(a^2+ab+b^2)$. (A papyrus dating to the year 1850 BCE (almost 4000 years ago!) indicates that Egyptian mathematicians had discovered this formula.)

(d) Confirm the answer in (c) by calculating the volume of the frustum a second time, as the difference of the volumes of two pyramids.

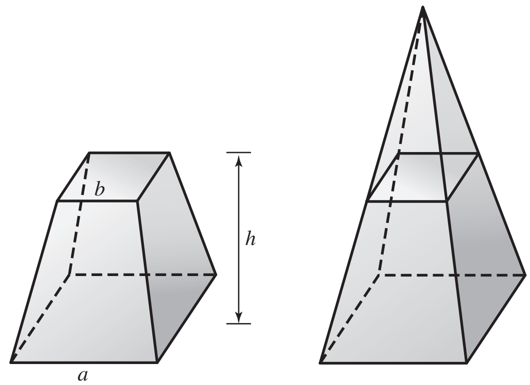


Figure 8

(a) Using similar triangles, we have the proportion $\frac{H}{a} = \frac{H-h}{b}$ which gives the height H of the full pyramid as $H = \frac{ha}{a-b}$.

(b) Again using similar triangles. we have the proportion $\frac{a}{H} = \frac{w}{H-x}$. Substituting the value for H from part (a) gives the width w of the square at height x as $w = \frac{a(h-x)+bx}{h}$.

(c) We have
$\int_0^h \left(\frac{1}{h}(a(h-x)+bx)\right)^2\,dx = \frac{1}{h^2}\int_0^h \left(a^2(h-x)^2 + 2ab(h-x)x + b^2x^2\right)\,dx =$
$\frac{1}{h^2}\left(-\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3\right)\Big|_0^h = \frac{h}{3}\left(a^2+ab+b^2\right)$.

(d) We have $\frac{a^2}{H^2}\int_0^H (H-x)^2\,dx - \frac{a^2}{H^2}\int_h^H (H-x)^2\,dx =$
$\left(-\frac{a^2}{3H^2}(H-x)^3\right)\Big|_0^H + \left(-\frac{a^2}{3H^2}(H-x)^3\right)\Big|_h^H = \frac{1}{3}a^2\left(H-(H-h)^3/H^2\right)$. Substituting the value for H from part (a) gives $\frac{h}{3}\left(a^2+ab+b^2\right)$.

23. Figure 10 shows the solid S obtained by intersecting two cylinders of radius r whose axes are perpendicular.

(a) The horizontal cross-section of one cylinder at a vertical distance y from the axis is a rectangular strip. Find the width of the strip.

(b) Determine the shape and area of the horizontal cross-section of the intersection of the cylinders at vertical distance y.

(c) Find the volume of S as a function of r.

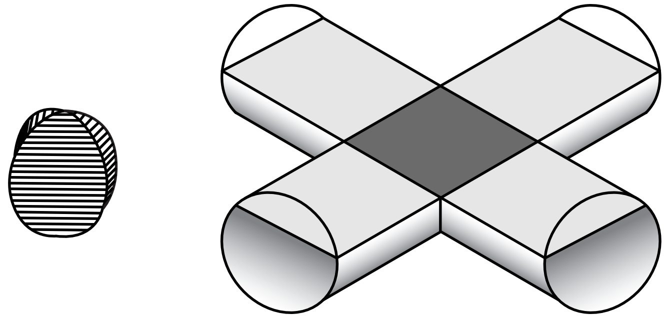


Figure 10 Intersection of two cylinders intersecting at right angles.

(a) The width w at vertical distance y is given by $w = 2\sqrt{r^2 - y^2}$.

(b) The intersection of the two cylinders at vertical distance y is a square with area $w^2 = 4(r^2 - y^2)$.

(c) We have the volume $S = 4\int_{-r}^{r} \left(r^2 - y^2\right)\,dy = 4\left(r^2 y - \frac{1}{3}y^3\right)\Big|_{-r}^{r} = \frac{16}{3}r^3$.

25. Find the total mass of a one-meter rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m.

We have $\int_0^1 \left(10(x+1)^{-2}\right)\,dx = \left(-10(x+1)^{-1}\right)\Big|_0^1 = 5$ kg.

27. A mineral deposit along a strip of length 6 cm has density $s(x) = .01x(6-x)$ g/cm. Calculate the total mass of the deposit.

We have $\int_0^6 (.01x(6-x))\,dx = \left(.03x^2 - \frac{.01}{3}x^3\right)\Big|_0^6 = 0.36$ g.

29. Calculate the population within a 10-mile radius of the city center if the radial population density is $\rho(r) = 8(1+r^2)^{1/3}$.

We have $2\pi\int_0^{10} \left(8r(1+r^2)^{1/3}\right)\,dr = 8\pi\int_0^{101} \left(u^{1/3}\right)\,du = \left(6\pi u^{4/3}\right)\Big|_0^{101} = 1474.531519$.

31. The population density (in people per hectare) as a function of distance (in km) from the city center in a medium-sized town is listed in Table 31. Estimate the total population within a 2-km radius of the center by taking the average of the left- and right-endpoint approximations.

r	$\rho(r)$	r	$\rho(r)$
0.0	125.0	1.2	37.6
0.2	102.3	1.4	30.8
0.4	83.8	1.6	25.2
0.6	68.6	1.8	20.7
0.8	56.2	2.0	16.9
1.0	46.0		

We have $\delta x = .2$. Thus
$R_{10} = .2\,(102.3 + 83.8 + 68.6 + 56.2 + 46 + 37.6 + 30.8 + 25.2 + 20.7 + 16.9) = 97.62$
and $L_{10} = .2\,(125 + 102.3 + 83.8 + 68.6 + 56.2 + 46 + 37.6 + 30.8 + 25.2 + 20.7) =$
119.24. The average is the left and right-endpoint approximations is then
$.5(L_{10} + R_{10}) = 108.43$.

In Exercises 33–39, calculate the average over the given interval.

33. x^3, $[0, 1]$

The average is $\frac{1}{1-0}\int_0^1 x^3\,dx = \int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$.

35. $\cos t$, $[0, \pi/2]$

$$\text{The average is } \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos t\,dt = \frac{2}{\pi}\left(\sin t\Big|_0^{\pi/2}\right) = \frac{2}{\pi}.$$

37. s^{-2}, $[1, 2]$

The average is $\frac{1}{2-1}\int_1^2 s^{-2}\,ds = \int_1^2 s^{-2}\,ds = -s^{-1}\Big|_1^2 = \frac{1}{2}$.

39. $2x^3 - 3x^2$, $[1, 2]$

The average is $\frac{1}{2-1}\int_1^2 \left(2x^3 - 3x^2\right)\,dx = \int_1^2 \left(2x^3 - 3x^2\right)\,dx = \left(\frac{1}{2}x^4 - x^3\right)\Big|_1^2 = \frac{1}{2}$.

41. Find the average of $f(x) = ax + b$ over the interval $[-M, M]$, where a, b, and M are arbitrary constants.

The average is
$\frac{1}{M-(-M)}\int_{-M}^{M} (ax + b)\,dx = \frac{1}{2M}\int_{-M}^{M} (ax + b)\,dx = \frac{1}{2M}\left(\frac{a}{2}x^2 + bx\right)\Big|_{-M}^{M} = b$.

43. A ball is thrown in the air vertically from ground level with an initial velocity of 64 ft/s. Find the average height of the ball over the time interval extending from the time of the ball's release to its return to ground level. Recall that the height at time t is $h(t) = 64t - 16t^2$.

The ball is at ground level at time $t = 0$ and $t = 4$. The average height of the ball is
$\frac{1}{4-0}\int_0^4 h(t)\,dt = \frac{1}{4}\int_0^4 \left(64t - 16t^2\right)\,dt = \frac{1}{4}\left(32t^2 - \frac{16}{3}t^3\right)\Big|_0^4 = 42.666667$.

45. An object with zero initial velocity accelerates at a constant rate of 10 m/sec^2. Find its average velocity during the first 15 seconds.

An acceleration $a(t) = 10$ gives $v(t) = 10t + c$ for some constant c and zero initial velocity implies $c = 0$. Thus the average velocity is given by $\frac{1}{15-0}\int_0^{15} 10t\,dt = \frac{1}{3}t^2\Big|_0^{15} = 75$ m/s.

47. Let M be the average value of $f(x) = x^3$ on $[0, 3]$. Find a value of c in $[0, 3]$ such that $f(c) = M$.

We have $M = \frac{1}{3-0}\int_0^3 x^3\,dx = \frac{1}{3}\int_0^3 x^3\,dx = \frac{1}{12}x^4\Big|_0^3 = \frac{27}{4}$. Then $M = f(c) = c^3 = \frac{27}{4}$ implies $c = \frac{3}{4^{1/3}} = 1.889882$.

49. Give an example of a function (necessarily discontinuous) which does not satisfy the conclusion of the Mean Value Theorem for integrals.

There are an infinite number of discontinuous function which do not satisfy the conclusion of the Mean Value Theorem for Integrals. Consider an function on $[-1, 1]$ such that for $x < 0$, $f(x) = -1$ and for $x \geq 0$, $f(x) = 1$. Clearly the average value is 0 but $f(c) \neq 0$ for all c in $[-1, 1]$.

Further Insights and Challenges

6.3 Volumes of revolution

Preliminary Questions

1. Which of the following is a solid of revolution?

(a) sphere
(b) pyramid
(c) cylinder
(d) cube

2. True or false: When a solid is formed by rotating the area under a graph about the x-axis, the cross-sections perpendicular to the x-axis are circular disks.

3. True or false: When a solid is formed by rotating the area between two curves about the x-axis, the cross-sections perpendicular to the x-axis are circular disks.

4. Which of the following integrals expresses the volume of the solid obtained by rotating the area between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around the x-axis (assume $f(x) \geq g(x) \geq 0$):

(a) $\pi \displaystyle\int_a^b (f(x) - g(x))^2\, dx$

(b) $\pi \displaystyle\int_a^b (f(x)^2 - g(x)^2)\, dx$

Exercises

In Exercises 1–4, (a) sketch the solid obtained by revolving the graph of the function about the x-axis over the given interval; (b) describe the cross-section perpendicular to the x-axis located at x; (c) calculate the volume of the solid.

1. $x + 1$, $[0, 3]$

(a) [FIGURE]
(b) The cross-section is a disk with radius $x + 1$.
(c) We have $\pi \int_0^3 (x+1)^2\, dx = \pi \int_0^3 (x^2 + 2x + 1)\, dx = \pi \left(\frac{1}{3}x^3 + x^2 + x\right)\Big|_0^3 = 21\pi$.

3. $\sqrt{x+1}$, $[1, 4]$

(a) [FIGURE]
(b) The cross-section is a disk with radius $\sqrt{x+1}$.
(c) We have $\pi \int_1^4 (\sqrt{x+1})^2\, dx = \pi \int_1^4 (x+1)\, dx = \pi \left(\frac{1}{2}x^2 + x\right)\Big|_1^4 = 10.5\pi$.

In Exercises 5–12, find the volume of the solid obtained by rotating the area under the graph of the function over the given interval about the x-axis.

5. $x^2 - 3x;\quad [0,3]$

We have
$\pi \int_0^3 (x^2 - 3x)^2\,dx = \pi \int_0^3 (x^4 - 6x^3 + 9x^2)\,dx = \pi \left(\frac{1}{5}x^5 - \frac{3}{2}x^3 + 3x^3\right)\Big|_0^3 = 25.4469.$

7. $x^{5/3};\quad [1, 8]$

We have

$$\pi \int_1^8 (x^{5/3})^2\,dx = \pi \int_1^8 x^{10/3}\,dx = \pi \frac{13}{3}\,x^{13/3}\Big|_1^8 = \pi \frac{13}{3}(2^{13} - 1) = \frac{13\pi}{3}(8192).$$

9. $\dfrac{2}{x+1};\quad [1, 3]$

We have $\pi \int_1^3 \left(\dfrac{2}{x+1}\right)^2 dx = 4\pi \int_1^3 (x+1)^{-2}\,dx = -4\pi\,(x+1)^{-1}\Big|_1^3 = \pi.$

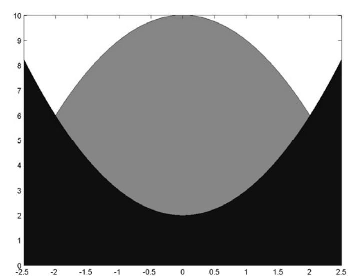
11. $\sqrt{\cos x + 1};\quad [0, \pi]$

We have $\pi \int_0^\pi (\sqrt{\cos x + 1})^2\,dx = \pi \int_0^\pi (\cos x + 1)\,dx = \pi\,(\sin x + x)\big|_0^\pi = \pi^2.$

In Exercises 13–20, sketch the region enclosed by the curves and find the volume of the solid obtained by rotating the region about the x-axis.

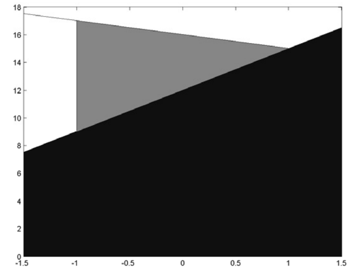
13. $y = x^2 + 2,\quad y = 10 - x^2$

The curves intersect at $x = \pm 2$. We have
$\pi \int_{-2}^2 (10 - x^2)^2 - (x^2 + 2)^2\,dx = \pi \int_{-2}^2 (96 - 24x^2)\,dx = \pi \left(96x - 8x^3\right)\Big|_{-2}^2 = 256\pi.$



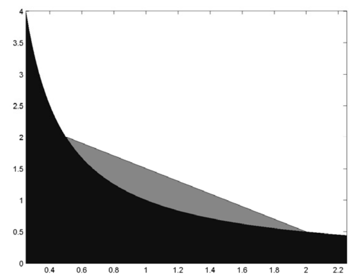
15. $y = 16 - x,\quad y = 3x + 12,\quad x = -1$

The lines intersect at $x = 1$. We have $\pi \int_{-1}^1 (16 - x)^2 - (3x + 12)^2\,dx =$
$\pi \int_{-1}^1 (112 - 104x - 8x^2)\,dx = \pi \left(112x - 52x^2 - \frac{8}{3}x^3\right)\Big|_{-1}^1 = 686.961593.$



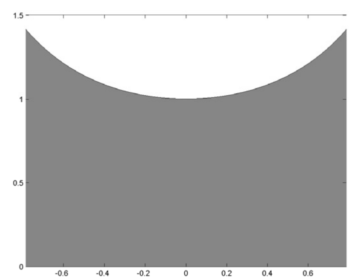
17. $y = \frac{1}{x}, \quad y = \frac{5}{2} - x$

The lines intersect at $x = 1/2$ and $x = 2$. We have
$\pi \int_{1/2}^{2} (\frac{5}{2} - x)^2 - (\dfrac{1}{x})^2 \, dx = \pi \int_{1/2}^{2} (\frac{25}{4} - 5x + x^2 - x^{-2}) \, dx =$
$\pi \left(\frac{25}{4}x - \frac{5}{2}x^2 + \frac{1}{3}x^3 + x^{-1} \right)\Big|_{1/2}^{2} = 21.125\pi = 66.366145.$



19. $y = \sec x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

We have $\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2 \, dx = \pi \, (\tan x)|_{-\pi/4}^{\pi/4} = 2\pi.$



In Exercises 21–26, make a rough sketch of the solids obtained by rotating region A in Figure 1 about the given axis and find its volume.

21. x-axis

We have
$\pi \int_0^2 (6)^2 - (x^2+2)^2 \, dx = \pi \int_0^2 (32 - 4x^2 - x^4) \, dx = \pi \left(32x - \frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = 147.445415.$

[FIGURE]

23. $y = 8$

We have $\pi \int_0^2 (8-(x^2+2))^2 - (2)^2\, dx = \pi \int_0^2 (32 - 12x^2 + x^4)\, dx =$
$\pi \left(32x - 4x^3 + \frac{1}{5}x^5\right)\Big|_0^2 = 80.424772.$

[FIGURE]

25. $x = -3$

We have $\pi \int_2^6 (3+\sqrt{y-2})^2 - (3)^2\, dy = \pi \int_2^6 (6\sqrt{y-2} + y - 2)\, dy =$
$\pi \left(4(y-2)^{3/2} + \frac{1}{2}y^2 - 2y\right)\Big|_2^6 = 40\pi.$

[FIGURE]

In Exercises 27–32, make a rough sketch of the solids obtained by rotating region B in Figure 1 about the given axis and find its volume.

27. x-axis

We have
$\pi \int_0^2 (x^2+2)^2\, dx = \pi \int_0^2 (x^4 + 4x^2 + 4)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right)\Big|_0^2 = 78.749256.$

[FIGURE]

29. $y = 8$

We have $\pi \int_0^2 (8)^2 - (8-(x^2+2))^2\, dx = \pi \int_0^2 (28 + 12x^2 - x^4)\, dx =$
$\pi \left(28x + 4x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = 256.35396.$

[FIGURE]

31. $x = 2$

We have $\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 (2-\sqrt{y-2})^2\, dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (2 + y - 4\sqrt{y-2})\, dy =$
$\pi\, (4y)\big|_0^2 + \pi \left(2y + \frac{1}{2}y^2 - \frac{8}{3}(y-2)^{3/2}\right)\Big|_2^6 = 33.510322.$

[FIGURE]

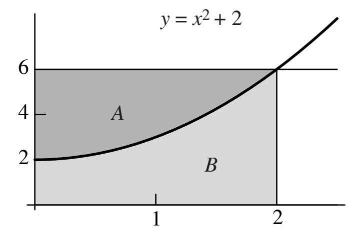


Figure 1

In Exercises 33–42, find the volume of the body obtained by rotating the region enclosed by the graphs about the given axis.

33. $y = x^2, \quad y = 12 - x, \quad x = 0 \quad \text{about } y = -2$

The curves intersect at $x = 3$. We have $\pi \int_0^3 (12 - x + 2)^2 - (x^2 + 2)^2 \, dx =$
$\pi \int_0^3 (192 - 28x - 3x^2 - x^4) \, dx = \pi \left(192x - 14x^2 - x^3 - \frac{1}{5}x^5\right)\Big|_0^3 = 1176.21229.$

35. $y = 16 - x, \quad y = 3x + 12, \quad x = 0, \quad \text{about } y\text{-axis}$

We have
$\pi \int_{12}^{15} (\frac{1}{3}(y - 12))^2 \, dy + \pi \int_{15}^{16} (16 - y)^2 \, dy = \pi \int_{12}^{15} \frac{1}{9}(y^2 - 24y + 144) \, dy + \pi \int_{15}^{16} (y^2 -$
$32y + 256) \, dy = \frac{\pi}{9} \left(\frac{1}{3}y^3 - 12y^2 + 144y\right)\Big|_{12}^{15} + \pi \left(\frac{1}{3}y^3 - 16y^2 + 256y\right)\Big|_{15}^{16} = \frac{4}{3}\pi.$

37. $y = \frac{1}{x}, \quad y = \frac{5}{2} - x \quad \text{about } y = -1$

The curves intersect at $x = 2$ and $x = \frac{1}{2}$. We have
$\pi \int_{1/2}^{2} (\frac{5}{2} - x + 1)^2 - (x^{-1} + 1)^2 \, dx = \pi \int_{1/2}^{2} (\frac{45}{4} - 7x + x^2 - x^{-2} - 2x^{-1}) \, dx =$
$\pi \left(\frac{45}{4}x - \frac{7}{2}x^2 + \frac{1}{3}x^3 + x^{-1} - 2\ln x\right)\Big|_{1/2}^{2} = 6.604919.$

39. $y = \frac{1}{x}, \quad y = \frac{5}{2} - x \quad \text{about } y\text{-axis}$

We have $\pi \int_{1/2}^{2} (\frac{5}{2} - y)^2 - (y^{-1})^2 \, dy = \pi \int_{1/2}^{2} (\frac{25}{4} - 5y + y^2 - y^{-2}) \, dy =$
$\pi \left(\frac{25}{4}y - \frac{5}{2}y^2 + \frac{1}{3}y^3 + y^{-1}\right)\Big|_{1/2}^{2} = 3.534292.$

41. $y = x^3, \quad y = x^{1/3}, \quad \text{about } y\text{-axis}$

We have
$\pi \int_{-1}^{1} (y^{1/3})^2 - (y^3)^2 \, dy = \pi \int_{-1}^{1} (y^{2/3} - y^6) \, dy = \pi \left(\frac{3}{5}y^{5/3} - \frac{1}{7}y^7\right)\Big|_{-1}^{1} = 2.872314.$

43. Find the volume of the solid obtained by revolving the hypocycloid $x^{2/3} + y^{2/3} = 1$ about the x-axis.

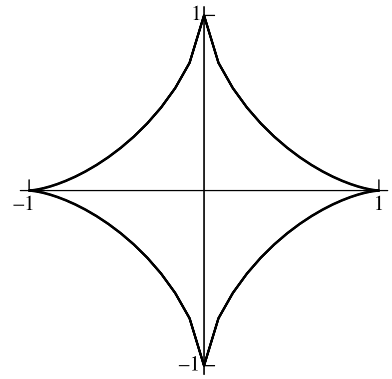


Figure 2 Hypocycloid $x^{2/3} + y^{2/3} = 1$.

We have $\pi \int_{-1}^{1} \left((1 - x^{2/3})^{3/2}\right)^2 \, dx = 2\pi \int_0^1 (1 - 3x^{2/3} + 3x^{4/3} - x^2)\, dx =$
$2\pi \left(x - \frac{9}{5}x^{5/3} + \frac{9}{7}x^{7/3} - \frac{1}{3}x^3\right)\Big|_0^1 = 0.957438.$

45. The volume generated by the rotating the hyperbola with equation $y^2 - x^2 = 1$ about the x-axis is called a *hyperboloid*. Find the volume of the portion of the hyperboloid with $-a \le x \le a$.

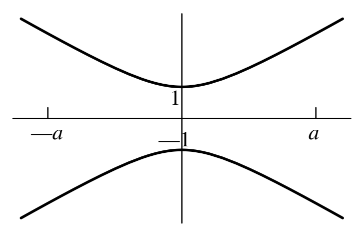


Figure 3 The hyperbola with equation $y^2 - x^2 = 1$.

We have $\pi \int_{-a}^{a} \left(\sqrt{1 + x^2}\right)^2 \, dx = \pi \int_{-a}^{a} (1 + x^2)\, dx = \pi \left(x + \frac{1}{3}x^3\right)\Big|_{-a}^{a} = \frac{8}{3}a\pi.$

Further Insights and Challenges

47. R & W Suppose that a *bead* is formed by removing a cylinder of radius r from the center sphere of radius R.

(a) Find the volume V of the bead in terms of r and R.

(b) Show that $V = \frac{\pi}{6} h^3$ where h is the height of the bead.

(c) The formula in (b) shows that V depends on the height but not on the radius of the bead. It follows that two beads of height 2 inches, one formed from a sphere the size of an orange and the other the size of the earth would have the same volume! (G. Alexanderson and L. Klosinski, "Some Surprising Volumes of Revolution," *Two-Year College Mathematics Journal*, v. 6, No. 3, pp. 13–15.) Can you explain intuitively how this is possible?

(a) We have $V = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\sqrt{R^2 - x^2}\right)^2 - r^2\,dx = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} ((R^2 - r^2) - x^2)\,dx = \pi \left((R^2 - r^2)x - \frac{1}{3}x^3\right)\Big|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} = \frac{4}{3}(R^2 - r^2)^{3/2}\pi.$

(b) We have $h = 2\sqrt{R^2 - r^2} = 2(R^2 - r^2)^{1/2}$ which gives $h^3 = 8(R^2 - r^2)^{3/2}$ and finally $(R^2 - r^2)^{3/2} = \frac{1}{8}h^3$. Substituting into the answer from part (a) gives $V = \frac{\pi}{6}h^3$.

(c) The beads may have the same volume but clearly the wall of earth-sized bead must be extremely thin while the orange-sized bead would be thicker.

49. A doughnut-shaped solid is called a *torus*. Use the Disk Method to calculate the volume of the torus obtained by rotating the circle with equation $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$).

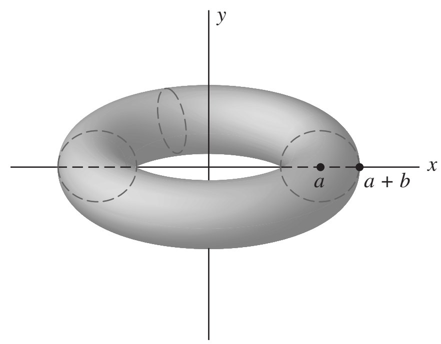


Figure 5 Torus obtained by rotating a circle about the y-axis.

We have $\pi \int_{-b}^{b} \left(a + \sqrt{b^2 - y^2}\right)^2 - \left(a - \sqrt{b^2 - y^2}\right)^2 dy = \pi \int_{-b}^{b} (4a\sqrt{b^2 - y^2})\,dy = 4a\pi \left(\frac{1}{2}y\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}(y/b)\right)\Big|_{-b}^{b} = 2\pi^2 ab^2.$