Applied Harmonic Analysis

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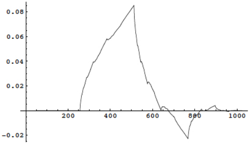

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Outline

References

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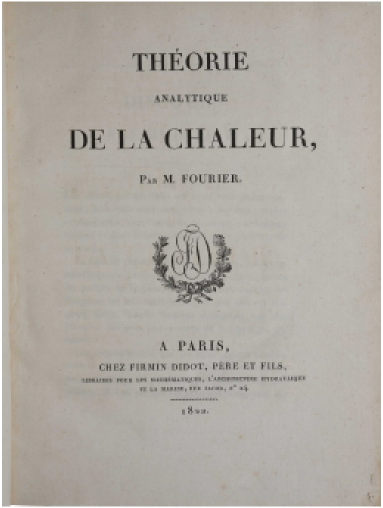
Joseph Fourier (1768-1830): *"... the first among the European scientists ..."* wote of him Giuseppe Lodovico Lagrangia (Joseph-Louis Lagrange).



THÉORIE

ANALYTIQUE

DE LA CHALEUR,

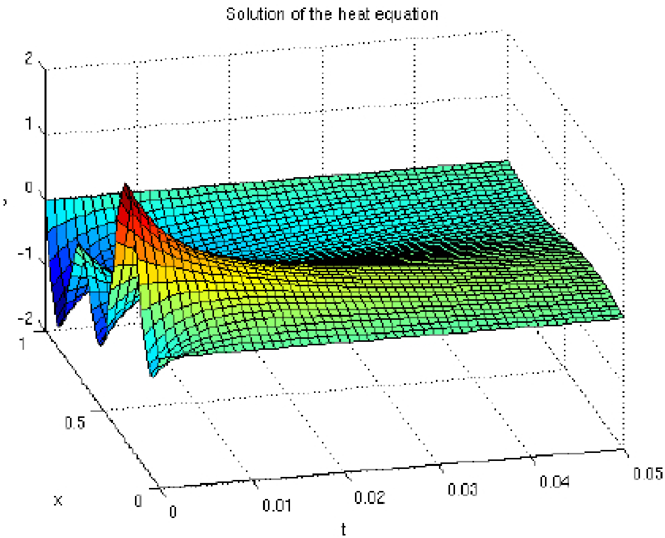
PAR M. FOURIER.

A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

1822.

Fourier's memoir on the theory of heat (1807)



Solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Fourier's idea

Every 2π-periodic function $f(t)$ (such as sin and cos) can be represented as superposition of fundamental waves of different frequency

$$f(t) \stackrel{?}{=} a_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)),$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(t)dt, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t)\cos(nt)dt, \\ a_b &= \frac{1}{\pi} \int_0^{2\pi} f(t)\sin(nt)dt. \end{aligned}$$

Complex Fourier series

From Euler's formula

$$e^{it} = \cos(t) + i\sin(t),$$

we deduce

- for a 2π-periodic function $f(t)$ its Fourier series is

$$f(t) \stackrel{?}{=} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \hat{f}_n e^{int}$$

- with Fourier coefficients

$$\hat{f}_n = \int_0^{2\pi} f(t) e^{-int} dt$$

- The functions $\{e^{int} : n \in \mathbb{Z}\}$ are orthonormal (we will see in a moment ...) with respect to the scalar product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)} dt.$$

Scalar product

Let $\mathcal{H}$ be a vector space. A scalar product $\langle u, v\rangle$ is a map from $\mathcal{H} \times \mathcal{H}$ and values in $\mathbb{C}$ such that

(i) $\langle au + bv, z\rangle = a\langle u, z\rangle + b\langle v, z\rangle$ for all $u, v, z \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in \mathcal{H}$.

(iii) $\langle u, u\rangle \in \mathbb{R}$, $\langle u, u\rangle \geq 0$ for all $u \in \mathcal{H}$ and $\langle u, u\rangle \neq 0$ if $u \neq 0$.

A Hilbert space is a vector space $\mathcal{H}$ endowed with the scalar product $\langle u, v\rangle$, which is also complete w.r.t. the norm $\|u\|_{\mathcal{H}} := \langle u, u\rangle^{1/2}$.

Examples

- Let $\Omega \subset \mathbb{R}^n$. The vector space $L^2(\Omega) = \{f : \Omega \to \mathbb{C} | \int_\Omega |f|^2 dx < \infty\}$ is a Hilbert space with the scalar product

$$\langle u, v \rangle = \int_\Omega u(x)\overline{v(x)} dx.$$

- Let $\Omega_d \subset \mathbb{Z}^n$. The vector space $\ell^2(\Omega_d) = \{f : \Omega_d \to \mathbb{C} | \sum_{k \in \Omega_d} |f(k)|^2 < \infty\}$ is a Hilbert space with the scalar product

$$\langle u, v \rangle = \sum_{k \in \Omega_d} u(k)\overline{v(k)}.$$

In particular if $|\Omega_d| = d < \infty$ then $\ell^2(\Omega_d) = \mathbb{C}^d$.

Spazi di Hilbert e basi ortonormali

A set $\{u_\alpha\}_{\alpha\in A}$ is orthonormal in $\mathcal{H}$ if $\langle u_\alpha, u_\beta\rangle = \delta_{\alpha,\beta}$ where $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

Theorem (Fourier)

Let $\{u_\alpha\}_{\alpha\in A}$ be an orthonormal set. Then the following conditions are equivalent:

(i) $x = \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha$ *for all* $x \in \mathcal{H}$.

(ii) *(Parseval identity)* $\langle x, y\rangle = \sum_{\alpha\in A}\langle x, u_\alpha\rangle\overline{\langle y, u_\alpha\rangle}$ *for all* $x, y \in \mathcal{H}$.
If $x = y$ *then it holds* $\|x\|^2_{\mathcal{H}} = \sum_{\alpha\in A}|\langle x, u_\alpha\rangle|^2$

(iii) *(Completeness) If* $x \in \mathcal{H}$ *and if* $\langle x, u_\alpha\rangle = 0$ *for all* α*, then* $x = 0$.

Zorn's lemma implies:

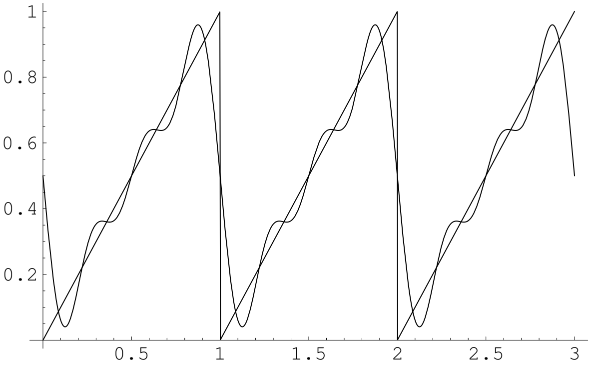
Theorem

Every Hilbert space has an orthonormal basis.

Again on the trigonometric series

The set $\{\frac{1}{\sqrt{\tau}}e^{2\pi int/\tau} : n \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space $\mathcal{H} = L^2(0,\tau)$ for any $\tau > 0$. Orthogonality:

$$\begin{aligned}
\langle \frac{1}{\sqrt{\tau}}e^{2\pi int/\tau}, \frac{1}{\sqrt{\tau}}e^{2\pi imt/\tau}\rangle &= \frac{1}{\tau}\int_0^\tau e^{2\pi int/\tau}\overline{e^{2\pi imt/\tau}}dt \\
&= \frac{1}{\tau}\int_0^\tau e^{2\pi i(n-m)t/\tau}dt \\
&= \int_0^1 e^{2\pi i(n-m)t}dt = \delta_{m,n}.
\end{aligned}$$



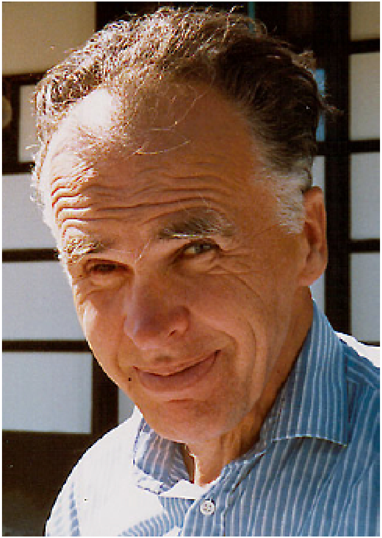
Partial Fourier series of $f(t) = t - \lfloor t \rfloor$.

1873: Paul du Bois-Reymond (1831-1889) constructed (discovered?) a continuous function whose Fourier series diverges in a point: it has form $f(t) = A(t)\sin(\omega(t)t)$ for a certain $A(t) \to \infty$ e $\omega(t) \to \infty$.

1903: Lipót Fejér (1880-1959) proved the convergence of the sum in the Cesàro sense (convergence of the means of the partial sums) of the Fourier series of continuous functions.

Henri Lebesgue (1875-1941) establishs the convergence in the square (mean) norm L^2 of the Fourier series of square summable/integrable functions on $[0, 2\pi]$.

1923/1926: Andrey Kolmogorov (1903-1987), at age 21 (!), constructs a summable/integrable function, i.e., it belongs to the Lebesgue space L^1, whose Fourier series diverges almost everywhere!



1966: Lennart Carleson (1928-, Abel Prize 2006) proved that the series of a square summable function, i.e. it belongs to the Lebesgue space L^2, converges almost everywhere!

Time-invariant linear operators

- We consider a function $u \to g(u)$ for $u \in \mathbb{R}$.
- Define a *time-invariant linear operator* $L : g \to Lg$ by means of the *convolution product*

$$Lg(u) = (f * g)(u) = \int_{-\infty}^{+\infty} f(t)g(u-t)dt.$$

- Here the function $f = L\delta_0$ is also called *impulse response* of L to the Dirac δ_0.

Fourier transform

- $g(t) = e^{i\omega t}$ is an *eigenfunction* (gen. eigenvector) of L

$$Lg(u) = \int_{-\infty}^{+\infty} f(t) e^{i\omega(u-t)} dt = \hat{f}(\omega) g(u).$$

- The eigenvalue

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt,$$

  is the so-called *Fourier transform* of f in $\omega \in \mathbb{R}$.
- The value $\hat{f}(\omega)$ is larger $f(t)$ the more similar f is to the complex wave $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ on a (of) large (measure) set.

Inverse Fourier transform and the convolution

- A reconstruction formula for f from its Fourier transform is

$$f(t) \stackrel{?}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

- Exercise: prove under suitable condition of summability of f and g that one has

$$\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega),$$

  per ogni $\omega \in \mathbb{R}$.
  Hint: use (the formula of the inverse Fourier transformation for g and) Fubini-Tonelli theorem, which allows for exchanging sequence of integrals (just give it for granted).
- Exercise: prove the (new) Parseval's identity

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega.$$

Fourier transform as a limit from the interval

Let $f \in L^2(-\overline{\omega}\pi, \overline{\omega}\pi)$ a square summable/integrable function on $(-\overline{\omega}\pi, \overline{\omega}\pi)$. By the Fourier theorem

$$f = \sum_{n \in \mathbb{Z}} \langle f, \frac{1}{\sqrt{2\overline{\omega}\pi}} e^{in\cdot/\overline{\omega}} \rangle \frac{1}{\sqrt{2\overline{\omega}\pi}} e^{int/\overline{\omega}} = \frac{1}{2\overline{\omega}\pi} \sum_{n \in \mathbb{Z}} \left(\int_{-\overline{\omega}\pi}^{\overline{\omega}\pi} f(\omega) e^{-in\omega/\overline{\omega}} d\omega \right) e^{int/\overline{\omega}}.$$

What happens (formally) if we let $\overline{\omega} \to \infty$? The last sum is in fact a Riemann sum. This makes us thinking that if f is summable/integrable over $\mathbb{R}$ then we could in fact write something like

$$\begin{aligned} f(t) &= \lim_{\overline{\omega} \to \infty} f(t) \chi_{[\overline{\omega}\pi, -\overline{\omega}\pi]}(t) \\ &\stackrel{?}{=} \lim_{\overline{\omega} \to \infty} \sum_{n \in \mathbb{Z}} \langle f, \frac{1}{\sqrt{2\overline{\omega}\pi}} e^{in\cdot/\overline{\omega}} \rangle \frac{1}{\sqrt{2\overline{\omega}\pi}} e^{int/\overline{\omega}} \\ &\stackrel{?}{=} \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi) e^{-i\xi\omega} d\xi \right) e^{it\omega} d\omega. \end{aligned}$$

What's the meaning of the Fourier transform?

What's the meaning of the Fourier transform? For what is it useful?

- The Fourier transform represents the *frequency content* of a function/signal. It tells us which are the important oscillatory constituents of a signal and their distinctive frequencies of oscillation.
- The "fortune" of Fourier analysis relies essentially on the fact that it is able to describe one of the fundamental and most frequent phenomena in nature: the oscillatory phenomena, many of which are rules by superpositions of laws of the type:

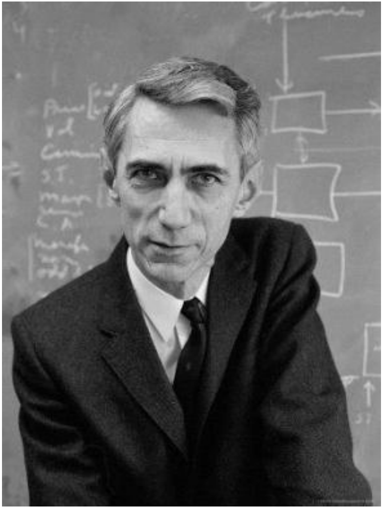
$$y_{\alpha,s_0}(t) = \begin{cases} e^{-\alpha t} e^{2\pi i s_0 t}, & t > 0 \\ 0, \quad t < 0. \end{cases}$$

- The Fourier transform of $y_{\alpha,s_0}(t)$ is called *Lorentzian*.

Significato della trasformata di Fourier

- For examples, when molecules are hit by an electromagnetic radiation, some damped overlapping oscillations are induced as described by the law $y_{\alpha,s_0}(t)$.
- Each molecule constituent has its own distinctive and unique oscillation. the Lorentzians are then called *molecular spectrum*.
- From these observations comes the idea which led to the Nobel prize to Richard Ernst[1] for chemistry (1991) for the development of a powerful tool for determining the molecular structure of complex organic molecules.

[1] He holds a Honorary Doctorate from the Technical University of Munich



1949: Claude Elwood Shannon (1916-2001) proves that a band-limited function can be recostructed from its samples and this observation is at the basis of our modern digital technology.

Analog and digital: Shannon's sampling theorem

Perhaps one of the most relevant applications of the Fourier transform is the analog$\leftrightarrow$digital conversion. A function f is called $\overline{\omega}$-band-limited if its Fourier transform $\hat{f}$ has compact support contained in the interval $[-\overline{\omega}\pi, \overline{\omega}\pi]$ for $\overline{\omega} > 0$.

Theorem (Whittacker-Shannon). *If $f \in L^2(\mathbb{R})$ is $\overline{\omega}$-band-limited, then for all* $0 < \tau \leq \overline{\omega}^{-1}$

$$f(t) = \sum_{n \in \mathbb{Z}} f(\tau n) \operatorname{sinc}(\tau^{-1} t - n),$$

ove

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}.$$

Analog and digital: Shannon's sampling theorem

Proof. Since $\mathrm{supp}(\hat{f}) \subset [-\overline{\omega}\pi, \overline{\omega}\pi]$, for the Fourier Theorem

$$\begin{aligned}
\hat{f}(\omega) &= \hat{f}(\omega)\chi_{[-\overline{\omega}\pi,\overline{\omega}\pi]}(\omega) \\
&= \frac{1}{2\overline{\omega}\pi}\sum_{n\in\mathbb{Z}}\langle \hat{f}\chi_{[-\overline{\omega}\pi,\overline{\omega}\pi]}, e^{in\cdot/\overline{\omega}}\rangle e^{in\omega/\overline{\omega}}\chi_{[-\overline{\omega}\pi,\overline{\omega}\pi]}(\omega)
\end{aligned}$$

By applying the inverse Fourier transform

$$\begin{aligned}
\frac{1}{2\overline{\omega}\pi}\langle \hat{f}\chi_{[-\overline{\omega}\pi,\overline{\omega}\pi]}, e^{in\cdot/\overline{\omega}}\rangle &= \frac{1}{2\overline{\omega}\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-in\omega/\overline{\omega}}d\omega \\
&= \frac{1}{\overline{\omega}}f\left(-\frac{n}{\overline{\omega}}\right).
\end{aligned}$$

The characteristic function $\chi_{[-1,1]}$ of the interval $[-1,1]$ has Fourier transform

$$\hat{\chi}_{[-1,1]}(t) = 2\frac{\sin(t)}{t}.$$

Analog and digital: Shannon's sampling theorem

Proof continues ...

The inverse Fourier transform of $e^{in\omega/\overline{\omega}}\chi_{[-\overline{\omega}\pi,\overline{\omega}\pi]}(\omega)$ is given by

$$\begin{aligned} e^{in\cdot/\overline{\omega}}\check{\chi}_{[-\overline{\omega}\pi,\overline{\omega}\pi]}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_{[-\overline{\omega}\pi,\overline{\omega}\pi]}(\omega)e^{i(t+n/\overline{\omega})\omega}d\omega \\ &\overset{\omega\leftrightarrow\pi\overline{\omega}\xi}{=} \frac{\overline{\omega}}{2}\int_{-\infty}^{\infty}\chi_{[-1,1]}(\xi)e^{i\pi\overline{\omega}(t+n/\overline{\omega})\xi}d\xi \\ &= \overline{\omega}\frac{\sin(\pi\overline{\omega}(t+n/\overline{\omega}))}{\pi\overline{\omega}(t+n/\overline{\omega})} \end{aligned}$$

Hence, we have

$$\begin{aligned} f(t) &= \sum_{n\in\mathbb{Z}} f\left(-\frac{n}{\overline{\omega}}\right)\frac{\sin(\pi\overline{\omega}(t+n/\overline{\omega}))}{\pi\overline{\omega}(t+n/\overline{\omega})} \\ &= \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\overline{\omega}}\right)\operatorname{sinc}(\overline{\omega}t-n) \end{aligned}$$

Analog and digital scalar product

With similar techniques to prove Shannon's sampling theorem, one can prove (difficult exercise!):

Theorem ("Plancherel" for band-limited functions). *If $f, g \in L^2(\mathbb{R})$ are both $\overline{\omega}$-band-limited, then for all $0 < \tau \leq \overline{\omega}^{-1}$ we have the identities*

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \tau \sum_{n\in\mathbb{Z}} f(\tau n)\overline{g(\tau n)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}dt.$$

Meta-Corollary ("Plancherel" for nearly band-limited functions). *if $f, g \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ are both functions which are* **NEARLY** *$\overline{\omega}$-band-limited, then for $0 < \tau \leq \overline{\omega}^{-1}$ we have the approximations*

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \approx \tau \sum_{n\in\mathbb{Z}} f(\tau n)\overline{g(\tau n)} \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}dt.$$

The entity of the (*aliasing*) approximation depends on the "tails" of the Fourier transforms $\hat{f}, \hat{g}$ out of the compact $[-\pi\tau^{-1}, \pi\tau^{-1}]$.

Operators of traslation, modulation and dilation

From the proof of Shannon's sampling theorem, we learned that there are fundamental operators, which are in a sort of duality (commutation rules) with respect to the Fourier transform. We define the operators of translation and modulation

$$T_{t_0} f(t) := f(t - t_0), \quad M_{\omega_0} f(t) = f(t) e^{i\omega t}, \quad t_0, \omega_0 \in \mathbb{R},$$

and the dilation

$$D_a f(t) := \frac{1}{|a|^{1/2}} f(t/a), \quad a \in \mathbb{R}_+.$$

They satisfy the commutation rules:

$$\widehat{T_{t_0} f}(\omega) = M_{-t_0} \hat{f}(\omega), \quad \widehat{M_{\omega_0} f}(\omega) = T_{\omega_0} \hat{f}(\omega), \quad \widehat{D_a f}(\omega) = D_{a^{-1}} \hat{f}(\omega)$$

The discrete Fourier transform (DFT)

- In the space $\ell^2(\mathbb{Z}_n) = \mathbb{C}^n$ (of the signals/vectors of n complex values) $(\frac{1}{\sqrt{n}}(e^{2\pi ik\ell/n})_{\ell\in\mathbb{Z}_n})_{k\in\mathbb{Z}_n}$ is an orthonormal basis, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$
- In fact, one can prove (exercise by induction!) that

$$1 + z + z^2 + \cdots + z^{n-1} = \begin{cases} n, & z = 1 \\ (z^n - 1)/(z - 1), & \text{otherwise.} \end{cases}$$

But then it is not hard to show that for $z = e^{2\pi i(k-l)/n}$ we have

$$\langle \frac{1}{\sqrt{n}}(e^{2\pi ikl/n})_{l\in\mathbb{Z}_n}, \frac{1}{\sqrt{n}}(e^{2\pi ilm/n})_{m\in\mathbb{Z}_n}\rangle = \frac{1}{n}\sum_{m=0}^{n-1} e^{2\pi im(k-l)/n} = \delta_{k,l}.$$

The discrete Fourier transform (DFT)

- For the Fourier Theorem any signal/vector $\mathbf{f}$ of length n can be written:

$$\mathbf{f} = \frac{1}{n}\sum_{k=0}^{n-1} \hat{\mathbf{f}}(k)(e^{2\pi i k\ell/n})_{\ell\in\mathbb{Z}_n},$$

  where

$$\hat{\mathbf{f}}(k) = \frac{1}{\sqrt{n}}\langle \mathbf{f}, (e^{2\pi i k\ell/n})_{\ell\in\mathbb{Z}_n}\rangle = \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} \mathbf{f}(\ell) e^{-2\pi i k\ell/n},$$

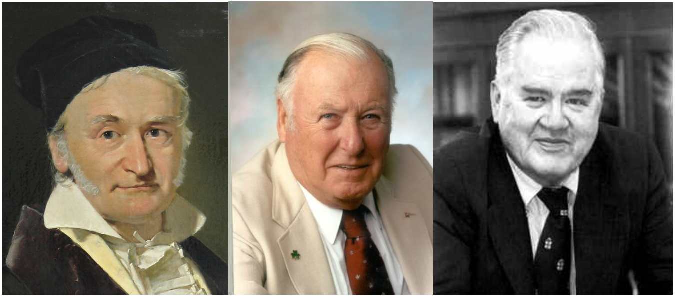
  define the components of the signal/vector *discrete Fourier transform* (DFT) $\hat{\mathbf{f}}$ of $\mathbf{f}$.

Complexity of a "naive" computation of DFT

- Assuming that an operation equals a sum or a multiplication, then the number of ops of a DFT is $2n$ sums and multiplications for n times, i.e., $2n^2$.
- Each complex operation costs double of a single real, the total computational cost is $\mathcal{C}(DFT)(n) = 4n^2$.
- Today a PC is able to execute 3×10^9 ops/sec and therefore it's able to produce a DFT of a signal of length $n = 1000$ in

  $$4 \cdot 1000^2 \cdot \frac{1}{3 \times 10^9} = 0.0013\,\text{sec}\,.$$

  Already for a signal of length $n = 1024 \times 1024 = 2^{20}$ the cost is $1466.02\,\text{sec}$.
- A simple digital image can be larger than $n = 2^{20}$ without difficulty.



1805: Carl Friedrich Gauss (1777-1855) invents the *Fast Fourier Transform* in its study of the interpolation of the trajectories of the asteroids 2 Pallas/Pallade and 3 Juno/Giunone; James Cooley and John Tukey re-invent/discolver the algorithm in 1965.

Operators of traslation, modulation, *upsampling*, duplication

Given a signal/vector $\mathbf{f} \in \mathbb{C}^n$ of length n, we define the operator of translation

$$T_m \mathbf{f}(k) = \mathbf{f}(k-m), \quad m \in \mathbb{Z}_n.$$

and the modulation operator

$$M_m \mathbf{f}(k) = e^{2\pi i m k/n} \mathbf{f}(k), \quad m \in \mathbb{Z}_n.$$

Moreover we define also the *upsampling* and duplication operators by

$$U\mathbf{f}(h) = \begin{cases} \mathbf{f}(h/2), & \mathrm{mod}(h,2) = 0 \\ 0, & \text{altrimenti}, \end{cases} \qquad D\mathbf{f}(h) = \frac{1}{2}(\mathbf{f}, \mathbf{f})(h),$$

for $h \in \mathbb{Z}_{2n}$. The action of the DFT on these operators yields new commutator rules

$$\widehat{T_m \mathbf{f}}(k) = M_{-m} \widehat{\mathbf{f}}(k), \quad \widehat{M_m \mathbf{f}}(k) = T_m \widehat{\mathbf{f}}(k), \quad \widehat{U\mathbf{f}}(h) = D\widehat{\mathbf{f}}(h).$$

Synthesis of a signal

Let's consider a signal of legnth $n = 2^2 = 4$ given by $\mathbf{f} = (f_0, f_1, f_2, f_3)$. Let us see how to assemble $\mathbf{f}$ from the single f_i

$$
\begin{array}{ccccc}
f_0 & f_2 & & f_1 & f_3 \\
\downarrow U & \downarrow U & & \downarrow U & \downarrow U \\
(f_0, 0) & (f_2, 0) & & (f_1, 0) & (f_3, 0) \\
\downarrow I & \downarrow T_{-1} & & \downarrow I & \downarrow T_{-1} \\
(f_0, 0) & (0, f_2) & & (f_1, 0) & (0, f_3) \\
\searrow & \downarrow & & \downarrow & \swarrow \\
 & (f_0, f_2) & & (f_1, f_3) & \\
 & \downarrow U & & \downarrow U & \\
 & (f_0, 0, f_2, 0) & & (f_1, 0, f_3, 0) & \\
 & \downarrow I & & \downarrow T_{-1} & \\
 & (f_0, 0, f_2, 0) & & (0, f_1, 0, f_3) & \\
 & \searrow & & \swarrow & \\
 & & \mathbf{f} & &
\end{array}
$$

The algorithm of the Fast Fourier Transform (FFT)

Noted that $\hat{f}_i = f_i$ for all $i = 0, \ldots f_{n-1}$ by applying the DFTto the previous diagram and substituting U, T_{-1}, resp. D, M_1 as given by the commutator rules, we generate a recursive algorithm to compute the DFT:

$$
\begin{array}{cccc}
f_0 & f_2 & f_1 & f_3 \\
\downarrow D & \downarrow D & \downarrow D & \downarrow D \\
(*,*) & (*,*) & (*,*) & (*,*) \\
\downarrow I & \downarrow M_1 & \downarrow I & \downarrow M_1 \\
(*,*) & (*,*) & (*,*) & (*,*) \\
\searrow & \downarrow & \downarrow & \swarrow \\
 & (*,*) & (*,*) & \\
 & \downarrow D & \downarrow D & \\
 & (*,*,*,*) & (*,*,*,*) & \\
 & \downarrow I & \downarrow M_1 & \\
 & (*,*,*,*) & (*,*,*,*) & \\
 & \searrow & \swarrow & \\
\multicolumn{4}{c}{\widehat{\mathbf{f}}}
\end{array}
$$

Complexity of the Fast Fourier Transform (FFT)

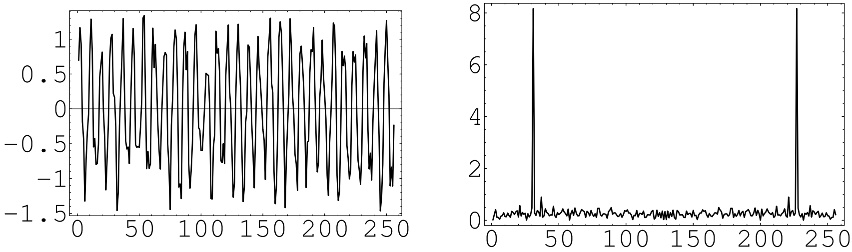
Assume $\mathcal{C}(I) = \mathcal{C}(D) = 0$. The cost of M_1 on a vector of length ℓ is $\ell - 1$. We assume $n = 2^m$. Starting from the bottom of the diagram, we execute only one M_1 and therefore a cost of $2^0(\frac{n}{2^0} - 1)$. This cost has to be summed up with that of the higher level, where we need to execute $2(\frac{n}{2} - 1)$ operations corresponding to 2 times M_1 on vectors of half length. And so on, for a total cost of

$$
\begin{aligned}
\mathcal{C}(FFT)(n) &= \sum_{k=0}^{m-1} 2^k(\frac{n}{2^k} - 1) \\
&= \sum_{k=0}^{m-1} \left(2^m - 2^k\right) = m2^m - 2^m + 1 = n\log_2(n) - n + 1.
\end{aligned}
$$

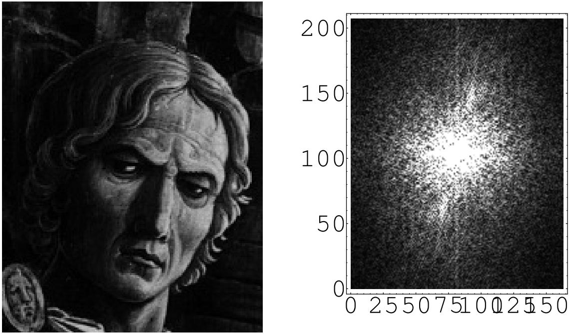
Hence a modern PC is able to produce an FFT of a signal of length $n = 2^{20}$ in

$$(2^{20}20 - 2^{20} + 1) \cdot \frac{1}{3 \times 10^9} = 0.0066\,\text{sec},$$

versus the 1466.02 sec which we would expect from the DFT!



Noisy sinusoidal signal and its Fourier Transform



A digital image and its Fourier Transform

Smothness in time = decay in frequency (and vice versa)

A function $f(t)$ is many times differentiable if $\hat{f}(\omega)$ tends rapidly to 0 for $|\omega| \to \infty$.

Theorem. *Let $r \geq 0$. If*

$$\int_{-\infty}^{+\infty} |\hat{f}(\omega)|(1 + |\omega|^r) d\omega < \infty,$$

then $f(t)$ is differentiable r times.

Since a flipped function is essentially the Fourier transform of its Fourier transform

$$f(-t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{-i\omega t} d\omega = \hat{\hat{f}}(t).$$

then the theorem can be re-formulated for $f \leftrightarrow \hat{f}$.

Singularity in time and loss of localization in frequency (and vice versa)

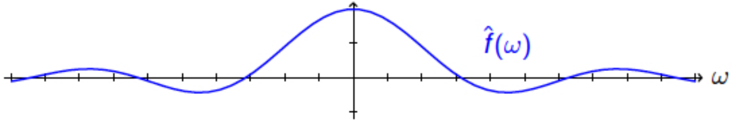
- The decay of $\hat{f}$ depends on the *worst* singular behavior of f
- The characteristic function $[-1, 1]$

$$f(t) = \begin{cases} 1 & t \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases}$$

has Fourier transform

$$\hat{f}(\omega) = 2\frac{\sin(\omega)}{\omega},$$

which has a slow decay (because of the "jumps" of f at $t = -1, 1$).



Time-frequency localization

- $e^{i\omega t}$ has “morally” the Dirac impulse δ_ω as Fourier transform, hence it’s very localized in frequency (an impulse is totally localized in ω) but it’s not localized in time.
- A time delay of f is not perceived by $|\hat{f}(\omega)|$
- In order to study transients, time-dependent phenomena, it would be better to substitute $e^{i\omega t}$ with functions $g(t)$ better localized both in time and frequency
- Can $g(t)$ and $\hat{g}(\omega)$ have simultaneously small support or decay rapidly?

Heisenberg uncertainty principle I

- Assume $\|g\|_2^2 = \int_{-\infty}^{+\infty} |g(t)|^2 dt = 1$, so that $|g(t)|^2$ is a probability density
- Plancherel: $\|\hat{g}\|_2^2 = \int_{-\infty}^{+\infty} |\hat{g}(\omega)|^2 d\omega = 2\pi$ (one gets it from Parseval)
- The mean value at t

$$\mu_t = \int_{-\infty}^{+\infty} t|g(t)|^2 dt$$

- The variance around μ_t

$$\sigma_t^2 = \int_{-\infty}^{+\infty} (t-\mu_t)^2 |g(t)|^2 dt.$$

Heisenberg uncertainty principle I

- The mean value in ω

$$\mu_\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega |\hat{g}(\omega)|^2 dt$$

- The variance around μ_ω

$$\sigma_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\omega - \mu_\omega)^2 |\hat{g}(\omega)|^2 dt.$$

Theorem. (Heisenberg (1927))

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$

Theoretical limit of simultaneous localization in time and frequency.

Heisenberg uncertainty principle II

Proposition (Heisenberg uncertainty principle for compactly supported functions).
If $0 \neq f \in C_c(\mathbb{R})$ then its Fourier transform $\hat{f}$ cannot have compact support as well.

Proof. If $f \in C_c(\mathbb{R})$ then $\hat{f}$ is an analytic function (as it has infinitely many derivatives suitably bounded). But a nonzero analytic function has at most countable number of zeros, hence it cannot have compact support.

Werner Heisenberg (1901-1976, Nobel prize in physics 1932)

Gabor atoms

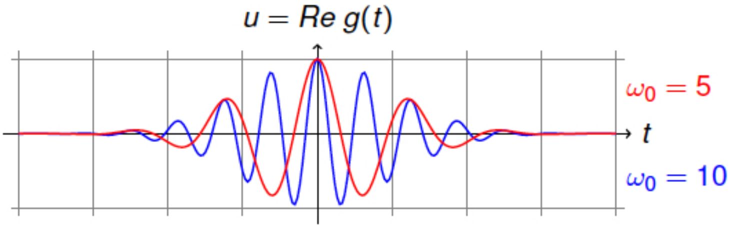
The minimal uncertainty $\sigma_t \cdot \sigma_\omega = \frac{1}{2}$ is obtain only for so-called Gabor atoms

$$g(t) = ae^{-bt^2}$$

for $a, b \in \mathbb{C}$ and their time-frequency shifts

$$g_{t_0,\omega_0}(t) = g(t - t_0) \cdot e^{i\omega_0 t},$$

obtained translating in time of t_0 and modulating in frequency of ω_0.



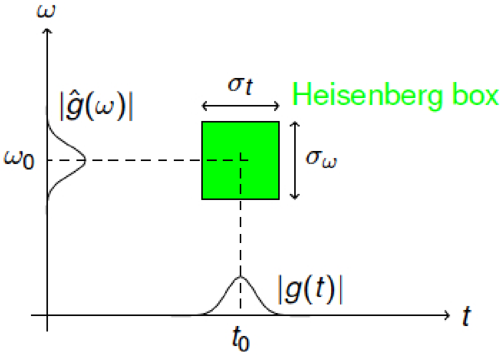
Dennis Gabor (1900-1979, Nobel prize in physics 1971)

Time-frequency support

The correlation of f and g

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega.$$

depends on f and $\hat{f}$ in (t, ω) where g and $\hat{g}$ are not too small.



Short-time Fourier Transform (STFT)

- Gabor introduced in 1946 the *short-time Fourier transform*

$$V_g(f)(t_0,\omega) = \int_{-\infty}^{+\infty} f(t)\overline{g(t-t_0)} \cdot e^{-i\omega_0 t} dt$$

  and proves the reconstruction of (audio signals) f by means of the inversion formula

$$f(t) \stackrel{?}{=} \frac{1}{\|g\|_2^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} V_g(f)(t_0,\omega) g(t-t_0) \cdot e^{i\omega_0 t} dt_0 d\omega_0,$$

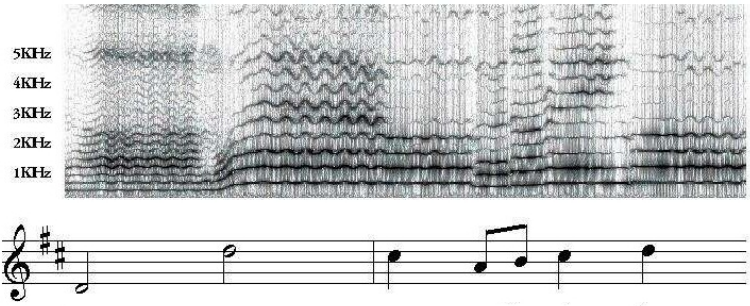
  it is in relationship with our way of perceiving sounds.
- He further conjectured that for $a = \Delta t, b = \Delta\omega > 0$

$$g_{ak,b\ell}(t) = g(t-ak)e^{ib\ell t}, \quad k,\ell \in \mathbb{Z}, \quad (a \cdot b \leq 2\pi(?))$$

  can build an orthonormal basis for $L^2(\mathbb{R})$, for which, by the Fourier Theorem, we would have

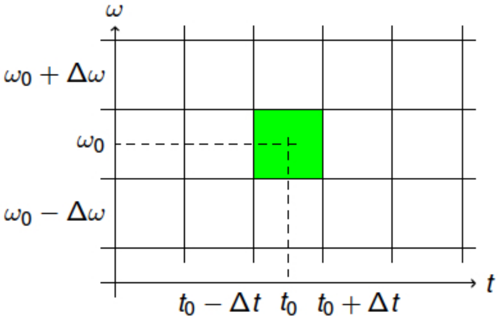
$$f = \sum_{k,\ell\in\mathbb{Z}} \langle f, g_{ak,b\ell}\rangle g_{ak,b\ell}.$$

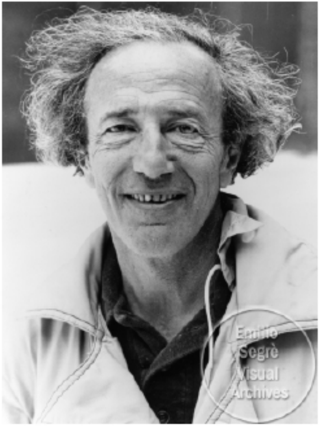
Time-Frequency Analysis



Balian and Low theorem

- Such an orthonormal basis would produce morally a covering of the time-frequency plane for translations g congruent with the "Heinsenberg box" of by integer multiples of $a = \Delta t, b = \Delta\omega > 0$.
- Roger Balian and Francis Low proved independently (around 1981) that an orthonormal basis cannot be obtained in this way by a function which is both localized and smooth.





Roger Balian (1933-) and Francis Low (1921-2007)

Heisenberg uncertainty principle III

Proposition (Weak uncertainty principle). *Let* $\|f\|_2^2 = \|g\|_2^2 = 1$*,* $U \subset \mathbb{R} \times \mathbb{R}$ *and* $C > 0$*, such that*

$$\int_U \mid V_g(f)(t,\omega) \mid^2 dt d\omega \geq C.$$

Then $|U| \geq C$*.*

Proof. By Cauchy-Schwarz $\mid V_g f(a,b) \mid \leq 1$. Hence

$$C \leq \int \int_U \mid V_g f(t,\omega) \mid^2 dt d\omega \leq \|V_g f\|_\infty^2 |U| \leq |U|. \quad .$$

- Because of uncertainty principle, if we use a "window" function g with large support, then $V_g(f)$ will have a good resolution in high frequency. In fact $\hat{g}$ will be highly localized.
- Vice versa if g is very localized, $V_g(f)$ will be have a good resolution in time and at low frequencies, but it would be blurred at high frequencies.

Jean Morlet (1931-2007)

Jean Morlet

- He worked in the '70s as geophysicist at the French company Elf-Aquitaine.
- He dealt with numerical processing of seismic signals in order to get information on geological layers.
- He found that the resolution at high frequency of the STFT is too rough to resolve the thin interfaces between layers.
- In 1981 he proposed *to dilate* (shorten the length) of a factor $a_0 > 1$ to translate of t_0 a mother window function ψ

  $$\psi\left(\frac{t - t_0}{a_0}\right),$$

  of constance shape of a *wavelet*.
- Balian suggested to Morlet the collaboration with Alexandre Grossmann of Marseille.

Alexandre Grossmann (1930-)

Again operators of translation, modulation, and dilation

We already introduced

$$T_{t_0} f(t) := f(t - t_0), \quad M_{\omega_0} f(t) = f(t) e^{i\omega t}, \quad t_0, \omega_0 \in \mathbb{R}.$$

With these operators we can define the STFT

$$V_g(f)(t_0, \omega_0) = \langle f, M_{\omega_0} T_{t_0} g \rangle = \int_{-\infty}^{+\infty} f(t) \overline{M_{\omega_0} T_{t_0} g(t)} dt.$$

Morlet proposed to introduce the dilation

$$D_a f(t) := \frac{1}{|a|^{1/2}} f(t/a), \quad a \in \mathbb{R}_+.$$

Continuous Wavelet Transform (CWT) - Time-Scale Analysis

So it was born the *continuous wavelet transform*

$$W_\psi(f)(t_0, a_0) = \langle f, D_{a_0} T_{t_0} \psi \rangle = \int_{-\infty}^{+\infty} f(t) \overline{D_{a_0} T_{t_0} \psi(t)} dt.$$

Theorem (Grossmann-Morlet (1984)). *One has the reproducing formula*

$$f(t) = \int_{\mathbb{R}_+ \times \mathbb{R}} W_\psi(f)(t_0, a_0) D_{a_0} T_{t_0} \psi(t) \frac{da_0}{a_0} dt_0.$$

- Grossmann recognized that the transformation proposed by Morlet as a "coherent state" of Lie group of affine motions $t \to a_0 t + t_0$, for $a_0 > 0$.
- The transformation was experimentally studied by Erik W. Aslaksen and John R. Klauder (1968/1969) also in quantum mechanics!

Continuous Wavelet Transform (CWT) - Time-Scale Analysis

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009

DECOMPOSITION OF HARDY FUNCTIONS INTO SQUARE INTEGRABLE WAVELETS OF CONSTANT SHAPE*

A. GROSSMANN† AND J. MORLET‡

Abstract. An arbitrary square integrable real-valued function (or, equivalently, the associated Hardy function) can be conveniently analyzed into a suitable family of square integrable wavelets of constant shape, (i.e. obtained by shifts and dilations from any one of them.) The resulting integral transform is isometric and self-reciprocal if the wavelets satisfy an "admissibility condition" given here. Explicit expressions are obtained in the case of a particular analyzing family that plays a role analogous to that of coherent states (Gabor wavelets) in the usual L_2-theory. They are written in terms of a modified Γ-function that is introduced and studied. From the point of view of group theory, this paper is concerned with square integrable coefficients of an irreducible representation of the nonunimodular $ax+b$-group.

1. Introduction.

1.1. It is well known that an arbitrary complex-valued square integrable function $\psi(t)$ admits a representation by Gaussians, shifted in direct and Fourier transformed space. If $g(t)=2^{-1/2}\pi^{-3/4}e^{-t^2/2}$ and t_0, ω_0 are arbitrary real, consider

$$g^{(t_0,\omega_0)}(t)=e^{-i\omega_0 t_0/2}e^{i\omega_0 t}g(t-t_0) \tag{1.1}$$

Grossmann-Morlet, SIAM J. MATH. ANAL., 1984)

Series of wavelets

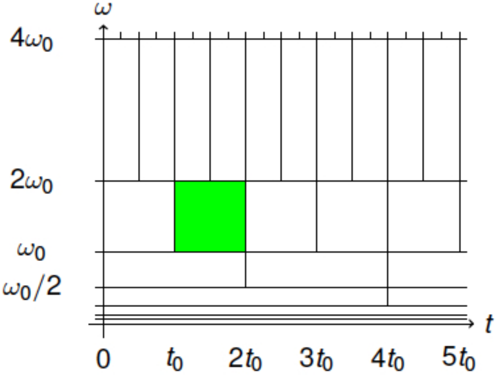
- For a dilation factor $s > 1$, Morlet searches ways of approximating the double integral $\mathbb{R}_+ \times \mathbb{R}$ by means of Riemann series of the type

$$f(t) \stackrel{?}{=} \sum_{j,k \in \mathbb{Z}} w_{jk}(f) s^{j/2} \psi(s^j t - k),$$

  where $D_{1/s, k/s^j} \psi = s^{j/2} \psi(s^j t - k)$.
- How to determine $w_{jk}(f)$ numerically?
- How large can s be taken? Can the Shannon limit $s = 2$ be possibile?

Dyadic covering of the time-frequency(scale) plane

- $s = 2$ corresponds to a dyadic covering of the time-frequency plabe by dilation and translations cof the Heisenberg box of ψ;
- one considers "shorter" times for higher frequencies.



Yves Meyer (1939-, Abel Prize 2017)

Calderón identity

- Yves Meyer recognizes that the reproducing identity by Grossmann-Morlet is the reproducing formula by Alberto Calderón (1964) studied in the context of singular integral operators:
$$f = \int_0^\infty Q_a(Q_a^* f)\frac{da}{a}$$
valid for all $f \in L^2(\mathbb{R})$.
- Here $\psi \in L^2(\mathbb{R})$ and one assumes
$$\int_0^\infty |\hat{\psi}(a\omega)|^2 \frac{da}{a} = 1$$
for almost every ω.
- The operator $Q_a : f \to \psi_a * f$ is a convolution of f with $\psi_a(t) = \frac{1}{a}\psi(\frac{t}{a})$, and Q_a^* is its adjoint operator.

The intuition

Yves Meyer:

“I recognized Calderón’s reproducing identity and I could not believe that it had something to do with signal processing.

I took the first train to Marseilles where I met Ingrid Daubechies, Alex Grossmann, and Jean Morlet. It was like a fairy tale.

This happened in 1984. I fell in love with signal processing. I felt I had found my homeland, something I always wanted to do”

The Marseille group: Ingrid Daubechies, Alex Grossmann, and Jean Morlet

Painless nonorthogonal expansions[a)]

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In a Hilbert space $\mathscr{H}$, discrete families of vectors $\{h_j\}$ with the property that $f = \Sigma_j \langle h_j | f \rangle h_j$ for every f in $\mathscr{H}$ are considered. This expansion formula is obviously true if the family is an orthonormal basis of $\mathscr{H}$, but also can hold in situations where the h_j are not mutually orthogonal and are "overcomplete." The two classes of examples studied here are (i) appropriate sets of Weyl–Heisenberg coherent states, based on certain (non-Gaussian) fiducial vectors, and (ii) analogous families of affine coherent states. It is believed, that such "quasiorthogonal expansions" will be a useful tool in many areas of theoretical physics and applied mathematics.

I. INTRODUCTION

A classical procedure of applied mathematics is to store some incoming information, given by a function $f(x)$ (where x is a continuous variable, which may be, e.g., the time) as a discrete table of numbers $\langle g_j | f \rangle = \int dx\, g_j(x) f(x)$ rather than in its original (sampled) form. In order to have a mathematical framework for all this, we shall assume that the possible functions f are elements of a Hilbert space $\mathscr{H}$ [we take here $\mathscr{H} = L^2(\mathbf{R})$]; the functions g_j are also assumed to be elements of this Hilbert space.

One can, of course, choose the functions g_j so that the family $\{g_j\}$ ($j \in J$, J a denumerable set) is an orthonormal basis of $\mathscr{H}$. The decomposition of f into the g_j is then quite straightforward: one has

$$J = \{(n,m);\ n,m \in \mathbf{Z},\ \text{the set of integers}\},$$

$$g_{n,m}(x) = \begin{cases} h_n(x - ma), & \text{for } ma < x < (m+1)a, \\ 0, & \text{otherwise.} \end{cases}$$

If now the function f undergoes a local change, confined to the interval $[ka, la]$, only the numbers $\langle g_{n,m} | f \rangle$ with $k < m < l - 1$ will be affected, reflecting the locality of the change. This choice for the g_j also has, however, its drawbacks: some of the functions g_j are likely to be discontinuous at the edges of the intervals, thereby introducing discontinuities in the analysis of f, which need not have been present in f itself. This is particularly noticeable if one takes the following natural choice for the h_n:

$$h_n(x) = a^{-1/2} e^{i2\pi x/a}.$$

Frames in Hilbert spaces

Let $\mathcal{H}$ be a separable Hilbert space.

Definition. *A set* $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is a frame for* $\mathcal{H}$ *if there exist* $A, B > 0$ *such that*

$$A \cdot \|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, g_n\rangle|^2 \leq B \cdot \|f\|^2, \quad \forall f \in \mathcal{H}.$$

An orthonormal basis is a frame with $A = B = 1$ by Parseval identity:

$$\|f\|^2 = \sum_{n\in\mathbb{N}} |\langle f, g_n\rangle|^2.$$

By the Fourier Theorem the operator $f \to \sum_{n\in\mathbb{N}} \langle f, g_n\rangle g_n$ is the identity, i.e., $f = \sum_{n\in\mathbb{N}} \langle f, g_n\rangle g_n$.

Exercise. If $\{g_n\}_{n\in\mathbb{N}}$ is a *thight* frame with $A = B = 1$ and if $\|g_n\| = 1$ for all n then $\{g_n\}_{n\in\mathbb{N}}$ is an orthonormal basis.
In general a frame is **not** orthonormal and in general its subsets are not linearly independent.

Frame operator

The *frame operator* is defined by $S : \mathcal{H} \to \mathcal{H}$

$$Sf = \sum_{n\in\mathbb{N}} \langle f, g_n \rangle g_n.$$

It does not coincide with the identity, but the frame condition implies that S is positive, self-adjoint, and **invertible**. Hence, one has the identities

$$f = SS^{-1}f = \sum_{n\in\mathbb{N}} \langle f, S^{-1}g_n \rangle g_n = S^{-1}Sf = \sum_{n\in\mathbb{N}} \langle f, g_n \rangle S^{-1}g_n.$$

The set $\{\tilde{g}_n = S^{-1}g_n\}_{n\in\mathbb{N}}$ is again a frame, the so-called *canonical dual* frame of $\{g_n\}_{n\in\mathbb{N}}$ with corresponding frame operator S^{-1}.

Example pf frame in in finite dimensions

Consider $\mathcal{H} = \mathbb{R}^2$, $f = (-1, 3)$ and $g_0 = (1, -1)$, $g_1 = (0, 1)$, $g_2 = (1, 1)$. The frame coefficients $c_n = \langle f, g_n \rangle$ are given by

$$\{c_n\}_{n=0,1,2} = \{-4, 3, 2\}$$

and its canonical dual is $\tilde{g} = \{(\frac{1}{2}, -\frac{1}{3}), (0, \frac{1}{3}), (\frac{1}{2}, \frac{1}{3})\}$ which gives the reconstruction of f as:

$$f = \sum_{i=0}^{2} c_n \tilde{g}_n = (-2, \frac{4}{3}) + (0, 1) + (1, \frac{2}{3}) = (-1, 3).$$

Do you remember Balian and Low?

Given $g \in L^2(\mathbb{R})$, let $a, b > 0$, and we say (g,a,b) that *generates a Gabor frame* for $L^2(\mathbb{R})$ if $\{M_{bm} T_{an} g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

The function g is called *Gabor atom*.

Balian and Low proved (1981) that there does not exist Gabor frames which are orthonormal basis if the Gabor atom is localized and smooth.

Gabor Frames

Theorem (Necessary condition). *For* $g \in L^2(\mathbb{R})$, $a, b > 0$, *if* (g, a, b) *generates a Gabor frame for* $L^2(\mathbb{R})$, *then* $ab \leq 2\pi$.

Teorema (Sufficient condition). *Let* $g \in L^2(\mathbb{R})$ *and* $a, b > 0$ *such that:*

(i) *there exists A,B s.t.* $0 < A \leq \sum_{n\in\mathbb{Z}} \mid g(t-na) \mid^2 \leq B < \infty$ *q.o.*

(ii) g *has compact support, with* $\operatorname{supp}(g) \subset I \subset \mathbb{R}$, *with* I *interval of length* $1/b$

Then (g, a, b) *generates a Gabor frame for* $L^2(\mathbb{R})$ *with frame bounds* $b^{-1}A, b^{-1}B$.

Gabor Frames

Proof. Fixed n, we note that the function $f(t)T_{na}\bar{g}(t)$ has support in $I_n = I - na = \{t - na : t \in I\}$, of length $1/b$. From (i) g in bounded, hence $fT_{na}\bar{g} \in L^2(I_n)$. The set

$$\{b^{1/2}e^{2\pi imb\chi_{I_n}}\}_{m\in\mathbb{Z}} = \{b^{1/2}M_{mb}\chi_{I_n}\}_{m\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(I_n)$, hence by the Fourier Theorem

$$\sum_{m\in\mathbb{Z}} |\langle fT_{na}\bar{g}, M_{mb}\chi_{I_n}\rangle|^2 = b^{-1}\int |f(t)|^2 |g(t-na)|^2\, dt.$$

Hence :

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} |\langle f, M_{mb}T_{-na}g\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} |\langle fT_{na}\bar{g}, M_{mb}\chi_{I_n}\rangle|^2 \\
&= b^{-1}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} |f(t)|^2 |g(t-na)|^2\, dt \\
&= b^{-1}\int_{\mathbb{R}} |f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-na)|^2\, dt.
\end{aligned}$$

Finite dimensional Gabor frames

Let us consider $a, b, L \in \mathbb{N}$ such that $a|L$ e $b|L$ and $a \cdot b \leq L$. Set $N = L/a$ e $M = L/b$.

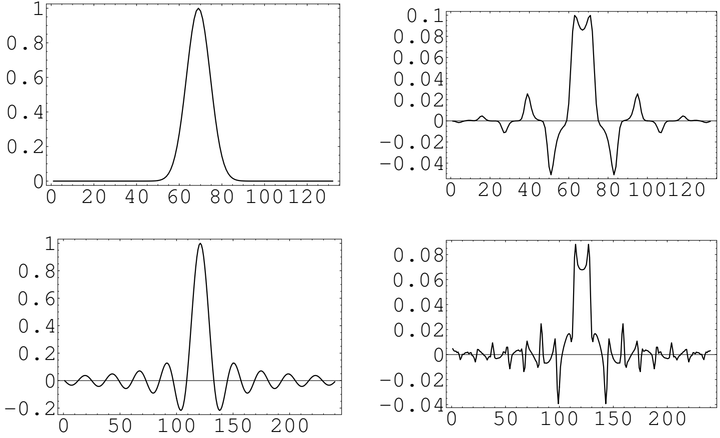
Then one defines the discrete Gabor frame

$$\mathbf{g}_{m,n} = M_{\frac{mb}{L}} T_{an}\mathbf{g}, \quad m = 0, ..., M-1, \quad n = 0, ..., N-1, \tag{1}$$

where $\mathbf{g} \in \mathbb{Z}_L$. Notice that $N \cdot M \geq L$.

Frames for $\ell^2(\mathbb{Z}_L)$ of the type

$$\mathcal{G}(\mathbf{g}, a, b) = \{\mathbf{g}_{m,n}\}_{m=0,\dots,M-1, n=0,\dots,N-1}.$$



Dual Gabor atoms. On the top the Gaussian with corresponding dual for $L = 132$, $a = b = 11$, and redundancy $L/(ab) = 1.09$. On the bottom the cadinal sin and respective dual for $L = 240$, $a = b = 15$, and redundancy $L/(ab) = 1.06$.

Frame frames?

Let us consider sets of the type

$$(\psi, a_0, b_0) := \{a_0^{-m/2}\psi(a_0^{-m}x - nb_0) : m, n \in \mathbb{Z}\}$$

where $\psi \in L^2(\mathbb{R})$ and $a_0, b_0 > 0$.

As a notation

$$\psi_{m,n}(x) := a_0^{-m/2}\psi(a_0^{-m}x - nb_0) = D_{a_0^m} T_{b_0 n}\psi.$$

Necessary condition: to be wave-like!!

Theorem, *If* (ψ, a_0, b_0) *defines a frame for* $L^2(\mathbb{R})$ *with constants* $A, B > 0$ *then*

$$\frac{b_0 \ln a_0}{2\pi} A \leq \int_0^\infty |\omega|^{-1} |\hat{\psi}(\omega)|^2 d\omega \leq \frac{b_0 \ln a_0}{2\pi} B,$$

and

$$\frac{b_0 \ln a_0}{2\pi} A \leq \int_{-\infty}^0 |\omega|^{-1} |\hat{\psi}(\omega)|^2 d\omega \leq \frac{b_0 \ln a_0}{2\pi} B,$$

Sufficient condition

Theorem. *If* ψ *and* a_0 *are such that*

$$\inf_{1 \leq |\omega| \leq a_0} \sum_{m=-\infty}^{\infty} |\hat{\psi}(a_0^m \omega)|^2 > 0,$$

$$\sup_{1 \leq |\omega| \leq a_0} \sum_{m=-\infty}^{\infty} |\hat{\psi}(a_0^m \omega)|^2 < \infty,$$

and if

$$\beta(s) = \sup_{\omega} \sum_{m=-\infty}^{\infty} |\hat{\psi}(a_0^m \omega)||\hat{\psi}(a_0^m \omega + s)|$$

decays at least as $(1+|s|)^{1+\varepsilon}$*, with* $\varepsilon > 0$*, then there exists* $b^0 > 0$ *such that* (ψ, a_0, b_0) *is a frame for* $L^2(\mathbb{R})$ *for all* $0 < b_0 \leq b^0$*.*

The conditions are fulfilled as soon as $|\hat{\psi}(\omega)| \leq C|\omega|^\alpha(1+|\omega|)^{-\gamma}$ *with* $\alpha > 0$*,* $\gamma > \alpha + 1$*.*

Orthonormal wavelets

- Let $\psi \in L^2(\mathbb{R})$. The functions

$$\psi_j(t) = 2^{j/2}\psi(2^j t)$$

  are dilated of ψ of a factor $a = 1/2^j$, and normalized.
- The functions

$$\psi_{j,k}(t) = \psi_j(t - 2^{-j}k),$$

  are translated $2^{-j}k$ of ψ_j.
- We say ψ is properly a wavelet if

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), \quad j, k \in \mathbb{Z},$$

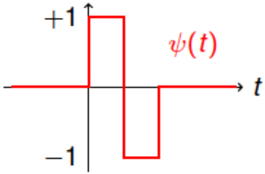
  is an orthonormal basis for $L^2(\mathbb{R})$.

Haar basis

- The most simple wavelet was proposed by Alfréd Haar (1909):

$$\psi(t) = \begin{cases} +1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \\ 0, & \text{otherwise} . \end{cases}$$

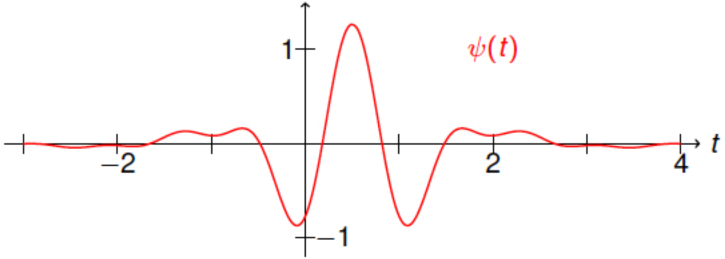
- Discontinuous. Localized in time but not in frequency



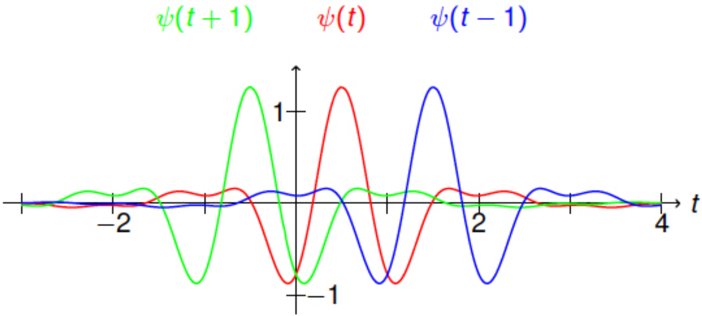
Alfred Haar (1885-1933)

Against Balian and Low: localized and smooth Meyer wavelets

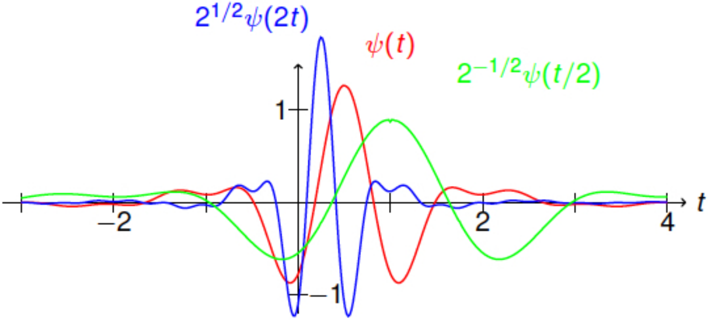
1985: Yves Meyer constructed (discovered?) a C^∞ wavelet with fast decay



Translation ...



… and dilations



Orthonormal Bases of Compactly Supported Wavelets

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Abstract

We construct orthonormal bases of compactly supported wavelets, with arbitrarily high regularity. The order of regularity increases linearly with the support width. We start by reviewing the concept of multiresolution analysis as well as several algorithms in vision decomposition and reconstruction. The construction then follows from a synthesis of these different approaches.

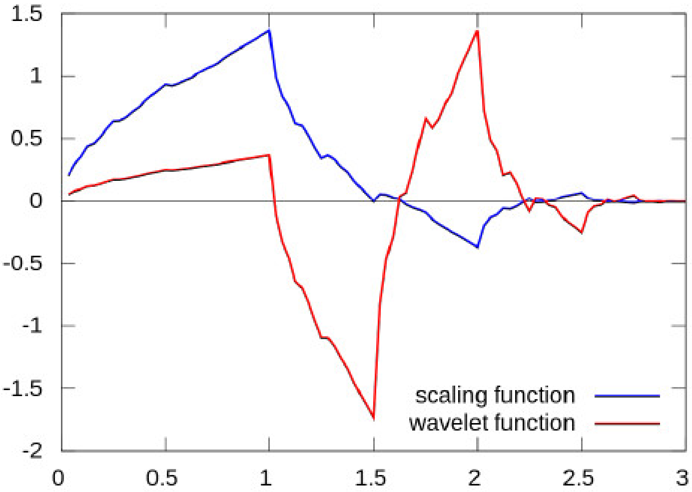
1. Introduction

In recent years, families of functions $h_{a,b}$,

$$h_{a,b}(x) = |a|^{-1/2} h\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbb{R},\ a \neq 0, \tag{1.1}$$

generated from one single function h by the operation of dilations and translations, have turned out to be a useful tool in many different fields of mathematics, pure as well as applied. Following Grossmann and Morlet [1], we shall call such families "wavelets".

Techniques based on the use of translations and dilations are certainly not new. They can be traced back to the work of A. Calderón [2] on singular integral operators, or to renormalization group ideas (see [3]) in quantum field theory and statistical mechanics. Even in these two disciplines, however, the explicit intro-



Scaling function ϕ and Daubechies wavelet ψ.

Applications of wavelets

- Generally the DWT is used for coding and compression (JPEG2000), while the CWT is used for signal analysis.
- Wavelet transform used (instead of Fourier transform): molecular dynamics, calculus ab initio, astrophysics, geophysics, optics, turbolence, quantum mechanics ....
- Applications: image processing, blood pressure, heart beat and ECG, DNA analysis, protein analysis, climatology, speach recognition, computational graphics, multifractal analysis ...
- Wavelets are playing a crucial role in the work of the Fields medalist Martin Hairer in this work on "regularity of structure", that provides an algebraic framework allowing to describe functions and/or distributions via a kind of "jet" or local Taylor expansion around each point. In particular, this allows to describe the local behaviour not only of functions but also of large classes of distributions.

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