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Anna M. Gil-Lafuente · José M. Merigó
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Editors

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Preface

The 24th International Conference of the 'Forum for Interdisciplinary Mathematics (FIM)' entitled *Applied Mathematics and Computational Intelligence* took place in Barcelona, Spain, November 18–20, 2015, and was co-organized by the *University of Barcelona* (Spain), the *Spanish Royal Academy of Economic and Financial Sciences* (Spain), and the *Forum for Interdisciplinary Mathematics* (India).

The Forum is a registered trust in India. It is, in effect, an India-based international society of scholars working in mathematical sciences and its partner areas (a partner area is defined as one where some knowledge of mathematical sciences is desirable to carry out research and development). The society was incepted in 1975 by a group of University of Delhi intellectuals led by Professor Bhu Dev Sharma. In 2015, the FIM is running into 42th year of active standing. Right from the beginning, FIM had the support and association of India's great mathematicians and also users of mathematics from different disciplines in the country and abroad.

The Forum began holding conferences right from the beginning. It started at the national level. The first conference was held in 1975 at 'Calcutta University, Calcutta (India).' The second conference was held at Rajasthan University, Jaipur (India), in 1976. With the General Secretary, Professor Bhu Dev Sharma taking-up a chair abroad, the holding of conferences at the national had a period of interruption. Later, it was decided to hold international conference every year alternating between India and outside.

The process of holding international conferences began in 1995 and is continuing unabated. In such a way, this 24th International Conference entitled *Applied Mathematics and Computational Intelligence* continues and extends the series of international conferences organized by FIM. Previous international conferences were held at Calcutta University, Calcutta, India (July 1995); Rajasthan University, Jaipur, India (June 1996); University of Southern Maine, USA (July 1997); Banaras Hindu University, India (December 1997); University of Mysore, India (December 1998); University of South Alabama, USA (December 1999); Indian Institute of Technology, Mumbai, India (December 2000); University of Wollongong, Australia (December 2001); University of Allahabad, Allahabad, India (December 2002); University of Southern Maine, USA (October 2003); Institute of

Engineering & Technology, Lucknow, India (December 2004); Auburn University, Auburn, AL, USA (December 2005); Tomar Polytechnic Institute, Tomar, Portugal (September 2006); IIT, Madras, Chennai, India (January 2007); University of Science & Technology of China, Shanghai (May 2007); Memphis University, USA (May 2008); University of West Bohemia, Czech Republic (May 2009); Jaypee University of Information Technology, Waknaghat, HP, India (August 2009); Patna University, Patna, Bihar, India (December 2010); Alcorn University, Montreal, Canada (June 2011); Panjab University, Chandigarh, India (December 2012); Waseda University, City of Kitakyushu, Japan (November 2013); NITK, Surathkal, Karnataka, India (December 2014).

Starting with the 8th International Conference at the University of Wollongong, Australia, the Forum has started organizing and funding a symposium solely for the purpose of encouraging and awarding young researchers consisting of new Ph.D. awardees and aspirants, also known as '***Professor R. S. Varma Memorial Student Competition***' (RSVMSC). These awards are well structured, critiqued, and judged by the leading scholars of various fields, and at the conclusion of which a certificate and cash award (presently Rs. 25,000.00) are provided to the winners. In a very short time, RSVMSC has become popular among young investigators in India as FIM has appreciably realized their participation at its conferences. Among other activities, FIM publishes the following scientific publications:

1. Journal of Combinatorics, Information and System Sciences
2. Research Monographs and Lecture Notes with Springer

This international conference aims to bring together the foremost experts from different disciplines, young researchers, academics, and students to discuss new research ideas and present recent advances in *interdisciplinary mathematics, statistics, computational intelligence, economics, and computer science.*

FIM-AMCI-2015 received a large number of papers from all over the world. They were carefully reviewed by experts, and only high-quality papers were selected for oral or poster presentation during conference days. This book comprises a selection of papers presented at the conference. We believe it is a good example of the excellent work of the associates and the significant progress about this line of research in recent times.

This book is organized according to four general tracks of the conference: Mathematical Foundations, Computational Intelligence and Optimization Techniques, Modeling and Simulation Techniques, Applications in Business and Engineering.

Finally, we would like to express our sincere thanks to all the plenary speakers, authors, reviewers, and participants at the conference, organizations, and institutional sponsors for their help, support, and contributions to the success of the event.

The AMCI 2015-FIM XXIV Conference is supported by:

November 2017

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Mathematical Foundations

Best Proximity Point Theorems for Generalized Contractive Mappings

S. Arul Ravi(✉) and A. Anthony Eldred

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Abstract. Recently, J. Calallero (Fixed Point Theory and Applications 2012, 2012:231) observed best proximity results for Geraghty-contractions by using the P-property. In this paper we introduce the notion of Boyd and wong result and Generalized weakly contractive mapping and show the existence and uniqueness of the best proximity point of such contractions in the setting of a metric space.

Keywords: Best proximity point · P-property
Boyd and Wong contraction · Generalized weakly contractive

1 Introduction

In nonlinear functional analysis, fixed point theory and best proximity point theory play an important role in the establishment of the existence of a certain differential and integral equations. As a consequence fixed point theory is very much useful for various quantitative sciences that involve such equations. The most remarkable paper in this field was reported by Banach in 1922 [3]. In his paper Banach proved that every contraction in a complete metric space has a unique fixed point. Following this paper many have extended and generalized this remarkable fixed point theorem of Banach by changing either the conditions of the mappings or the construction of the space. In particular, one of the notable generalizations of Banach fixed point theorem was introduced by Geraghty [7].

Theorem 1 [7]**.** Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Suppose that there exists $\beta : [0, \infty) \to (0, 1)$ satisfying if f satisfies the following inequality:

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for any } x, y \in X,$$

then f has unique fixed point.

It is natural that some mapping, especially non-self mappings defined on a complete metric space (X, d), do not necessarily possess a fixed point, that is $d(x, f(x)) > 0$ for all $x \in X$. In such situations it is reasonable to search for

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the existence and uniqueness of a point $x^* \in X$ such that $d(x^*, f(x^*))$ is an approximation of an $x \in X$ such that $d(x, f(x)) = 0$.
In other words one speculates to determine an approximate solution x^* that is optimal in the sense that the distance between x^* and $f(x^*)$ is minimum. Here the point x^* is called the best proximity point. In this paper we generalize and improve certain results of Caballero et al. [6].

2 Prelimineries

Let (X, d) be a metric space and A and B be nonempty subsets of a metric space X. A mapping $f : A \to B$ is called a k-contraction if there exists $k \in (0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y) \text{ for any } x, y \in A.$$

It is clear that a k-contraction coincides with the celebrated Banach fixed point theorem if one takes $A = B$ where A is a complete subset of X.

Let A and B be nonempty subsets of a metric space (X, d). we denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B), \quad forsome \quad y \in B\}$$
$$B_0 = \{y \in B : d(x, y) = d(A, B), \quad forsome \quad x \in A\} \text{ where}$$
$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 1 [8]**.** Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ $d(x_1, y_1) = d(A, B)$ and $d(x_2, y_2) = d(A, B)$ implies that $d(x_1, x_2) = d(y_1, y_2)$.

It can be easily seen that for any nonempty subset A of (X, d), the pair (A, A) has the P-property. In [11] Sankarraj has proved that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies P-property.

Now we introduce the class of those functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition: $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

Definition 2 [6]**.** Let A and B be nonempty subsets of a metric space (X, d). A mapping $f : A \to B$ is said to be a Geraghty contraction if there exists $\beta \in F$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for any } x, y \in A.$$

Remark 1. Notice that since $\beta : [0, \infty) \to (0, 1)$, we have

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) < d(x, y) \text{ for any } x, y \in A \text{ with } x \neq y.$$

Theorem 2 [6]**.** Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $f : A \to B$ be a continuous Geraghty contraction satisfying $f(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.

We would like to extend the result of Caballero and explore the best proximity point based on the well known result of Boyd and Wong [5].

Theorem 3 [1]**.** Let X be a complete metric space and let $f : X \to X$ satisfy

$$d(f(x), f(y)) \leq \psi(d(x, y))$$

where $\psi : R^+ \to R^+$ is upper semi-continuous from the right and satisfies $0 \leq \psi(t) < t$. Then f has a unique fixed point. Further if $x_0 \in X$ and $x_{n+1} = f(x_n)$, then $\{x_n\}$ converges to the fixed point.

A mapping $f : X \to X$ is said to be contractive if

$$d(f(x), f(y)) < d(x, y) \quad foreach \quad x, y \in X \quad with \quad x \neq y. \tag{1}$$

3 Main Results

Theorem 4 Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $f : A \to B$ be such that $f(A_0) \subseteq B_0$. Suppose

$$d(f(x), f(y)) \leq \psi(d(x, y)) \text{ for each } x, y \in A,$$

where $\psi : R^+ \to [0, \infty)$ is upper semi-continuous from the right satisfies $0 \leq \psi(t) < t$ for $t > 0$. Furthermore the pair (A, B) has the P-property. Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.

Proof. Regarding that A_0 is nonempty, we take $x_0 \in A_0$.

Since $f(x_0) \in f(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, f(x_0)) = d(A, B)$. Analogously regarding the assumption $f(x_1) \in f(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that $d(x_2, f(x_1)) = d(A, B)$.

Recursively we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, f(x_n)) = d(A, B) \quad forany \quad n \in N \tag{2}$$

Since (A, B) has the P-property we derive that

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \quad forany \quad n \in N \tag{3}$$

If there exists $n_0 \in N$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is completed. Indeed

$$0 = d(x_{n_0}, x_{n_0+1}) = d(f(x_{n_0-1}), f(x_{n_0})) \tag{4}$$

and consequently $f(x_{n_0-1}) = f(x_{n_0})$.

On the other hand due to 2 we have $d(x_{n_0}, f(x_{n_0-1})) = d(A, B)$.

Therefore we conclude that

$$d(A, B) = d(x_{n_0}, f(x_{n_0-1})) = d(x_{n_0}, f(x_{n_0})) \tag{5}$$

For the rest of the proof we suppose that $d(x_n, x_{n+1}) > 0$ for any $n \in N$.

Since f is contractive, for any $n \in N$, we have that

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}) \tag{6}$$

consequently $\{d(x_n, x_{n+1})\}$ is monotonically decreasing sequence and bounded below and so we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$ exists.

Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0$.

Assume that $r > 0$. Then from 1 we have $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ which implies that $r \leq \psi(r) \Rightarrow r = 0$.

That is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{7}$$

Notice that since $d(x_{n+1}, f(x_n)) = d(A, B)$ for any $n \in N$, for fixed $p, q \in N$, we have $d(x_p, f(x_{p-1})) = d(x_q, f(x_{q-1})) = d(A, B)$ and since (A, B) satisfies the P-property, $d(x_p, x_q) = d(f(x_{p-1}), f(x_{q-1}))$.

In what follows, we prove that $\{x_n\}$ is cauchy sequence.

On the contrary, assume that we have

$$\epsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0 \tag{8}$$

Then there exists $\epsilon > 0$, such that for any $k \in N$, there exists $m_k > n_k \geq k$, such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{9}$$

Furthermore assume that for each k, m_k is the smallest number greater than n_k for which 9 holds. In view of 6, there exists k_0 such that $k \geq k_0$ implies that $d(x_k, x_{k+1}) \geq \epsilon$.

For such k, we have

$$\begin{aligned}
\epsilon &\leq d(x_{m_k}, x_{n_k}) \\
&\leq d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{n_k}) \\
&\leq d(x_{m_k}, x_{m_{k-1}}) + \epsilon \\
&\leq d(x_k, x_{k-1}) + \epsilon.
\end{aligned}$$

This proves $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.

On the other hand

$$\begin{aligned}
d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) \\
&\leq 2d(x_k, x_{k+1}) + \psi(d(x_{m_k}, x_{n_k})).
\end{aligned}$$

Since $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$,
the above inequality yields

$$\epsilon \leq \limsup_{m,n\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{m,n\to\infty} \psi(d(x_{m_k}, x_{n_k})) \leq \psi(\epsilon).$$

It follows that $\epsilon \leq \psi(\epsilon)$, a contradiction.

Therefore $\{x_n\}$ is a cauchy sequence.

Since $\{x_n\} \subset A$ and A is closed subset of the complete metric space (X, d), we can find $x^* \in A$ such that $x_n \to x^*$.

Since the mapping is contractive and continuous, we have $f(x_n) \to f(x^*)$.

This implies that $d(x_n, x_{n+1}) \to d(x^*, f(x^*))$.

Taking into consideration that the sequence $\{d(x_{n+1}, f(x_n))\}$ is a constant sequence with the value $d(A, B)$, we deduce that $d(x^*, f(x^*)) = d(A, B)$.

This means that x^* is a best proximity point of f.

This proves the existence of our result.

For the uniqueness, suppose that x_1 and x_2 are two best proximity points of f with $x_1 \neq x_2$. This means that $d(x_i, f(x_i)) = d(A, B)$ for $i = 1, 2$.

Using the P-property, we have $d(x_1, x_2) = d(f(x_1), f(x_2))$

Using the fact that f is contractive and continuous, we have

d(x$_1, x_2)$
$= d(f(x_1), f(x_2))$
$\leq \psi(d(x_1, x_2))$
$< d(x_1, x_2)$ which is a contradiction.

Therefore $x_1 = x_2$.

This completes the proof.

In the following result we introduce the concept of generalized weakly contractive mapping and find best proximity point based on the work of Choudhury [4].

Definition 3 [10]**.** A mapping $f : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if for any $x, y \in X$, then

$$d(f(x), f(y) \leq d(x, y) - \phi(d(x, y)) \tag{10}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1 - k)t$, where $0 < k < t$, a weak contraction reduces to a Banach contraction.

In [2] Alber and Guerre proved that if $f : \Omega \to \Omega$ is a weakly contractive self-map, where Ω is a closed convex subset of a Hilbert space, then f has a unique fixed point in Ω. Later, in [10] Rhodes proved that the existence of a unique fixed point for a weakly contractive self-map could be achieved even in a complete metric space setting.

Definition 4 [12]**.** Let A, B be nonempty subsets of a metric space X.

A map $f : A \to B$ is said to be weakly contractive mapping if

$$d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)), \text{ for all } x, y \in A,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty), \psi(0) = 0$ and $\lim_{n\to\infty} \psi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

Note that

$$d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)) < d(x, y) \text{ if } x, y \in A \text{ with } x \neq y.$$

That is f is a contractive map. The notion called the P-property was introduced in [11] and was used to prove a extended version of Banach's contraction principle.

Theorem 5 [12]**.** Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $f : A \to B$ be a weakly contractive mapping satisfying $f(A_0) \subseteq B_0$. Assume that the pair (A, B) has the p-property. Then there exists a unique $x^* \in A$ such that $d(x^*, f(x^*)) = d(A, B)$.

Definition 5 [9]**.** A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering function if the following properties are satisfied:

(a) ψ is monotone increasing and continuous
(b) $\psi(t) = 0$ if and only if $t = 0$.

Definition 6 [4]**.** Let (X, d) be a metric space, f a self-mapping of X. We shall call f a generalized weakly contractive mapping if for all $x, y \in X$, then

$$\psi(d(f(x), f(y)) \leq \psi(m(x, y)) - \phi(\max\{d(x, y), d(y, f(y))\})$$

where

$$m(x, y) = \max\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\}$$

and ψ is an altering distance function also $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. A generalized weakly contractive mapping is more general than that satisfying $d(f(x), f(y)) \leq km(x, y)$ for some constant $0 \leq k < 1$ and is included in those mappings which satisfy

$$d(f(x), f(y)) < m(x, y).$$

Definition 7. Let A, B be nonempty subsets of a metric space X. A map $f : A \to B$ is said to be a generalized weakly contractive mapping if for all $x, y \in A$, then

$$\psi(d(f(x), f(y)) \leq \psi(m(x, y)) - \phi(\max\{d(x, y), d(y, f(y)) - d(A, B))\})$$

where

$$m(x, y) = \max\{d(x, y), d(x, f(x)) - d(A, B), d(y, f(y)) - d(A, B), \\ \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - d(A, B)\}.$$

A generalized weakly contractive mapping is more general than that satisfying $d(f(x), f(y)) \leq km(x, y)$ for some constant $0 \leq k < 1$ and is included in those mappings which satisfy

$$d(f(x), f(y)) < m(x, y).$$

Theorem 6. Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $f : A \to B$ be such that $f(A_0) \subseteq B_0$. Suppose

$$\psi(d(f(x), f(y)) \leq \psi(m(x, y)) - \phi(\max\{d(x, y), d(y, f(y)) - d(A, B))\}) \quad (11)$$

Furthermore the pair (A, B) has the p-property. Then there exists a unique x^* in A such that $d(x^*, f(x^*)) = d(A, B)$.

Proof. Choose $x_0 \in A$.

Since $f(x_0) \in f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, f(x_0)) = d(A, B)$.

Analogously regarding the assumption, $f(x_1) \in f(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that $d(x_2, f(x_1)) = d(A, B)$.

Recursively we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, f(x_n)) = d(A, B) \quad forany \quad n \in N \quad (12)$$

Claim: $d(x_n, x_{n+1}) \to 0$.

If $x_N = x_{N+1}$, then x_N is a best proximity point.

By the P-property, we have

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})).$$

Hence we assume that $x_n \neq x_{n+1}$ for all $n \in N$.

Since $d(x_{n+1}, f(x_n)) = d(A, B)$, from (11) we have for all $n \in N$

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(f(x_n), f(x_{n+1}))) \\
&\leq \psi(max\{d(x_n, x_{n+1}), d(x_n, f(x_n)) - d(A, B), d(x_{n+1}, f(x_{n+1})) - d(A, B), \\
&\quad \frac{1}{2}[d(x_n, f(x_{n+1})) + d(x_{n+1}, f(x_n))] - d(A, B)\}) \\
&\quad -\phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A, B)\}) \\
&\leq \psi(max\{d(x_n, x_{n+1}), d(x_n, f(x_n)) - d(A, B), d(x_{n+1}, f(x_{n+1})) - d(A, B), \\
&\quad \frac{1}{2}(d(x_n, f(x_{n+1}))) - d(A, B)\}) \\
&\quad -\phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A, B)\})
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2}(d(x_n, f(x_{n+1}))) - d(A, B) &\leq \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, f(x_{n+1}))) - d(A, B) \\
&\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A, B)\} \\
d(x_n, f(x_n)) - d(A, B) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, f(x_n)) - d(A, B) \\
&= d(x_n, x_{n+1})
\end{aligned}$$

It follows that

$$\begin{aligned}
\psi(d(f(x_n), f(x_{n+1})) &\leq \psi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A, B)\}) \\
&\quad -\phi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, f(x_{n+1})) - d(A, B)\})
\end{aligned}$$

$$\psi(d(x_{n+1},x_{n+2})) \leq \psi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}) \\ -\phi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}) \tag{13}$$

Suppose that $d(x_n,x_{n+1}) \leq d(x_{n+1},x_{n+2})$, for some positive integer n.

Then from 13 we have

$$\psi(d(x_{n+1},x_{n+2}) \leq \psi(d(x_{n+1},x_{n+2})) - \phi(d(x_{n+1},x_{n+2})),$$

that is

$$\phi(d(x_{n+1},x_{n+2})) \leq 0,$$

which implies that $d(x_{n+1},x_{n+2}) = 0$, contradicting our assumption.

Therefore $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ for any $n \in N$ and hence $\{d(x_n,x_{n+1})\}$ is monotone decreasing sequence of non-negative real numbers, hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. In view of the facts from 13 for any $n \in N$, we have

$$\psi(d(x_{n+1},x_{n+2})) \leq \psi(d(x_n,x_{n+1})) - \phi(d(x_n,x_{n+1})),$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ψ and ϕ we have $\psi(r) \leq \psi(r) - \phi(r)$ which implies $\phi(r) = 0$.

Hence

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \tag{14}$$

Next we show that $\{x_n\}$ is a cauchy sequence.

If otherwise there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for all positive integers $k, n_k > m_k > k$,

$$d(x_{m_k},x_{n_k}) \geq \epsilon \text{ and } d(x_{m_k},x_{n_k-1}) < \epsilon.$$

Now

$$\epsilon \leq d(x_{m_k},x_{n_k}) \leq d(x_{m_k},x_{n_k-1}) + d(x_{n_k-1},x_{n_k}).$$

that is

$$\epsilon \leq d(x_{m_k},x_{n_k}) < \epsilon + d(x_{n_k-1},x_{n_k}),$$

Taking the limit as $k \to \infty$ in the above inequality and using 14 we have

$$\lim_{k\to\infty} d(x_{m_k},x_{n_k}) = \epsilon \tag{15}$$

Again

$$d(x_{m_k},x_{n_k}) \leq d(x_{m_k},x_{m_k+1}) + d(x_{m_k+1},x_{n_k+1}) + d(x_{n_k+1},x_{n_k})$$

Taking the limit as $k \to \infty$ in the above inequalities and using 14 and 15 we have

$$\lim_{k\to\infty} d(x_{m_k+1},x_{n_k+1}) = \epsilon \tag{16}$$

Again

$$d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \text{ and}$$
$$d(x_{m_k}, x_{n_k+1}) \le d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})$$

Letting $k \to \infty$ in the above inequalities and using 14 and 15 we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon \tag{17}$$

Similarly

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon \tag{18}$$

For $x = x_{m_k}, y = y_{m_k}$, we have

$$\begin{aligned} d(x_{m_k}, f(x_{m_k})) - d(A,B) &\le d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, f(x_{m_k})) - d(A,B) \\ &= d(x_{m_k}, x_{m_k+1}) \end{aligned}$$

similarly

$$d(x_{n_k}, f(x_{n_k})) - d(A,B) = d(x_{n_k}, x_{n_k+1}).$$

Also

$$d(x_{m_k}, f(x_{n_k})) - d(A,B) = d(x_{m_k}, x_{n_k+1}) \text{ and}$$
$$d(x_{n_k}, f(x_{m_k})) - d(A,B) = d(x_{n_k}, x_{m_k+1}).$$

From 11 we have

$$\begin{aligned} \psi(d(x_{m_k+1}, x_{n_k+1})) &= \psi(d(f(x_{m_k}), f(x_{n_k}))) \\ &\le \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, f(x_{m_k})) - d(A,B), d(x_{n_k}, f(x_{n_k})) - d(A,B), \\ &\quad \frac{1}{2}[d(x_{m_k}, f(x_{n_k})) + d(x_{n_k}, f(x_{m_k})) - d(A,B)]\}) \\ &\quad -\phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, f(x_{n_k})) - d(A,B)\}) \\ &\le \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \\ &\quad \frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})]\}) \\ &\quad -\phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\}) \end{aligned}$$

It follows that

$$\begin{aligned} \psi(d(f(x_{m_k}), f(x_{n_k}))) &\le \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, f(x_{n_k+1})), \\ &\quad \frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})]\}) \\ &\quad -\phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, f(x_{n_k+1}))\}) \\ \psi(d(x_{m_k+1}, x_{n_k+1}) &\le \psi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\}) \\ &\quad -\phi(\max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, x_{n_k+1})\}) \end{aligned}$$

From 14, 15, 17, 18 and Letting $k \to \infty$ in the above inequalities and using the continuities of ψ and ϕ,
we have $\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon)$
which is contradiction by virtue of property of ϕ.

Hence $\{x_n\}$ is a cauchy sequence.

Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space (X, d), there exists x^* in A such that $x_n \to x^*$.

Putting $x = x_n$ and $y = x^*$ in 11 and since

$$d(x_n, f(x^*)) \le d(x_n, x^*) + d(x^*, f(x_n)) \text{ and}$$
$$d(x^*, f(x_n)) \le d(x^*, f(x^*)) + d(f(x^*), f(x_n))$$

we have

$$\begin{aligned}\psi(d(x_{n+1}, f(x^*)) - d(A,B) &\le \psi(d(f(x_n), f(x^*)))\\ &\le \psi(\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)) - d(A,B),\\ &\quad \frac{1}{2}[d(x_n, f(x^*)) + d(x^*, f(x_n))] - d(A,B)\})\\ &\quad -\phi(\max\{d(x_n, x^*), d(x^*, f(x^*)) - d(A,B)\})\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ψ and ϕ, we have $\psi(d(x^*, f(x^*)) - d(A,B)) \le \psi(d(x^*, f(x^*)) - d(A,B)) - \phi(d(x^*, f(x^*)) - d(A,B)$. Which implies that $d(x^*, f(x^*)) = d(A,B)$.

Hence x^* is a best proximity point of f.

For the uniqueness

Let p and q be two best proximity points of f and suppose that $p \ne q$.

Then putting $x = p$ and $y = q$ in (11) we obtain

$$\begin{aligned}\psi(d(f(p), f(q))) \le\ & \psi(\max\{d(p,q), d(p, f(p)) - d(A,B), d(q, f(q)) - d(A,B),\\ & \frac{1}{2}[d(p, f(q)) + d(q, f(p))] - d(A,B)\})\\ & -\phi(\max\{d(p,q), d(q, f(q)) - d(A,B)\})\end{aligned}$$

That is

$$\psi(d(p,q)) \le \psi(d(p,q)) - \phi(d(p,q))$$

which is contradiction by virtue of a property of ϕ.

Therefore $p = q$.

This completes the proof.

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The Method of Optimal Nonlinear Extrapolation of Vector Random Sequences on the Basis of Polynomial Degree Canonical Expansion

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Abstract. The given work is dedicated to the solving of important scientific and technical problem of forming of the method of the optimal (in mean-square sense) extrapolation of the realizations of vector random sequences for the accidental quantity of the known values used for prognosis and for various order of nonlinear stochastic relations. Prognostic model is synthesized on the basis of polynomial degree canonical expansion of vector random sequence. The formula for the determination of the mean-square error of the extrapolation which allows us to estimate the accuracy of the solving of the prognostication problem with the help of the introduced method is obtained. The block diagrams of the algorithms of the determination of the parameters of the introduced method are also presented in the work. Taking into account the recurrent character of the processes of the estimation of the future values of the investigated sequence the method is quite simple in calculating respect. The introduced method of extrapolation as well as the vector canonical expansion assumed as its basis doesn't put any essential limitations on the class of prognosticated random sequences (linearity, Markovian property, stationarity, scalarity, monotony etc.).

Keywords: Optimal nonlinear · Extrapolation · Vector random sequences
Polynomial canonical expansion

1 Introduction

The peculiarity of the wide range of applied problems in different spheres of science and techniques is the probabilistic nature of the investigated phenomenon or the presence of the influence of random factors on the investigated object as a result of what the process of changing of its state also takes probabilistic character. The objects of such a class which relate to the objects with randomly variable conditions of functioning (RVCF) are investigated, for example, during the solving of the problems

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of technical diagnostics [4], radiolocation, medical diagnostics [5], robotics and automation [13, 20], forecasting control of reliability [15], weather forecasting [17], information security, synthesis of the models of chemical kinetics, management of technological processes, motion control [14], etc. The characteristic peculiarity of these problems is the presence of the preliminary stage of gathering of the information about the object of investigation. Random character of external influence and coordinates (input and output) of the objects with RVCF under the conditions of sufficient statistic data volume determines the necessity and reasonability of the usage of deductive [10] methods of random sequences prognosis for their solving.

It is known that the most general extrapolation form for the solving of the problems of the prognosis is the mathematical model in the form of Kolmogorov-Gabor polynomial [9]. Such a model allows taking into account the accidental number of random sequence measurements and the order of degree nonlinearity. But its practical application is limited with significant difficulties connected with the forming of the large quantity of equations for the determination of the extrapolator parameters. Existing optimal methods which are used during the solving of applied problems are obtained for the definite classes of random sequences, in particular, the methods of Kolmogorov [12] and Wiener [21] are for stationary processes, Kalman's filter-extrapolator [11, 18] is for markovian random sequences, methods of Pugachev [19], Kudritsky [16] are for non-stationary gaussian sequences etc. It should be mentioned that their application allows to obtain optimal results only for the sequences with definite a priori known characteristics.

Thus the theoretically substantiated solutions of the problem of the prognosis of random sequences exist but the known methods and models are based on the usage of appropriate limitations which don't permit to obtain maximal accuracy of extrapolation and can't be used in practice for the objects with RVCF under the most general assumptions concerning the degree of nonlinear stochastic relations and the quantity of measurements used for the prognosis.

2 Statement of the Problem

Vector random sequence $\left\{\vec{X}\right\} = X_h(i),\ h = \overline{1,H}$ describing the time change of H interconnected parameters of a certain object with randomly changeable conditions of functioning is completely designated in the discrete series of points $t_i,\ i = \overline{1,I}$ by moment functions $M\left[X_l^\nu(i)\right],\ M\left[X_l^\nu(i)X_h^\mu(j)\right],\ i,j = \overline{1,I};\ l,h = \overline{1,H};\ \nu,\mu = \overline{1,N}$. It is necessary to get the optimal estimations $x_h^*(i), i = \overline{k+1,I}, h = \overline{1,H}$ of the future values of the investigated random sequence for each its constituent $X_h(i)$ provided that the values $x_h^\mu(j),\ j = \overline{1,k},\ \mu = \overline{1,N},\ h = \overline{1,H}$ in the first k points of observation are known.

3 Solution

The most universal approach to the solving of a stated problem from the point of view of the limitations put on a random process is the usage of the apparatus of canonical expansions [16, 19]. For the vector case such an expansion with full account of correlated relations between the constituents is of the form [1]:

$$X_h(i) = M[X_h(i)] + \sum_{\nu=1}^{i}\sum_{\lambda=1}^{H} V_\nu^{(\lambda)}\varphi_{h\nu}^{(\lambda)}(i),\ i = \overline{1,I}, \tag{1}$$

where

$$\begin{aligned} V_\nu^{(\lambda)} = X_\lambda(\nu) - M[X_\lambda(\nu)] - \sum_{\mu=1}^{\nu-1}\sum_{j=1}^{H} V_\mu^{(j)}\varphi_{\lambda\mu}^{(j)}(\nu) \\ - \sum_{j=1}^{\lambda-1} V_\nu^{(j)}\varphi_{\lambda\nu}^{(j)}(\nu),\ \nu = \overline{1,I}; \end{aligned} \tag{2}$$

$$\begin{aligned} \varphi_{h\nu}^{(\lambda)}(i) = \frac{M\left[V_\nu^{(\lambda)}(X_h(i) - M[X_h(i)])\right]}{M\left[\left\{V_\nu^{(\lambda)}\right\}^2\right]} = \frac{1}{D_\lambda(\nu)}(M[X_\lambda(\nu)X_h(i)] \\ - M[X_\lambda(\nu)]M[X_h(i)] - \sum_{\mu=1}^{\nu-1}\sum_{j=1}^{H} D_j(\mu)\varphi_{\lambda\mu}^{(j)}(\nu)\varphi_{h\mu}^{(j)}(i) \\ - \sum_{j=1}^{\lambda-1} D_j(\nu)\varphi_{\lambda\nu}^{(j)}(\nu)\varphi_{h\nu}^{(j)}(i),\ \lambda = \overline{1,h}, \nu = \overline{1,i}. \end{aligned} \tag{3}$$

$$\begin{aligned} D_\lambda(\nu) = M\left[\left\{V_\nu^{(\lambda)}\right\}^2\right] = M\left[\{X_\lambda(\nu)\}^2\right] - M^2[X_\lambda(\nu)] \\ - \sum_{\mu=1}^{\nu-1}\sum_{j=1}^{H} D_j(\mu)\left\{\varphi_{\lambda\mu}^{(j)}(\nu)\right\}^2 - \sum_{j=1}^{\lambda-1} D_j(\nu)\left\{\varphi_{\lambda\nu}^{(j)}(\nu)\right\}^2,\ \nu = \overline{1,I}; \end{aligned} \tag{4}$$

Coordinate functions $\varphi_{h\nu}^{(\lambda)}(i),\ h,\lambda = \overline{1,H}; \nu, i = \overline{1,I}$ have the following characteristics:

$$\varphi_{h\nu}^{(\lambda)}(i) = \begin{cases} 1, & \text{for } (h = \lambda) \wedge (\nu = i); \\ 0, & \text{for } (\mathrm{i} < \nu) \vee ((h < \lambda) \wedge (\nu = i)). \end{cases} \tag{5}$$

The algorithm of the prognosis on the basis of canonical expansion (1) is of the form [6]: