Applied Functional Analysis
Lecture Notes
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1 Introduction to Functional Analysis

1.1 Goals of this Course

In these lectures, we shall present functional analysis for partial differential equations (PDE's) or distributed parameter systems (DPS) as the basis of modern PDE techniques. This is in contrast to classical PDE techniques such as separation of variables, Fourier transforms, Sturm-Louiville problems, etc. It is also somewhat different from the emphasis in usual functional analysis courses where one learns functional analysis as a subdiscipline in its own right. Here we treat functional analysis as a **tool** to be used in understanding and treating distributed parameter systems. We shall also motivate our discussions with numerous application examples from biology, electromagnetics and materials/mechanics.

1.2 Uses of Functional Analysis for PDEs

As we shall see, functional analysis techniques can often provide powerful tools for insight into a number of areas including:

- Modeling
- Qualitative analysis
- Inverse problems
- Control
- Engineering analysis
- Computation (such as finite element and spectral methods)

1.3 Example 1: Heat Equation

$$\frac{\partial y}{\partial t}(t,\xi) = \frac{\partial}{\partial \xi}\left(D(\xi)\frac{\partial y}{\partial \xi}(t,\xi)\right) + f(t,\xi) \qquad 0 < \xi < l, \quad t > 0 \tag{1}$$

where $y(t,\xi)$ denotes the temperature in the rod at time t and position ξ and $f(t,\xi)$ is the input from a source, e.g. heat lamp, laser, etc.
B.C.

$\xi = 0:$	$y(t,0) = 0$	Dirichlet B.C. This indicates temperature is held constant.
$\xi = l:$	$D(l)\frac{\partial y}{\partial \xi}(t,l) = 0$	Neumann B.C. This indicates an insulated

end and results in heat flux being zero.

Here $j(t,\xi) = D(\xi)\frac{\partial y}{\partial \xi}(t,\xi)$ is called the heat flux.

I.C.

$$y(0,\xi) = \Phi(\xi) \qquad 0 < \xi < l$$

Here Φ denotes the initial temperature distribution in the rod.

1.4 Some Preliminary Operator Theory

Let X, Y be normed linear spaces. Then

$$\mathcal{L}(X,Y) = \{T : X \to Y | \text{bounded, linear}\}$$

is also a normed linear space, and

$$|T|_{\mathcal{L}(X,Y)} = \sup_{|x|_X = 1} |Tx|_Y.$$

Recall that linear spaces are closed under addition and scalar multiplication. A normed linear space X is complete if for any Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Note: $\mathcal{L}(X,Y)$ is complete if Y is complete.

If X is a Hilbert or Banach space (complex), then T is a bounded linear operator from $X_1 \to X_2$, i.e., $T \in \mathcal{L}(X_1, X_2) \iff T$ is continuous.

Differentiation in Normed Linear Spaces [HP]

Suppose $f : X \to Y$ is a (nonlinear) transformation (mapping).

Definition 1 If $\lim_{\epsilon\to 0^+} \frac{f(x_0+\epsilon z)-f(x_0)}{\epsilon}$ exists, we say f has a directional derivative at x_0 in direction z. This is denoted $\delta f(x_0; z)$, and is called the *Gateaux differential* at x_0 in direction z.

If the limit exists for any direction z, we say f is Gateaux differentiable at x_0 and $z \to \delta f(x_0; z)$ is Gateaux derivative. Note that $z \to \delta f(x_0; z)$ is not necessarily linear, however it is homogeneous of degree one (i.e., $\delta f(x_0; \lambda z) = \lambda \delta f(x_0; z)$ for scalars λ). Moreover, it need not be continuous in z.

The following definition of $o(|z|)$ will be useful in defining the Fréchet derivative:

Definition 2 $g(z)$ is $o(|z|)$ if $\frac{|g(z)|}{|z|} \to 0$ as $z \to 0$.

Definition 3 If there exists $df(x_0; \cdot) \in \mathcal{L}(X, Y)$ such that $|f(x_0 + z) - f(x_0) - df(x_0; z)|_Y = o(|z|_X)$, then $df(x_0; \cdot)$ is the *Fréchet derivative* of f at x_0. Equivalently, $\lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon z) - f(x_0)}{\epsilon} = df(x_0; z)$ for every $z \in X$ with $z \to df(x_0; z) \in \mathcal{L}(X, Y)$. We write $df(x_0; z) = f'(x_0)z$.

Results

1. If f has a Fréchet derivative, it is unique.

2. If f has a Fréchet derivative at x_0, then it also has a Gateaux derivative at x_0 and they are the same.

3. If $f : D_{open} \subset X \to Y$ has a Fréchet derivative at x_0, then f is continuous at x_0.

Examples

1. $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $f : X \to Y$ such that each component of f has partials at x_0 in the form $\frac{\partial f^i}{\partial x^j}(x_0)$. Then

$$f'(x_0) = \left(\frac{\partial f^i}{\partial x^j}(x_0) \right)$$

   and hence

$$df(x_0; z) = \left(\frac{\partial f^i}{\partial x^j}(x_0) \right) z.$$

2. Now we consider the case where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^1$. Then

$$df(x_0; z) = \nabla f(x_0) \cdot z.$$

3. We also look at the case where $X = Y = \mathbb{R}^1$. Here,

$$df(x_0; z) = f'(x_0)z.$$

   Normally $z = 1$ because that is the *only* direction in $\mathbb{R}^1$. Then $f'(x_0)$ is called the derivative, but in actuality $z \to f'(x_0)z$ is the derivative. Note that $f'(x_0) \in \mathcal{L}(\mathbb{R}^1, \mathbb{R}^1)$, but the elements of $\mathcal{L}(\mathbb{R}^1, \mathbb{R}^1)$ are just numbers.

Homework Exercises

- Ex. 1 : Consider $f : \mathbb{R}^2 \to \mathbb{R}^1$.

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2^2}{x_1^2 + x_2^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

  Show f is Gateaux differentiable at $x = (x_1, x_2) = 0$, but not continuous at $x = 0$, (and hence cannot be Fréchet differentiable).

1.5 Transforming the Initial Boundary Value Problem

If we can transform the initial boundary value problem (IBVP) for equation (1) into something of the form

$$\begin{cases} \dot{x}(t) &= Ax(t) + F(t) \\ x(0) &= x_0 \end{cases} \tag{2}$$

then conceptually it might be an easier problem with which to work. That is, formally it looks like an ordinary differential equation problem for which e^{At} is a solution operator.

To rigorously make this transformation and develop a corresponding conceptual framework, we need to undertake several tasks:

1. Find a space X of functions and an operator $A : \mathcal{D}(A) \subset X \to X$ such that the IBVP can be written in the form of equation (2). We may find $X = L_2(0, l)$ or $C(0, l)$.

   We want a solution $x(t) = y(t, \cdot)$ to (2), but in what sense - mild, weak, strong, classical? This will answer questions about regularity (i.e., smoothness) of solutions.

2. We also want solution operators or "semigroups" $T(t)$ which play the role of e^{At}. But what does "e^{At}" mean in this case? If $A \in R^n \times R^n$ is a constant matrix, then

$$e^{At} \equiv \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{A^2 t^2}{2!} + \dots$$

   where the series has nice convergent properties. In this case, by the "variation of constants" or "variation of parameters" representation, we can then write

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} F(s) ds.$$

We want a similar variation of constants representation in X with operators $T(t) \in \mathcal{L}(X) = \mathcal{L}(X, X)$ such that $T(t) \sim e^{At}$, so that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F(s)ds.$$

holds and represents the PDE solutions in some appropriate sense and can be used for qualitative (stability, asymptotic behavior, control) and quantitative (approximation and numerics) analyses.

2 Semigroups and Infinitesimal Generators

2.1 Basic Principles of Semigroups [HP, Pa, Sh, T]

Definition 4 A *semigroup* is a one parameter set of operators $\{T(t) : T(t) \in \mathcal{L}(X)\}$, where X is a Banach or Hilbert space, such that $T(t)$ satisfies

1. $T(t+s) = T(t)T(s)$ (semigroup or Markov or translation property)
2. $T(0) = I$. (identity property)

Classification of semigroups by continuity

- $T(t)$ is *uniformly continuous* if $\lim_{t \to 0^+} |T(t) - I| = 0$.

  This is not of interest to us, because $T(t)$ is uniformly continuous if and only if $T(t) = e^{At}$ where A is a bounded linear operator.

- $T(t)$ is *strongly continuous*, denoted C_0, if for each $x \in X$, $t \to T(t)x$ is continuous on $[0, \delta)$ for some positive δ.

Note 1: All continuity statements are in terms of continuity from the right at zero. For fixed t

$$\begin{aligned} T(t+h) - T(t) &= T(t)[T(h) - T(0)] \\ &= T(t)[T(h) - I] \end{aligned}$$

and

$$T(t) - T(t-\epsilon) = T(t-\epsilon)[T(\epsilon) - I]$$

so that continuity from the right at zero is equivalent to continuity at any t for operators that are uniformly bounded on compact intervals.

Note 2: $T(t)$ uniformly continuous implies $T(t)$ strongly continuous ($|T(t)x - x| \leq |T(t) - I|\ |x|$), but not conversely.

2.2 Return to Example 1 : Heat Equation

To begin the process of writing the system of Example 1 in the form of equation (2), take $X = L_2(0, l)$ and define

$$\mathcal{D}(A) = \{\varphi \in H^2(0, l) | \varphi(0) = 0, \varphi'(l) = 0\}$$

to be the domain of A. (Assume D is smooth for now, e.g., D is at least H^1.) Then we can define $A : \mathcal{D}(A) \subset X \to X$ by

$$A[\varphi](\xi) = (D(\xi)\varphi'(\xi))'. \tag{3}$$

Notation: In general, $W^{k,p}(a,b) = \{\varphi \in L^p | \varphi', \varphi'', \ldots, \varphi^{(k)} \in L^p\}$. If $p = 2$, we write $W^{k,2}(a,b) = H^k(a,b)$. $AC(a,b) = \{\varphi \in L_2(a,b) | \varphi$ is absolutely continuous (AC) on $[a,b]\}$.

Homework Exercises

- Ex. 2 : Show that $A : \mathcal{D}(A) \subset X \to X$ of the heat example (Example 1) is not bounded $X \to X$.
- Ex. 3 : Give an example when X is a Hilbert space, but $\mathcal{L}(X)$ is not a Hilbert space.
- Ex. 4 : Consider $H^1(a,b) = W^{1,2}(a,b), W^{1,1}(a,b)$ and $AC(a,b)$. Find the relationships for each pair of spaces in terms of $\subset$. That is, establish $X \subset Y$ or $X \subsetneq Y$, etc.

2.3 Example 2 : General Transport Equation

Problems in which the transport equation is used

- Insect dispersion
- Growth and decline of population-a special case-see Example 4 below
- Flow problems (convective/diffusive transport)

Specific applications as described in [BKa, BK] include the "cat brain problem" (which involves in vivo labeled transport) and population dispersal problems.

Description of various fluxes

The quantity $y(t,\xi)$ is the population or amount of a substance at location ξ in the habitat.

- $j_1(t,\xi)$ is the flux due to dispersion (random foraging or moments; random molecular collisions).

$$j_1(t,\xi) = D(\xi)\frac{\partial y}{\partial \xi}(t,\xi)$$

- $j_2(t,\xi)$ is the advective/convective/directed bulk movement.

$$j_2(t,\xi) = \nu(\xi)y(t,\xi)$$

- $j_3(t,\xi)$ is the net loss due to "birth/death"; immigration/emigration, label decay.

$$j_3(t,\xi) = \mu(\xi)y(t,\xi)$$

General transport equation for a 1-dimensional habitat

$$\frac{\partial y}{\partial t}(t,\xi) + \frac{\partial}{\partial \xi}(\nu(\xi)y(t,\xi)) = \frac{\partial}{\partial \xi}(D(\xi)\frac{\partial y}{\partial \xi}(t,\xi)) - \mu(\xi)y(t,\xi) \qquad (4)$$

$$0 < \xi < l, \quad t > 0$$

B.C.

$\xi = 0$: $\quad y(t,0) = 0 \quad$ (essential)

$\xi = l$: $\quad \left[D(\xi)\frac{\partial y}{\partial \xi}(t,\xi) - \nu(\xi)y(t,\xi)\right]_{\xi=l} = 0 \quad$ (natural)

I.C. $y(0,\xi) = \Phi(\xi)$

To write this example in operator form (2), we first let $X = L_2(0,l)$ be the usual complex Hilbert space of square integrable functions, and the domain of A be defined by $\mathcal{D}(A) = \{\varphi \in H^2(0,l) | \varphi(0) = 0, [D\varphi' - \nu\varphi](l) = 0\}$ where

$$A\varphi(\xi) = (D(\xi)\varphi'(\xi))' - (\nu(\xi)\varphi(\xi))' - \mu\varphi(\xi). \qquad (5)$$

Note: We will assume D and ν are smooth for now.

2.4 Example 3 : Delay Systems–Insect/Insecticide Models

Delay systems have been of interest for the past 70 or more years, arising in applications ranging from aerospace engineering to biology (biochemical pathways, etc.), population models, ecology, HIV and other disease progression models to viscoelastic and smart hysteretic materials, as well as network models [BBH, BBJ, BBPP, BKurW1, BKurW2, BRS, Hutch, KP, Ma, Vis, Warga, Wright]. We describe here one arising in insect/insecrticide investigations [BBJ].

Mathematical models that are suitable for field data with mixed populations should consider reproductive effects and should also account for multiple generations, containing neonates (juveniles) and adults and their interconnectedness. This suggests the need at the minimum for a coupled system of equations describing two separate age classes. Additionally, due to individual differences within the insect population, it is biologically unrealistic to assume that all neonate aphids born on the same day reach the adult age class at the same time. In fact, the age at which the insects reach adulthood varies from as few as five to as many as seven days. Hence one must include a term in any model to account for this variability, leading one to develop a coupled delay differential equation model for the insect population dynamics. We consider the delay between birth and adulthood for neonate pea aphids and present a first mathematical model that treats this delay as a random variable. For a careful derivation of models with similar structure in HIV progression at the cellular level, see [BBH].

Let $a(t)$ and $n(t)$ denote the number of adults and neonates, respectively, in the population at time t. We lump the mortality due to insecticide into one parameter p_a for the adults, p_n for the neonates, and denote by d_a and d_n the background or natural mortalities for adults and neonates, respectively. We let b be the rate at which neonates are born into the population.

We suppose that there is a time delay for maturation of a neonate to adult life stage. We further assume that this time delay varies across the insect population according to a probability distribution $P(\tau)$ for $\tau \in [-T_n, 0]$ with corresponding density $m(\tau) = \frac{dP(\tau)}{d\tau}$. Here we tacitly assume an upper bound on T_n for the maturation period of neonates into adults. Thus, we have that $m(\tau)$, $\tau < 0$, is the probability per unit time that a neonate who has been in the population $-\tau$ time units becomes an adult. Then the rate at which such neonates become adults is $n(t+\tau)m(\tau)$. Summing over all such τ's, we obtain that the rate at which neonates become adults is $\int_{-T_n}^{0} n(t+\tau)m(\tau)d\tau$. Using the biological knowledge that the maturation process varies between five and seven days (i.e., m vanishes outside $[-7,-5]$), we obtain the functional differential equation (FDE) (see [JKH1, JKH2, JKH3] for the widespread interest and use of such systems) system

$$\begin{array}{rcl} \frac{da}{dt}(t) &=& \int_{-7}^{-5} n(t+\tau)m(\tau)d\tau - (d_a+p_a)\,a(t) \\ \frac{dn}{dt}(t) &=& ba(t) - (d_n+p_n)\,n(t) - \int_{-7}^{-5} n(t+\tau)m(\tau)d\tau \\ a(\theta) &=& \Phi(\theta), \quad n(\theta) = \Psi(\theta), \quad \theta \in [-7,0) \\ a(0) &=& a^0, \qquad n(0) = n^0, \end{array} \tag{6}$$

where m is now a probability density kernel which we have assumed has the property $m(\tau) \geq 0$ for $\tau \in [-7, -5]$ and $m(\tau) = 0$ for $\tau \in (-\infty, -7) \cup (-5, 0]$.

The system of functional differential equations described in (6) can be written in terms of semigroups [BBu1, BBu2, BKa].

Let

$$x(t) = (a(t),\, n(t))^T$$

and

$$x_t(\tau) = x(t + \tau), -7 \leq \tau \leq 0. \tag{7}$$

We define the Hilbert space $Z \equiv \mathbb{R}^2 \times L^2(-7, 0; \mathbb{R}^2)$ with inner product

$$|(\eta, \varphi)|_Z = \left(|\eta|^2 + \int_{-7}^{0} |\varphi(\theta)|^2 d\theta \right)^{1/2}, \ (\eta, \varphi) \in Z,$$

and let $z(t) = (x(t), x_t) \in Z$. Then our system (6) can be written as

$$\begin{array}{l} \frac{dx}{dt}(t) = L\left(x(t), x_t\right) \ \text{ for } 0 \leq t \leq T, \\ (x(0), x_0) = (\Phi(0), \Phi) \in Z, \quad \Phi \in \mathcal{C}(-7, 0; \mathbb{R}^2), \end{array} \tag{8}$$

where $T < \infty$ and for $\eta = (\psi^0, \zeta^0)^T \in \mathbb{R}^2$ and $\varphi = (\psi, \zeta)^T \in \mathcal{C}(-7, 0; \mathbb{R}^2)$

$$L(\eta, \varphi) = \begin{bmatrix} -d_a - p_a & 0 \\ b & -d_n - p_n \end{bmatrix} \eta + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \int_{-7}^{-5} \varphi(\tau) m(\tau) d\tau. \tag{9}$$

We now define a linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \to Z$ with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (\eta, \varphi) \in Z \,|\, \varphi \in H^1(-7, 0; \mathbb{R}^2) \text{ and } \eta = \varphi(0) \right\} \tag{10}$$

by

$$\mathcal{A}(\eta, \varphi) = \left(L(\eta, \varphi), \dot{\varphi} \right). \tag{11}$$

Then the delay system (6) can be formulated as

$$\begin{array}{rcl} \dot{z}(t) & = & \mathcal{A} z(t) \\ z(0) & = & z_0, \end{array} \tag{12}$$

where $z_0 = \left((a^0, n^0)^T, (\Phi, \Psi)^T \right)$.

2.5 Example 4 : Probability Measure Dependent Systems - Maxwell's Equations

We consider Maxwell's equations in a complex, heterogeneous material (see [BBL] for details):

$$\begin{aligned} \nabla \times E &= \frac{-\partial B}{\partial t} \\ \nabla \times H &= \frac{\partial D}{\partial t} + J \\ \nabla \cdot D &= \rho \\ \nabla \cdot B &= 0 \end{aligned}$$

where E is the electric field (force), H is the magnetic field (force), D is the electric flux density (also called the electric displacement), B is the magnetic flux density, and ρ is the density of charges in the medium.

To complete this system, we need constituitive (material dependent) relations:

$$\begin{aligned} D &= \epsilon_0 E + P \\ B &= \mu_0 H + M \\ J &= J_c + J_s \end{aligned}$$

where P is electric polarization, M is magnetization, J_c is the conduction current density, J_s is the source current density, ϵ_0 is the dielectric permittivity, and μ_0 is the magnetic permeability.

General Polarization:

$$P(t, \bar{x}) = [g * E](t, \bar{x}) = \int_0^t g(t-s, \bar{x}) E(s, \bar{x}) ds$$

Here, g is the polarization susceptibility kernel, or dielectric response function (DRF).

Several examples of polarization are widely used:

- **Debye Model for Polarization**
  This describes reasonably well a polar material. This is also called dipolar or orientational polarization. The DRF is defined by

$$g(t-s, \bar{x}, \tau) = e^{\frac{-(t-s)}{\tau}} \left(\frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{\tau} \right)$$

and corresponds to

$$\dot{P} + \frac{1}{\tau} P = \epsilon_0(\epsilon_s - \epsilon_\infty)E.$$

It is important to note that P represents macroscopic polarization, and when we refer to microscopic polarization we will instead use p.

- **Lorentz Polarization**
  This is also called electronic polarization or the electronic cloud model. The DRF is given by

$$g(t - s, \bar{x}, \tau) = \frac{\epsilon_0 \omega_p^2}{\nu_0} e^{\frac{-(t-s)}{2\tau}} \sin(\nu_0(t - s)),$$

  and corresponds to

$$\ddot{P} + \frac{1}{\tau}\dot{P} + \omega_0^2 P = \epsilon_0 \omega_p^2 E$$

  where $\omega_p = \omega_0\sqrt{\epsilon_s - \epsilon_\infty}$ and $\nu_0 = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}}$.

Note: When we allow for instantaneous polarization, we find that $D = \epsilon_0\epsilon_r E + P$ where $\epsilon_r = 1 + \chi \geq 1$ is a relative permittivity.

For complex composite materials, the standard Debye or Lorentz polarization model is not adequate, e.g., we need multiple relaxation times τ's in some kind of distribution [BBo, BG1, BG2]. Then the multiple Debye model becomes

$$P(t, \bar{x}; F) = \int_{\mathcal{T}} p(t, \bar{x}; \tau) dF(\tau)$$

where $\mathcal{T}$ is a set of possible relaxation parameters τ and

$$F \in \mathcal{F}(\mathcal{T}) = \{F : \mathcal{T} \to \mathbb{R}^1 | F \text{ is a probability distribution } \mathcal{T}\}.$$

Note: Here the microscopic polarization is given by

$$p(t, \bar{x}; \tau) = \int_0^t g(t - s, \bar{x}; \tau)E(s, \bar{x})ds$$

where $g(t - s, \bar{x}; \tau) = \frac{\epsilon_0(\epsilon_s - \epsilon_\infty)}{\tau} e^{\frac{-(t-s)}{\tau}}$ which corresponds to

$$\dot{p} + \frac{1}{\tau} p = \epsilon_0(\epsilon_s - \epsilon_\infty)E.$$

Here, E is the total electric field. Thus,

$$\begin{aligned}
P(t,\bar{x};F) &= \int_{\mathcal{T}}\int_0^t g(t-s,\bar{x};\tau)E(s,\bar{x})dsdF(\tau) \\
&= \int_0^t \left[\int_{\mathcal{T}} g(t-s,\bar{x};\tau)dF(\tau)\right] E(s,\bar{x})ds \\
&= \int_0^t G(t-s,\bar{x};F)E(s,\bar{x})ds
\end{aligned}$$

Assuming $M = 0$ (nonmagnetic materials), we find this system becomes

$$\begin{aligned}
\nabla \times E &= -\frac{\partial}{\partial t}(\mu_0 H) \\
\nabla \times H &= \frac{\partial}{\partial t}\left[\epsilon_0\epsilon_r E + \int_0^t G(t-s,\bar{x};F)E(s,\bar{x})ds\right] + J \\
\nabla \cdot D &= 0 \\
\nabla \cdot H &= 0
\end{aligned} \tag{13}$$

where $F \in \mathcal{F}(\mathcal{T})$ and $J = J_c + J_s$. Note: J_c is usually also a convolution on E although Ohm's law uses $J_c = \sigma E$ where σ is the conductivity of the material. In general, one should treat J_c as a convolution, e.g.,

$$J_c = \sigma_c * E = \int_0^t \sigma_c(t-s,\bar{x})E(s,\bar{x})ds.$$

For the measure dependent system (13), existence and uniqueness via a semigroup formulation have not been established. Comparison of solutions via semigroups versus weak solution have not been done. Nothing has yet been done in two or three dimensions. For the one-dimensional case only, existence, uniqueness, and continuous dependence have been established via a weak formulation (see [BG1]). For continuous dependence of solutions on F, a metric is needed on $\mathcal{F}(\mathcal{T})$. More generally, we may need to treat other material parameters $q = (\tau, \epsilon_s, \epsilon_\infty, \sigma)$, where $q \in Q \subset \mathbb{R}^4$ and we look for $F \in \mathcal{F}(Q)$.

- **Special Case**

We can consider a physically meaningful special case of the Maxwell system in a dielectric material which has a general polarization convolution relationship. Detailed derivations given in Section 2.3 of [BBL] lead to a one-dimensional version we next present and will use for an example in our subsequent discussions. Among the assumptions are some homogeneity in the medium (in planes parallel to that of an interrogating polarized planar wave) and a polarized sheet antenna source J_s. The resulting model leads to only nontrivial E fields in the x direction, and H fields in the y direction, each depending only on t and z. Assuming Ohm's law for J_c, we find the system reduces to

$$\begin{aligned} \frac{\partial E}{\partial z} &= -\mu_0 \frac{\partial H}{\partial t} \\ \frac{\partial H}{\partial z} &= \frac{\partial D}{\partial t} + \sigma E + J_s \\ D &= \epsilon E + P \end{aligned} \tag{14}$$

or in second order form for $E(t, z)$:

$$\mu_0 \epsilon \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P}{\partial t^2} + \mu_0 \sigma \frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial z^2} = -\mu_0 \frac{\partial J_s}{\partial t}$$

or

$$\epsilon_r \frac{\partial^2 E}{\partial t^2} + \frac{1}{\epsilon_0} \frac{\partial^2 P}{\partial t^2} + \frac{\sigma}{\epsilon_0} \frac{\partial E}{\partial t} - c^2 \frac{\partial^2 E}{\partial z^2} = \frac{-1}{\epsilon_0} \frac{\partial J_s}{\partial t},$$

where $c^2 = \frac{1}{\mu_0 \epsilon_0}$ and ϵ_r is a relative permittivity that is material and geometry dependent. Typical boundary conditions (say on $\Omega = [0, 1]$) are

$$\begin{aligned} \left[\frac{1}{c} \frac{\partial E}{\partial t} - \frac{\partial E}{\partial z} \right]_{z=0} &= 0 \qquad \text{(absorbing B.C. at } z = 0\text{)} \\ \text{and} \qquad E|_{z=1} &= 0 \qquad \text{(supraconductive B.C. at } z = 1\text{).} \end{aligned}$$

If we assume a general polarization relationship

$$P(t, z) = \int_0^t g(t - s, z) E(s, z) ds$$

and initial conditions

$$E(0, z) = \Phi(z), \qquad \frac{\partial E(0, z)}{\partial t} = \Psi(z),$$

then it can be argued that the system becomes

$$\frac{\partial^2 E}{\partial t^2} + \gamma \frac{\partial E}{\partial t} + \beta E + \int_{-\infty}^0 \tilde{g}(s) E(t + s) ds - c^2 \frac{\partial^2 E}{\partial z^2} = \mathcal{J}(t),$$

(without loss of generality we may take $\epsilon_r = 1$ for theoretical conditions) where we have tacitly assumed $E(t,z) = 0$ for $t < 0$. If we approximate the "memory" term by assuming only a finite past history is significant, we obtain the integro-partial differential equation

$$\frac{\partial^2 E}{\partial t^2} + \gamma\frac{\partial E}{\partial t} + \beta E + \int_{-r}^{0} \tilde{g}(s)E(t+s)ds - c^2\frac{\partial^2 E}{\partial z^2} = \mathcal{J}(t). \qquad (15)$$

As with the first two examples, this example can also be written as $\frac{dx}{dt} = Ax + F$ in an appropriate function space setting. In this direction we define an operator A in appropriate state spaces. In this example one might choose the electric field E and the magnetic field (H in (14)) as "natural" state spaces along with some type of hysteresis state to account for the memory in (15). However, in second order (in time) systems, it is also sometimes natural to choose the state E and velocity $\frac{\partial E}{\partial t}$ as states. We do that in this example. We first define an auxiliary variable $w(t)$ in $W \equiv L^2_{\tilde{g}}(-r,0;L^2(\Omega))$ by

$$w(t)(\theta) = E(t) - E(t+\theta), \qquad -r \leq \theta \leq 0.$$

For the inner product in W, we choose the weighted inner product

$$\langle \eta, w\rangle_W \equiv \int_{-r}^{0} \tilde{g}(\theta)\langle \eta(\theta), w(\theta)\rangle_{L^2(\Omega)} d\theta$$

and then (15) can be written as

$$\frac{\partial^2 E}{\partial t^2} + \gamma\frac{\partial E}{\partial t} + (\beta + g_{11})E - \int_{-r}^{0} \tilde{g}(s)w(t)(s)ds - c^2\frac{\partial^2 E}{\partial z^2} = \mathcal{J}(t),$$

where $g_{11} \equiv \int_{-r}^{0} \tilde{g}(s)ds$ and $w(t)(s) = E(t) - E(t+s)$, $-r \leq s \leq 0$. For a semigroup formulation we consider the state space

$$Z = V \times H \times W = H^1_R(\Omega) \times L^2(\Omega) \times L^2_{\tilde{g}}(-r,0;H)$$

(here $H^1_R(\Omega) = \{\phi \in H^1 | \phi(1) = 0\}$) with states

$$(\phi, \psi, \eta) = (E(t), \frac{\partial E(t)}{\partial t}, w(t)) = (E(t), \frac{\partial E(t)}{\partial t}, E(t) - E(t+\cdot)).$$

To define what we will later see is an infinitesimal generator of a C_0 semigroup, we first define several component operators. Let $\hat{A} \in \mathcal{L}(V, V^\star)$ be defined by

$$\hat{A}\phi \equiv c^2\phi'' - (\beta + g_{11})\phi + c^2\phi'(0)\delta_0$$

where δ_0 is the Dirac operator $\delta_0\psi = \psi(0)$. Here $V^\star$ is the dual space of V (e.g. the space of continuous linear functionals on V, to be discussed in detail later). We also define $B \in \mathcal{L}(V, V^\star)$ and $\hat{K} \in \mathcal{L}(W, H)$ by

$$B\phi = -\gamma\phi - c\phi(0)\delta_0$$

and, for $\eta \in W$,

$$\hat{K}\eta(z) = \int_{-r}^{0} \tilde{g}(s)\eta(s,z)ds, \qquad z \in \Omega.$$

Moreover, we introduce the operator $C : \mathcal{D}(C) \subset W \to W$ defined on

$$\mathcal{D}(C) = \{\eta \in H^1(-r, 0; L^2(\Omega)) \mid \eta(0) = 0\}$$

by

$$C\eta(\theta) = \frac{\partial}{\partial\theta}\eta(\theta).$$

We then define the operator $\mathcal{A}$ on

$$\mathcal{D}(\mathcal{A}) = \{(\phi, \psi, \eta) \in Z | \psi \in V, \eta \in \mathcal{D}(C), \hat{A}\phi + B\psi \in H\}$$

by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ \hat{A} & B & \hat{K} \\ 0 & I & C \end{pmatrix}.$$

That is,

$$\mathcal{A}\Phi = (\psi, \hat{A}\phi + B\psi + \hat{K}\eta, \psi + C\eta)$$

for $\Phi = (\phi, \psi, \eta) \in \mathcal{D}(\mathcal{A})$.

2.6 Example 5: Structured Population Models

We consider the special case of "transport" models or Example 2 with $D = 0$ but with so-called renewal or recruitment boundary conditions. The "spatial" variable ξ is actually "size" in place of spatial location (see [BT]) and such models have been effectively used to model marine populations such as mosquitofish [BBKW, BF, BFPZ] and, more recently, shrimp [GRD-FP, BDEHAD, GRD-FP2]. Such models have also been the basis of labeled cell proliferation models in which ξ represents label intensity [BSTBRSM]. The early versions of these size structured population models were first proposed by Sinko and Streifer [SS] in 1967. Cell population versions were proposed by Bell and Anderson [BA] almost simultaneously.

Other versions of these models called "age structured models", where age can be "physiological age" as well as chronical age, are discussed in [MetzD].

The *Sinko-Streifer model (SS)* for size-structured mosquitofish populations is given by

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial}{\partial \xi}(gv) &= -\mu v, \quad \xi_0 < \xi < \xi_1, \quad t > 0 \\ v(0,\xi) &= \Phi(\xi) \\ g(t,\xi_0)v(t,\xi_0) &= \int_{\xi_0}^{\xi_1} K(t,s)v(t,s)ds \\ g(t,\xi_1) &= 0. \end{aligned} \tag{16}$$

Here $v(t,\xi)$ represents number density (given in numbers per unit length) or population density, where t represents time and ξ represents the length of the mosquitofish. The growth rate of individual mosquitofish is assumed given by $g(t,\xi)$, where

$$\frac{d\xi}{dt} = g(t,\xi) \tag{17}$$

for each individual (all mosquitofish of a given size have the same growth rate).

In the SS $\mu(t,\xi)$ represents the *mortality rate* of mosquitofish, and the function $\Phi(\xi)$ represents initial size density of the population, while K represents the *fecundity kernel.* The boundary condition at $\xi = \xi_0$ is *recruitment*, or *birth rate*, while the boundary condition at $\xi = \xi_1 = \xi_{max}$ ensures the maximum size of the mosquitofish is ξ_1. The SS model *cannot* be used as formulated above to model the mosquitofish population because it *does not predict dispersion or bifurcation* of the population in time under biologically reasonable assumptions [BBKW, BF, BFPZ]. As we shall see, we will subsequently replace the growth rate g by a family $\mathcal{G}$ of growth rates and reconsider the model with a probability distribution P on this family. The population density is then given by summing "cohorts" of subpopulations where individuals belong to the same subpopulation if they have the same growth rate [BD, BDTR, GRD-FP, BDEHAD, GRD-FP2]. Thus, in the so-called Growth Rate Distribution (GRD) model, the population density $u(t,\xi;P)$, first discussed in [BBKW] and developed in [BF], is actually given by

$$u(t,\xi;P) = \int_{\mathcal{G}} v(t,\xi;g)dP(g), \tag{18}$$

where $\mathcal{G}$ is a collection of admissible growth rates, P is a probability measure on $\mathcal{G}$, and $v(t, \xi; g)$ is the solution of the (SS) with growth rate g. This model assumes the population is made up of *collections of subpopulations* with individuals in the same subpopulation having the same size dependent growth rate. This example can also be formulated in terms of semigroups [BKa, BKW1, BKW2], but the details are somewhat more difficult than those for the first three examples. In some cases it is advantageous to use a *weak formulation* (to be developed below) instead of a semigroup formulation.

2.7 Infinitesimal Generators

Definition 5 Assume $\{T(t) \in \mathcal{L}(X); 0 \leq t < \infty\}$ is a C_0 semigroup in X or on X, then the *infinitesimal generator* A of the semigroup $T(t)$ is defined by

$$\mathcal{D}(A) = \{x \in X : \lim_{t \to 0+} \frac{T(t)x - x}{t} \text{ exists in X}\}$$

and

$$Ax = \lim_{t \to 0+} \frac{T(t)x - x}{t}$$

with $A : \mathcal{D}(A) \subset X \to X$.

Theorem 1 *Let $T(t)$ be a C_0 semigroup in X. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$|T(t)| \leq Me^{\omega t} \qquad t \geq 0.$$

See [Pa]: Theorem 2.2.

Outline of proof: We want to show there exists $\eta > 0$ and $M > 0$ such that $|T(t)| \leq M$ on $[0, \eta]$. If not, then there exists $t_n \to 0^+$ such that $|T(t_n)| \geq n$, which implies $|T(t_n)x|$ is unbounded for some $x \in X$ (by the uniform boundedness principle). This contradicts the fact that $t \to T(t)x$ continuously on $[0, \delta]$ for each $x \in X$. Then define $\omega \equiv \frac{1}{\eta} ln(M)$. Then $e^{\omega t} = M^{t/\eta}$ for any $t > 0$.

Let $t = \delta + k\eta$ for some integer k and $\delta \in [0, \eta]$. Then $|T(t)| = |T(\delta)T(\eta)^k| \leq MM^k \leq MM^{t/\eta} = Me^{\omega t}$.

<u>Notation</u>: Write $A \in G(M, \omega)$. For $M = 1$ and $\omega = 0, A \in G(1, 0)$ is the infinitesimal generator of the contraction semigroup: $|T(t)x - T(t)y| \leq |x - y|$.

Theorem 2 *Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. Then*

1. *For $x \in X$,*
$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x ds = T(t)x.$$

2. *For $x \in X$,*
$$\int_0^t T(s)xds \in \mathcal{D}(A) \text{ and } A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

3. *If $x \in \mathcal{D}(A)$, then $T(t)x \in \mathcal{D}(A)$ and*
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

   In other words, $\mathcal{D}(A)$ is invariant under $T(t)$, and on $\mathcal{D}(A)$ at least, $T(t)x_0$ is a solution of $\begin{cases} \dot{x}(t) &= Ax(t) \\ x(0) &= x_0. \end{cases}$

4. *For $x \in \mathcal{D}(A)$,*
$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

See [Pa]: Theorem 2.4.

Corollary 1 *$A \in G(M,\omega) \Rightarrow \mathcal{D}(A)$ is dense in X and A is a closed linear operator.*

See [Pa]: Corollary 2.5.

Recall: By definition, a linear operator A is closed is equivalent to the property that A has a closed graph in $X \times X$.

That is, $Gr(A) = \{(x,y) \in X \times X | x \in \mathcal{D}(A), y = Ax\}$ is closed or for any $(x_n, y_n) \in Gr(A)$, if $x_n \to x$ and $y_n \to y$, then $x \in \mathcal{D}(A)$ and $y = Ax$.

Question: Does there exist a one-to-one relationship between a semigrouup and its infinitesimal generator?

Theorem 3 *Let $T(t)$ and $S(t)$ be C_0 semigroups on X with infinitesimal generators A and B respectively. If $A = B$, then $T(t) = S(t)$, i.e., the infinitesimal generator uniquely determines the semigroup on all of X.*

See [Pa]: Theorem 2.6.

3 Generators

3.1 Introduction to Generation Theorems

How do we tell when A, derived from a PDE (recall Example 1: the heat equation), is actually a generator of a C_0 semigroup? This is important, because it leads to the idea of well-posedness and continuous dependence of solutions for an IBVPDE.

Well-posedness of a PDE is equivalent to saying that a unique solution exists in some sense and is continuous with respect to data. In other words,

$$\begin{cases} \dot{x}(t) &= Ax(t) + F(t) \\ x(0) &= x_0, \end{cases}$$

is satisfied in some sense and the corresponding semigroup generated solution $x(t) = T(t)x_0 + \int_0^t T(t-s)F(s)ds$, yields the map $(x_0, F) \to x(\cdot; x_0, F)$, that is then continuous in some sense (depending on the spaces used).

Probably the best known generation theorem is the *Hille-Yosida* theorem. First, however, we review resolvents. In the study of C_0-semigroups, one frequently encounters the operators $\lambda I - A$, for $\lambda \in \mathbb{C}$ (also denoted by $\lambda - A$), and their inverse $R_\lambda(A) = (\lambda - A)^{-1}$. Here $\mathbb{C}$ is the field of complex scalars. For a linear operator A in X, we denote the *resolvent set* by $\rho(A) = \{\lambda \in \mathbb{C} | \lambda - A$ has range $\mathcal{R}(\lambda - A)$ dense in X and $\lambda - A$ has a continuous inverse on $\mathcal{R}(\lambda - A)\}$. We recall that any continuous densely defined linear operator in X can be extended continuously to all of X. For $\lambda \in \rho(A)$, we denote the resolvent operator in $\mathcal{L}(X)$ by $R_\lambda(A) = (\lambda - A)^{-1}$. The *spectrum* $\sigma(A)$ of a linear operator A is the complement in $\mathbb{C}$ of the resolvent set $\rho(A)$.

3.2 *Hille-Yosida* Theorems [HP, Pa, Sh, T]

Theorem 4 *(Hille - Yosida) For $M \geq 1, \omega \in \mathbb{R}$, we have $A \in G(M, \omega)$ if and only if*

1. *A is closed and densely defined (i.e., A closed and $\overline{D(A)} = X$).*
2. *For real $\lambda > \omega$, we have $\lambda \in \rho(A)$ and $R_\lambda(A)$ satisfies*

$$|R_\lambda(A)^n| \leq \frac{M}{(\lambda - \omega)^n}, \qquad n = 1, 2, \ldots.$$

Note: The equivalent version of *Hille-Yosida* in [Pa] is stated differently:

Theorem 5 *(Hille - Yosida)* $A \in G(1, 0) \iff$

1. *A is closed and densely defined.*
2. $R^+ \subset \rho(A)$ *and for every* $\lambda > 0$, $|R_\lambda(A)| \leq \frac{1}{\lambda}$.

See [Pa]: Theorem 3.1.

Homework Exercises

- Ex. 5 :

  a) Study the proof of Theorem 3.1 in [Pa] and read Section 1.3 carefully.

  b) Show that the version of *Hille-Yosida* in [Pa] is completely equivalent to the version stated previously. (Hint: This is not an exercise in proving *Hille-Yosida*. It is an exercise in comparing $A \in G(M, \omega)$ and $A \in G(1, 0)$ with regard to exponential rates and norms.)

3.3 Results from the *Hille-Yosida* proof

Results of the necessity portion of the proof

Out of the necessity portion of the proof of *Hille-Yosida* we find the representation:

$$R_\lambda(A)x = R(\lambda, A)x \equiv \int_0^\infty e^{-\lambda t} T(t)x dt \qquad \text{for } x \in X, \quad \lambda > \omega$$

This says that the Laplace transform of the semigroup is the resolvent operator.

We now ask the question: can we recover the semigroup $T(t)$ from the resolvent $R_\lambda(A)$ through some kind of inverse Laplace transform? Yes - later!!

Homework Exercises

- Ex. 6 : Show that the heat equation operator of Example 1 generates a C_0 semigroup in $X = L_2(0, l)$.

  (That is, show that $A\varphi = (D(\xi)\varphi'(\xi))'$ on $\mathcal{D}(A) = \{\varphi \in H^2(0, l) | \varphi(0) = 0, \varphi'(l) = 0\}$ is the infinitesimal generator of a C_0 semigroup in X.)

  Note: If you want to, you can take $D(\xi) = D$ constant for this exercise.

Results of the sufficiency portion of the proof

In the sufficiency proof of *Hille-Yosida*, we encounter the Yosida approximation:

For $\lambda > \omega$,

$$A_\lambda \equiv \lambda A R_\lambda(A) = \lambda^2 R_\lambda(A) - \lambda I.$$

From this definition, we can see that A_λ is bounded and defined on all of X. To see the above relationship, note that

$$\lambda A R_\lambda(A) = \lambda[(A - \lambda I)R_\lambda(A) + \lambda R_\lambda(A)].$$

We also found that for A satisfying the hypothesis of *Hille-Yosida*, we have

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \qquad \text{for all } x \in \mathcal{D}(A).$$

This says that for large λ, A_λ acts like A. Moreover, under this hypothesis, A_λ is the infinitesimal generator of the uniformly continuous semigroup of contraction operators : e^{tA_λ}. Then we can argue that $T(t)x \equiv \lim_{\lambda\to\infty} e^{tA_\lambda}x$, for all $x \in X$, is the desired semigroup that A generates. (The proof is constructive.) See Corollary 3.5 in [Pa].

3.4 Corollaries to ***Hille-Yosida***

Corollary 2 *Let A be the infinitesimal generator of a C_0 semigroup in X, and A_λ be the Yosida approximation, then*

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x \qquad \textit{for all } x \in X.$$

Corollary 3 *If A is the infinitesimal generator of a C_0 semigroup in X, then we actually have $\{\lambda | Re(\lambda) > \omega\} \subset \rho(A)$ and for such λ*

$$|R_\lambda(A)^n| \leq \frac{M}{(Re\lambda - \omega)^n}.$$

Corollary 4 *$A \in G(M, \omega)$ implies $\mathcal{D}(A)$ is dense in X, and moreover, we find $\bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$ is also dense in X.*

4 Example 6: Cantilever Beam

4.1 The Beam Equation

We consider the beam equation given by

$$\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2}{\partial \xi^2} M = f(t, \xi) \tag{19}$$

where the term $\gamma \frac{\partial y}{\partial t}$ represents the external damping (the air or viscous damping), M is the internal moment, and f is the external force applied. For a development of this equation from basic principles, see [BSW, BT].

Boundary Conditions

Fixed end:

$$\begin{array}{rcll} y(t,0) & = & 0 & \text{No displacement.} \\ \frac{\partial y}{\partial \xi}(t,0) & = & 0 & \text{No slope.} \end{array}$$

Free end:

$$\begin{array}{rcll} M(t,l) & = & 0 & \text{No moment.} \\ \frac{\partial M}{\partial \xi}(t,l) & = & 0 & \text{No shear.} \end{array}$$

where the shear force is represented by $\frac{\partial M}{\partial \xi}(t, \xi)$.

Initial Conditions

$$\begin{array}{rcl} y(0,\xi) & = & \Phi(\xi) \\ \dot{y}(0,\xi) & = & \Psi(\xi) \end{array}$$

Now we have an equation given with two unknowns; however we have the constitutive relationship given by

$$M = M(y).$$

Using Hooke's law, we get

$$\sigma = E\epsilon = I \frac{\partial^2 y}{\partial \xi^2}$$

where I represents the cross-sectional area. Note: I may be a function of ξ, $I = I(\xi)$. From this we have

$$M = EI \frac{\partial^2 y}{\partial \xi^2}$$

where E is Young's modulus, and E could also depend on ξ, $E = E(\xi)$.

Note: We used the assumptions of linear elasticity and small displacements. (Without small displacements, M could be a nonlinear function of y.)

We also have internal damping involved for which we must account. We will assume Kelvin-Voigt damping so that $M(t,\xi)$ is given by

$$M(t,\xi) = EI\frac{\partial^2 y}{\partial \xi^2} + c_D I \frac{\partial^3 y}{\partial \xi^2 \partial t}. \tag{20}$$

Combining equations (19) and (20), we have the beam equation given by

$$\rho\frac{\partial^2 y}{\partial t^2} + \gamma\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial \xi^2}\left(EI\frac{\partial^2 y}{\partial \xi^2} + c_D I\frac{\partial^3 y}{\partial \xi^2 \partial t}\right) = f(t,\xi).$$

with the modified boundary conditions

$$\begin{aligned} \left(EI\frac{\partial^2 y}{\partial \xi^2} + c_D I\frac{\partial^3 y}{\partial \xi^2 \partial t}\right)|_{\xi=l} &= 0 \\ \left[\frac{\partial}{\partial \xi}\left(EI\frac{\partial^2 y}{\partial \xi^2} + c_D I\frac{\partial^3 y}{\partial \xi^2 \partial t}\right)\right]_{\xi=l} &= 0. \end{aligned}$$

4.2 Writing the Beam Equation in the form $\dot{x} = Ax + F$

We want the beam equation to be of the form $\dot{x} = Ax + F$. Let x be represented by

$$x(t,\cdot) = \begin{pmatrix} y(t,\cdot) \\ \dot{y}(t,\cdot) \end{pmatrix}$$

and let our space $X = H_L^2(0,l) \times L_2(0,l)$ where

$$H_L^2(0,l) = \{\varphi \in H^2(0,l) |\, \varphi(0) = 0, \varphi'(0) = 0\}.$$

In other words, the function and the derivative both vanish at the left boundary. Now, let's rewrite our equation using the abbreviation $\partial = \frac{\partial}{\partial \xi}$.

$$\ddot{y} = \frac{1}{\rho}\left[-\partial^2(EI\partial^2 y) - \partial^2(c_D I \partial^2 \dot{y})\right] - \frac{\gamma}{\rho}\dot{y} + \frac{1}{\rho}f.$$

We expect to have the following:

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\rho}f \end{pmatrix}.$$

Let

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}$$

where we define A and B in the following way:

$$\begin{aligned} A &= \tfrac{1}{\rho}\partial^2(EI\partial^2\cdot) \\ B &= \tfrac{1}{\rho}\partial^2(c_D I\partial^2\cdot) + \tfrac{\gamma}{\rho}. \end{aligned}$$

Then we have the form $\dot{x} = \mathcal{A}x + F$ with $\mathcal{D}(\mathcal{A}) = \{(\varphi,\psi) \in X |\ \psi \in H_L^2(0,l),$ $A\varphi + B\psi \in L_2(0,l)$, $[EI\partial^2\varphi + c_D I\partial^2\psi]_l = 0$, $[\partial(EI\partial^2\varphi + c_D I\partial^2\psi)]_l = 0\}$.

4.3 Show that $\mathcal{A}$ is an Infinitesimal Generator

We claim that $\mathcal{A}$ is an infinitesimal generator of a C_0 semigroup in $X_\mathcal{E}$ where $X_\mathcal{E}$ is the energy space. We define $X_\mathcal{E}$ as X with the energy product

$$\begin{aligned} \langle(\varphi_1,\psi_1),(\varphi_2,\psi_2)\rangle_\mathcal{E} &= \textstyle\int_0^l EI\partial^2\varphi_1\partial^2\varphi_2 d\xi + \int_0^l \rho\psi_1\psi_2 d\xi \\ &= \textstyle\int_0^l EI\partial^2\varphi_1\partial^2\varphi_2 d\xi + \langle\rho\psi_1,\psi_2\rangle. \end{aligned}$$

If $0 < \rho_1 \leq \rho(\epsilon) \leq \rho_2 < \infty$, then $\langle\rho\psi_1,\psi_2\rangle$ is equivalent to the norm in $L_2(0,l)$ where the norm for $H^2(0,l)$ is defined by

$$|\varphi|_{H^2}^2 = |\varphi|_{L^2}^2 + |\varphi'|_{L^2}^2 + |\varphi''|_{L^2}^2.$$

An equivalent norm is

$$|\varphi|_\sim^2 = |\varphi(a)| + |\varphi'(b)| + |\varphi''|_{L^2}^2.$$

In other words, on H_L^2, we can write $|\varphi|_{H^2}^2 \equiv |\varphi(0)| + |\varphi'(0)| + |\varphi''|_{L^2}^2$. This defines a Hilbert space which is topologically equivalent to X so that we thus have that the semigroup that $\mathcal{A}$ generates is also a C_0 semigroup in X.

4.4 Poincare (First) Inequality

Suppose there exists a set Ω which is a bounded subset of R^n, a $c = c(\Omega)$, and for all $\varphi \in W_0^{l,2}(\Omega) = H_0^l(\Omega)$. Then

$$|\varphi|_{H^l(\Omega)}^2 \leq c(\Omega) \sum_{|s|=l} \int_\Omega |\partial^s\varphi|^2$$

where $s = (s_1,\ldots,s_n)$.

4.5 Dissipativeness of $\mathcal{A}$

We argue that $\mathcal{A}$ is dissipative in $X_{\mathcal{E}}$:

$$\begin{aligned}
\langle \mathcal{A}(\varphi,\psi),(\varphi,\psi)\rangle_{\mathcal{E}} &= \int_0^l EI\partial^2\psi\partial^2\varphi + \int_0^l \left[-\partial^2(EI\partial^2\varphi + c_D I\partial^2\psi) - \gamma\psi\right]\psi \\
&= \int_0^l EI\partial^2\psi\partial^2\varphi - \int_0^l \partial^2(EI\partial^2\varphi + c_D I\partial^2\psi)\psi - \int_0^l \gamma\psi^2) \\
&= \int_0^l EI\partial^2\psi\partial^2\varphi - \int_0^l \left((EI\partial^2\varphi + c_D I\partial^2\psi)\partial^2\psi + \gamma\psi^2\right) \\
&\quad - \partial M\psi|_0^l + M\partial\psi|_0^l \\
&= -\int_0^l c_D I|\partial^2\psi|^2 - \int_0^l \gamma|\psi|^2 \\
&\leq -k|\psi|^2_{H^2(0,l)} \\
&\leq 0
\end{aligned}$$

for $k =$min $\{|c_D I|, |\gamma|\}$. Here we have integrated by parts twice and used the fact that $(\varphi,\psi) \in \mathcal{D}(\mathcal{A})$ so that $\psi(0) = \partial\psi(0) = 0$ and $M(t,l) = \partial M(t,l) = 0$.

4.6 Show $\mathcal{R}(\lambda I - \mathcal{A}) = X$ for some λ

The range statement of Lumer Phillips means that we need to show

$$\lambda(\varphi,\psi) - \mathcal{A}(\varphi,\psi) = (h,g). \tag{21}$$

for $(h,g) \in X$. However, (21) reduces to finding, for any $(h,g) \in X = H_L^2(0,l) \times L_2(0,l)$, a solution (φ,ψ) in $\mathcal{D}(\mathcal{A})$ to

$$-\psi + \lambda\varphi = h \tag{22}$$

$$A\varphi + B\psi + \lambda\psi = g \tag{23}$$

where $h \in H_L^2$ and $g \in L^2$. We can rewrite (22) to get

$$\psi = \lambda\varphi - h \tag{24}$$

By substituting (24) into(23), we can reduce the above system to solving

$$\lambda^2\varphi + A\varphi + \lambda B\varphi = g + \lambda h + Bh \tag{25}$$

for φ, given any (h,g) in $H_L^2(0,l) \times L_2(0,l)$. After solving for φ, we can then solve for ψ. If $(\varphi,\psi) \in \mathcal{D}(\mathcal{A})$, then we have proven the range statement of Lumer-Phillips.

5 Gelfand Triple

An easy way to solve the above problem is to use Lax-Milgram; however, we must first talk about Gelfand triples. For relevant material, see also [Sh, T, W].

5.1 Concept of Gelfand Triple

The usual notation for a Gelfand triple is "$V \hookrightarrow H \hookrightarrow V^*$ with pivot space H". This notation stands for V, H, complex Hilbert spaces, such that $V \subset H$ and V is densely and continuously embedded in H. That is, V is a dense subset of H and

$$|v|_H \leq k|v|_V$$

for all $v \in V$ and some constant k. Therefore, you can identify elements in V with elements in H with an injection operator, i, where i is continuous and $i(V)$ is a dense subset of H.

We denote by V^* the conjugate dual of V. That is, V^* consists of all conjugate linear continuous functionals on V. (Note that we use V' to denote the algebraic dual of V and V^* to denote the topological dual. Frequently one encounters exactly the opposite notation in the literature.)

For $h \in H$, we define $\varphi(h) \in V^*$ by

$$\varphi(h)(v) = \langle h, v \rangle_H$$

for $v \in V$. We claim that $\varphi : H \to \varphi(H) \subset V^*$ is continuous, linear, one-to-one and onto. Moreover, $\varphi(H)$ is dense in V^* in the V^* topology.

By the Riesz theorem, every $h^* \in H^*$ can be represented by

$$h^*(h) = \langle \hat{h}^*, h \rangle_H \qquad h \in H$$

for some $\hat{h}^* \in H$. Hence, it is readily argued that H^* is isomorphic to H, $H^* \simeq H$. That is, we may identify H^* with H through $\varphi(H) = \tilde{H} \simeq H^*$ and write $h(v) = \langle h, v \rangle_H$ for $h \in H = H^*$.

This construction is commonly written as $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ for the pivot space H.

5.2 Duality Pairing

With a Gelfand triple, one frequently utilizes the *duality pairing* denoted by $\langle \cdot, \cdot \rangle_{V^*,V}$ given by the extension by continuity of the H inner product from $H \times V$ to $V^* \times V$. That is, for $v^* \in V^*$,

$$v^*(v) = \langle v^*, v \rangle_{V^*,V} = \lim_n \langle h_n, v \rangle_H$$

where $h_n \in H, h_n \to v^*$ in V^*. Note that we have $\langle h, v \rangle_{V^*,V} = \langle h, v \rangle_H$ if $h \in V^*$ also satisfies $h \in H$.

6 Sesquilinear Forms

Definition 6 Let H_1 and H_2 be two complex Hilbert spaces, and let $\sigma : H_1 \times H_2 \to C$. Then we call σ a *sesquilinear form* if it satisfies

1. σ is linear/conjugate linear.
2. σ is continuous. In other words,

   $$|\sigma(x, y)| \leq \gamma |x|_1 |y|_2$$

   for $x \in H_1$ and $y \in H_2$.

6.1 Norm of a Sesquilinear Form

The norm of a sesquilinear form σ is defined by

$$|\sigma| = \sup_{x,y \neq 0} \frac{|\sigma(x, y)|}{|x|_1 |y|_2}$$

for $x \in H_1$ and $y \in H_2$.

6.2 Representation

We'll consider two different representations:

1. For $y \in H_2$, $x \to \sigma(x, y)$ is continuous and linear on H_1. Therefore, by the Riesz Theorem, there exists a unique $z \in H_1$ such that

   $$\sigma(x, y) = \langle x, z \rangle_{H_1}.$$

   In other words, there exists a mapping $B \in \mathcal{L}(H_2, H_1)$ defined by $By = z$. Therefore,

   $$\sigma(x, y) = \langle x, z \rangle_{H_1} = \langle x, By \rangle_{H_1}.$$

   Claim $|\sigma| = |B|_{\mathcal{L}(H_2, H_1)}$

   Proof

For x in H_1 and y in H_2, we have

$$\begin{aligned}|\sigma| &= \sup_{x,y\neq 0} \tfrac{|\sigma(x,y)|}{|x|_1|y|_2} \\ &= \sup \tfrac{|\langle x,By\rangle_{H_1}|}{|x|_1|y|_2} \\ &\leq \sup \tfrac{|x|_1|By|_1}{|x|_1|y|_2} \\ &= \sup_{y\neq 0} \tfrac{|By|_1}{|y|_2} \\ &= |B|_{\mathcal{L}(H_1,H_2)}.\end{aligned}$$

Therefore, we have $|\sigma| \leq |B|_{\mathcal{L}(H_2,H_1)}$. Now we need to prove that $|\sigma| \geq |B|_{\mathcal{L}(H_2,H_1)}$. However, we have

$$\begin{aligned}|\sigma| &= \sup_{x,y\neq 0} \tfrac{|\sigma(x,y)|}{|x|_1|y|_2} \\ &= \sup \tfrac{|\langle x,By\rangle_{H_1}|}{|x|_1|y|_2} \\ &\geq \sup_{x=By} \tfrac{|\langle By,By\rangle_{H}|}{|By|_1|y|_2} \\ &= \sup \tfrac{|By|_1}{|y|_2} \\ &= |B|_{\mathcal{L}(H_2,H_1)}.\end{aligned}$$

Therefore, $|\sigma| = |B|_{\mathcal{L}(H_2,H_1)}$.

2. For $x \in H_1$, $y \to \sigma(x,y)$ is continuous and antilinear on H_2, i.e. $\sigma(x,\cdot) \in H_2^*$. Therefore, by the Riesz Theorem, there exists a unique $z \in H_2$ such that
$$\sigma(x,y) = \langle z,y\rangle_{H_2}.$$
In other words, there exists a mapping $A \in \mathcal{L}(H_1,H_2)$ defined by $Ax = z$. Therefore,
$$\sigma(x,y) = \langle z,y\rangle_{H_2} = \langle Ax,y\rangle_{H_2}.$$
with $|A| = |\sigma|$. Define $\tilde{\sigma}(y,x) = \overline{\sigma(x,y)}$, and interchange the roles of H_1 and H_2 in the above arguments to prove $|A| = |\sigma|$.

We want to consider elliptic equations of the form $Ax = f$. However, first, we want to start by discussing operators of the form $A : H_1 \to H_2$ such that A is bounded and continuous. This is very useful in integral equations. We will need to modify this to account for the unbounded nature of PDE's. We can think of our equation $Ax = f$ in the form $\sigma(x, y) = \langle f, y \rangle_H$ for all y in H where σ is equivalent to A. In other words, $\langle Ax - f, y \rangle_H = 0$ for all y in H.

7 Lax-Milgram(bounded form)

Theorem 6 *(Lax-Milgram Theorem (bounded form)) Suppose $\sigma : H \times H \to C$ is continuous and linear/conjugate linear, i.e., it is a sesquilinear form with*

$$|\sigma(x,y)| \leq \gamma |x|_H |y|_H$$

for all x and y in H, and σ is strictly positive, i.e

$$|\sigma(x,x)| \geq \delta |x|^2$$

for all x in H where γ and δ are positive constants. Then there exists $A : H \to H, A \in \mathcal{L}(H,H)$ defined by

$$\sigma(x,y) = \langle Ax, y \rangle$$

with $|A| = |\sigma| \leq \gamma$ for all x and y in H. Moreover,

$$\sigma(A^{-1}x, y) = \langle x, y \rangle$$

with $|A^{-1}| \leq \frac{1}{\delta}$ for all x and y in H.

Proof of Theorem

By the Riesz Theorem, we know there exists $A \in \mathcal{L}(H)$ such that $|A| = |\sigma| \leq \gamma$ where $\sigma(x,y) = \langle Ax, y \rangle_H$ for all x and y in H.

Claim A is one-to-one.

We need to show that $Ax = 0$ implies $x = 0$. Suppose $Ax = 0$. Then, by the definition of A and the assumptions given, we have

$$\begin{aligned} 0 &= |\langle Ax, x \rangle| \\ &= |\sigma(x,x)| \\ &\geq \delta |x|^2 \end{aligned}$$

Therefore, $\delta |x|^2 \leq 0$ implies $|x| = 0$, and hence $x = 0$. In other words, A is one-to-one. Since A is one-to-one, we know that A^{-1} exists on $\mathcal{R}(A) \subset H$.

Claim A^{-1} is bounded on $\mathcal{R}(A)$.

By the proposition that σ was strictly positive and the definition of A, we have

$$\begin{aligned} \delta |x|^2 &\leq |\sigma(x,x)| \\ &= |\langle Ax, x \rangle| \\ &\leq |Ax||x| \end{aligned}$$

Therefore, $|Ax| \geq \delta |x|$ for all x in H. In particular, for $x = A^{-1}y$, we have

$$|y| \geq \delta |A^{-1}y|.$$

Therefore,

$$|A^{-1}y| \leq \frac{1}{\delta}|y|$$

for all y in $\mathcal{R}(A)$. In other words, A^{-1} is bounded.
Now all we have to show is that $\mathcal{R}(A) = H$.

Claim $\mathcal{R}(A) = H$.

By assumption, we know that A is a continuous operator and we have argued that A^{-1} is bounded on $\mathcal{R}(A)$. Therefore, $\mathcal{R}(A)$ is closed. Now suppose $\mathcal{R}(A) \neq H$. This implies there exists $z \neq 0$ such that $z \perp \mathcal{R}(A)$. In other words,

$$\langle Ax, z \rangle = 0$$

for all x in H. In particular, if $x = z$,

$$0 = \langle Az, z \rangle = \sigma(z, z) \geq \delta |z|^2.$$

This implies $z = 0$; therefore, we have a contradiction. So, $\mathcal{R}(A) = H$.

7.1 Discussion of $Ax = f$ with A bounded

The bounded form of Lax-Milgram is very useful in linear integral equations. The Fredholm integral equations with kernel $k \in L^2([a, b] \times [a, b])$:

$$\begin{aligned} \int_a^b k(x,y)\varphi(y)dy &= f(x) \qquad x \in [a,b] \qquad \text{(first kind)} \\ \varphi(x) - \int_a^b k(x,y)\varphi(y)dy &= f(x) \qquad x \in [a,b] \qquad \text{(second kind)} \end{aligned}$$

can be written as operator equations in $H = L^2(a, b)$,

$$\begin{aligned} A\varphi &= f \\ \varphi - A\varphi &= f \end{aligned}$$

to be solved for φ, given f, where A is the set of bounded linear operators in H. It also has applications in scattering theory (acoustic and electromagnetic radiation), for field radiation, and single and double layer potential.

For relevant material, see also [K], [CK1], and [CK2].

The bounded form of Lax-Milgram is adequate if one wants to solve $Ax = f$ in H where A is bounded. It's okay for an integral operator; however, most applications of interest in PDE's require an unbounded operator A. Let's consider $k \in L^2(\Omega)$ with $\Omega = [0,1] \times [0,1]$ with $H = L^2(0,1)$. Let's define $A : H \to H$ by

$$(A\varphi)(t) = \int_0^1 k(t,\xi)\varphi(\xi)d\xi.$$

We want to know if this operator is bounded, i.e, is

$$\int_0^1 \left(\int_0^1 k(t,\xi)\varphi(\xi)d\xi \right)^2 dt < \gamma \int_0^1 |\varphi(\xi)|^2 d\xi$$

We can readily argue this. Therefore, there is a σ such that $A \leftrightarrow \sigma$ with

$$\begin{aligned} \sigma(\varphi,\psi) &= \textstyle\int_0^1 \left(\int_0^1 k(t,\xi)\varphi(\xi)d\xi \right) \psi(t)dt \\ &= \langle A\varphi, \psi \rangle_{L_2(0,1)}. \end{aligned}$$

We can show that all the requirements of Lax-Milgram(bounded form) are met with this operator A; therefore, Lax-Milgram(bounded form) is applicable.

7.2 Example - Steady State Heat Equation

Let $\Omega = [0,1] \times [0,1]$. Then the heat equation is given by

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathcal{D}\nabla u) + f.$$

We assume boundary conditions that require $u \in H_0^1(\Omega)$ where $\nabla = \frac{\partial}{\partial \xi_1}\hat{\imath} + \frac{\partial}{\partial \xi_2}\hat{\jmath}$ with $(\xi_1, \xi_2) \in \Omega$. Then the steady state equation is given by

$$-\nabla \cdot (\mathcal{D}\nabla u) = f \tag{26}$$

or

$$\frac{\partial}{\partial \xi_1}(\mathcal{D}\frac{\partial u}{\partial \xi_1}) + \frac{\partial}{\partial \xi_2}(\mathcal{D}\frac{\partial u}{\partial \xi_2}) = f(\xi_1, \xi_2).$$

In the weak or variational form, we have

$$\langle \mathcal{D}\nabla u, \nabla \varphi \rangle = \langle f, \varphi \rangle \tag{27}$$

for a test function φ. Let's take $\sigma(\psi, \varphi) = \langle \mathcal{D}\nabla\psi, \nabla\varphi\rangle$ with $|\mathcal{D}| \le \gamma$, i.e, $\mathcal{D}$ is bounded. Then σ is not continuous on $H = L_2(\Omega)$. On the other hand,

$$\begin{aligned} |\sigma(\psi, \varphi)| &\le \gamma |\nabla\psi|_{L_2} |\nabla\varphi|_{L_2} \\ &\le \gamma |\psi|_{H_0^1} |\varphi|_{H_0^1} \end{aligned}$$

implies σ is continuous on $V = H_0^1(\Omega)$.

Moreover, do we have some type of positivity? If $|\mathcal{D}(\xi_1, \xi_2)| \ge \delta$, then

$$\begin{aligned} |\sigma(\varphi, \varphi)| &= |\langle \mathcal{D}\nabla\varphi, \nabla\varphi\rangle| \\ &\ge \delta |\nabla\varphi|_{L_2}^2 \\ &\ge \tilde{\delta} |\varphi|_{H_0^1(\Omega)} \end{aligned}$$

In other words, in the $V = H_0^1(\Omega)$ norm, we would have both continuity and strictly positive. But, we don't have continuity and strictly positive in the $H = L^2(\Omega)$ sense. However, if we choose our H space to be $H_0^1(\Omega)$, we would get a solution of $\langle Au - f, \varphi\rangle = 0$. This requires our f to be in $H_0^1(\Omega)$ which is a strong requirement for f. Therefore, it would not be useful to choose our H space to be $H_0^1(\Omega)$. Instead, we need an extension of Lax-Milgram to treat this case.

8 Lax-Milgram (unbounded form)

Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triple.

Definition 7 A sesquilinear form σ is said to be V *continuous* if

$$|\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V$$

for all φ and ψ in V.

Consequences of V-continuous

For a fixed φ in V, let's consider the mapping $\psi \to \sigma(\varphi, \psi)$. This mapping is continuous by the definition of continuous from above, and its a conjugate linear mapping into C. Therefore, $\psi \to \sigma(\varphi, \psi)$ is in V^*. This implies there exists an operator $\mathcal{A} \in \mathcal{L}(V, V^*)$ such that $\sigma(\varphi, \psi) = \langle \mathcal{A}\varphi, \psi \rangle_{V^*,V}$.

Conversely, if $\mathcal{A} \in \mathcal{L}(V, V^*)$, we can define $\sigma : V \times V \to C$ by $\sigma(\varphi, \psi) = (\mathcal{A}\varphi)\psi$ where σ is V-continuous and linear/conjugate linear.

In other words, if we have a V-continuous sesquilinear form σ, there is a one-to-one correspondence between σ and $\mathcal{A} \in \mathcal{L}(V, V^*)$. So the operator is not going to be bounded in V. Therefore, $Au = f$ in V^* will be useful.

Definition 8 A sesquilinear form σ is V*-coercive* if there exists a constant $\delta > 0$ such that

$$|\sigma(\varphi, \varphi)| \geq \delta |\varphi|_V^2$$

for $\varphi \in V$.

Theorem 7 *(Lax-Milgram Theorem (unbounded form)) Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triple. Let $\sigma : V \times V \to C$ be a V continuous, V-coercive sesquilinear form. Then $\mathcal{A} : V \to V^*$ given by*

$$\sigma(\varphi, \psi) = \langle \mathcal{A}\varphi, \psi \rangle_{V^*,V}$$

is a linear (topological) isomorphism between V and V^*. $\mathcal{A}^{-1}$ is continuous from V^* to V with

$$|\mathcal{A}^{-1}|_{\mathcal{L}(V^*,V)} \leq \frac{1}{\delta}$$

Proof of Theorem

Let $R : V^* \to V$ be a Riesz isomorphism, in other words,

$$v^* \in V^* \to v^*(v) = \langle v^*, v\rangle_{V^*,V} = \langle Rv^*, v\rangle_V. \tag{28}$$

Then $R^{-1} : V \to V^*$ is continuous.

Let $\sigma : V \times V \to C$ be a V continuous, V-coercive sesquilinear form. In other words, $\psi \to \sigma(\varphi, \psi)$ is in V^* implies there exists a z in V such that $\sigma(\varphi, \psi) = \langle z, \psi\rangle_V$. From the bounded version of Lax-Milgram, this implies there exists $A \in \mathcal{L}(V)$ such that

$$\sigma(\varphi, \psi) = \langle A\varphi, \psi\rangle_V \tag{29}$$

for all φ and ψ in V, with A one-to-one, A onto, $|A|_{\mathcal{L}(V)} \leq \gamma$ and $|A^{-1}|_{\mathcal{L}(V)} \leq \frac{1}{\delta}$. In other words, A is an isomorphism $V \to V$.

We have that $R^{-1} : V \to V^*$ is also an isomorphism; therefore, $R^{-1}A : V \to V^*$ is an isomorphism. The claim is that $\mathcal{A} = R^{-1}A$. By (28)

$$\langle R^{-1}A\varphi, \psi\rangle_{V^*,V} = \langle A\varphi, \psi\rangle_V.$$

However, by (29), this implies

$$\langle R^{-1}A\varphi, \psi\rangle_{V^*,V} = \sigma(\varphi, \psi).$$

Therefore, by definition of $\mathcal{A} : V \to V^*$, $\mathcal{A}$ must be given by $\mathcal{A} = R^{-1}A$. So, we have

$$\langle \mathcal{A}\varphi, \psi\rangle_{V^*,V} = \sigma(\varphi, \psi) \leq \gamma|\varphi|_V|\psi|_V \tag{30}$$

and

$$\langle \mathcal{A}\varphi, \varphi\rangle_{V^*,V} \geq \delta|\varphi|^2_V. \tag{31}$$

However, $\sup_{\substack{|\psi|\leq 1\\ \psi\neq 0}} |\langle \mathcal{A}\varphi, \psi\rangle_{V^*,V}| = |\mathcal{A}\varphi|_{V^*}$. Now, let $\varphi = \mathcal{A}^{-1}\psi$. By combining this with (30) and (31), we have

$$\delta|\varphi|^2_V \leq |\mathcal{A}\varphi|^2_{V^*} \leq \gamma|\varphi|_V. \tag{32}$$

Therefore, we have

$$|\mathcal{A}|_{\mathcal{L}(V,V^*)} \leq \gamma$$

and

$$|\mathcal{A}^{-1}|_{\mathcal{L}(V^*,V)} \leq \frac{1}{\delta}.$$

Implications of Lax-Milgram(unbounded form)

Consider $\mathcal{A}u = f$ in V^*, i.e. $f \in V^*$. Then Lax-Milgram implies there exists a unique solution $u = \mathcal{A}^{-1}f$ in V that depends continuously on f. In other words,

$$|u|_V = |\mathcal{A}^{-1}f|_V \leq \delta|f|_{V^*}.$$

Now revisit the steady-state heat equation (26) in the example above with $\mathcal{D} \in L^\infty(\Omega)$. This, of course, allows discontinuous coefficients. Then for any f in $V^* = H^{-1}(\Omega)$ satisfying $\mathcal{A}u = f$. That is, $\langle \mathcal{A}u - f, \varphi\rangle_{V^*,V} = \sigma(u,\varphi) - \langle f, \varphi\rangle_{V^*,V} = \langle \mathcal{D}\nabla u, \nabla\varphi\rangle - \langle f,\varphi\rangle_{V^*,V} = 0$ for all $\varphi \in V$. Thus we say $u \in V$ satisfies $\mathcal{A}u = f$ in the sense of V^*. Hence (26) holds in the V^* sense which is precisely (27). This is also sometimes referred to as u satisfying (26) in the "sense of distributions."

8.1 The concept of D_A

Definition 9 Now we assume that we have a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ and a continuous sesquilinear form $\sigma : V \times V \to C$. As usual, $f(\psi) = \langle f, \psi\rangle_H$ defines, for $f \in H$, an element $f \in V^*$. Considering $(\mathcal{A}\varphi)(\psi) = \sigma(\varphi, \psi)$ for φ and ψ in V, we define

$$D_A = \{\varphi \in V | \mathcal{A}\varphi \in H\}.$$

That is, D_A is the set of $\varphi \in V$ such that $\mathcal{A}\varphi \in V^*$ has the representation $(\mathcal{A}\varphi)(\psi) = \langle\tilde{\varphi}, \psi\rangle_H$, for ψ in V and for some $\tilde{\varphi}$ in H.

We denote this element $\tilde{\varphi}$ by $-A\varphi = \tilde{\varphi}$, i.e, A is linear from $D_A \subset V$ into H and given by

$$\sigma(\varphi, \psi) = \langle -A\varphi, \psi\rangle_H$$

for ψ in V and φ in D_A.

We note that the above can be interpreted as: $\varphi \in D_A \subset V$ if and only if $\varphi \in V$ and $\mathcal{A}\varphi \in H^* \cong H$ so that $(\mathcal{A}\varphi)(\psi) = \langle\tilde{\varphi}, \psi\rangle_H$, for all ψ in V, and for some $\tilde{\varphi} \in H$. Alternatively, we may write $D_A = \{\varphi \in V || (\mathcal{A}\varphi)(\psi)| = |\sigma(\varphi,\psi)| \leq k|\psi|_H, \psi \in V\}$ for some k. Moreover, we could write $D_A = \{\varphi \in V | \psi \to \sigma(\varphi,\psi)$ is in H^*, i.e, continuous on H (continuous in the H sense) $\}$.

Note that we also have

$$\sigma(\varphi,\psi) = (\mathcal{A}\varphi)(\psi) = \langle \mathcal{A}\varphi, \psi\rangle_{V^*,V}$$

for all φ and ψ in V. If we restrict $\varphi \in \mathcal{D}_A$ then this also equals $\langle -A\varphi, \psi\rangle_H$.

Theorem 8 *If σ is a continuous V-coercive sesquilinear form on V, then D_A is dense in V and, hence, dense in H.*

Proof of Theorem

Define $\tilde{\sigma}(\varphi,\psi) = \overline{\sigma(\psi,\varphi)}$ (called the "adjoint" sesquilinear form)
If σ is V-coercive and V continuous, then $\tilde{\sigma}$ is also V-coercive and V continuous. In other words, there exists an operator $\tilde{\mathcal{A}} : V \to V^*$ such that $\tilde{\mathcal{A}} \in \mathcal{L}(V, V^*)$ with

$$\tilde{\sigma}(\varphi,\psi) = \langle \tilde{\mathcal{A}}\varphi, \psi\rangle_{V^*,V} = (\tilde{\mathcal{A}}\varphi)(\psi)$$

for all φ and ψ in V. Hence, $\mathcal{R}(\tilde{\mathcal{A}}) = V^*$ by the Lax-Milgram theorem.

Now we need to show that D_A is dense in V. It suffices to show that if $f \in V^*$ and $f(v) = 0$ for all $v \in D_A$, then $f \equiv 0$.
Let $f \in V^*$ such that $f(v) = 0$ for all $v \in D_A$. Since $f \in V^*$ and $\mathcal{R}(\tilde{\mathcal{A}}) = V^*$, then there exists $\varphi \in V$ such that $f = \tilde{\mathcal{A}}\varphi$.
For $v \in D_A$, we have

$$\begin{aligned}
(-Av)(\varphi) &= (\mathcal{A}v)(\varphi) \\
&= \sigma(v,\varphi) \\
&= \overline{\tilde{\sigma}(\varphi, v)} \\
&= \overline{\langle \tilde{\mathcal{A}}\varphi, v\rangle_{V^*,V}} \\
&= \overline{\langle f, v\rangle_{V^*,V}} \\
&= \overline{f(v)} \\
&= 0
\end{aligned}$$

Therefore, $(\mathcal{A}v)\varphi = 0$ for all $v \in D_A$.
But, we know $\mathcal{R}(\mathcal{A}|_{D_A}) = H \cong H^*$. Then, for every $h \in H^*$, $h(\varphi) = 0$. However, as H^* is dense in V^*, then this implies $\varphi = 0$. Moreover, $\tilde{\mathcal{A}}\varphi = f$ implies $f = 0$.

Thus we find that any continuous V-coercive sesquilinear form on V gives rise to a densely defined operator A on $\mathcal{D}(A) = D_A$ with

$$\begin{aligned}
\sigma(\varphi,\psi) &= \langle \mathcal{A}\varphi, \psi\rangle_{V^*,V} \qquad && \varphi,\psi \in V \\
&= \langle -A\varphi, \psi\rangle_H && \varphi \in D_A, \psi \in V.
\end{aligned}$$

8.2 V-elliptic

Definition 10 A sesquilinear form σ on V is V-*elliptic* if there exists a constant $\delta > 0$ such that

$$\text{Re } \sigma(\varphi, \varphi) \geq \delta |\varphi|_V^2 \quad \varphi \in V.$$

Discussion of V-elliptic

We note that σ is V-elliptic implies σ is V-coercive. Of course, if we are working in real spaces, Re $\sigma = \sigma$ and V-elliptic is equivalent to V-coercive. We also remark that the terminology among various authors is not standard. Some authors (e.g. Wloka in [W]) use our definition for V-coercive as the definition of V-elliptic and then use

$$|\sigma(\varphi, \varphi) + k|\varphi|_H^2| \geq \delta |\varphi|_V^2 \quad \varphi \in V, \delta > 0$$

as the definition of V-coercive.

Some of the terminology and usage of sesquilinear forms derives directly from that for PDE's of the form

$$\frac{\partial y}{\partial t} = \sum_{i,j} \frac{\partial}{\partial \xi_i}(a_{ij}\frac{\partial y}{\partial \xi_j}) + \sum_j b_j \frac{\partial y}{\partial \xi_j}, \quad t > 0, \xi \in G \subset R^n.$$

Definition 11 In classical PDE's, an "operator" $\{a_{ij}\}$ is said to be *strongly elliptic* on G if there exists $\delta > 0$ such that for $\xi \in G$

$$\text{Re } \sum_{i,j} a_{ij}(\xi) q_i \bar{q}_j \geq \delta \sum_i |q_i|^2$$

for all $q \in C^n$.

An associated sesquilinear form on $V = H^1(G)$ can be defined by

$$\sigma(\varphi, \psi) = \int_G \left[\sum_{i,j} a_{ij}(\xi) \frac{\partial \varphi}{\partial \xi_i} \frac{\partial \bar{\psi}}{\partial \xi_j} + \sum_k b_k \frac{\partial \varphi}{\partial \xi_k} \bar{\psi} \right] d\xi$$

Discussion of Strongly elliptic

It is a standard result that $\{a_{ij}\}$ strongly elliptic implies there exists $\delta > 0$ such that for λ sufficiently large

$$\text{Re } \sigma(\varphi, \varphi) + \lambda |\varphi|_H^2 \geq \delta |\varphi|_V^2, \quad \varphi \in V.$$

Hence if $\{a_{ij}\}$ is strongly elliptic, then for some λ_0 sufficiently large,

$$\tilde{\sigma}(\varphi, \psi) = \sigma(\varphi, \psi) + \lambda_0 \langle \varphi, \psi \rangle_H$$

is V-elliptic and hence V-coercive.

A most useful result is that continuous V-elliptic forms give rise to operators that are infinitesimal generators of C_0 (actually analytic) semigroups. (We say that $T(t)$ is an analytic semigroup if $t \to T(t)\varphi$ is analytic for $\varphi \in H$.)

9 Necessary Theorems

Theorem 9 *Let V, H be complex Hilbert spaces with $V \hookrightarrow H \hookrightarrow V^*$ and suppose that $\sigma : V \times V \to C$ is continuous and V-elliptic; i.e.*

$$|\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V \quad \varphi, \psi \in V,$$

and

$$Re\ \sigma(\varphi, \varphi) \geq \delta |\varphi|_V^2 \quad \delta > 0, \varphi \in V.$$

Define $A : \mathcal{D}(A) \subset V \to H$ by

$$\mathcal{D}(A) = \{\varphi \in V | \text{ there exists } K_\varphi > 0 \text{ such that } \sigma(\varphi, \psi) \leq K_\varphi |\psi|_H, \psi \in V\}$$

and

$$\sigma(\varphi, \psi) = \langle -A\varphi, \psi \rangle_H, \quad \varphi \in \mathcal{D}(A), \psi \in V.$$

Then $\mathcal{D}(A)$ is dense in H and A is the infinitesimal generator of a contraction semigroup in H that actually is an analytic semigroup.

The proof of this theorem will come later.

Definition 12 $T(t)$ is called an *analytic semigroup* if $t \to T(t)\varphi$ is analytic for each φ in H.

Theorem 10 *Suppose all the assumptions of Theorem 18 hold except that the V-ellipticity condition for σ is replaced by*

$$Re\ \sigma(\varphi, \varphi) + \lambda_0 |\varphi|_H^2 \geq \delta |\varphi|_V^2$$

for some $\lambda_0, \delta > 0, \varphi \in V$. Then defining A as in Theorem 18, we have that A is densely defined and is the infinitesimal generator of an analytic semigroup in H.

9.1 Example 1

The system is given by

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial \xi}\left(D(\xi)\frac{\partial y}{\partial \xi}\right)$$

$$\begin{aligned} y(t,0) &= 0 \\ \tfrac{\partial y}{\partial \xi}(t,l) &= 0 \\ y(0,\xi) &= \Phi(\xi). \end{aligned}$$

We choose the state space $X = L_2(0,l)$ as before. To obtain the weak variational form and the space V, we work backwards by multiplying the equation by a "test" function φ and integrating.

$$\begin{aligned} \textstyle\int_0^l \dot{y}\varphi &= \textstyle\int_0^l (Dy')'\varphi \\ &= \textstyle\int_0^l -Dy'\varphi' + Dy'\varphi|_0^l \end{aligned}$$

Therefore we have

$$\langle \dot{y}(t), \varphi\rangle + \langle Dy'(t), \varphi'\rangle - Dy'(t)\varphi|_0^l = 0 \tag{33}$$

However, (36) is equivalent to

$$\langle \dot{y}(t), \varphi\rangle + \langle Dy'(t), \varphi'\rangle = 0$$

if $\varphi \in H_L^1(0,l) = \{\varphi \in H^1(0,l) | \varphi(0) = 0\}$ and $Dy'(t,l) = 0$.

Defining $V = H_L^1(0,l)$ and σ on $V \times V$ by

$$\sigma(\varphi, \psi) = \langle D\varphi', \psi'\rangle,$$

we may write the equation in weak form as: find $y(t) \in V$ satisfying

$$\langle \dot{y}(t), \varphi\rangle + \sigma(y(t), \varphi) = 0$$

for all $\varphi \in V$. This equation is equivalent to the original system whenever $y(t) \in V \bigcap H^2(0,l)$ by using the reverse of the above arguments.

What about the flux boundary condition of the original problem? Suppose $y(t)$ is a weak solution, i.e.,

$$\langle \dot{y}(t), \varphi\rangle + \sigma(y(t), \varphi) = 0 \quad \forall\, \varphi \in V$$

$$y(0) = \Phi(\xi)$$

and y in $H^2(0,l)$. Then

$$\langle \dot{y}(t), \varphi\rangle + \int_0^l Dy'\varphi' = 0$$

Integrating by parts, the above equation is equivalent to

$$\int_0^l (\dot{y} - (Dy')')\varphi + Dy'(t,l)\varphi(l) = 0 \tag{34}$$

for all $\varphi \in H_L^1$. However, $H_0^1 \subset H_L^1$; therefore,

$$\int_0^l (\dot{y} - (Dy')')\varphi = 0 \tag{35}$$

for all $\varphi \in H_0^1$. Since H_0^1 is dense in $L_2(0,l)$, (38) implies $\dot{y} - (Dy')' = 0$. However, if we choose $\varphi \in H_L^1$ such that $\varphi(l) \neq 0$, then (37) implies $Dy'(t,l) = 0$, i.e., the flux boundary condition is satisfied.

If we define the V-inner product as

$$\langle \varphi, \psi \rangle_V = \int_0^l \varphi' \psi',$$

and set $H = X = L_2(0,l)$, then we readily see $V \hookrightarrow H \hookrightarrow V^*$. Note that the V norm is equivalent to the usual H^1 norm on $H_L^1(0,l)$. Furthermore, we have

$$\begin{aligned} |\sigma(\varphi,\psi)| &= |\langle D\varphi', \psi' \rangle| \\ &\leq |D|_\infty |\varphi'|_{L^2} |\psi'|_{L^2} \\ &= |D|_\infty |\varphi|_V |\psi|_V. \end{aligned}$$

Also,

$$\text{Re } \sigma(\varphi,\varphi) = \text{ Re } \langle D\varphi', \varphi' \rangle \geq \delta |\varphi'|_{L^2}^2 = \delta |\varphi|_V^2$$

so that σ is bounded and V-elliptic.

We can define $\mathcal{A} : V \to V^*$ by

$$\langle \mathcal{A}\varphi, \psi \rangle_{V^*,V} = \sigma(\varphi,\psi) = \langle D\varphi', \psi' \rangle$$

Note that $\mathcal{A}\varphi \in H \hookrightarrow V^*$ if and only if $\langle D\varphi', \psi' \rangle = \langle w, \psi \rangle$ for all $\psi \in V$ for some $w \in H$. However, integrating by parts we have

$$\begin{aligned} \textstyle\int_0^l D\varphi'\psi' &= -\textstyle\int_0^l (D\varphi')'\psi + D\varphi'\psi|_0^l \\ &= \langle -(D\varphi')', \psi \rangle + D(l)\varphi'(l)\psi(l) \\ &= \langle -(D\varphi')', \psi \rangle \end{aligned}$$

if $\varphi'(l) = 0$ and $(D\varphi')' \in L_2(0,l)$. Thus we may define

$$A\varphi = (D\varphi')'$$

on

$$\mathcal{D}(A) = \{\varphi \in H_L^1(0,l) | (D\varphi')' \in L_2(0,l), \varphi'(l) = 0\}$$

and obtain $\mathcal{A}\varphi = -A\varphi \in H$ exactly whenever $\varphi \in \mathcal{D}(A)$.

The above results hence guarantee that A generates a C_0-semigroup (actually an analytic semigroup) $T(t)$ on $H = X = L_2(0,l)$.

9.2 Example 2

Let's consider the transport equation given by

$$\frac{\partial y}{\partial t} + \frac{\partial}{\partial \xi}(\nu y) = \frac{\partial}{\partial \xi}(D\frac{\partial y}{\partial \xi}) - \mu y$$

$$\begin{aligned} y(t,0) &= 0 \\ (D\tfrac{\partial y}{\partial \xi} - \nu y)|_{\xi=l} &= 0 \\ y(0,\xi) &= \Phi(\xi). \end{aligned}$$

We can rewrite the transport equation by

$$y_t = (Dy' - \nu y)' - \mu y.$$

Multiplying by a test function and integrating from 0 to l, we have

$$\begin{aligned} \langle y_t, \varphi \rangle &= \textstyle\int_0^l ((Dy' - \nu y)'\varphi - \mu y \varphi)\, d\xi \\ &= -\langle Dy' - \nu y, \varphi' \rangle + (Dy' - \nu y)\varphi|_0^l - \langle \mu y, \varphi \rangle. \end{aligned}$$

If we choose $H = X = L_2(0,l)$ and $V = H_L^1(0,l)$ as in Example 1, with the same V- inner product, we have

$$\langle y_t, \varphi \rangle = -\langle Dy' - \nu y, \varphi' \rangle - \langle \mu y, \varphi \rangle.$$

As before, we have $V \hookrightarrow H \hookrightarrow V^*$. Then we can define the sesquilinear form $\sigma : V \times V \to C$ by

$$\sigma(\varphi, \psi) = \langle D\varphi' - \nu\varphi, \psi' \rangle + \langle \mu\varphi, \psi \rangle.$$

Therefore, we have the equation

$$\langle \dot{y}, \varphi \rangle + \sigma(y, \varphi) = 0.$$

Briefly, we'll discuss the various possibilities for boundary conditions and the effects on the choice of V. If we had a no flux boundary condition at

$\xi = 0$, we would choose $V = H^1_R(0,l)$. On the other hand, if we had essential boundary conditions at both boundaries, i.e, $y = 0$ at $\xi = 0$ and $\xi = l$, we would need to choose $V = H^1_0(0,l)$. A third possibility is if we had the no flux boundary conditions at both boundaries, $\xi = 0$ and $\xi = l$. In that case, as both boundary conditions were natural, we would choose $V = H^1(0,l)$.

The V-continuity of σ is established by arguing

$$\begin{aligned}
|\sigma(\varphi,\psi)| &\leq |D|_\infty|\varphi'|_H|\psi'|_H + |\nu|_\infty|\varphi|_H|\psi'|_H + |\mu|_\infty|\varphi|_H|\psi|_H \\
&\leq |D|_\infty|\varphi|_V|\psi|_V + |\nu|_\infty k|\varphi|_V|\psi|_V + |\mu|_\infty k^2|\varphi|_V|\psi|_V \\
&= (|D|_\infty + k|\nu|_\infty + k^2|\mu|_\infty)|\varphi|_V|\psi|_V.
\end{aligned}$$

As σ is V-continuous, we have

$$\begin{aligned}
\sigma(\varphi,\psi) &= \langle \mathcal{A}\varphi,\psi\rangle_{V^*,V} \qquad \varphi \in V \\
&= \langle -A\varphi,\psi\rangle_H \qquad \varphi \in \mathcal{D}(A)
\end{aligned}$$

where $\mathcal{D}(A)$ is defined by

$$\mathcal{D}(A) = \{\varphi \in H^2(0,l) | \varphi(0) = 0, (D\varphi' - \nu\varphi) \in H^1(0,l), (D\varphi' - \nu\varphi)(l) = 0\}.$$

Note that V carries the essential boundary conditions, while the natural boundary conditions are found in $\mathcal{D}(A)$.

To show that σ is V-coercive, if we assume $D \geq c_1 > 0$ and $\langle \mu\varphi,\varphi\rangle \geq -|\mu|_\infty|\varphi|^2_H$, then we have

$$\begin{aligned}
\text{Re }\sigma(\varphi,\varphi) &\geq c_1|\varphi|^2_V - \tfrac{|\nu|^2_\infty}{4\epsilon}|\varphi|^2_H - \epsilon|\varphi|^2_V - |\mu|_\infty|\varphi|^2_H \\
&= (c_1 - \epsilon)|\varphi|^2_V - (\tfrac{|\nu|^2_\infty}{4\epsilon} + |\mu|_\infty)|\varphi|^2_H
\end{aligned}$$

Hence, setting $\epsilon = \frac{c_1}{2}$, we have

$$\text{Re }\sigma(\varphi,\varphi) \geq \frac{c_1}{2}|\varphi|^2_V - \lambda_0|\varphi|^2_H$$

for some $\lambda_0 = \frac{|\nu|^2_\infty}{2c_1} + |\mu|_\infty$. Thus we see that $\tilde{\sigma}$ given by

$$\begin{aligned}
\tilde{\sigma}(\varphi,\psi) &= \sigma(\varphi,\psi) + \lambda_0\langle\varphi,\psi\rangle \\
&= \langle -A\varphi,\psi\rangle + \lambda_0\langle\varphi,\psi\rangle \\
&= \langle -(A - \lambda_0)\varphi,\psi\rangle
\end{aligned}$$

is V-elliptic (indeed it is V coercive). We thus find that $A - \lambda_0$, and hence A, is the generator of an analytic semigroup in $H = X = L_2(0,l)$.

9.3 Example 6

We return to the beam equation. Recall the system is given by

$$\rho y_{tt} + \gamma y_t + \partial^2 M = f \qquad 0 < \xi < l$$

with

$$y(t,0) = 0 = \frac{\partial y}{\partial \xi}(t,0)$$
$$M(t,l) = 0 = \partial M(t,l)$$

where $M(t,\xi) = EI\partial^2 y + c_D I \partial^2 y_t$. We choose as our basic space $H = L_2(0,l)$ with the weighted inner product $\langle \varphi, \psi \rangle_H = \langle \rho\varphi, \psi \rangle_{L^2(0,l)}$. Then the weak form becomes

$$\langle y_{tt} + \frac{\gamma}{\rho} y_t, \varphi \rangle_H + \langle \frac{EI}{\rho} \partial^2 y, \partial^2 \varphi \rangle_H + \langle \frac{c_D I}{\rho} \partial^2 y_t, \partial^2 \varphi \rangle_H = \langle \frac{1}{\rho} f, \varphi \rangle_H$$

for all $\varphi \in V = H_L^2(0,l)$. We choose the weighted inner product for V given by $\langle \varphi, \psi \rangle_V = \int_0^l EI \varphi'' \psi''$.

We define the sesquilinear forms σ_1 and σ_2 on $V \times V \to C$ by

$$\sigma_1(\varphi, \psi) = \langle \frac{EI}{\rho} \varphi'', \psi'' \rangle_H = \int_0^l EI \varphi'' \psi''$$

$$\sigma_2(\varphi, \psi) = \langle c_D \frac{I}{\rho} \varphi'', \psi'' \rangle_H + \langle \frac{\gamma}{\rho} \varphi, \psi \rangle_H.$$

The weak form of the equation is then

$$\langle y_{tt}, \varphi \rangle_H + \sigma_1(y, \varphi) + \sigma_2(y_t, \varphi) = \langle \frac{f}{\rho}, \varphi \rangle_H$$

for $\varphi \in V$. To write this in first order vector form, we use the state space $X_E = \mathcal{H} = V \times H$ with the space $\mathcal{V} = V \times V$, noting that $V \hookrightarrow H \hookrightarrow V^*$ and $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ form Gelfand triples.

Homework Exercises

- Ex. 9 : Explain why we have $\mathcal{V}^* = V \times V^*$ in the Gelfand triple instead of $\mathcal{V}^* = V^* \times V^*$.

We define the sesquilinear form $\sigma : \mathcal{V} \times \mathcal{V} \to C$ by (for $\chi = (\varphi, \psi), \zeta = (g, h)$ in $\mathcal{V}$)

$$\sigma(\chi, \zeta) = \sigma((\varphi, \psi), (g, h)) = -\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h).$$

Using the state variable $w(t) = (y(t, \cdot), y_t(t, \cdot))$ in $X_E = \mathcal{H}$, we can rewrite the equation as

$$\langle \dot{w}(t), \chi \rangle_{\mathcal{H}} + \sigma(w(t), \chi) = \langle F(t), \chi \rangle_{\mathcal{H}}$$

for $\chi \in \mathcal{V}$, where $F(t) = (0, \frac{1}{\rho} f(t))$.

We readily argue that σ is bounded (continuous) and $\mathcal{V}$-elliptic (actually, $\sigma - \lambda_0 |\cdot|^2_{X_E}$ is $\mathcal{V}$-elliptic). Consider first the boundedness argument:

$$\begin{aligned}
|\sigma(\chi, \zeta)| &= |\sigma((\varphi, \psi), (g, h))| = |-\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h)| \\
&\leq |\psi|_V |g|_V + \gamma_1 |\varphi|_V |h|_V + \gamma_2 |\psi|_V |h|_V \\
&\leq |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}} + \gamma_1 |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}} + \gamma_2 |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}} \\
&= (1 + \gamma_1 + \gamma_2) |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}}
\end{aligned}$$

for $\chi, \zeta \in \mathcal{V}$. The arguments for $\mathcal{V}$-ellipticity are also simple: for $\chi = (\varphi, \psi) \in \mathcal{V}$ we find

$$\begin{aligned}
\text{Re } \sigma(\chi, \chi) &= \text{Re } \{-\langle \psi, \varphi \rangle_V + \sigma_1(\varphi, \psi) + \sigma_2(\psi, \psi)\} \\
&= \text{Re } \{-\overline{\langle \varphi, \psi \rangle}_V + \langle \varphi, \psi \rangle_V + \sigma_2(\psi, \psi)\} \\
&= \text{Re } \sigma_2(\psi, \psi) \\
&\geq \delta_2 |\psi|_V^2 \\
&= \delta_2 (|\varphi|_V^2 + |\psi|_V^2) - \delta_2 |\varphi|_V^2 \\
&\geq \delta_2 (|\varphi|_V^2 + |\psi|_V^2) - \delta_2 (|\varphi|_V^2 + |\psi|_H^2) \\
&= \delta_2 |\chi|_{\mathcal{V}}^2 - \delta_2 |\chi|_{\mathcal{H}}^2.
\end{aligned}$$

We thus find that $\sigma(\chi, \zeta) = \langle \tilde{\mathcal{A}} \chi, \zeta \rangle_{\mathcal{V}, \mathcal{V}^*}$ gives rise to the infinitesimal generator $\mathcal{A}$ of a C_0 (indeed, analytic) semigroup on $X_E = \mathcal{H}$. It is readily argued that $\sigma(\chi, \zeta) = \langle -\mathcal{A} \chi, \zeta \rangle_{\mathcal{H}}$ for $\chi \in \mathcal{D}(\mathcal{A}) = \{\chi = (\varphi, \psi) \in \mathcal{H} | \psi \in V =$

$H^2_L(0,l), A_1\varphi + A_2\psi \in H, (EI\varphi'' + c_D I\psi'')(l) = 0, (EI\varphi'' + c_D I\psi'')'(l) = 0\}$
where

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -A_2 \end{pmatrix}$$

with $A_1\varphi = \delta^2(\frac{EI}{\rho}\delta^2\varphi)$ and $A_2\varphi = \delta^2(\frac{c_D I}{\rho}\delta^2\varphi)$.

<u>Homework Exercises</u>

- <u>Ex. 10</u> : Some books define $\mathcal{D}(\mathcal{A})$ by

  $$\tilde{\mathcal{D}}(\mathcal{A}) = (H^4(0,l) \cap H^2_L(0,l)) \times (H^4(0,l) \cap H^2_L(0,l))$$

  plus boundary conditions. We know $\mathcal{A}|_{\mathcal{D}(\mathcal{A})}$ is an infinitesimal generator of a C_0 semigroup which, in turn, implies $\mathcal{A}|_{\mathcal{D}(\mathcal{A})}$ is a closed operator. You can show $\mathcal{A}|_{\tilde{\mathcal{D}}(\mathcal{A})}$ is not closed. Therefore, we claim that $\mathcal{D}(\mathcal{A}) \neq \tilde{\mathcal{D}}(\mathcal{A})$. Is this true? Look at both the damped and undamped cases.

10 Examples in Applying the Previous Theorems

10.1 Example 1

The system is given by

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial \xi}\left(D(\xi)\frac{\partial y}{\partial \xi}\right)$$

$$\begin{aligned} y(t,0) &= 0 \\ \tfrac{\partial y}{\partial \xi}(t,l) &= 0 \\ y(0,\xi) &= \Phi(\xi). \end{aligned}$$

We choose the state space $X = L_2(0,l)$ as before. To obtain the weak variational form and the space V, we work backwards by multiplying the equation by a "test" function φ and integrating.

$$\begin{aligned} \textstyle\int_0^l \dot{y}\varphi &= \textstyle\int_0^l (Dy')'\varphi \\ &= \textstyle\int_0^l -Dy'\varphi' + Dy'\varphi|_0^l \end{aligned}$$

Therefore we have

$$\langle \dot{y}(t), \varphi\rangle + \langle Dy'(t), \varphi'\rangle - Dy'(t)\varphi|_0^l = 0 \tag{36}$$

However, equation 36 is equivalent to

$$\langle \dot{y}(t), \varphi\rangle + \langle Dy'(t), \varphi\rangle = 0$$

if $\varphi \in H_L^1(0,l) = \{\varphi \in H^1(0,l) | \varphi(0) = 0\}$ and $Dy'(t,l) = 0$.

Defining $V = H_L^1(0,l)$ and σ on $V \times V$ by

$$\sigma(\varphi, \psi) = \langle D\varphi', \psi\rangle,$$

we may write the equation in weak form as: find $y(t) \in V$ satisfying

$$\langle \dot{y}(t), \varphi\rangle + \sigma(y(t), \varphi) = 0$$

for all $\varphi \in V$. This equation is equivalent to the original system whenever $y(t) \in V \bigcap H^2(0,l)$ by using the reverse of the above arguments.

What about the flux boundary condition of the original problem? Suppose $y(t)$ is a weak solution, i.e.,

$$\langle \dot{y}(t), \varphi \rangle + \sigma(y(t), \varphi) = 0 \quad \forall\, \varphi \in V$$

$$y(0) = \Phi(\xi)$$

and y in $H^2(0,l)$. Then

$$\langle \dot{y}(t), \varphi \rangle + \int_0^l Dy'\varphi' = 0$$

Integrating by parts, the above equation is equivalent to

$$\int_0^l (\dot{y} - (Dy')')\varphi + Dy'(t,l)\varphi(l) = 0 \tag{37}$$

for all $\varphi \in H_L^1$. However, $H_0^1 \subset H_L^1$; therefore,

$$\int_0^l (\dot{y} - (Dy')')\varphi = 0 \tag{38}$$

for all $\varphi \in H_0^1$. Since, H_0^1 is dense in $L_2(0,l)$, equation 38 implies $\dot{y} = (Dy')' = 0$. However, if we choose $\varphi \in H_L^1$ such that $\varphi(l) \neq 0$, then equation 37 implies $Dy'(t,l) = 0$, i.e., the flux boundary condition is satisfied.

If we define the V-inner product as

$$\langle \varphi, \psi \rangle_V = \int_0^l \varphi'\psi',$$

and set $H = X = L_2(0,l)$, then we readily see $V \hookrightarrow H \hookrightarrow V^*$. Note that the V norm is equivalent to the usual H^1 norm on $H_L^1(0,l)$. Furthermore, we have

$$\begin{aligned} |\sigma(\varphi,\psi)| &= |\langle D\varphi', \psi' \rangle| \\ &\leq |D|_\infty |\varphi'|_{L^2} |\psi'|_{L^2} \\ &= |D|_\infty |\varphi|_V |\psi|_V. \end{aligned}$$

Also,

$$\text{Re}\ \sigma(\varphi,\varphi) = \ \text{Re}\ \langle D\varphi', \varphi' \rangle \geq \delta |\varphi'|_{L^2}^2 = \delta |\varphi|_V^2$$

so that σ is bounded and V-elliptic.

We can define $\mathcal{A} : V \to V^*$ by

$$\langle \mathcal{A}\varphi, \psi\rangle_{V^*,V} = \sigma(\varphi, \psi) = \langle D\varphi', \psi'\rangle$$

Note that $\mathcal{A}\varphi \in H \hookrightarrow V^*$ if and only if $\langle D\varphi', \psi'\rangle = \langle w, \psi\rangle$ for all $\psi \in V$ for some $w \in H$. However, integrating by parts we have

$$\begin{aligned} \textstyle\int_0^l D\varphi'\psi' &= -\textstyle\int_0^l (D\varphi')'\psi + D\varphi'\psi|_0^l \\ &= \langle -(D\varphi')', \psi\rangle + D(l)\varphi'(l)\psi(l) \\ &= \langle -(D\varphi')', \psi\rangle \end{aligned}$$

if $\varphi'(l) = 0$ and $(D\varphi')' \in L_2(0, l)$. Thus we may define

$$A\varphi = (D\varphi')'$$

on

$$\mathcal{D}(A) = \{\varphi \in H_L^1(0, l) | (D\varphi')' \in L_2(0, l), \varphi'(l) = 0\}$$

and obtain $\mathcal{A}\varphi = -A\varphi \in H$ exactly whenever $\varphi \in \mathcal{D}(A)$.

The above results hence guarantee that A generates a C_0-semigroup (actually an analytic semigroup) $T(t)$ on $H = X = L_2(0, l)$.

10.2 Example 2

Let's consider the transport equation given by

$$\frac{\partial y}{\partial t} + \frac{\partial}{\partial \xi}(\nu y) = \frac{\partial}{\partial \xi}(D\frac{\partial y}{\partial \xi}) - \mu y$$

$$\begin{aligned} y(t, 0) &= 0 \\ (D\tfrac{\partial y}{\partial \xi} - \nu y)|_{\xi=l} &= 0 \\ y(0, \xi) &= \Phi(\xi). \end{aligned}$$

We can rewrite the transport equation by

$$y_t = (Dy' - \nu y)' - \mu y.$$

Multiplying by a test function and integrating from 0 to l, we have

$$\begin{aligned} \langle y_t, \varphi\rangle &= \textstyle\int_0^l ((Dy' - \nu y)'\varphi - \mu y\varphi)\, d\xi \\ &= -\langle Dy' - \nu y, \varphi'\rangle + (Dy' - \nu y)\varphi|_0^l - \langle \mu y, \varphi\rangle. \end{aligned}$$

If we choose $H = X = L_2(0,l)$ and $V = H^1_L(0,l)$ as in Example 1, with the same V- inner product, we have

$$\langle y_t, \varphi\rangle = -\langle Dy' - \nu y, \varphi'\rangle - \langle \mu y, \varphi\rangle.$$

As before, we have $V \hookrightarrow H \hookrightarrow V^*$. Then we can define the sesquilinear form $\sigma : V \times V \to C$ by

$$\sigma(\varphi, \psi) = \langle D\varphi' - \nu\varphi, \psi'\rangle + \langle \mu\varphi, \psi\rangle.$$

Therefore, we have the equation

$$\langle \dot{y}, \varphi\rangle + \sigma(y, \varphi) = 0.$$

Briefly, we'll discuss the various possibilities for boundary conditions and the effects on the choice of V. If we had a no flux boundary condition at $\xi = 0$, we would choose $V = H^1_R(0,l)$. On the other hand, if we had essential boundary conditions at both boundaries, i.e, $y = 0$ at $\xi = 0$ and $\xi = l$, we would need to choose $V = H^1_0(0,l)$. A third possibility is if we had the no flux boundary conditions at both boundaries, $\xi = 0$ and $\xi = l$. In that case, as both boundary conditions were natural, we would choose $V = H^1(0,l)$.

The V-continuity of σ is established by arguing

$$\begin{aligned} |\sigma(\varphi,\psi)| &\leq |D|_\infty |\varphi'|_H |\psi'|_H + |\nu|_\infty |\varphi|_H |\psi'|_H + |\mu|_\infty |\varphi|_H |\psi|_H \\ &\leq |D|_\infty |\varphi|_V |\psi|_V + |\nu|_\infty k |\varphi|_V |\psi|_V + |\mu|_\infty k^2 |\varphi|_V |\psi|_V \\ &= (|D|_\infty + k|\nu|_\infty + k^2 |\mu|_\infty)|\varphi|_V |\psi|_V. \end{aligned}$$

As σ is V-continuous, we have

$$\begin{aligned} \sigma(\varphi,\psi) &= \langle \mathcal{A}\varphi, \psi\rangle_{V^*,V} \qquad \varphi \in V \\ &= \langle -A\varphi, \psi\rangle_H \qquad \varphi \in \mathcal{D}(A) \end{aligned}$$

where $\mathcal{D}(A)$ is defined by

$$\mathcal{D}(A) = \{\varphi \in H^2(0,l) | \varphi(0) = 0, (D\varphi' - \nu\varphi) \in H^1(0,l), (D\varphi' - \nu\varphi)(l) = 0\}.$$

Make a note that V carries the essential boundary conditions, while the natural boundary conditions are found in $\mathcal{D}(A)$.

To show that σ is V-coercive, if we assume $D \geq c_1 > 0$ and $\langle \mu\varphi, \varphi\rangle \geq -|\mu|_\infty |\varphi|^2_H$, then we have

$$\begin{aligned} \text{Re } \sigma(\varphi,\varphi) &\geq c_1 |\varphi'|^2_V - \tfrac{|\nu|^2_\infty}{4\epsilon} |\varphi|^2_H - \epsilon |\varphi|^2_V - |\mu|_\infty |\varphi|^2_H \\ &= (c_1 - \epsilon)|\varphi|^2_V - (\tfrac{|\nu|^2_\infty}{4\epsilon} + |\mu|_\infty)|\varphi|^2_H \end{aligned}$$

Hence, setting $\epsilon = \frac{c_1}{2}$, we have

$$\text{Re } \sigma(\varphi, \varphi) \geq \frac{c_1}{2}|\varphi|_V^2 - \lambda_0|\varphi|_H^2$$

for some λ_0. Thus we see that $\tilde{\sigma}$ given by

$$\begin{aligned} \tilde{\sigma}(\varphi, \psi) &= \sigma(\varphi, \psi) + \lambda_0\langle\varphi, \psi\rangle \\ &= \langle -A\varphi, \psi\rangle + \lambda_0\langle\varphi, \psi\rangle \\ &= \langle -(A - \lambda_0)\varphi, \psi\rangle \end{aligned}$$

is V-elliptic (indeed it is V coercive). We thus find that $A - \lambda_0$, and hence A, is the generator of an analytic semigroup in $H = X = L_2(0, l)$.

10.3 Example 6

We return to the beam equation. Recall the system is given by

$$\rho y_{tt} + \gamma y_t + \partial^2 M = f \quad\quad 0 < \xi < l$$

with

$$\begin{aligned} y(t, 0) = 0 = \frac{\partial y}{\partial \xi}(t, 0) \\ M(t, l) = 0 = \partial M(t, l) \end{aligned}$$

where $M(t, \xi) = EI\partial^2 y + c_D I\partial^2 y_t$. We choose as our basic space $H = L_2(0, l)$ with the weighted inner product $\langle\varphi, \psi\rangle_H = \langle\rho\varphi, \psi\rangle_{L^2(0,l)}$. Then the weak form becomes

$$\langle y_{tt} + \frac{\gamma}{\rho}y_t, \varphi\rangle_H + \langle\frac{EI}{\rho}\partial^2 y, \partial^2\varphi\rangle_H + \langle\frac{c_D I}{\rho}\partial^2 y_t, \partial^2\varphi\rangle_H = \langle\frac{1}{\rho}f, \varphi\rangle_H$$

for all $\varphi \in V = H_L^2(0, l)$. We choose the weighted inner product for V given by $\langle\varphi, \psi\rangle_V = \int_0^l EI\varphi''\psi''$.

We define the sesquilinear forms σ_1 and σ_2 on $V \times V \to C$ by

$$\sigma_1(\varphi, \psi) = \langle\frac{EI}{\rho}\varphi'', \psi''\rangle_H = \int_0^l EI\varphi''\psi''$$

$$\sigma_2(\varphi, \psi) = \langle c_D\frac{I}{\rho}\varphi'', \psi''\rangle_H + \langle\frac{\gamma}{\rho}\varphi, \psi\rangle_H.$$

The weak form of the equation is then

$$\langle y_{tt}, \varphi\rangle_H + \sigma_1(y, \varphi) + \sigma_2(y_t, \varphi) = \langle\frac{f}{\rho}, \varphi\rangle_H$$

for $\varphi \in V$. To write this in first order vector form, we use the state space $X_E = \mathcal{H} = V \times H$ with the space $\mathcal{V} = V \times V$, noting that $V \hookrightarrow H \hookrightarrow V^*$ and $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ form Gelfand triples.

Homework Exercises

- Ex. 9 : Explain why we have $\mathcal{V}^* = V \times V^*$ in the Gelfand triple instead of $\mathcal{V}^* = V^* \times V^*$.

We define the sesquilinear form $\sigma : \mathcal{V} \times \mathcal{V} \to C$ by (for $\chi = (\varphi, \psi), \zeta = (g, h)$ in $\mathcal{V}$)

$$\sigma(\chi, \zeta) = \sigma((\varphi, \psi), (g, h)) = -\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h).$$

Using the state variable $w(t) = (y(t, \cdot), y_t(t, \cdot))$ in $X_E = \mathcal{H}$, we can rewrite the equation as

$$\langle \dot{w}(t), \chi \rangle_{\mathcal{H}} + \sigma(w(t), \chi) = \langle F(t), \chi \rangle_{\mathcal{H}}$$

for $\chi \in \mathcal{V}$, where $F(t) = (0, \frac{1}{\rho} f(t))$.

We readily argue that σ is bounded (continuous) and $\mathcal{V}$-elliptic (actually, $\sigma - \lambda_0 |\cdot|^2_{X_E}$ is $\mathcal{V}$-elliptic). Consider first the boundedness argument:

$$\begin{aligned}
|\sigma(\chi, \zeta)| &= |\sigma((\varphi, \psi), (g, h))| = |-\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h)| \\
&\leq |\psi|_V |g|_V + \gamma_1 |\varphi|_V |h|_V + \gamma_2 |\psi|_V |h|_V \\
&\leq |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}} + \gamma_1 |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}} \ + gamma_2 |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}} \\
&= (1 + \gamma_1 + \gamma_2) |\chi|_{\mathcal{V}} |\zeta|_{\mathcal{V}}
\end{aligned}$$

for $\chi, \zeta \in \mathcal{V}$. The arguments for $\mathcal{V}$-ellipticity are also simple: for $\chi = (\varphi, \psi) \in \mathcal{V}$ we find

$$\begin{aligned}
\text{Re } \sigma(\chi, \chi) &= \text{Re } \{-\langle \psi, \varphi \rangle_V + \sigma_1(\varphi, \psi) + \sigma_2(\psi, \psi)\} \\
&= \text{Re } \{-\overline{\langle \varphi, \psi \rangle}_V + \langle \varphi, \psi \rangle_V + \sigma_2(\psi, \psi)\} \\
&= \text{Re } \sigma_2(\psi, \psi) \\
&\geq \delta_2 |\psi|^2_V \\
&= \delta_2 (|\varphi|^2_V + |\psi|^2_V) - \delta_2 |\varphi|^2_V \\
&\geq \delta_2 (|\varphi|^2_V + |\psi|^2_V) - \delta_2 (|\varphi|^2_V + |\psi|^2_H) \\
&= \delta_2 |\chi|^2_{\mathcal{V}} - \delta_2 |\chi|^2_{\mathcal{H}}
\end{aligned}$$

We thus find that $\sigma(\chi,\zeta) = \langle \tilde{\mathcal{A}}\chi, \zeta\rangle_{\mathcal{V},\mathcal{V}^*}$ gives rise to the infinitesimal generator $\mathcal{A}$ of a C_0 (indeed, analytic) semigroup on $X_E = \mathcal{H}$. It is readily argued that $\sigma(\chi,\zeta) = \langle -\mathcal{A}\chi, \zeta\rangle_{\mathcal{H}}$ for $\chi \in \mathcal{D}(\mathcal{A}) = \{\chi = (\varphi,\psi) \in \mathcal{H} | \psi \in V = H_L^2(0,l), A_1\varphi + A_2\psi \in H, (EI\varphi'' + c_D I\psi'')(l) = 0, (EI\varphi'' + c_D I\psi'')'(l) = 0\}$ where

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -A_2 \end{pmatrix}$$

with $A_1\varphi = \delta^2(\frac{EI}{\rho}\delta^2\varphi)$ and $A_2\varphi = \delta^2(\frac{c_D I}{\rho}\delta^2\varphi)$.

Homework Exercises

- Ex. 10 : Some books define $\mathcal{D}(\mathcal{A})$ by

$$\tilde{\mathcal{D}}(\mathcal{A}) = (H^4(0,l) \cap H_L^2(0,l)) \times (H^4(0,l) \cap H_L^2(0,l))$$

plus boundary conditions. We know $\mathcal{A}|_{\mathcal{D}(\mathcal{A})}$ is an infinitesimal generator of a C_0 semigroup which, in turn, implies $\mathcal{A}|_{\mathcal{D}(\mathcal{A})}$ is a closed operator. You can show $\mathcal{A}|_{\tilde{\mathcal{D}}(\mathcal{A})}$ is not closed. Therefore, we claim that $\mathcal{D}(\mathcal{A}) \neq \tilde{\mathcal{D}}(\mathcal{A})$. Is this true? Look at both the damped and undamped cases.

11 Summary of Results on Analytic Semigroup Generation by Sesquilinear Forms

Let V and H be complex Hilbert spaces with the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. Let $\langle \cdot, \cdot \rangle_{V^*,V}$ be the duality product, and $\sigma : V \times V \to C$ be a sesquilinear form such that σ is

1. V continuous, i.e., $|\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V$.
2. V-elliptic, i.e., Re $\sigma(\varphi, \varphi) \geq \delta |\varphi|_V^2$. (We can replace this by a shift: Re $\sigma(\varphi, \varphi) + \lambda_0 |\varphi|_H^2 \geq \delta |\varphi|_V^2$.)

As before, let $\hat{A} \in \mathcal{L}(V, V^*)$ (note that this is $-\mathcal{A}$ in our old notation) and $A : D_A \subset H \to H$ be defined such that

$$\begin{aligned} \sigma(\varphi, \psi) &= \langle -\hat{A}\varphi, \psi \rangle_{V^*,V} && \forall\, \varphi, \psi \in V \\ &= \langle -A\varphi, \psi \rangle_H && \varphi \in D_A, \psi \in V. \end{aligned}$$

Then we have $\mathcal{R}(\hat{A}) = V^*$, $\mathcal{R}(A) = H$, and $0 \in \rho(\hat{A})$. We can also note

$$\text{Re } \sigma(\varphi, \varphi) = \text{Re } \langle -\hat{A}\varphi, \varphi \rangle_{V^*,V} \geq \delta |\varphi|_V^2.$$

for all $\varphi \in V$. In other words, $\text{Re}\langle \hat{A}\varphi, \varphi \rangle \leq -\delta |\varphi|_V^2 \leq 0$. Similarly, for $\varphi \in D_A$, Re $\langle A\varphi, \varphi \rangle_H \leq 0$ which implies A is dissipative. By Lumer Phillips, as A is dissipative and $\mathcal{R}(A) = H$, A is an infinitesimal generator of a C_0 semigroup of contractions $S(t)$ on H.

Recall the definition of dissipativeness in a Banach space X. An operator $B \in \mathcal{D} \subset X \to X$ is *dissipative* if for each $x \in \mathcal{D}(B)$ there exists $x^* \in F(x) \subset X^*$ such that $\text{Re}\langle x^*, Bx \rangle_{X^*,X} \leq 0$ where $F(x)$ is the duality set. Let's apply this definition to $X = V^*$, which is a reflexive Banach space in its own right, with the operator $B = \hat{A}$, $\hat{A} : V \subset V^* \to V^*$. We have $\hat{A}$ being dissipative in the Banach space V^* means for $x \in V$ there exists $x^* \in F(X) \subset X^* = V^{**} = V$ such that $\text{Re}\langle x^*, \hat{A}x \rangle_{V,V^*} \leq 0$ or $\text{Re}\langle \hat{A}x, x^* \rangle_{V^*,V} \leq 0$. However, we have this holding for every $x^* \in V \subset V^*$. (In particular, we can find such a x^* in the duality set.) Therefore, $\hat{A} : V = \mathcal{D}(\hat{A}) \subset V^* \to V^*$ is dissipative. Using Lumer Phillips again we have $\hat{A}$ is an infinitesimal generator of a C_0 semigroup of contractions $\hat{S}(t)$ on V^* where $\hat{S}(t)|_H = S(t)$.

Recall, $D_A = \{x \in V | \hat{A}x \in H\}$. Let's define $\hat{\hat{D}}_A = \{x \in V | \hat{A}x \in V\}$ and look at the operator $\hat{\hat{A}} = A|_{\hat{\hat{D}}_A}$. We have $\mathcal{R}(\hat{\hat{A}}) = V$; therefore the range statement needed for Lumer Phillips holds for $\hat{\hat{A}}$. However, for $\hat{\hat{A}}$

to be dissipative in V, we must have for each $x \in \hat{\hat{D}}_A \subset V$ there exists $x^* \in F(x) \subset V^*$ such that $\mathrm{Re}\langle x^*, \hat{\hat{A}}x\rangle_{V^*,V} \leq 0$. We do not directly have that $\hat{\hat{A}}$ is dissipative in V. We need to first look at the Tanabe estimates.

11.1 Tanabe Estimates (on "Regular Dissipative Operators")

Suppose a sesquilinear form $\sigma(\sim \hat{A})$ is V continuous and V-elliptic. Then for $\mathrm{Re}\lambda \geq 0$ and $\lambda \neq 0, R_\lambda(\hat{A}) = (\lambda I - \hat{A})^{-1} \in \mathcal{L}(V^*, V)$, and

1. $|R_\lambda(\hat{A})\varphi|_V \leq \frac{1}{\delta}|\varphi|_{V^*}$ for $\varphi \in V^*$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(V^*,V)} \leq \frac{1}{\delta}$.)
2. $|R_\lambda(\hat{A})\varphi|_H \leq \frac{M_0}{|\lambda|}|\varphi|_H$ for $\varphi \in H$ where $M_0 = 1 + \frac{\gamma}{\delta}$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(H)} \leq \frac{M_0}{|\lambda|}$.)
3. $|R_\lambda(\hat{A})\varphi|_{V^*} \leq \frac{M_0}{|\lambda|}|\varphi|_{V^*}$ for $\varphi \in V^*$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(V^*)} \leq \frac{M_0}{|\lambda|}$.)
4. $|R_\lambda(\hat{A})\varphi|_V \leq \frac{M_0}{|\lambda|}|\varphi|_V$ for $\varphi \in V$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(V)} \leq \frac{M_0}{|\lambda|}$.)

Proof of 4

Define the dual or adjoint operator in the usual manner: define $\hat{A}^* \in \mathcal{L}(V, V^*)$ by $\sigma(\varphi, \psi) = \langle \varphi, -\hat{A}^*\psi\rangle_{V,V^*}$ for $\varphi, \psi \in V$. Then

$$\sigma^*(\varphi, \psi) = \overline{\sigma(\varphi, \psi)}$$

and

$$\sigma^*(\varphi, \psi) = \langle \varphi, -\hat{A}^*\psi\rangle_{V,V^*}$$

implies

$$\langle \varphi, -\hat{A}^*\psi\rangle_{V,V^*} = \overline{\langle \varphi, -\hat{A}^*\psi\rangle}_{V,V^*}.$$

Therefore, $\hat{A}^*$ also satisfies 1-3 as σ^* is V continuous and V-elliptic. Applying 3 to $\hat{A}^*$ gives us for $\mathrm{Re}\lambda \geq 0$, $\lambda \neq 0, \varphi, \psi \in V$

$$\begin{aligned} |\langle R_\lambda(\hat{A})\varphi, \psi\rangle_{V,V^*}| &= |\langle \varphi, R_\lambda(\hat{A}^*)\psi\rangle_{V,V^*}| \\ &\leq |\varphi|_V |R_\lambda(\hat{A}^*)\psi|_{V^*} \\ &\leq |\varphi|_V \tfrac{M_0}{|\lambda|}|\psi|_{V^*}. \end{aligned}$$

Therefore, $|R_\lambda(\hat{A})\varphi|_V \leq \frac{M_0}{|\lambda|}|\varphi|_V$.

11.2 Infinitesimal Generators in a General Banach Space

Recall that if A is an infinitesmal generator of a C_0 semigroup $T(t)$ in a Hilbert space X, then $S(t) = T^*(t)$ is a C_0 semigroup in X with infinitesimal generator A^*. Thus, if A is an infinitesimal generator, $\mathcal{D}(A)$ is dense in X. Similarly, if A^* is an infinitesimal generator, $\mathcal{D}(A^*)$ is also dense in X. We can generalize this result in a general Banach space.

Theorem 11 *If X is a reflexive Banach space and A is an infinitesimal generator of a C_0 semigroup $T(t)$ in X, then A^* is an infinitesimal generator of a C_0 semigroup $S(t)$ in X^* and $S(t) = T^*(t) = (T(t))^*$. In other words, $(e^{A^*t}$ on $X^*)^* = e^{At}$ on X.*

Corollary 5 *If $\hat{A}$ is an infinitesimal generator of a C_0 semigroup on V^*, then $\hat{A}^*$ is an infinitesimal generator on $V^{**} = V$.*

We know $\hat{A} \in \mathcal{L}(V, V^*)$ and $\hat{A}^* \in \mathcal{L}(V^{**}, V^*) = \mathcal{L}(V, V^*)$ are infinitesimal generators of C_0 semigroups of contractions on V^*. In other words, $\hat{S}^*(t) = e^{\hat{A}^*(t)}$ is a C_0 semigroup of contractions on V^*. Applying the previous corollary, we have $(\hat{S}^*(t)$ on $V^*)^* = \hat{S}(t)$ on V. However, $\hat{S}^*(t) \in \mathcal{L}(V^*, V^*)$ implies $(\hat{S}^*(t))^* \in \mathcal{L}(V^{**}, V^{**}) = \mathcal{L}(V, V)$. Since V is a reflexive Hilbert space, $\hat{\hat{S}}(t)$ is exactly $\hat{S}(t)|_V$.

We can show that the C_0 semigroups from above are actually analytic. The theorem below gives the condition for analyticity.

Theorem 12 *Let $T(t)$ be a C_0 semigroup on X with infinitesimal generator A, $0 \in \rho(A)$. Then a semigroup is analytic on X if there exists a constant c such that*

$$|R_{\mu+i\tau}(A)|_{\mathcal{L}(X)} \leq \frac{c}{|\tau|}$$

for $\mu > 0, \tau \neq 0$ where $\lambda = \mu + i\tau$.

See Pazy - Theorem II.5.2(b).

From the Tanabe estimates, we have $|R_\lambda(A)|_{\mathcal{L}(X)} \leq \frac{c}{|\lambda|} = \frac{c}{\sqrt{\mu^2+\tau^2}} \leq \frac{c}{|\tau|}$. Therefore, our estimates suffice to give analyticity. Thus we have the following theorem.

Theorem 13 *Let* $V \hookrightarrow H \hookrightarrow V^*$ *be a Gelfand triple. Assume the sesquilinear form* σ *is* V *continuous and* V*-elliptic. Let* $\hat{A}, A,$ *and* $\hat{\hat{A}}$ *be defined as above. Then*

- $\hat{A}$ *is an infinitesimal generator of an analytic semigroup* $\hat{S}(t)$ *of contractions on* V^*.
- A *is an infinitesimal generator of an analytic semigroup* $S(t)$ *of contractions on* H.
- $\hat{\hat{A}}$ *is an infinitesimal generator of an analytic semigroup* $\hat{\hat{S}}(t)$ *of contractions on* V.

We also have

- $dom_{V^*}(\hat{A}) = V$.
- $dom_H(\hat{A}) = D_A = \{x \in V | \hat{A}x \in H\}$.
- $dom_V(\hat{A}) = \hat{\hat{D}}_A = \{x \in V | \hat{A}x \in V\}$.

This is usually stated as A or $\hat{A}$ generate an analytic semigroup of contractions on V,H, V^*.

12 General Second Order Systems

12.1 Introduction to Second Order Systems

The ideas in Example 6 can be used to treat general second order systems. Consider the general abstract second order system

$$\ddot{y}(t) + \mathcal{A}_2\dot{y}(t) + \mathcal{A}_1 y(t) = f(t)$$

or, in variational form

$$\langle \ddot{y}(t), \varphi \rangle_H + \sigma_1(y(t), \varphi) + \sigma_2(\dot{y}(t), \varphi) = \langle f(t), \varphi \rangle_H \tag{39}$$

where H is a complex Hilbert space. As usual, we assume that σ_1 and σ_2 are sesquilinear forms on V where $V \hookrightarrow H \hookrightarrow V^*$ is a Gelfand triple. We also assume that σ_1 is continuous, V-elliptic and symmetric ($\sigma_1(\varphi, \psi) = \overline{\sigma_1(\psi, \varphi)}$). We assume that σ_2 is continuous and satisfies a weakened ellipticity condition which we formally call H-semiellipticity.

Definition 13 A sesquilinear form σ on V is *H-semielliptic* if there is a constant $b \geq 0$ such that

$$\text{Re } \sigma(\varphi, \varphi) \geq b|\varphi|_H^2 \quad \text{for all } v \in V.$$

Note that $b = 0$ is allowed in this definition.

Since σ_1 and σ_2 are continuous, we have that there exists $\mathcal{A}_i \in \mathcal{L}(V, V^*)$, $i = 1, 2$, such that

$$\sigma_i(\varphi, \psi) = \langle \mathcal{A}_i \varphi, \psi \rangle_{V^*,V} \quad \text{for all } \varphi, \psi \in V, \quad i = 1, 2.$$

Following the ideas of Example 6, we define spaces $\mathcal{V} = V \times V$ and $\mathcal{H} = V \times H$ and rewrite our second order system as a first order vector system. Defining, for $\chi = (\varphi, \psi), \zeta = (g, h) \in \mathcal{V}$, the sesquilinear form

$$\sigma(\chi, \zeta) = \sigma((\varphi, \psi), (g, h)) = -\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h),$$

we can write our system for $x(t) = (y(t), \dot{y}(t))$ as

$$\langle \dot{x}(t), \chi \rangle_{\mathcal{H}} + \sigma(x(t), \chi) = \langle F(t), \chi \rangle_{\mathcal{H}} \quad \chi \in \mathcal{V}$$

where $F(t) = (0, f(t))$. This is formally equivalent to the system

$$\dot{x}(t) = \mathcal{A}x(t) + F(t)$$

where $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \{x = (\varphi, \psi) \in \mathcal{H} | \psi \in V \text{ and } \mathcal{A}_1\varphi + \mathcal{A}_2\psi \in H\} \tag{40}$$

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -\mathcal{A}_1 & \mathcal{A}_2 \end{pmatrix}. \tag{41}$$

We first note that σ is $\mathcal{V}$ continuous. To see this, we observe that σ_1 and σ_2 being V continuous implies

$$\sigma_1(\varphi, h) \le \gamma_1 |\varphi|_V |h|_V$$

and

$$\sigma_2(\varphi, h) \le \gamma_2 |\psi|_V |h|_V.$$

We also have $|\chi|_{\mathcal{V}}^2 = |\varphi|_V^2 + |\psi|_V^2$ and $|\zeta|_{\mathcal{V}}^2 = |g|_V^2 + |h|_V^2$. Putting all of this together, we have

$$\begin{aligned} |\sigma((\varphi,\psi),(g,h))| &\le |\psi|_V|g|_V + \gamma_1|\varphi|_V|h|_V + \gamma_2|\psi|_V|h|_V \\ &\le |\chi|_{\mathcal{V}}|\zeta|_{\mathcal{V}} + \gamma_1|\chi|_{\mathcal{V}}|\zeta|_{\mathcal{V}} + \gamma_2|\chi|_{\mathcal{V}}|\zeta|_{\mathcal{V}} \\ &= (1+\gamma_1+\gamma_2)|\chi|_{\mathcal{V}}|\zeta|_{\mathcal{V}}. \end{aligned}$$

This indeed implies that σ is $\mathcal{V}$ continuous.

As σ is $\mathcal{V}$ continuous, $\mathcal{A}$ is the negative of the restriction to $\mathcal{D}(\mathcal{A})$ of the operator $\tilde{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ defined by $\sigma(\chi, \zeta) = \langle \tilde{\mathcal{A}}\chi, \zeta\rangle_{\mathcal{V}^*,\mathcal{V}}$ so that $\sigma(\chi,\zeta) = \langle -\mathcal{A}\chi, \zeta\rangle_{\mathcal{H}}$ for $\chi \in \mathcal{D}(\mathcal{A}), \zeta \in \mathcal{V}$.

12.2 Results for σ_2 V-elliptic

If both σ_1 and σ_2 are V-elliptic and σ_1 is the same as the V inner product, then we have exactly the case of Kelvin-Voigt damping in Example 3. We proved with these assumptions, σ is $\mathcal{V}$-elliptic. (Actually, we proved $\sigma(\cdot,\cdot) + \lambda_0\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is $\mathcal{V}$-elliptic.) Therefore, we have $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup (not of contractions) on $\mathcal{H}$.

Even if the V inner product and σ_1 are not the same, this result is true. Since σ_1 is continuous, we have $|\sigma_1(\varphi,\varphi)| \le \gamma_1|\varphi|_V^2$ while σ_1 is symmetric (i.e. $\sigma_1(\varphi,\psi) = \overline{\sigma_1(\psi,\varphi)}$) implies Re $\sigma_1(\varphi,\varphi) = \sigma_1(\varphi,\varphi)$. Thus, σ_1 being V-elliptic is equivalent to σ_1 is V-coercive: $\sigma_1(\varphi,\varphi) \ge \delta|\varphi|_V^2$. Hence, σ_1 and the inner product are equivalent. We may thus define V_1 as the space V with σ_1 as inner product, obtaining a space that is setwise equal and topologically equivalent to V. In the space $\mathcal{H}_1 = V_1 \times H$ the operator $\mathcal{A}$ is now associated with the $\mathcal{V}_1 = V_1 \times V_1$-elliptic form $\sigma^{(1)}(\chi,\zeta) = \langle -\mathcal{A}\chi, \zeta\rangle_{\mathcal{H}_1}$ that (as we argued in Example 6) satisfies the conditions of our theorem. Hence, $\mathcal{A}$ generates an analytic semigroup in $\mathcal{H}_1$ and hence an analytic semigroup in the equivalent space $\mathcal{H}$.

Theorem 14 *Let $V \hookrightarrow H \hookrightarrow V^*$ and suppose that σ_1 and σ_2 of (39) are V continuous and V-elliptic sesquilinear forms on V and that σ_1 is symmetric. Then the operator $\mathcal{A}$ defined in (40) and (41) is the infinitesimal generator of an analytic semigroup in $\mathcal{H} = V \times H$.*

12.3 Results for σ_2 H-semielliptic

If σ_2 is not V-elliptic, then we will not, in general, obtain an analytic solution semigroup for our system. We will obtain a C_0 semigroup, but must work a little more to obtain such. So assume that σ_2 is only H-semielliptic. Then we have $\mathcal{A}$ defined in (40) and (41) is dissipative in $\mathcal{H}_1$ since

$$\begin{aligned}
\text{Re } \langle \mathcal{A}x, x\rangle_{\mathcal{H}_1} &= \text{Re } \{\sigma_1(\psi,\varphi) - \sigma_1(\varphi,\psi) - \sigma_2(\psi,\psi)\} \\
&= \text{Re } \{\overline{\sigma_1(\varphi,\psi)} - \sigma_1(\varphi,\psi) - \sigma_2(\psi,\psi)\} \\
&= -\text{Re } \sigma_2(\psi,\psi) \\
&\leq -b|\psi|_H^2 \\
&\leq 0.
\end{aligned}$$

To argue that $\mathcal{A}$ is a generator, we use the Lumer Phillips theorem; thus we need to argue that for some $\lambda > 0$, the range of $\lambda I - \mathcal{A}$ is $\mathcal{H}_1$. Thus, given $\zeta = (g,h) \in \mathcal{H}_1$, we wish to solve $(\lambda - \mathcal{A})\chi = \zeta$ for $\chi = (\varphi,\psi) \in \mathcal{D}(\mathcal{A})$.

So we consider the equation

$$(\lambda - \mathcal{A})(\varphi,\psi) = (g,h) \quad \text{for } (g,h) \in V_1 \times H.$$

This is equivalent to

$$\left\{ \begin{aligned} \lambda\varphi - \psi &= g \\ \lambda\psi + \mathcal{A}_1\varphi + \mathcal{A}_2\psi &= h. \end{aligned} \right. \tag{42}$$

If we formally solve the first equation for $\psi = \lambda\varphi - g$ and substitute this into the second equation, we obtain

$$\lambda^2\varphi - \lambda g + \mathcal{A}_1\varphi + \mathcal{A}_2(\lambda\varphi - g) = h$$

or

$$\lambda^2\varphi + \mathcal{A}_1\varphi + \lambda\mathcal{A}_2\varphi = h + \lambda g + \mathcal{A}_2 g. \tag{43}$$

This equation must be solved for $\varphi \in V_1$ (and then ψ defined by $\psi = \lambda\varphi - g$ will also be in V_1).

These formal calculations suggest that we define for $\lambda > 0$ the associated sesquilinear form on $V \times V \to C$

$$\sigma_\lambda(\varphi, \psi) = \lambda^2\langle\varphi, \psi\rangle_H + \sigma_1(\varphi, \psi) + \lambda\sigma_2(\varphi, \psi).$$

Since σ_1 is V-elliptic and σ_2 is H-semielliptic we have

$$\begin{aligned} \text{Re } \sigma_\lambda(\varphi, \varphi) &= \lambda^2|\varphi|_H^2 + \text{Re } \sigma_1(\varphi, \varphi) + \lambda \text{ Re } \sigma_2(\varphi, \varphi) \\ &\geq \lambda^2|\varphi|_H^2 + c_1|\varphi|_V^2 + \lambda b|\varphi|_H^2 \\ &= \lambda(\lambda + b)|\varphi|_H^2 + c_1|\varphi|_V^2 \\ &> c_1|\varphi|_V^2 \end{aligned}$$

for $\tilde{\lambda} = \lambda(\lambda + b) > 0$. Hence σ_λ is V-elliptic and (43) is solvable for $\varphi \in V$ by Lax-Milgram. It follows that (42) is solvable for $(\varphi, \psi) \in \mathcal{D}(\mathcal{A})$, i.e. $\mathcal{R}(\lambda - \mathcal{A}) = \mathcal{H}_1$. Thus we have that $\mathcal{A}$ generates a contraction semigroup in $\mathcal{H}_1$ and a C_0 semigroup in $\mathcal{H}$.

Theorem 15 *Let $V \hookrightarrow H \hookrightarrow V^*$ and suppose that σ_1 and σ_2 of (39) satisfy: σ_1 is V-elliptic, V continuous and symmetric, σ_2 is V continuous and H-semielliptic. Then $\mathcal{A}$ defined by (40) and (41) generates a C_0 semigroup in $\mathcal{H}$.*

12.4 Stronger Assumptions for σ_2

If we strengthen the assumption on the damping form, we can obtain a stronger result.

Theorem 16 *Suppose σ_1 is V-elliptic, V continuous, and symmetric and σ_2 is H-elliptic, V continuous, and symmetric.Then $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ in $\mathcal{H} = V \times H$ that is exponentially stable, i.e. $|T(t)\chi|_{\mathcal{H}} \leq Me^{-\omega t}|\chi|_{\mathcal{H}}$ for $\omega > 0$.*

To motivate the arguments used to establish this result, we consider for $\omega > 0$ the change of dependent variable $y(t) - e^{\omega t} r(t)$ in the equation

$$\ddot{y} + \mathcal{A}_2 \dot{y}(t) + \mathcal{A}_1 y(t) = 0. \tag{44}$$

Upon substitution, we obtain

$$\ddot{r}(t) + \hat{\mathcal{A}}_2 \dot{r}(t) + \hat{\mathcal{A}}_1 r(t) = 0 \tag{45}$$

where

$$\begin{aligned} \hat{\mathcal{A}}_1 &= \mathcal{A}_1 - \omega \mathcal{A}_2 + \omega^2 I \\ \hat{\mathcal{A}}_2 &= \mathcal{A}_2 - 2\omega I. \end{aligned}$$

This suggests that we define the sesquilinear forms

$$\hat{\sigma}_1(\varphi, \psi) = \sigma_1(\varphi, \psi) - \omega \sigma_2(\varphi, \psi) + \omega^2 \langle \varphi, \psi \rangle_H$$

$$\hat{\sigma}_2(\varphi, \psi) = \sigma_2(\varphi, \psi) - 2\omega \langle \varphi, \psi \rangle_H$$

so that $\hat{\sigma}_i(\varphi, \psi) = \langle \hat{\mathcal{A}}_i \varphi, \psi \rangle_{V^*, V}, i = 1, 2$, and the transformed variational form of (44) is

$$\langle \ddot{r}(t), \varphi \rangle_H + \hat{\sigma}_1(r(t), \varphi) + \hat{\sigma}_2(\dot{r}(t), \varphi) = 0$$

for $\varphi \in V$.

We observe that $\hat{\sigma}_1, \hat{\sigma}_2$ are continuous and $\hat{\sigma}_1$ is symmetric since both σ_1 and σ_2 are. Since σ_2 is symmetric (hence $\sigma_2(\varphi, \varphi)$ is real) and continuous with $\sigma_2(\varphi, \varphi) \le k_2 |\varphi|_V^2$, we have for $\varphi \in V$

$$\begin{aligned} \text{Re } \hat{\sigma}_1(\varphi, \varphi) &= \hat{\sigma}_1(\varphi, \varphi) \\ &= \sigma_1(\varphi, \varphi) - \omega \sigma_2(\varphi, \varphi) + \omega^2 |\varphi|_H^2 \\ &\ge c_1 |\varphi|_V^2 - \omega \gamma_2 |\varphi|_V^2 + \omega^2 |\varphi|_H^2 \\ &\ge (c_1 - \omega \gamma_2) |\varphi|_V^2. \end{aligned}$$

Hence $\hat{\sigma}_1$ is V-elliptic if $\omega > 0$ is chosen so that $\omega < \frac{c_1}{\gamma_2}$.

Moreover, we find that $\hat{\sigma}_2$ is H-semielliptic if ω is chosen properly since

$$\text{Re } \hat{\sigma}_2(\varphi, \varphi) = \text{Re } \sigma_2(\varphi, \varphi) - 2\omega |\varphi|_H^2 \ge (b - 2\omega) |\varphi|_H^2.$$

Therefore, $\hat{\sigma}_2$ is H-semielliptic if $\omega < \frac{b}{2}$.

Thus, if we choose $\omega > 0$ as $\omega = \frac{1}{2} \min \{\frac{b}{2}, \frac{c_1}{\gamma_2}\}$, we find that $\hat{\sigma}_1$ and $\hat{\sigma}_2$ satisfy the assumptions of Theorem 15. By the arguments preceding that theorem, we see that

$$\hat{A} = \begin{pmatrix} o & I \\ -\hat{A}_1 & -\hat{A}_2 \end{pmatrix}$$

(see 40 and 41) generates a contraction semigroup $\hat{T}(t)$ on $\hat{\mathcal{H}}_1 = \hat{V}_1 \times H$ where $\hat{V}_1$ is V taken with $\hat{\sigma}_1$ as inner product ($\hat{V}_1$ is equivalent to V).

Now let $T(t)$ be the C_0-semigroup generated by $\mathcal{A}$ (see (40), (41) and Theorem 15). If $x(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}$ and $w(t) = \begin{pmatrix} r(t) \\ \dot{r}(t) \end{pmatrix}$ are solutions of (44) and (45) respectively, we have $x(t) = T(t)x_0$ where $x_0 = \begin{pmatrix} y_0 \\ w_0 \end{pmatrix}$ and $w(t) = \hat{T}(t)w_0$. Since $y(t) = e^{-wt}r(t)$ and $\dot{y}(t) = -we^{wt}r(t) + e^{wt}\dot{r}(t)$, we see that $x(t) = e^{wt}\Gamma w(t)$ where

$$\Gamma = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}$$

and $w_0 = \Gamma^{-1}x_0$. It follows since $|\hat{T}(t)|_{\hat{\mathcal{H}}_1} \leq 1$ that

$$\begin{aligned} |T(t)x_0|_{\hat{\mathcal{H}}_1} &\leq e^{-wt}|\Gamma\hat{T}(t)\Gamma^{-1}x_0|_{\hat{\mathcal{H}}_1} \\ &\leq Me^{-wt}|x_0|_{\hat{\mathcal{H}}_1}. \end{aligned}$$

Since $\hat{\mathcal{H}}_1$ and $\mathcal{H} = V \times H$ are norm equivalent, we thus find that the semigroup $T(t)$ is exponentially stable in $\mathcal{H}$.

13 Abstract Cauchy Problem

It is of interest to know when, and in what sense, solutions of the abstract equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t) \\ x(0) &= x_0 \end{aligned} \tag{46}$$

exist. Moreover, representations of such solutions in terms of a variation of parameters formula and the semigroup generated by A will play a fundamental role. We begin by summarizing results available in the standard literature on linear semigroups and abstract Cauchy problems.

Consider the abstract Cauchy problem (ACP) given by(46) where A is the infinitesimal generator of a C_0-semigroup $T(t)$ in a Hilbert space H. We define a *mild solution* x_m of (46) as functions given by

$$x_m(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds \tag{47}$$

whenever this entity is well defined (i.e., f is sufficiently smooth).

We say that $x : [0,T] \to H$ is a *strong solution* of (ACP) if $x \in C([0,T],H) \bigcap C^1((0,T],H), x(t) \in \mathcal{D}(A)$ for $t \in (0,T]$, and x satisfies (46) on $[0,T]$.

We have the following series of results.

Theorem 17 *If $f \in L_1((0,T),H)$ and $x_0 \in H$, there is at most one strong solution of (46). If a strong solution exists, it is given by (47).*

Theorem 18 *If $x_0 \in \mathcal{D}(A)$ and $f \in C^1([0,T],H)$, then x_m given by (47) provides the unique strong solution of (46).*

Theorem 19 *If $x_0 \in \mathcal{D}(A), f \in C([0,T],H), f(t) \in \mathcal{D}(A)$ for each $t \in [0,T]$ and $Af \in C([0,T],H)$, then (47) provides the unique strong solution of (46).*

Theorem 20 *Suppose A is the infinitesimal generator of an analytic semigroup $T(t)$ on H. Then if $x_0 \in H$ and f is Hölder continuous (i.e. $|f(t) - f(s)| \leq k|t-s|^\gamma$ for some $\gamma \leq 1$), then x_m of (47) provides the unique strong solution of (46).*

Unfortunately, all of these powerful results are too restrictive for use in many applications, including control theory, where typically $f(t) = Bu(t)$ is not continuous, let alone Hölder continuous on C^1. For this reason, a weaker formulation is more appropriate. For this, we follow the presentations of Lions, Wolka, and Tanabe which are developed in the context of sesquilinear forms and Gelfand triples, $V \hookrightarrow H \hookrightarrow V^*$, where V, H, V^* are Hilbert spaces.

We define the solution space $\mathcal{W}(0,T)$ by

$$\mathcal{W}(0,T) = \{g \in L_2((0,T),V) : \frac{dg}{dt} \in L_2((0,T),V^*)\}$$

with scalar product

$$\langle g,h\rangle_{\mathcal{W}} = \int_0^T \langle g(t),h(t)\rangle_V dt + \int_0^T \langle \frac{dg}{dt}(t), \frac{dh}{dt}(t)\rangle_{V^*} dt.$$

Then it can be shown that $\mathcal{W}(0,T)$ is a Hilbert space which embeds continuously into $C([0,T],H)$.

Assume $\sigma : V \times V \to C$ satisfies for $\varphi, \psi \in V$

$$\text{Re } \sigma(\varphi,\varphi) \geq c_1|\varphi|_V^2 - \lambda_0|\varphi|_H^2 \quad c_1 \geq 0, \lambda_0 \text{ real, for all } \varphi \in V,$$

$$|\sigma(\varphi,\psi)| \leq \gamma|\varphi|_V|\psi|_V \quad \text{for all } \varphi,\psi \in V.$$

Then, as usual, we have $\mathcal{A} \in \mathcal{L}(V,V^*)$ such that $\sigma(\varphi,\psi) = \langle \mathcal{A}\varphi,\psi\rangle_{V^*,V} = \langle -A\varphi,\psi\rangle_H$ where A is the densely defined restriction of $-\mathcal{A}$ to the set $\mathcal{D}_A = \{\varphi \in V | \mathcal{A}\varphi \in H\}$. We have moreover, that A is the infinitesimal generator of an analytic semigroup $T(t)$ on H. In fact, it turns out that $-\mathcal{A}$ is the generator of an analytic semigroup $\mathcal{T}(t)$ in V, H and V^* and $\mathcal{T}(t)$ agrees with $T(t)$ on V and H.

We may consider solutions of (46) in the sense of V^*, i.e., in the sense

$$\begin{aligned} \langle \dot{x}(t),\psi\rangle + \sigma(x(t),\psi) &= \langle f(t),\psi\rangle_{V^*,V} \quad \text{for } \psi \in V, \\ x(0) &= x_0. \end{aligned} \tag{48}$$

By a *strong solution of (46) in the V^* sense* (weak or variational sense), we shall mean a function $x \in L_2((0,T),V)$ such that $\dot{x} \in L_2((0,T),V^*)$ and (48) (or equivalently $\dot{x}(t) + \mathcal{A}x(t) = f(t)$) holds almost everywhere on $(0,T)$. Similarily, *mild solutions* $x_m \in V^*$ are given by the analogue of (47)

$$x_m(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)ds. \tag{49}$$

We then have the fundamental existence and uniqueness theorem.

Theorem 21 *Suppose $x_0 \in H$ and $f \in L_2((0,T),V^*)$. Then (48) has a unique strong solution and this is given by the mild solution (49).*

Proof

Let $\{\phi_i\}_1^\infty \subset V$ be a linearly independent total subset of V. We define the "Galerkin" approximations by $x_k(t) = \sum_{i=1}^{k} w_i(t)\varphi_i$ where the coefficients $\{w_i\}$ are chosen so that

$$\langle \dot{x}_k(t), \varphi_j \rangle_H + \sigma(x_k(t), \varphi_j) = \langle f(t), \varphi_j \rangle_{V^*,V} \tag{50}$$

for $j = 1, \ldots, k$, satisfying the intial condition

$$x_k(0) = x_{k_0}$$

where

$$x_{k_0} = \sum_{i=1}^{k} w_{i_0}\varphi_i \to x_0$$

in H as $k \to \infty$. Equivalently, (50) can be written as

$$\sum_{i=1}^{k} \dot{w}_i(t)\langle \varphi_i, \varphi_j \rangle + \sum_{i=1}^{k} w_i(t)\sigma(\varphi_i, \varphi_j) = F_j(t)$$

where $F_j(t) = \langle f(t), \varphi_j \rangle_{V^*,V}$ for $j = 1, \ldots, k$. Therefore, $w_1, \ldots, w_k$ are unique solutions to a vector ordinary differential equation system.

Now, multiplying (50) by w_j and summing over $j = 1, \ldots, k$, we obtain

$$\langle \dot{x}_k(t), x_k(t) \rangle_H + \sigma(x_k(t), x_k(t)) = \langle f(t), x_k(t) \rangle_{V^*,V}$$

with $x_k(0) = x_{k_0} \to x_0$ in H. Therefore,

$$\frac{1}{2}\frac{d}{dt}|x_k(t)|_H^2 + \sigma(x_k(t), x_k(t)) = \langle f(t), x_k(t) \rangle_{V^*,V}. \tag{51}$$

Integrating (51), we obtain

$$\frac{1}{2}|x_k(t)|_H^2 - \frac{1}{2}|x_k(0)|_H^2 + \int_0^t \sigma(x_k(s), x_k(s))ds = \int_0^t \langle f(s), x_k(s) \rangle_{V^*,V} ds.$$

Using the fact that σ is V-elliptic, we have

$$\begin{aligned} \frac{1}{2}|x_k(t)|_H^2 + c_1 \int_0^t |x_k(s)|_V^2 ds &\leq \frac{1}{2}|x_k(0)|_H^2 + \int_0^t |\langle f(s), x_k(s) \rangle_{V^*,V}| ds \\ &\leq \frac{1}{2}|x_k(0)|_H^2 + \int_0^t (\frac{1}{4\epsilon}|f(s)|_{V^*}^2 + \epsilon |x_k(s)|_V^2) ds. \end{aligned}$$

Therefore,

$$\frac{1}{2}|x_k(t)|_H^2 + (c_1 - \epsilon)\int_0^t |x_k(s)|_V^2 ds \le \frac{1}{2}|x_k(0)|_H^2 + \int_0^t \frac{1}{4\epsilon}|f(s)|_{V^*}^2 ds \quad (52)$$

or

$$\frac{1}{2}|x_k(t)|_H^2 + (c_1 - \epsilon)\int_0^t |x_k(s)|_V^2 ds \le \frac{1}{2}|x_k(0)|_H^2 + \frac{1}{4\epsilon}|f|^2_{L_2((0,t),V^*)}.$$

This implies we have $\{x_k\}$ bounded in $C((0,T);H)$ and in $L_2((0,T),V)$. Since $L_2((0,T),V)$ is a Hilbert space, we can choose $\{x_{k_n}|x_{k_n} \rightharpoonup \tilde{x} \in L_2(0,T)\}$ to be a convergent subsequence of x_k. Without loss of generality, we denote x_{k_n} by x_k. Then $\tilde{x}$ is our candidate for a solution where $x_k \rightharpoonup \tilde{x}$ in $L_2((0,T),V)$.

Now, let $\chi(t) \in C^1(0,T)$ with $\chi(T) = 0$ and $\chi(0) = 0$ and $\Psi_j(t,\cdot)$ is defined by $\Psi_j(t,\cdot) = \chi(t)\varphi_j$. Multiplying (50) by $\chi(t)$ and integrating, we have

$$\int_0^T (\langle \dot{x}_k(t), \varphi_j\rangle_H \chi(t) + \sigma(x_k(t),\varphi_j)\chi(t) - \langle f(t),\varphi_j\rangle_{V^*,V}\chi(t))\, dt = 0. \quad (53)$$

Integrating by parts, we find that (53) becomes

$$-\int_0^T \langle x_k(t),\varphi_j\rangle \dot{\chi}(t)dt + \int_0^T \sigma(x_k(t),\varphi_j)\chi(t) - \int_0^T \langle f(t),\varphi_j\rangle_{V^*,V}\chi(t)dt = 0.$$

We can now let $k \to \infty$ and pass the limit through term by term to obtain

$$\int_0^T -\langle \tilde{x}(t),\varphi_j\rangle \dot{\chi}(t)dt + \int_0^T \sigma(\tilde{x}(t),\varphi_j\rangle\chi(t)dt - \int_0^T \langle f(t),\varphi_j\rangle_{V^*,V}\chi(t)dt = 0, \quad (54)$$

holding for all $\varphi_j \in V$. We can rewrite $\sigma(\tilde{x}(t),\varphi)\chi(t)$ as $\mathcal{A}\tilde{x}(t)\Psi$ and $\langle f(t),\varphi\rangle_{V^*,V}\chi(t)$ as $f(\Psi)$.

Therefore, (54) becomes

$$\langle \frac{d}{dt}\tilde{x}(t),\Psi\rangle_{V^*,V} + (\mathcal{A}\tilde{x} - f)\Psi = 0$$

where $\Psi \in L_2((0,T),V)$. We can also write $\frac{d\tilde{x}}{dt} + \mathcal{A}\tilde{x} - f = 0$ in the $L_2((0,T),V)^*$ sense. However, we have the following theorem.

Theorem 22 *Let X be a reflexive Banach space. Then*

$$L_p((0,T),X)^* \cong L_q((0,T),X^*)$$

where $\frac{1}{p}+\frac{1}{q}=1, 1<p<\infty$.

Relevant material can be found in [E].

Therefore, the solution exists in the $L_2((0,T),V^*)$ sense. To obtain $\tilde{x}(0)=x_0$, we may use the same arguments with $\chi \in C^1(0,T)$, $\chi(T)=0$, but $\chi(0)\neq 0$.

To prove uniqueness of the solution, it suffices to argue that the solution corresponding to $x_0=0, f=0$ is identically zero. With these specific values for f and x_0, (48) can be written as

$$\langle \dot{x}(t),\varphi\rangle_{V^*,V}+\sigma(x(t),\varphi)=0, \tag{55}$$

$$x(0)=0.$$

Let $\varphi = x(t)$. Then (55) becomes

$$\frac{1}{2}\frac{d}{dt}|x(t)|_H^2+\sigma(x(t),x(t))=0.$$

Integrating by parts and using the V-ellipticity of σ, we obtain

$$\frac{1}{2}|x(t)|_H^2+\int_0^t c_1|x(s)|_V^2 ds \leq 0.$$

Therefore, $x(t)=0$. In other words, the solution is unique.

To show continuous dependence of the solution, define

$$x(\cdot;x_0,f):(x_0,f)\in H\times L_2((0,T),V^*)\to x\in L_2((0,T),V)\bigcap C((0,T),H).$$

Therefore, $x\in L_2((0,T),V^*)$.Taking the limits in (52) and using the property that the norms are weakly lower semi-continuous, we obtain the following relation:

$$\frac{1}{2}|x(t)|_H^2+(c_1-\epsilon)\int_0^t |x(s)|_V^2 ds \leq \frac{1}{2}|x_0|^2+\frac{1}{4\epsilon}\int_0^t |f(s)|_{V^*}^2 ds.$$

Recall $(x_0,f)\to x$ is a linear map. Suppose $(x_0,f)\to(0,0)$ in the $H\times L_2((0,T),V^*)$ sense. Then

$$\sup_t \frac{1}{2}|x(t)|_H^2+(c_1-\epsilon)\int_0^T |x(s)|_V^2 ds \leq \frac{1}{2}|x_0|_H^2+\frac{1}{4\epsilon}\int_0^T |f(s)|_{V^*}^2 ds$$

where the right side (and hence the left side) goes to zero as $(x_0, f) \to 0$. Therefore, x is continuous in $C((0,T), H)$, $L_2((0,T), V^*)$ and $L_2((0,T), V)$.

Finally, we want to prove the equivalence between this solution and the mild solution given by (49). Claim: $(x_0, f) \to x(\cdot, x_0, f)$ is continuous from $H \times L_2((0,T), V^*) \to L_2((0,T), V^*)$. Let $x(\cdot, x_0, f)$ be a weak solution. Then x and x_m are both continuous in the above sense. If two functions agree on a dense subset of the whole set, then the solutions will agree on the whole set. Therefore, if there is a dense subset of $H \times L_2((0,T), V^*)$ in which x and x_m agree, then they will agree on the whole set.

Choose $x_0 \in D_A$ and $f \in C^1((0,T), H)$. Then Theorem 18 guarantees that x_m is the unique solution in the H sense. However, if x_m is a strong solution in the H sense, then it must also be a weak solution (i.e. a strong solution in the V^* sense). However, the mild solution being unique means $x_m(\cdot, x_0, f) = x_{var}(\cdot, x_0, f)$ for $(x_0, f) \in D_A \times C^1((0,T), H)$. But $D_A \times C^1((0,T), H)$ is dense in $H \times L_2((0,T), V^*)$. Therefore, we have the equivalence between the solutions.

14 "Weak" or "Variational Form"

We consider the origin of the terms "weak or variational form" as opposed to strong or closed form of PDE's. We use the beam equation (Example 3) to illustrate ideas.

Recall Example 6, the cantilever beam. This example, given in classical form (which can be derived in a straight forward manner using force and moment balance) is

$$\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2}{\partial \xi^2}\left(EI \frac{\partial^2 y}{\partial \xi^2} + c_D I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) = f(t, \xi)$$

with boundary conditions

$$\begin{array}{rcl} y(t,0) & = & 0 \\ \frac{\partial y}{\partial \xi}(t,0) & = & 0 \end{array} \tag{56}$$

$$\begin{array}{rcl} \left(EI \frac{\partial^2 y}{\partial \xi^2} + c_D I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) |_{\xi = l} & = & 0 \\ \frac{\partial}{\partial \xi}\left[\left(EI \frac{\partial^2 y}{\partial \xi^2} + c_D I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) \right] |_{\xi = l} & = & 0 \end{array} \tag{57}$$

and initial conditions

$$\begin{array}{rcl} y(0,\xi) & = & \Phi(\xi) \\ \frac{\partial y}{\partial t}(0,\xi) & = & \Psi(\xi) \end{array}$$

To facilitate our discussions, we consider an undamped and unforced version (i.e. $\gamma = c_D I = 0, f = 0$) of the above system. Rather than force and moment balance, we consider energy formulations for the beam. For a segment of the beam in $[\xi, \xi + \Delta\xi]$, one can argue that the kinetic energy (at a given time t) is given by

$$KE = T = \frac{1}{2} \int_{\xi}^{\xi + \Delta \xi} \rho (\frac{\partial y}{\partial t}(t,s))^2 ds,$$

and hence the kinetic energy of the entire beam is given by

$$T = \frac{1}{2} \int_0^l \rho \dot{y}^2 d\xi.$$

Similarly, the potential (or strain) energy U of the beam at any given time t is given by

$$PE = U = \frac{1}{2} \int_0^l EI (\frac{\partial^2 y}{\partial \xi^2})^2 d\xi.$$

A fundamental tenant of the mechanics of rigid or elastic bodies is Hamilton's "Principle of Stationary Action" (often, in a misnomer, referred to as Hamilton's principle of "least action") which postulates that any system undergoing motion during a period $[t_0, t_1]$ will exhibit motion $y(t, \xi)$ that provides the "least action" for the system with a stationary value. The "Action" is defined by

$$\begin{aligned} A &= \int_{t_0}^{t_1}(KE - PE)dt \\ &= \int_{t_0}^{t_1}[T - U]dt. \end{aligned}$$

For the beam of Example 6, this means that the vibrations $y(t, \xi)$ must provide a stationary value to the action

$$A[y] = \int_{t_0}^{t_1}\int_0^l [\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}EI(y'')^2]d\xi dt.$$

Through the calculus of variations (a field of mathematics that was the precursor to modern control theory), this leads to an equation of motion for the vibrations y that the beam motion must satisfy.

To further explore this, we consider $y(t, \xi)$ as the motion of the beam and consider a family of variations $y(t, \xi) + \epsilon\eta(t, \xi)$ where η is chosen so that $y + \epsilon\eta$ is an "admissible variation", i.e., $y + \epsilon\eta$ must satisfy the essential boundary conditions (56).

We define $V = H^2_L(0, l) = \{\varphi \in H^2(0, l)|\varphi(0) = \varphi'(0) = 0\}$. Let $\psi \in C^2(t_0, t_1)$ with $\psi(t_0) = \psi(t_1) = 0$. Then $\eta \in \mathcal{N} = \{\eta|\eta = \psi\varphi, \varphi \in V\}$ satisfies η is C^2 in t, H^2 in ξ with $\eta(t_0, \xi) = \eta(t_1, \xi) = 0$ and $\eta(t, 0) = \eta'(t, 0) = 0$. Then by Hamilton's principle, we must have that $A[y + \epsilon\eta]$ for $\epsilon > 0, \eta \in \mathcal{N}$, must have a stationary value at $\epsilon = 0$. That is,

$$\frac{d}{d\epsilon}A[y + \epsilon\eta]|_{\epsilon=0} = 0.$$

Since

$$A[y + \epsilon\eta] = \int_{t_0}^{t_1}\int_0^l [\frac{1}{2}\rho(\dot{y} + \epsilon\dot{\eta})^2 - \frac{1}{2}EI(y'' + \epsilon\eta'')^2]d\xi dt,$$

we find

$$0 = \int_{t_0}^{t_1}\int_0^l [\rho\dot{y}\dot{\eta} - EIy''\eta'']d\xi dt \tag{58}$$

for all $\eta \in \mathcal{N}$. We integrate by parts in the first term (with respect to t) to obtain

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^l \rho\dot{y}\dot{\eta}d\xi dt &= -\int_{t_0}^{t_1}\int_0^l \rho\ddot{y}\eta d\xi dt + \int_0^l \rho\dot{y}\eta d\xi|_{t=t_0}^{t=t_1} \\ &= -\int_{t_0}^{t_1}\int_0^l \rho\ddot{y}\eta d\xi dt\end{aligned}$$

since $\eta(t_0,\xi) = \eta(t_1,\xi) = 0$. Since η has the form $\eta = \psi\varphi$, equation (58) has the form

$$\int_{t_0}^{t_1}\int_0^l [\rho\ddot{y}\varphi + EIy''\varphi'']\psi d\xi dt = 0 \tag{59}$$

for all $\psi \in C^2[t_0,t_1]$ with $\psi(t_0) = \psi(t_1) = 0$, and all $\varphi \in V$. Since this holds for arbitrary ψ, we must have in the $L_2(t_0,t_1)$ sense

$$\int_0^l [\rho\ddot{y}\varphi + EIy''\varphi'']d\xi = 0 \quad \text{for all } \varphi \in V.$$

In our former notation of Gelfand triples with $V = H_L^2(0,l)$ and $H = L_2(0,l)$, this may be written

$$\langle \rho\ddot{y}, \varphi\rangle_{V^*,V} + \langle EIy'', \varphi''\rangle_H = 0 \quad \text{for all } \varphi \in V$$

in the $L_2(t_0,t_1)$ sense, which is exactly the "weak" or "variational" form of the beam equation we have encountered previously. Note that in fact the true variational form was given in (58); that is,

$$\int_{t_0}^{t_1} [-\langle \rho\dot{y}, \varphi\rangle\dot{\psi} + \langle EIy'', \varphi''\rangle\psi]dt = 0$$

for all $\varphi \in V$ and $\psi \in C^2[t_0,t_1]$ with $\psi(t_0) = \psi(t_1) = 0$. (See the proofs and our remarks concerning solutions in the $L_2(t_0,t_1;V)^* \cong L_2(t_0,t_1;V^*)$ sense in the well posedness (existence) results above.

We note that if the variational solution y has additional smoothness so that $y \in V \bigcap H^4(0,l)$ (more precisely $EIy'' \in H^2(0,l)$), then we can integrate by parts twice (with respect to ξ) in the second term of (59) to obtain in place of (59):

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^l [\rho\ddot{y}\varphi + (EIy'')''\varphi]\psi d\xi dt + \int_{t_0}^{t_1} -(EIy'')\varphi'|_{\xi=0}^{\xi=l}\psi dt \\ + \int_{t_0}^{t_1} (EIy'')'\varphi|_{\xi=0}^{\xi=l}\psi dt = 0\end{aligned}$$

for $\varphi \in V, \psi \in C^2[t_0, t_1]$ with $\psi(t_0) = \psi(t_1) = 0$. This can be written

$$\int_{t_0}^{t_1} \left[\int_0^l [\rho \ddot{y} \varphi + (EIy'')'' \varphi] d\xi + -EIy'' \varphi'|_{\xi=l} + (EIy'')' \varphi|_{\xi=l} \right] \psi dt = 0$$

for arbitrary $\varphi \in V$. We note once again that this results in the strong or classical form of the equations

$$\rho \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial \xi^2}(EI \frac{\partial^2 y}{\partial \xi^2}) = 0$$

with the essential boundary conditions

$$y(t, 0) = y'(t, 0) = 0$$

as well as the natural boundary conditions

$$EI \frac{\partial^2 y}{\partial \xi^2}(t, l) = \frac{\partial}{\partial \xi}(EI \frac{\partial^2 y}{\partial \xi^2})(t, l) = 0$$

holding.

The weak or variational form may be thought of as Euler's equations in calculus of variations. If

$$J(y) = \int F(t, y, \dot{y}) dt,$$

then the condition

$$\frac{d}{d\epsilon} J(y + \epsilon \eta)|_{\epsilon=0} = 0$$

implies

$$\int (F_y \eta + F_{\dot{y}} \dot{\eta}) dt = 0$$

which is the "true" Euler's equation. If we assume enough smoothness and integrate by parts, we obtain the strong form of Euler's equation:

$$-\frac{d}{dt} F_{\dot{y}}(t, y, \dot{y}) + F_y(t, y, \dot{y}) = 0.$$

In other words,

$$\frac{\partial F}{dy} = \frac{d}{dt} \frac{\partial F}{d\dot{y}}$$

by applying duBois Raymond's lemma given below.

Lemma 1 *(duBois Raymond's Lemma)*
If

$$\int (f_1\eta + f_2\dot{\eta}) = 0$$

for all η, *then*

$$\frac{d}{dt} f_2 = f_1.$$

In other words, f_1 *is the distibutional derivative of* f_2.

In the derivation above, we assumed the undamped and unforced version of the equation. In the case of the forced beam, we can add a conservative force term $W = fy$ in our derivation, and we will obtain the desired $\langle f, \varphi \rangle$ term in our result. However, there is no known way to derive the weak form with damping. In other words, Hamilton's principle is essentially valid for conservative forces, but it doesn't conveniently handle nonconservative forces (damping).

15 Finite Element Approximations

We will now consider finite element approximations or Galerkin approximations for parabolic systems. Consider

$$\begin{cases} \dot{x}(t) = Ax + F & \text{in } V^* \ \ (H \text{ if possible}) \\ x(0) = x_0 \end{cases} \tag{60}$$

where $V \hookrightarrow H \hookrightarrow V^*$ is the usual gelfand triple. We can write the above system in the weak or variational form as

$$\begin{cases} \langle \dot{x}(t), \varphi \rangle_{V^*,V} + \sigma(x(t), \varphi) = \langle F(t), \varphi \rangle_{V^*,V} \\ x(0) = x_0 \end{cases}$$

for $\varphi \in V$. If σ is V continuous and V-elliptic, and $T(t) \sim e^{At}$, we can write

$$x(t) = T(t)x_0 + \int_0^t T(t-\xi)F(\xi)d\xi \tag{61}$$

where $x \in L_2(0,T;V) \bigcap C(0,T;H)$ and $\dot{x} \in L_2(0,T;V^*)$. We can use this formulation to give a nice treatment of finite element approximations of Galerkin type.

In general, this is an infinite dimensional space; therefore, we want to project the system into a finite dimensional space in which we can compute. Let $H^N = span\{B_1^N, B_2^N, ..., B_N^N\} \subset V$ be the approximation of H. The idea is to replace (60) by

$$\begin{cases} \dot{x}^N(t) = A^N x^N(t) + F^N(t) & \text{in } H^N \\ x^N(0) = x_0^N \end{cases}$$

or equivalenty, replace (61) by

$$x^N(t) = T^N(t)x_0^N + \int_0^t T^N(t-\xi)F^N(\xi)d\xi$$

where $T^N(t) \sim e^{A^N t}$.

One of the key constructs we need is $P^N : H \to H^N$ which is called the *orthogonal projection* of H onto H^N. In other words, P^N is defined by

$$\langle P^N\varphi - \varphi, \psi \rangle = 0 \quad \forall \psi \in H^N$$

or

$$|P^N\varphi - \varphi|_H = \inf_{\psi \in H^N} |\psi - \varphi|_H.$$

We would like $F^N \to F$ and $x_0^N \to x_0$, so let's take $x_0^N = P^N x_0$ and $F^N(t) = P^N F(t)$. We also want $A^N \in \mathcal{L}(H^N)$ and $A^N \approx A$. However, we have defined $T^N(t) = e^{A^N t}$ and $T(t) = e^{At}$; therefore, if we had $T^N(t) \to T(t)$, then we would be done. This is considered in the Trotter-Kato theorem which will be discussed later.

Now, to relate this to finite elements, let's restrict

$$\langle \dot{x}(t), \varphi \rangle + \sigma_1(x(t), \varphi) = \langle F(t), \varphi \rangle \quad \forall \varphi \in V \tag{62}$$

to $H^N \times H^N$. In other words, let

$$x^N(t) = \sum_{j=1}^{N} w_j^N B_j^N$$

be a trial solution with

$$x^N(0) = \sum_{j=1}^{N} w_{0j}^N B_j^N.$$

Plugging this into (62), we have

$$\langle \sum_{j=1}^{N} \dot{w}_j^N(t) B_j^N, \varphi \rangle + \sigma_1(\sum_{j=1}^{N} w_j^N(t) B_j^N, \varphi) = \langle F(t), \varphi \rangle \tag{63}$$

for $\varphi \in H^N$. Let $\varphi = B_1^N, B_2^N, ...B_N^N$. From this we obtain an $N \times N$ vector system for $w^N(t) = (w_1^N, .., w_N^N)^T$ given by

$$\sum_{j=1}^{N} \dot{w}^N(t) \langle B_j^N, B_i^N \rangle + \sum_{j=1}^{N} w_j^N(t) \sigma(B_j^N, B_i^N) = \langle F(t), B_i^N \rangle \tag{64}$$

for $i = 1, 2, ..., N$.

Let's define the mass matrix $M^N = (\langle B_i^N, B_j^N \rangle)$, the stiffness matrix $K^N = (\sigma(B_i^N, B_j^N))$, and the column vector $F^N(t) = (\langle F(t), B_i^N \rangle)$. Then (64) becomes

$$\begin{cases} M^N \dot{w}^N(t) + K^N w^N(t) = F^N(t) \\ w^N(0) = w_0^N \end{cases} \tag{65}$$

or

$$\begin{cases} \dot{w}^N(t) = -(M^N)^{-1} K^N w^N(t) + (M^N)^{-1} F^N(t) \\ w^N(0) = w_0^N \end{cases}.$$

Now considering w_0^N, we have $x^N(0) = P^N x_0$ which implies $\langle P^N x_0 - x_0, B_i^N \rangle = 0$ for $i = 1, ..., N$. However, $x_0^N = \sum_{j=1}^N w_{0j}^N B_j^N$. Therefore,

$$\langle \sum_{j=1}^{N} w_{0j}^N B_j^N - x_0, B_i^N \rangle = 0$$

for $i = 1, ..., N$ which gives

$$\sum_{j=1}^{N} w_{0j}^N \langle B_j^N, B_i^N \rangle = \langle x_0, B_i^N \rangle.$$

Let's define $w_0^N = col(w_{01}^N, ..., w_{0N}^N)$. Then we have

$$w_0^N = (M^N)^{-1} col(\langle x_0, B_i^N \rangle).$$

From this, our system for w becomes

$$\begin{cases} \dot{w}^N(t) = -(M^N)^{-1} K^N w^N(t) + (M^N)^{-1} F^N(t) \\ w_0^N = (M^N)^{-1} col(\langle x_0, B_i^N \rangle) \end{cases}$$

However, we normally do not solve the system in this form. If $\langle B_i, B_j \rangle = 0$ for $i \neq j$, then M^N is diagonal and the system of the form (65) is an easier system with which to work. More generally, the (finite element) system is solved in the form

$$\begin{aligned} M^N \dot{w}^N(t) &= -K^N w^N(t) + F^N(t) \\ M^N w_0^N &= col(\langle x_0, B_i^N \rangle). \end{aligned}$$

16 Trotter-Kato Approximation Theorem

The Trotter-Kato Approximation Theorem is the functional analysis version of the Lax Equivalence Principle used in finite difference approximation for PDE's which dates back to the 1960's. The ideas of the Lax Equivalence Principle is that "consistency" and "stability" are achieved if and only if we have "convergence" of our system. If we have a PDE

$$u_{tt} = Au$$

and an approximation

$$u_{tt}^N = A^N u^N$$

then consistency refers to $A^N \to A$ in some sense. Stability refers to $|e^{A^N t}| \leq Me^{\omega t}$, and convergence means $e^{A^N t} \to e^{At}$ in some sense.

For relevant material, see [RM].

There are two different versions of the Trotter-Kato theorem which will be considered. We will first consider the Operator Convergence form of the Trotter-Kato theorem.

Theorem 23 *Let X and X^N be Hilbert spaces such that $X^N \subset X$. Let $P^N : X \to X^N$ be an orthogonal projection of X onto X^N. Assume $P^N x \to x$ for all $x \in X$. Let A^N, A be infinitesimal generators of C_0 semigroups $S^N(t), S(t)$ on X^N, X respectively satisfying*

(i) there exists M, ω such that $|S^N(t)| \leq Me^{\omega t}$ for each N

(ii) there exists $\mathcal{D}$ dense in X such that for some λ, $(\lambda I - A)\mathcal{D}$ is dense in X and $A^N P^N x \to Ax$ for all $x \in \mathcal{D}$.

Then for each $x \in X$, $S^N(t)P^N x \to S(t)x$ uniformly in t on compact intervals $[0, T]$.

See Theorem 4.5 in Chapter 3 of Pazy.

Next, we will examine the Resolvent Convergence Form of the Trotter-Kato theorem. This form is a modification of the previous form.

Theorem 24 *Replace (ii) in the above theorem by $(\tilde{ii})$.*

$(\tilde{ii})$ There exists $\lambda \in \rho(A) \bigcap_{N=1}^{\infty} \rho(A^N)$ with Re(λ) $> \omega$ so that $R_\lambda(A^N)P^N x \to R_\lambda(A)x$ for each $x \in X$.

See Theorem 4.2,4.3,4.4 in Chapter 3 of Pazy. See also Theorem 1.14 of [BK]. For convergence rates, see Theorem 1.16 of [BK].

For certain problems, it is not necessary for $H^N \subset dom(A)$ which carries both the essential and natural boundary conditions. We may need to only choose an appropriate approximation H^N such that $H^N \subset V$ which carries just the essential boundary conditions. If we restrict ourselves to first order systems in the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with $H^N \approx V$, then we have a special case of the Trotter-Kato theorem.

Let the condition (C1) be denoted by
(C1) For each $z \in V$, there exists $\hat{z}^N \in H^N$ such that $|z - \hat{z}^N|_V \to 0$ as $N \to \infty$.

Suppose σ is V-elliptic, i.e. $\text{Re}\sigma(\varphi,\varphi) \geq \delta|\varphi|_V^2$. Also assume σ is V continuous, i.e. $|\sigma(\varphi,\psi)| \leq \gamma|\varphi|_V|\psi|_V$. Let $P^N : H \to H^N$ be an orthogonal projection. Then

$$|P^N z - z| = inf\{|z^N - z|_H | z^N \in H^N\}.$$

Under (C1), we have $|P^N z - z|_H \leq |\hat{z}^N - z|_H \leq |\hat{z}^N - z|_V \to 0$ as $N \to \infty$. Therefore, under (C1), $P^N z \to z$ for $z \in H$.

Now, we need to define A^N. We have $\sigma(\varphi,\psi) = \langle -A\varphi, \psi\rangle_H$ for $\varphi \in dom(A) = \{\psi \in V | \mathcal{A}\psi \in H\}, \psi \in V$ where A is an infinitesimal generator of a C_0 semigroup of contractions on H, i.e. $|e^{At}| \leq 1$. Let's define A^N through the restriction of σ to $H^N \times H^N$. Therefore, $A^N : H^N \to H^N$ is defined by

$$\begin{aligned} \sigma(\varphi^N, \psi^N) &= \langle -A^N \varphi^N, \psi^N\rangle_H \quad \varphi^N, \psi^N \in H^N \\ &= \langle -\mathcal{A}\varphi^N, \psi^N\rangle_{V^*,V}. \end{aligned}$$

By V-ellipticity, we obtain A^N is an infinitesimal generator of a contraction semigroup on H^N, i.e. $|e^{A^N t}| \leq 1$.

Theorem 25 *If σ is V-elliptic, V continuous and (C1) holds, then*

$$R_\lambda(A^N)P^N z \to R_\lambda(A)z$$

in the V norm for $z \in H$ and $\lambda = 0$.

See [BI] for relevant material.

Proof

Let $z \in H$, and take $\lambda = 0$. Now define $w^N = R_\lambda(A^N)P^N z$ and $w = R_\lambda(A)z$ where $\sigma(\varphi, \psi) = \langle -A\varphi, \psi\rangle_H, \varphi \in dom(A), \psi \in V$. By definition, we have $w \in dom(A)$.

By (C1), there exists $\hat{w}^N \in H^N$ such that $|\hat{w}^N - w|_V \to 0$ as $N \to \infty$. Let $z^N = w^N - \hat{w}^N$. We need to show $z^N \to 0$ in V.

Since $R_\lambda(A) = (\lambda I - A)^{-1}$,

$$\begin{aligned} \sigma(w, z^N) &= \langle -AR_\lambda(A)z, z^N\rangle_H \\ &= \langle z, z^N\rangle \end{aligned}$$

and

$$\begin{aligned} \sigma(w^N, z^N) &= \langle -A^N R_\lambda(A^N)z, z^N\rangle_H \\ &= \langle z, z^N\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \delta|z^N|_V^2 &\leq \sigma(z^N, z^N) \\ &\leq \sigma(w^N, z^N) - \sigma(\hat{w}^N, z^N) \\ &= \sigma(w, z^N) - \sigma(\hat{w}^N, z^N) \\ &= \sigma(w - \hat{w}^N, z^N) \\ &\leq \gamma|w - \hat{w}^N|_V|z^N|_V. \end{aligned}$$

Therefore, we have $\delta|z^N|_V \leq \gamma|w - \hat{w}^N|_V$. However, $|\hat{w}^N - w|_V \to 0$ implies $|z^N|_V \to 0$.

Remark: Theorem 2.2 of [BI] is a parameter dependent version of this, i.e. $A = A(q), A^N = A^N(q^N), q, q^N \in Q$.

Theorem 26 *Suppose σ is V-elliptic, V continuous, and (C1) holds. Then $T^N(t)P^N z \to T(t)z$ in the V norm for each $z \in H$ uniformly in t on compact intervals.*

Proof

Let $X = H, X^N = H^N, P^N : H \to H^N$ be an orthogonal projection. Then $P^N z \to z$ for all $z \in H$ by (C1). To obtain V convergence is a little more work and more delicate (see Theorem 2.3 of [BI]).

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