APSC 174 Final Exam Solutions Manual

Proof of all statements may be found in the online lecture notes

Section 0: Sets, Quantifiers, Mappings & Functions, Complex Numbers

Basic set notation:

$$B = \{x \in \mathbb{N} : x \geq 5\} \quad \text{OR} \quad B = \{x \in \mathbb{N} \mid x \geq 5\}$$

Read as: "the set of all elements x in N such that $x \geq 5$"

Definition 2. Let A and B be two sets. A is said to be a **subset** of B if every element of A is also an element of B; we write this as $A \subset B$.

Ex: Every element of N is also an element of $\mathbb{R}$, i.e. $\mathbb{N}$ is a subset of $\mathbb{R}$, and we can therefore write $\mathbb{N} \subset \mathbb{R}$

Definition 3: Let S and T be sets. We denote by $S \cap T$ the set of all elements which are both in S and in T. We call $S \cap T$ the **intersection** of the sets S and T .

Definition 4: Let S and T be sets. We denote by $S \cup T$ the set of all elements which are in either S or T or both. We call $S \cup T$ the **union** of the sets S and T .

Definition 5: Let S and T be sets. We denote by $S \setminus T$ the set of all elements of S which are not in T. We call $S \setminus T$ the **set difference** of the sets S and T (in that order).

Definition 6: Let S and T be sets. We denote by $S \times T$ the set of all pairs of the form (s, t) where $s \in S$ and $t \in T$. We call $S \times T$ the **Cartesian product** of the sets S and T

Ex: Let A = {0, 1, 2, 3}, B = {2, 3, 4, 5}, and C = {5, 6, 7}, then we can write:

Intersection	**Union**	**Difference**
$A \cap B = \{2, 3\}$, $B \cap C = \{5\}$, $A \cap C = \emptyset$	$A \cup B = \{0, 1, 2, 3, 4, 5\}$, $B \cup C = \{2, 3, 4, 5, 6, 7\}$, $A \cup C = \{0, 1, 2, 3, 5, 6, 7\}$.	$A \setminus B = \{0, 1\}$, $B \setminus A = \{4, 5\}$, $A \setminus C = A$, $C \setminus A = C$

Cartesian Product: E = {0,1,2}, F = {4,5}

$$E \times E = \{(0, 0),(0, 1),(0, 2),(1, 0),(1, 1),(1, 2),(2, 0),(2, 1),(2, 2)\},$$
$$E \times F = \{(0, 4),(0, 5),(1, 4),(1, 5),(2, 4),(2, 5)\},$$
$$F \times E = \{(4, 0),(5, 0),(4, 1),(5, 1),(4, 2),(5, 2)\},$$
$$F \times F = \{(4, 4),(4, 5),(5, 4),(5, 5)\}.$$

Section 1: Systems of Linear Equations

A **system of linear equations** (with real coefficients), is a set of equations of the form:

$$(E)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \cdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $a_{11}, a_{12}, .., a_{1n}, a_{21}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn}$ and $b_1, \ldots, b_m$ are given real numbers and we are trying to solve for the real numbers $x_1, \ldots, x_n$. The **solution** to the system of equations will take the form $(x_1, x_2, \ldots, x_n)$.

Systems of equations will either have a unique solution, no solution or infinitely many solutions.
Note: m is the number of equations and n is the number of unknowns.

Section 2: Real Vector Spaces

Definition 12: Let V be a set, with two operations defined on it:

(i) An operation denoted by "+" and called **addition**, defined formally as a mapping + : V × V → V which maps a pair (v, w) in V × V to the element v + w of V

(ii) An operation denoted by "·" and called **scalar multiplication**, defined formally as a mapping · : ℝ × V → V which maps a pair (α,v) in ℝ × V (i.e. α ∈ ℝ and v∈V) to the element α·v of ℝ

Definition: For V to be a **Real Vector Space**, it must satisfy these 8 conditions:

1. Operation "+" is **associative** meaning ∀x, y, z ∈ V we have:
$$x + (y + z) = (x + y) + z$$
2. There exists an element in V, the **zero vector** denoted by **0**, such that ∀x∈V we have:
$$x + \mathbf{0} = \mathbf{0} + x = x$$
3. There exists an element in V, denoted by –x (and is the **opposite** or **inverse** of x) such that ∀x∈V we have:
$$x + (-x) = (-x) + x = \mathbf{0}$$
4. Operation "+" is **commutative** meaning ∀x, y∈V we have:
$$x + y = y + x$$
5. ∀ α, β ∈ ℝ (for any real scalars) and ∀x∈V we have:
$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$$
6. ∀ α ∈ ℝ and ∀x, y ∈ V we have:
$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$$
7. ∀ α, β ∈ ℝ and ∀x ∈ V we have:
$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$
8. ∀x ∈ V we have:
$$1 \cdot x = x$$

Note: We can denote this vector space by (V,+,·) or simply V.

Note: Every element of V ($v \in V$) is a **vector**.

Section 3: Vector Subspaces

Definition 13: Let (V, +, ·) be a real vector space, and let W be a subset of V ($W \subset V$). W is said to be a **vector subspace** of (V, +, ·) if the following properties hold:

(i) The zero element **0** of V is also in W, $\mathbf{0} \in W$.
(ii) $\forall x, y \in W$, we have $x+y \in W$ (elements in W added together are also in W)
(iii) $\forall \alpha \in \mathbb{R}$ and $\forall x \in W$, we have $\alpha x \in W$ (element in W multiplied by a scalar is also in W)

In order to prove that something is a vector subspace, simply show that all three properties hold **FOR ALL** elements in W ($\forall w \in W$) and **ALL** scalars ($\forall \alpha \in \mathbb{R}$). If one or more properties do not hold, then it can be concluded that W is not a vector subspace.

Theorem 3: Let (V,+,·) be a real vector space, and let $W_1 \subset V$ and $W_2 \subset V$ be two vector subspaces of V; then their intersection $W_1 \cap W_2$ is also a vector subspace of V.

Remark 1: If (V, +, ·) is a real vector space and W_1, W_2 two vector subspaces of V, then their union $W_1 \cup W_2$ is in general not a vector subspace of V.

Problem 3a, 2015 Final Exam

3. Consider the vector space $(\mathbf{W}_3, +, \cdot)$ with

$$\mathbf{W}_3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R} \text{ and } x > 0, y > 0, z > 0\}$$

under the following addition and scalar multiplication operations:

– **Addition:** For any (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$,

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 x_2, y_1 y_2, z_1 z_2).$$

– **Scalar multiplication:** For any scalar $\alpha \in \mathbb{R}$ and (x, y, z) in $\mathbf{W}_3$,

$$\alpha \cdot (x, y, z) = (x^\alpha, y^\alpha, z^\alpha).$$

Let $\mathbf{X}_3$ be a subset of $\mathbf{W}_3$ given by

$$\mathbf{X}_3 = \{(x, y, z) \in \mathbf{W}_3 \ : \ xy = 1\}.$$

(a) Show that $\mathbf{X}_3$ is a vector subspace of $(\mathbf{W}_3, +, \cdot)$. **(3 pts)**

Solution: We need to verify that the 3 properties of a vector subspace hold:

1. The zero vector $\mathbf{0}_{W3}$ of W_3 must be in X_3:
   If we denote the zero vector as $\mathbf{0}_{W3}$ = (x′,y′,z′), we need to find this vector such that
   $\forall(x,y,z)\in W_3$, (x,y,z) + (x′,y′,z′) = (x,y,z)
   Addition gives: (xx′,yy′,zz′) = (x,y,z)
   Thus clearly, $\mathbf{0}_{W3}$ = (x′,y′,z′) = (1,1,1)
   To see if $\mathbf{0}_{W3}\in X_3$, check the property of X_3 that xy = 1. Evidently 1·1 = 1, therefore the zero vector of W_3 is in X_3.
2. $\forall(x_1,y_1,z_1),(x_2,y_2,z_2)\in X_3$, we must have that $(x_1,y_1,z_1)+(x_2,y_2,z_2)\in X_3$:
   Letting $(x_1,y_1,z_1),(x_2,y_2,z_2)\in X_3$, we know that $x_1y_1=1$ and $x_2y_2=1$. Therefore $y_1=1/x_1$ and $y_2=1/x_2$ so we can write:
   $(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1, 1/x_1, z_1)+(x_2, 1/x_2, z_2) = (x_1x_2, 1/x_1 1/x_2, z_1z_2)$
   Now to check if xy=1: $(x_1x_2)(1/x_1 1/x_2) = (x_1/x_1)(x_2/x_2) = 1$, thus the condition holds and $(x_1,y_1,z_1)+(x_2,y_2,z_2)\in X_3$.
3. $\forall\alpha\in R$ and $\forall(x,y,z)\in X_3$ we must have that $\alpha(x,y,z)\in X_3$:
   Letting $\alpha\in R$ and $(x,y,z)\in X_3$, we know that (x,y,z) = (x, 1/x, z). We have that:
   $\alpha(x,y,z) = \alpha(x, 1/x, x) = (x^\alpha, (1/x)^\alpha, z^\alpha)$
   Now to check if xy=1: $(x^\alpha)(1/x^\alpha) = x^{\alpha/} x^\alpha = 1$, thus the condition holds and $\alpha(x,y,z)\in X_3$.

All three conditions hold, therefore X_3 is a vector subspace of W_3. □

Section 4: Linear Combinations & Span

Definition 14: Let (V,+,·) be a real vector space, and let $v_1, \ldots, v_p$ be a finite number of elements of V (with p ≥ 1). We call the expression

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p, \qquad \text{with } \alpha_1, \ldots, \alpha_p \in \mathbb{R}$$

a **linear combination** of the vectors $v_1, \ldots, v_p$ or "the linear combination of the vectors $v_1, \ldots, v_p$ with respective scalar coefficients $\alpha_1, \ldots, \alpha_p$".

If an element v of V can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p,$$

where $\alpha_1, \ldots, \alpha_p$ are all real numbers, then we say that **v is a linear combination** of the vectors $v_1, \ldots, v_p$.

Notation: Denote $S_{(v1, v2, \ldots, vp)}$ the **set of all linear combinations** of the vectors $v_1, v_2, \ldots, v_p$:

$$S_{(v1, v2, \ldots, vp)} = \{ \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p \mid \alpha_1, \alpha_2, \ldots, \alpha_p \in R \}$$

Definition 15: The vector subspace $S_{(v1, v2, \ldots, vp)}$ of V is called the **linear span** of the vectors v_1, v_2, …, v_p, or the subspace of V generated by the vectors v_1, v_2, …, v_p. What this means is that any vector $v \in V$ can be created from a linear combination of the vectors v_1, v_2, …, v_p:

$\forall v \in V$, there exists real numbers α_1, …, α_p such that we can write $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p$

Problem 1d), 2016 Midterm 1

Consider the real vector space $(R^2, +, \cdot)$, and let $v_1 = (1,1)$, $v_2 = (2,2)$. Using the definition of linear span, prove whether or not $S_{(v1, v2)} = R^2$.

Solution: The definition tells us that the span of v_1 and v_2 is the set of all linear combinations of

v_1 and v_2 $S_{(v1, v2)} = \{ \alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1, \alpha_2 \in R \}$

$= \{ \alpha_1(1,1) + \alpha_2(2,2) \mid \alpha_1, \alpha_2 \in R \}$ which we can simplify to

$= \{ (\alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2) \mid \alpha_1, \alpha_2 \in R \}$

The vector $(\alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2)$ will always have the same values in both entries (be of the form (x,x)). Thus, all vectors with distinct entries (of the form (x,y)) cannot be created. For example we know that $(0,1) \in R^2$, however $(0,1) = (\alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2)$ has no solution as no such scalars α_1, α_2 exist. Thus, the span of v_1 and v_2 cannot create any vector R^2. Therefore $S_{(v1, v2)} \neq R^2$.

□

Section 5: Linear Dependence & Independence

Note: The **scalar multiple** of a vector is simply the multiplication of each entry by the same scalar:

Ex. $\frac{1}{2}(4,12) = (\frac{1}{2}\cdot 4, \frac{1}{2}\cdot 12) = (2,6)$

Definition 16: Let $(V,+,\cdot)$ be a real vector space, and let $S = \{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V.

(i) The subset S is said to be **linearly independent** if for any $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, the relation $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p = \mathbf{0}$ implies that $\alpha_1 = \ldots = \alpha_p = 0$

(ii) The subset S is said to be **linearly dependent** if it is not linearly independent (if there exist real numbers α_1, …, α_p <u>not all</u> 0, but for which $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p = \mathbf{0}$)

Theorem 4: Let $(V,+,\cdot)$ be a real vector space, and let $S = \{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V . Then:

(i) If S is a linearly dependent subset of V, then at least one of the elements of S can be written as a linear combination of the other elements of S. This means that there exists an element $v_i \in S$ such that $v_i = \alpha_1 v_1 + \ldots + \alpha_{i-1}v_{i-1} + \alpha_{i+1}v_{i+1} + \ldots + \alpha_p v_p$ where $1 \le i \le p$ and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are real numbers not all zero (this is simply rearranging the equation $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p = \mathbf{0}$)

(ii) If at least one of the elements of S can be written as a linear combination of the other elements of S, then S is a linearly dependent subset of V.

Lemma 4: Let (V, +, ·) be a real vector space, and S, T two finite subsets of V such that $S \subset T$. If S is linearly dependent, then T is also linearly dependent.

<u>Problem 2d), 2014 Final Exam</u>

2. Let $(V, +, \cdot)$ be a real vector space.

(d) Suppose now $\{v_1, v_2, v_3, v_4\}$ is a subset of V such that the subsets $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are each linearly independent; suppose furthermore that there exists a non-zero vector $e \in V$ such that $e \in S_{(v_1,v_2)} \cap S_{(v_3,v_4)}$ (i.e. e is both in the linear span of $\{v_1, v_2\}$ and in the linear span of $\{v_3, v_4\}$). Determine whether or not the subset $\{v_1, v_2, v_3, v_4\}$ is linearly independent. (5 pts)

Solution: Since e is in the linear span of $\{v_1, v_2\}$, there exist $\alpha_1, \alpha_2 \in R$ such that $e = \alpha_1 v_1 + \alpha_2 v_2$. Since $e \neq 0_V$ it follows that α_1, α_2 cannot both be zero. Similarly, since e is in the linear span of $\{v_3, v_4\}$, there exist $\alpha_3, \alpha_4 \in R$ such that $e = \alpha_3 v_3 + \alpha_4 v_4$. Since $e \neq 0_V$ it follows that α_3, α_4 cannot both be zero. We thus have $e = \alpha_1 v_1 + \alpha_2 v_2 = \alpha_3 v_3 + \alpha_4 v_4$, therefore $\alpha_1 v_1 + \alpha_2 v_2 - \alpha_3 v_3 - \alpha_4 v_4 = 0_V$.

We then have that because $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not all zero it follows that $\{v_1, v_2, v_3, v_4\}$ is a linearly <u>dependent</u> subset of V. ☐

Section 6: Relating Linear Combinations to Linear Independence

Theorem 5: Let (V,+,·) be a real vector space, and let $S = \{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V. Let $v \in V$ be in the linear span of $v_1, v_2, \ldots, v_p$. If S is a linearly independent subset of V, then v can be expressed only in a unique way as a linear combination of $v_1, v_2, \ldots, v_p$; i.e., the values of $(\alpha_1, \ldots, \alpha_p)$ of real numbers will be unique such that:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p$$

Theorem 6: Let (V,+,·) be a real vector space, and let $S = \{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V.

Assume that any $v \in V$ that is in the linear span of $v_1, v_2, \ldots, v_p$ can be expressed only in a unique way as a linear combination of $v_1, v_2, \ldots, v_p$; i.e., for any $v \in S_{(v1, v2, \ldots, vp)}$ there is a unique p−tuple $(\alpha_1, \ldots, \alpha_p)$ of real numbers such that:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_p v_p$$

Then, S is a linearly independent subset of V.

Section 7: Matrices of Systems of Equations

Definition 17: Consider the system of m linear equations and n unknowns given by

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots a_{2,n}x_n = b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots a_{m,n}x_n = b_m$$

The **augmented matrix** of this system is the table of real numbers with m rows and $n+1$ columns given by

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & | & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & | & b_2 \\ & & \vdots & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & | & b_m \end{pmatrix}$$

Definition 18: The **augmented matrix** is said to be in **row-echelon form** if the following two conditions are met:

1. Each row with all entries equal to 0 is below every row having at least one nonzero entry
2. The leftmost non-zero entry on each row is to the right of the leftmost non-zero entry of the preceding row

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{5} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{0} \end{bmatrix} \boldsymbol{Row\ Echelon} \qquad \begin{bmatrix} \mathbf{1} & \mathbf{5} & \mathbf{6} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} & \mathbf{1} \end{bmatrix} \boldsymbol{Not\ Row\ Echelon}$$

Rules for Row Echelon:

- Add/subtract rows
- Multiplication by a non-zero scalar
- Exchange two rows
- Any combination of the above

DO NOT:

- Multiply two rows together
- Exchange two columns
- Do anything that is not listed to the left ←

Gaussian Elimination Steps:

Step 1: Write down the augmented matrix of the system of linear equations;

Step 2: Transform the augmented matrix in row-echelon form through a sequence of elementary row operations;

Step 3: Solve the system corresponding to the row-echelon augmented matrix obtained in Step 2 by back-substitution.

<u>Problem 1, 2015 Final Exam</u>

1. Given a real number a, consider the system of linear equations given by:

$$\begin{aligned} x_1 + x_2 + 2x_4 &= a, \\ x_1 - x_2 + 2x_4 &= 2, \\ -2x_1 - x_2 + 5x_3 + x_4 &= 0, \\ x_1 + x_2 - 2x_3 &= 2, \end{aligned}$$

where we wish to solve for the quadruple (x_1, x_2, x_3, x_4) of real numbers.

(a) Write the augmented matrix for this system. (**4 pts**)

(b) Transform the augmented matrix to row-echelon form using a sequence of elementary row operations (clearly indicate which operation you perform at each step). (**6 pts**)

(c) Using (b), determine all the values of a for which the system has no solution. (**2 pts**)

(d) Using (b), determine all the values of a for which the system has a solution. (**2 pts**)

(e) For those values of a obtained in (d) for which the system has a solution, determine the set of all solutions to the original system of linear equations. (**6 pts**)

Solution:

(a) The augmented matrix for this system is given by:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 1 & -1 & 0 & 2 & 2 \\ -2 & -1 & 5 & 1 & 0 \\ 1 & 1 & -2 & 0 & 2 \end{array}\right)$$

(b) Exchanging rows 1 and 4 ($R1 \leftrightarrow R4$) yields:

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 1 & -1 & 0 & 2 & 2 \\ -2 & -1 & 5 & 1 & 0 \\ 1 & 1 & 0 & 2 & a \end{array}\right)$$

Adding $-1\times$ row 1 to row 2 ($-R1 + R2 \to R2$) yields

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 0 \\ -2 & -1 & 5 & 1 & 0 \\ 1 & 1 & 0 & 2 & a \end{array}\right).$$

Adding twice row 1 to row 3 ($2R1 + R3 \to R3$) yields

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 1 & 0 & 2 & a \end{array}\right).$$

Adding $\frac{1}{2}\times$ row 2 to row 3 ($\frac{1}{2}R2 + R3 \to R3$) yields

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 \\ 1 & 1 & 0 & 2 & a \end{array}\right)$$

Adding $-1\times$ row 1 to row 4 $(-R1+R4 \to R4)$ yields

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 2 & 2 & a-2 \end{array}\right)$$

Adding $-1\times$ row 3 to row 4 $(-R3+R4 \to R4)$ yields

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & a-6 \end{array}\right)$$

The above matrix is now in row-echelon form.

(c) It can be seen from the last row of the augmented matrix in row echelon form that the given system of linear equations has no solution if and only if $a - 6 \neq 0$, that is, if and only if $a \neq 6$.

d) Similarly it can be seen from part (c) that the system of linear equations has a solution if and only if $a = 6$.

(e) For a = 6, the augmented matrix in row-echelon form is given by

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 2 \\ 0 & -2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & a-6 \end{array}\right)$$

The system of linear equation corresponding to this augmented matrix in row echelon form is given by:

$$\begin{cases} x_1 + x_2 - 2x_3 = 2 \\ -2x_2 + 2x_3 + 2x_4 = 0 \\ 2x_3 + 2x_4 = 4 \end{cases}$$

Solving for x_3 using the last equation we get: $x_3 = 2 - x_4$

Substituting this into the second equation we find: $x_2 = 2$

Then finally substituting what we found for x_2 and x_3 into the first equation we get: $x_1 = 4-2x_4$

We can now state that when we have a=6, the system of equations has infinitely many solutions given by the set S, where $S = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1 = 4-2x_4,\ x_2=2,\ x_3=2-x_4\}$.

Or we could write $(x_1,x_2,x_3,x_4) = (4-2x_4, 2, 2-x_4, x_4)$ where x_4 is a real number.

Section 8: Generating Sets & Bases

Definition 20: Let $(V, +, \cdot)$ be a real vector space, and let $S = \{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V. S is said to be a **generating set** for the vector space V if any $v \in V$ can be written as a linear combination of $v_1, v_2, \ldots, v_p$.

Ex: {[1,0,0], [0,1,0], [0,0,1]} is a generating set for $\mathbb{R}^3$.

Theorem 11: Let $(V, +, \cdot)$ be a real vector space, and let $S = \{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V such that:

(i) S is a generating set for V

(ii) S is linearly independent

Then, any element $v \in V$ can be expressed in a **unique** way as a linear combination of elements of S.

Definition 21: Let $(V, +, \cdot)$ be a real vector space, and let $v_1, v_2, \dots, v_p \in V$. The p−tuple $(v_1, v_2, \dots, v_p)$ is said to be a **basis** of V if

(i) $\{ v_1, v_2, \dots, v_p \}$ is a generating set for V, and
(ii) The vectors $(v_1, v_2, \dots, v_p)$ are linearly independent.

Ex: {[1,0,0], [0,1,0], [0,0,1]} is a basis set for $\mathbb{R}^3$.

Note: A basis for a vector space is not unique. {[1,1,0], [0,5,1], [0,0,1]} is also a basis for $\mathbb{R}^3$.

Definition 22: Let $(V, +, \cdot)$ be a real vector space, let $B = (v_1, v_2, \dots, v_p)$ be a basis of V, and let $v \in V$. The p–tuple $(\alpha_1, \alpha_2, \dots, \alpha_p)$ of real numbers is called the **component vector** or **coordinate vector** of v with respect to the basis B if we have:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p$$

Problem 2b), 2015 Midterm 2

(b) Consider the real vector space $(\mathbf{W}_3, +, \cdot)$ with

$$\mathbf{W}_3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R} \text{ and } x > 0, y > 0, z > 0\}$$

under the following addition and scalar multiplication operations:

* **Addition:** For any (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$,

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 x_2, y_1 y_2, z_1 z_2).$$

* **Scalar multiplication:** For any scalar $\alpha \in \mathbb{R}$ and (x, y, z) in $\mathbf{W}_3$,

$$\alpha \cdot (x, y, z) = (x^\alpha, y^\alpha, z^\alpha).$$

Recall that the zero vector $\mathbf{0}$ of $\mathbf{W}_3$ is given by $\mathbf{0} = (1, 1, 1)$. Consider the following vectors in $\mathbf{W}_3$: $\mathbf{v_1} = (e, 1, 1)$, $\mathbf{v_2} = (1, e, 1)$ and $\mathbf{v_3} = (1, 1, e)$ where $e = 2.718\cdots$ is Euler's number. **Show that $(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ is a basis for $\mathbf{W}_3$.** [5 pts]

Solution: To show that (v_1, v_2, v_3) is a basis for W_3, we must show two properties:

I) (v_1, v_2, v_3) is a generating set.
For any vector $v = (x, y, z)$ in W_3, we need to show that there exist properly chosen scalars α_1, α_2 and α_3 such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. Thus we need to solve for α_1, α_2 and α_3 in

$$(x, y, z) = \alpha_1 \cdot (e, 1, 1) + \alpha_2 \cdot (1, e, 1) + \alpha_3 \cdot (1, 1, e).$$

Thus

$$(x, y, z) = (e^{\alpha 1}, 1^{\alpha 1}, 1^{\alpha 1}) + (1^{\alpha 2}, e^{\alpha 2}, 1^{\alpha 2}) + (1^{\alpha 3}, 1^{\alpha 3}, e^{\alpha 3}) = (e^{\alpha 1}, e^{\alpha 2}, e^{\alpha 3})$$

Therefore,

$$x = e^{\alpha 1}, y = e^{\alpha 2}, z = e^{\alpha 3}$$

Taking the natural log of both sides of the above equations gives:

$$\alpha_1 = \ln(x), \alpha_2 = \ln(y), \alpha_3 = \ln(z)$$

Substituting these values of α back into our original equation gives:

$$(x, y, z) = \ln(x) \cdot (e, 1, 1) + \ln(y) \cdot (1, e, 1) + \ln(z) \cdot (1, 1, e)$$

Which implies that (v_1, v_2, v_3) is a generating set for W_3

II) (v_1, v_2, v_3) is linearly independent

We must show that for the equation

$$\beta_1 \cdot (e, 1, 1) + \beta_2 \cdot (1, e, 1) + \beta_3 \cdot (1, 1, e) = (1, 1, 1)$$

$\beta_1, \beta_2, \beta_3$, must all be zero. Thus

$$(e^{\beta 1}, e^{\beta 2}, e^{\beta 3}) = (1, 1, 1)$$

$\beta_1 = \ln(1) = 0, \beta_2 = \ln(1) = 0, \beta_3 = \ln(1) = 0$

Thus, (v_1, v_2, v_3) is linearly independent.

Therefore (v_1, v_2, v_3) is a basis for W_3. ☐

Section 9: Finite Dimensional Vector Spaces

Definition 23: Let (V, +, ·) be a real vector space.

• V is said to be **finite-dimensional** if there exists an integer N ≥ 0 such that any subset of V containing N + 1 elements is linearly dependent. The smallest integer N for which this holds is then called the dimension of V (equivalently, V is said to have dimension N).

• V is said to be **infinite-dimensional** if it is not finite-dimensional.

Examples:

- The vector spaces $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots \mathbb{R}^n$ are **all finite-dimensional**
- The vector space $\mathbb{R}[x]$, i.e. the vector space of polynomial functions of one variable, is **infinite-dimensional.**

Theorem 12: Let (V, +, ·) be a finite-dimensional real vector space of dimension N. Let $\{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V containing p vectors. If $\{ v_1, v_2, \ldots, v_p \}$ is a linearly independent subset of V then p ≤ N.

Theorem 13: Let (V, +, ·) be a finite-dimensional real vector space of dimension N. Let $\{ v_1, v_2, \ldots, v_p \}$ be a finite subset of V containing p vectors. If $\{ v_1, v_2, \ldots, v_p \}$ is a generating set for V of V then p ≥ N.

Theorem 14: Let (V, +, ·) be a finite-dimensional real vector space of dimension N. Let B = (v_1, v_2, ..., v_p) be a basis for V (i.e. linearly independent and a generating set). Then, according to Theorem 12 and Theorem 13, p ≤ N and p ≥ N, so we must have p = N.

To compute the dimension of a real vector space, it is enough to find a basis for that vector space. The dimension of the vector space is then equal to the number of elements of that basis.

<u>Problem 1, 2017 Midterm 2</u>

Problem 1

Let $(\mathcal{M}_{2,2}(\mathbb{R}), +, \cdot)$ denote the real vector space of real 2×2 matrices, endowed with the usual addition and multiplication by scalars operations that we have defined for real matrices.

(a) **Find a basis for $\mathcal{M}_{2,2}(\mathbb{R})$ and use it to compute the dimension of $\mathcal{M}_{2,2}(\mathbb{R})$.** [5 pts]

(b) **Let $sl_2(\mathbb{R}) = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}) \mid a_{1,1} + a_{2,2} = 0 \right\}$ (i.e. $sl_2(\mathbb{R})$ is the subset of $\mathcal{M}_{2,2}(\mathbb{R})$ consisting of all real 2×2 matrices with the diagonal entries adding up to 0). Show that $sl_2(\mathbb{R})$ is a vector subspace of $\mathcal{M}_{2,2}(\mathbb{R})$.** [5 pts]

(c) **Find a basis for $sl_2(\mathbb{R})$ and use it to compute the dimension of $sl_2(\mathbb{R})$.** [5 pts]

SOLUTION:

(a) Any matrix $\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ in $\mathcal{M}_{2,2}(\mathbb{R})$ can be written (under the operations of $\mathcal{M}_{2,2}(\mathbb{R})$) as follows

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{1,2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{2,1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{2,2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, setting

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have shown that $\{A_1, A_2, A_3, A_4\}$ is a generating set for $\mathcal{M}_{2,2}(\mathbb{R})$. We next show that $\{A_1, A_2, A_3, A_4\}$ is a linearly independent subset of $\mathcal{M}_{2,2}(\mathbb{R})$: for scalars α_1, α_2, α_3 and α_4, we have that

$$\begin{aligned} \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = \mathbf{0} &\implies \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &\implies \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \end{aligned}$$

and thus $\{A_1, A_2, A_3, A_4\}$ is linearly independent. Since $\{A_1, A_2, A_3, A_4\}$ is both a generating set for $\mathcal{M}_{2,2}(\mathbb{R})$ and is linearly independent, we thus conclude that (A_1, A_2, A_3, A_4) is a basis for $\mathcal{M}_{2,2}(\mathbb{R})$. Finally, since the above basis have four components, we deduce that the dimension of $\mathcal{M}_{2,2}(\mathbb{R})$ is **4**. □

SOLUTION:

(b) We will show that $sl_2(\mathbb{R})$ satisfies the three properties of the vector subspace to conclude that $sl_2(\mathbb{R})$ is a vector subspace of $\mathcal{M}_{2,2}(\mathbb{R})$.

- *Zero vector property of $sl_2(\mathbb{R})$:* The zero vector of $\mathcal{M}_{2,2}(\mathbb{R})$ is given by $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $\mathbf{0}_{1,1} + \mathbf{0}_{2,2} = 0 + 0 = 0$ (with $\mathbf{0}_{1,1}$ denoting the entry of $\mathbf{0}$ on row 1 and column 1, and $\mathbf{0}_{2,2}$ denoting the entry of $\mathbf{0}$ on row 2 and column 2), it follows that the zero vector $\mathbf{0}$ is an element of $sl_2(\mathbb{R})$. Hence, we can write $\mathbf{0} \in sl_2(\mathbb{R})$.
- *Closure property of $sl_2(\mathbb{R})$ under addition:* Let

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

be two elements of $sl_2(\mathbb{R})$. Thus we have that $a_{1,1} + a_{2,2} = 0$ and $b_{1,1} + b_{2,2} = 0$. Now

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \end{pmatrix},$$

and since

$$\begin{aligned} (a_{1,1} + b_{1,1}) + (a_{2,2} + b_{2,2}) &= (a_{1,1} + a_{2,2}) + (b_{1,1} + b_{2,2}) \\ &= 0 + 0 = 0, \end{aligned}$$

it follows that $\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \in sl_2(\mathbb{R})$.
- *Closure property of $sl_2(\mathbb{R})$ under scalar multiplication:* Let $\alpha \in \mathbb{R}$ be a scalar and let

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in sl_2(\mathbb{R}).$$

– *Closure property of $sl_2(\mathbb{R})$ under scalar multiplication:* Let $\alpha \in \mathbb{R}$ be a scalar and let

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in sl_2(\mathbb{R}).$$

Thus we have that $a_{1,1} + a_{2,2} = 0$. Now

$$\alpha \cdot \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} \alpha a_{1,1} & \alpha a_{1,2} \\ \alpha a_{2,1} & \alpha a_{2,2} \end{pmatrix},$$

and since

$$\alpha a_{1,1} + \alpha a_{2,2} = \alpha(a_{1,1} + a_{2,2}) = \alpha(0) = 0,$$

it follows that $\alpha \cdot \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in sl_2(\mathbb{R})$.

We conclude: $sl_2(\mathbb{R})$ is a vector subspace of $\mathcal{M}_{2,2}(\mathbb{R})$. □

(c) For any matrix $\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ in $sl_2(\mathbb{R})$, we have that $a_{1,1} + a_{2,2} = 0$, and hence $a_{2,2} = -a_{1,1}$., and therefore we can write

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_{1,2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{2,1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, setting

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we have shown that $\{B_1, B_2, B_3\}$ is a generating set for $sl_2(\mathbb{R})$. We next show that $\{B_1, B_2, B_3\}$ is also linearly independent: for scalars β_1, β_2 and β_3, we have that

$$\begin{aligned} \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = \mathbf{0} \implies & \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & -\beta_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \implies & \beta_1 = \beta_2 = \beta_3 = 0 \end{aligned}$$

and thus $\{B_1, B_2, B_3\}$ is linearly independent. Since $\{B_1, B_2, B_3\}$ is both a generating set for $sl_2(\mathbb{R})$ and is linearly independent, we conclude that (B_1, B_2, B_3) is a basis for $sl_2(\mathbb{R})$. Therefore, the dimension of $sl_2(\mathbb{R})$ is **3**. □

Section 10: Linear Transformations, Dimension, & Kernel

Note: A mapping, function, and transformation are equivalent terms.

Definition 24: Let V and W be two real vector spaces, and let L : V → W be a mapping from V to W. L is said to be a **linear mapping** if the following two properties hold:

1. For any v_1, $v_2 \in V$, $L(v_1 + v_2) = L(v_1) + L(v_2)$
2. For any $\alpha \in R$ and any $v \in V$, $L(\alpha v) = \alpha L(v)$

Problem 3c), 2015 Final Exam (uses the same "Weird Space" W_3 as in Problem 3a in

(c) Consider the following mapping $L : \mathbf{X}_3 \to \mathbb{R}^2$ given by

$$L((x,y,z)) = (\ln(x), \ln(yz)) \qquad \forall (x,y,z) \in \mathbf{X}_3,$$

where $\ln(\cdot)$ denotes the natural logarithm and $(\mathbb{R}^2, +\cdot)$ is the vector space of real-valued ordered pairs under the traditional (component-wise) addition and scalar multiplication operations.

Determine whether or not the mapping L is linear. (**6 pts**)

Solution:

(c) Recall that in order to show that the mapping $L : \mathbf{X}_3 \to \mathbb{R}^2$ is linear we have to show the following two properties:

(i) For any (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{X}_3$,

$$L((x_1,y_1,z_1) + (x_2,y_2,z_2)) = L((x_1,y_1,z_1)) + L((x_2,y_2,z_2)).$$

(ii) For any $(x,y,z) \in \mathbf{X}_3$ and any scalar $\alpha \in \mathbb{R}$,

$$L(\alpha(x,y,z)) = \alpha L((x,y,z)).$$

Let us separately examine each of the above linearity properties:

(i) For any $(x_1,y_1,z_1) = \left(x_1, \frac{1}{x_1}, z_1\right)$ and $(x_2,y_2,z_2) = \left(x_2, \frac{1}{x_2}, z_2\right)$ in $\mathbf{X}_3$, by properly using the operations of $\mathbf{W}_3$ and $\mathbb{R}^2$, we have the following

$$\begin{aligned}
L((x_1,y_1,z_1) + (x_2,y_2,z_2)) &= L\left(\left(x_1, \frac{1}{x_1}, z_1\right) + \left(x_2, \frac{1}{x_2}, z_2\right)\right) \\
&= L\left(\left(x_1x_2, \frac{1}{x_1}\frac{1}{x_2}, z_1z_2\right)\right) \\
&= \left(\ln(x_1x_2), \ln\left(\frac{z_1z_2}{x_1x_2}\right)\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
L((x_1,y_1,z_1)) + L((x_2,y_2,z_2)) &= L\left(\left(x_1, \frac{1}{x_1}, z_1\right)\right) + L\left(\left(x_2, \frac{1}{x_2}, z_2\right)\right) \\
&= \left(\ln(x_1), \ln\left(\frac{z_1}{x_1}\right)\right) + \left(\ln(x_2), \ln\left(\frac{z_2}{x_2}\right)\right) \\
&= \left(\ln(x_1) + \ln(x_2), \ln\left(\frac{z_1}{x_1}\right) + \ln\left(\frac{z_2}{x_2}\right)\right) \\
&= \left(\ln(x_1x_2), \ln\left(\frac{z_1z_2}{x_1x_2}\right)\right)
\end{aligned}$$

Comparing the above expressions, we directly get that

$$L((x_1, y_1, z_1) + (x_2, y_2, z_2)) = L((x_1, y_1, z_1)) + L((x_2, y_2, z_2))$$

and hence this property holds.

(ii) For any $(x, y, z) = \left(x, \frac{1}{x}, z\right) \in \mathbf{X}_3$ and scalar $\alpha \in \mathbb{R}$, again by properly using the operations of $\mathbf{W}_3$ and $\mathbb{R}^2$, we have the following

$$\begin{aligned} L(\alpha(x, y, z)) &= L\left(\alpha\left(x, \frac{1}{x}, z\right)\right) \\ &= L\left(\left(x^\alpha, \frac{1}{x^\alpha}, z^\alpha\right)\right) \\ &= \left(\ln(x^\alpha), \ln\left(\frac{z^\alpha}{x^\alpha}\right)\right) \\ &= \left(\alpha \ln(x), \alpha \ln\left(\frac{z}{x}\right)\right) \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha L((x, y, z)) &= \alpha L\left(\left(x, \frac{1}{x}, z\right)\right) \\ &= \alpha\left(\ln(x), \ln\left(\frac{z}{x}\right)\right) \\ &= \left(\alpha \ln(x), \alpha \ln\left(\frac{z}{x}\right)\right) \end{aligned}$$

Comparing the above expressions, we directly get that

$$L(\alpha(x, y, z)) = \alpha L((x, y, z))$$

and hence this property holds.

We conclude that the mapping L is *linear*.

Theorem 15: Let V and W be real vector spaces; let $\mathbf{0}_V$ denote the zero vector of V, and let $\mathbf{0}_W$ denote the zero vector of W. Let L : V→W be a linear mapping. Then, we have:

$$L(\mathbf{0}_V) = \mathbf{0}_W$$

Note: Use this fact to prove a mapping is **not** linear if the zero vector of V does not map to the zero vector of W. You cannot use this fact to prove that a mapping is linear.

Theorem 16: Let V, W, and Z be real vector spaces, and let $L_1 : V \to W$ and $L_2 : W \to Z$ be linear mappings. Then, the mapping $L_2 \circ L_1 : V \to Z$ (called the **composition**) is also linear.

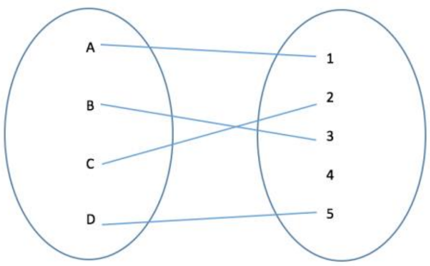
Note: $L_2 \circ L_1(v) = L_2(L_1(v))$, which means that we apply L_1 to v and then apply L_2 to this result.

Definition: Let V and W be two real vector spaces, and let L : V → W be a mapping from V to W. The **Kernel** or **Null Space** of L (denoted ker(L)) is the set of all vectors in V that map to the zero vector of W.

$$\ker(L) = \{ v \in V \mid L(v) = \mathbf{0}_W \}$$

Theorem 17: Let V and W be real vector spaces, and let $L : V \to W$ be a linear mapping. Then, the kernel ker(L) of L is a vector subspace of V.

Definition (from section 0): Let f: $S \to T$ be a mapping from V to W. f is said to be **injective** (or **one-to-one**) if $\forall x, y \in S$, $x \neq y$ implies that $f(x) \neq f(y)$.



Injective mapping

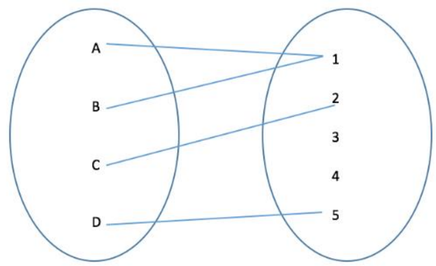


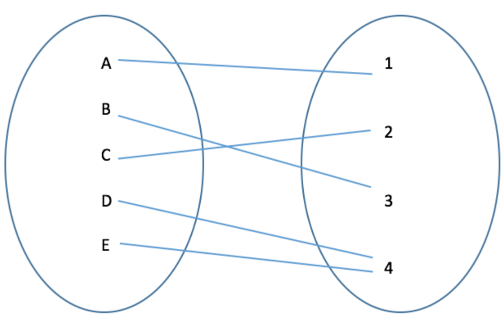
Figure 1: Not injective mapping

Theorem 18: Let $L : V \to W$ be a linear mapping from the real vector space V to the real vector space W. Then L is **injective** if and only if the kernel, ker(L), of L is equal to $\{\mathbf{0}_V\}$, i.e. $\ker(L) = \{\mathbf{0}_V\}$.

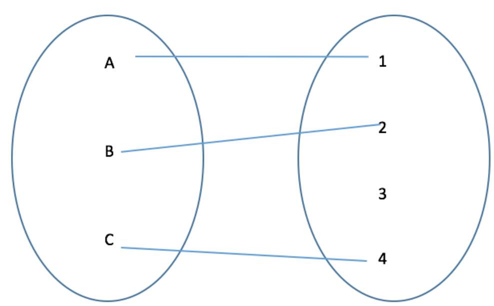
Definition: Let V and W be two real vector spaces, and let $L : V \to W$ be a mapping from V to W. The **Range** or **Image** of L (denoted Im(L)) is the set of values in W that L maps to.

$$\mathrm{Im}(L) = \{L(v) \in W \mid v \in V \}$$

Definition (from section 0): Let f: $S \to T$ be a mapping from S to T. f is said to be **surjective** (or **onto**) if $\forall x \in T$, $\exists\, y \in S : x = f(y)$. i.e. $\mathrm{Im}(f) = T$.



Surjective mapping



Not surjective mapping

Theorem 19: Let V and W be real vector spaces, and let $L : V \to W$ be a linear mapping. Then, the range Im(L) of L is a vector subspace of W.

Definition 25: Let V, W be real vector spaces, assume V is finite-dimensional, and let $L : V \to W$ be a linear mapping.

(i) The **rank** of L, denoted by rank(L), is defined to be the dimension of Im(L). (This is the number of elements in a basis of the Image)
(ii) The **nullity** of L, denoted by nullity(L), is defined to be the dimension of ker(L). (This is the number of elements in a basis of the Kernel)

Theorem 20 (Rank-Nullity Theorem): Let V, W be real vector spaces, and let L : V → W be a linear mapping. Assume V is finite-dimensional, and let N denote the dimension of V. Then:

$$\text{rank}(L) + \text{nullity}(L) = N$$

Section 11: Matrices & Linear Transformations

Definition 26: Let m and n be integers ≥ 1. A **real matrix** with m rows and n columns (also called a real $m \times n$ matrix) is a table (or array) of the form:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & -- & a_{1,n} \\ a_{2,1} & a_{2,2} & -- & a_{2,n} \\ | & | & -- & | \\ a_{m,1} & a_{m,2} & -- & a_{m,n} \end{bmatrix}$$

This can be used to represent a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ ($\mathbb{R}^{(\text{\#of columns})}$ to $\mathbb{R}^{(\text{\# of rows})}$).

- **Addition:** Two $m \times n$ matrices can be added together if and only if $m_1 = m_2$ and $n_1 = n_2$ (i.e. same size).

$$A + B = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots\cdots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & & a_{2,n} + b_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}$$

- **Scalar Multiplication:** When multiplying a matrix by a scalar α, multiply each entry of the matrix by α.

$$\alpha \cdot B = \begin{bmatrix} \alpha \cdot b_{1,1} & \cdots & \alpha \cdot b_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha \cdot b_{m,1} & \cdots & \alpha \cdot b_{m,n} \end{bmatrix}$$

Theorem 24: Let A be a real $m \times n$ matrix; the range Im(A) of the matrix A (i.e. the range of the linear mapping $L_A : \mathbb{R}^n \to \mathbb{R}^m$) is the linear span of the column vectors $A_1, A_2, \ldots, A_n$ of A.

Theorem 26: Let $A \in M_{m,n}(\mathbb{R})$ (i.e. A is a real m×n matrix). We have ker(A) = $\{0_R\}$ if and only if the column vectors of A are linearly independent.

Theorem 27: Let $A \in M_{m,n}(\mathbb{R})$, then: rank(A) + nullity(A) = n.

i.e. rank(A) + nullity(A) is equal to the number of columns of A.

Theorem 28: Let $A \in M_{m,n}(\mathbb{R})$ represent the linear map $L_A : \mathbb{R}^n \to \mathbb{R}^m$. We have the following:

a) If, for each vector $b \in \mathbb{R}^m$, there exists a **unique** $v \in \mathbb{R}^n$ which satisfies $L_A(v) = b$, then:
   i. $m = n$
   ii. The column vectors of A are linearly independent
b) Conversely, if $m = n$ and the column vectors of A are linearly independent, then for each $b \in \mathbb{R}^m$, there exists a unique $v \in \mathbb{R}^n$ which satisfies $L_A(v) = b$.

Problem 1, 2012 Midterm 3

1. (**25 pts**) For the real the matrix A given by $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ -5 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$, do the following:

(a) **Specify** the **linear transformation** L_A that it defines. (**5 pts**)

(b) Specify its **kernel** $\ker(A)$ (i.e. $\ker(L_A)$) and **find a basis** for $\ker(A)$. (**10 pts**)

(c) Specify its **range** $\mathrm{Im}(A)$ (i.e. $\mathrm{Im}(L_A)$) and **find a basis** for $\mathrm{Im}(A)$. (**10 pts**)

d) Verify the Rank-Nullity Theorem for the linear mapping A

(a) The linear transformation L_A defined by matrix A is the mapping from $\hat{\mathbb{R}}^3$ (since A has **3** columns) to $\hat{\mathbb{R}}^4$ (since A has **4** rows) defined by:

$$\begin{aligned} L_A \; : \; \hat{\mathbb{R}}^3 \; &\rightarrow \; \hat{\mathbb{R}}^4 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; &\mapsto \; L_A\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + 2y + z \\ -x - 2y - z \\ -5x \\ 2x \end{pmatrix} \end{aligned}$$

(b) We have: $\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \hat{\mathbb{R}}^3$:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(L_A) &\Leftrightarrow L_A(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \mathbf{0}_{\hat{\mathbb{R}}^4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \begin{pmatrix} x+2y+z \\ -x-2y-z \\ -5x \\ 2x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \begin{cases} x+2y+z=0 \\ -x-2y-z=0 \\ -5x=0 \\ 2x=0 \end{cases} \\
&\Leftrightarrow \begin{cases} x+2y+z=0 \\ x=0 \end{cases} \\
&\Leftrightarrow \begin{cases} 2y+z=0 \\ x=0 \end{cases} \\
&\Leftrightarrow \begin{cases} z=-2y \\ x=0 \end{cases}
\end{aligned}$$

Hence, $\ker(L_A)$ is given by:

$$\ker(L_A) = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \hat{\mathbb{R}}^3 \ : \ z=-2y \text{ and } x=0\}.$$

We now compute a basis for $\ker(L_A)$; first, we try to find a generating set for $\ker(L_A)$. Using our characterization of $\ker(L_A)$, we can write: $\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \hat{\mathbb{R}}^3$,

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(L_A) &\Leftrightarrow \begin{cases} z=-2y \\ x=0 \end{cases} \\
&\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ -2y \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}
\end{aligned}$$

Letting $\mathbf{v}_1 \in \hat{\mathbb{R}}^3$ be defined by $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$, we have therefore shown that, $\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \hat{\mathbb{R}}^3$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(L_A) \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \cdot \mathbf{v}_1.$$

This shows that any element in $\ker(L_A)$ is in the linear span of $\mathbf{v}_1$, and, conversely, any element in the linear span of $\mathbf{v}_1$ is in $\ker(L_A)$; in other words, the linear span $\mathcal{S}_{(\mathbf{v}_1)}$ of $\mathbf{v}_1$ is equal to $\ker(L_A)$. Hence, $\{\mathbf{v}_1\}$ is a generating set for $\ker(L_A)$. Furthermore, since $\mathbf{v}_1 \neq \mathbf{0}_{\hat{\mathbb{R}}^3}$, it follows that $\{\mathbf{v}_1\}$ is a linearly independent subset of $\ker(L_A)$. Hence, $(\mathbf{v}_1)$ is a basis for $\ker(L_A)$. (Note that since this basis has one element, it follows that $\ker(L_A)$ has dimension 1).

c) Recall that $\mathrm{Im}(L_A)$ is the linear span of the column vectors of A. Let $A_{;1}, A_{;2}, A_{;3} \in \hat{\mathbb{R}}^4$ be the first, second, and third column vectors of A, respectively; i.e., we have:

$$A_{;1} = \begin{pmatrix} 1 \\ -1 \\ -5 \\ 2 \end{pmatrix}, \quad A_{;2} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad A_{;3} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Writing that $\mathrm{Im}(L_A)$ is the linear span of $A_{;1}, A_{;2}, A_{;3}$, and then using the definition of linear span, we have:

$$\mathrm{Im}(L_A) = \mathcal{S}_{(A_{;1}, A_{;2}, A_{;3})} = \{\alpha_1 \cdot A_{;1} + \alpha_2 \cdot A_{;2} + \alpha_3 \cdot A_{;3} \in \hat{\mathbb{R}}^4 \ : \ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\},$$

and noting that $A_{;2} = 2 \cdot A_{;3}$, we obtain:

$$\begin{aligned} \mathrm{Im}(L_A) &= \{\alpha_1 \cdot A_{;1} + 2\alpha_2 \cdot A_{;3} + \alpha_3 \cdot A_{;3} \in \hat{\mathbb{R}}^4 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} \\ &= \{\alpha_1 \cdot A_{;1} + (2\alpha_2 + \alpha_3) \cdot A_{;3} \in \hat{\mathbb{R}}^4 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} \\ &= \{\alpha \cdot A_{;1} + \beta \cdot A_{;3} \in \hat{\mathbb{R}}^4 \mid \alpha, \beta \in \mathbb{R}\} \\ &= \mathcal{S}_{(A_{;1}, A_{;3})}, \end{aligned}$$

which shows that $\{A_{;1}, A_{;3}\}$ is a generating set for $\mathrm{Im}(L_A)$. Let us now prove that the subset

$\{A_{;1}, A_{;3}\}$ of $\mathrm{Im}(L_A)$ is also linearly independent. We have, $\forall \alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}
\alpha \cdot A_{;1} + \beta \cdot A_{;3} = \mathbf{0}_{\hat{\mathbb{R}}^4} \quad &\Leftrightarrow \quad \alpha \cdot \begin{pmatrix} 1 \\ -1 \\ -5 \\ 2 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \begin{pmatrix} \alpha + \beta \\ -\alpha - \beta \\ -5\alpha \\ 2\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \quad \alpha + \beta = 0 \text{ and } \alpha = 0 \\
&\Rightarrow \quad \alpha = \beta = 0.
\end{aligned}$$

This proves that $\{A_{;1}, A_{;3}\}$ is a linearly independent subset of $\mathrm{Im}(L_A)$. Hence, $(A_{;1}, A_{;3})$ is a basis for $\mathrm{Im}(L_A)$. (Note that since this basis has 2 elements, it follows that $\mathrm{Im}(L_A)$ has dimension 2).

d) dim(V) = rank(L) + nullity(L)
3 = 2 + 1
3 =3

Matrix Multiplication:

When multiplying two matrices together, consider the following:

$$\begin{array}{ccc} A & \cdot & B \\ 3 \cdot \mathbf{3} & & \mathbf{3} \cdot 2 \end{array}$$

In order to multiply matrices A and B, the number of columns in the first matrix must match the number of rows of the second. See in the example above, A has 3 columns, and B has 3 rows, therefore the multiplication A · B is possible. However, B · A is not possible since B has 2 columns and A has 3 rows.

Exercise:

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}$. Compute A·B

Solution: A•B = $\begin{bmatrix} 11 & 4 \\ 2 & 9 \\ 15 & 18 \end{bmatrix}$

Theorem 29: Matrix multiplication satisfies the following properties:

(i) Let m, n, p, q be integers ≥ 1. $\forall A \in M_{m,n}(\mathbb{R})$, $\forall B \in M_{n,p}(\mathbb{R})$, and $\forall C \in M_{p,q}(\mathbb{R})$, we have: (AB)C = A(BC), i.e. matrix multiplication is associative.

(ii) Let m, n, p be integers ≥ 1. $\forall A \in M_{m,n}(\mathbb{R})$, $\forall B, C \in M_{n,p}(\mathbb{R})$, we have:

$$A(B + C) = AB + AC$$

(iii) Let m, n, p be integers ≥ 1. $\forall A, B \in M_{m,n}(\mathbb{R})$, $\forall C \in M_{n,p}(\mathbb{R})$, we have:

$$(A + B)C = AC + BC$$

(iv) Let m, n, p be integers ≥ 1. $\forall A \in M_{m,n}(\mathbb{R})$, $\forall B \in M_{n,p}(R)$, $\forall\, \alpha \in \mathbb{R}$, we have:

$$A(\alpha B) = (\alpha A)B = \alpha (AB).$$

Section 12: Invertible Square Matrices

Definition 30: Let $A \in M_n(\mathbb{R})$ be a real $n \times n$ matrix. A is said to be invertible if there exists a real $n \times n$ matrix B such that $AB = BA = I$, where I (shown below) is the $n \times n$ identity matrix. This matrix be is called the **inverse** of A, or A^{-1}

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note:

(i) The definition of invertible matrices applies only to **square matrices**.

(ii) It does not make any sense to talk about invertibility of an $m \times n$ matrix with $m \neq n$.

Theorem 31: Let $A \in M_n(\mathbb{R})$ be a real $n \times n$ matrix. We have that A is **invertible** if and only if the column vectors of A are linearly independent. Equivalently, A is invertible if and only if A has rank n.

Definition 33: Let $A \in M_n(\mathbb{R})$ be a square $n \times n$ real matrix. The **determinant**, denoted det(A), of A is the real number defined as follows:

(i) If $n = 1$, i.e. $A = (a)$ for some real number a, then $\det(A) = a$;

(ii) If $n > 1$, then det(A) is recursively defined as follows:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \det([A]_{1,j})$$

Where $[A]_{1,j}$ is the matrix A without the i'th row and j'th column. (*note that it doesn't have to be the first row, it can be any row or any column).

Note: For a 2x2 matrix: $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Theorem 32: Let $A \in M_n(\mathbb{R})$ be a real $n \times n$ matrix. We have that A is invertible if and only if $\det(A) \neq 0$.

Definition 34: Let $M \in M_{m,n}(\mathbb{R})$ be a real $m \times n$ matrix. The **transpose** of M, denoted M^T, is the $n \times m$ real matrix defined by: $(M^T)_{i,j} = (M)_{j,i}$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, where by the notation

$(C)_{a,b}$ we mean the entry of matrix C on row a and column b. (ie: flip the matrix along the diagonal)

Theorem 34: Let $A \in M_n(\mathbb{R})$ be a real n × n matrix. We have: $\det(A) = \det(A^T)$.

Theorem 35: Let $A, B \in M_n(\mathbb{R})$ be real n × n matrices. We have: $\det(AB) = \det(A)\det(B)$

Note: For a diagonal, upper triangular, or lower triangular matrix, the determinant is the product of the entries on the diagonal.

Example: Compute the determinant of $\begin{bmatrix} 0 & 1 & 1 \\ 3 & 3 & 0 \\ 1 & 2 & 9 \end{bmatrix}$

Solution: $det(\begin{bmatrix} 0 & 1 & 1 \\ 3 & 3 & 0 \\ 1 & 2 & 9 \end{bmatrix}) = 0 - 1 * \det\begin{pmatrix} 3 & 0 \\ 1 & 9 \end{pmatrix} + \det\begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix} = -1(27) + (3) = -24$

Theorem 38: $\forall A \in M_n(\mathbb{R})$, $\forall \alpha \in \mathbb{R}$, we have: $\det(\alpha A) = \alpha$ n $\det(A)$.

Problem 6, 2015 Final Exam

6. Answer the following questions.

(a) Suppose that A and B are invertible $n \times n$ matrices. Is AB invertible? (Provide an argument if your answer is yes, and a counterexample if your answer is no.) (**5 pts**)

Let A and B be the matrices

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 \\ 1 & -4 \\ -2 & 1 \end{pmatrix}.$$

(b) Are either of A or B invertible matrices? (Be sure to give reasons). (**2 pts**)

(c) Compute the product AB. (**4 pts**)

(d) Is AB an invertible matrix? (**4 pts**)

Solution:

(a) For these two $n \times n$ matrices, if A is invertible with inverse A^{-1} and B is invertible with inverse B^{-1}, then by associativity of matrix multiplication and the property of the identity matrix I_n (of size $n \times n$), we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.$$

Similarly,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Thus AB is invertible and its inverse is given by

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(b) Since invertibility is defined only for square matrices, noting that A and B are both rectangular matrices (with sizes 2×3 and 3×2, respectively), we directly conclude that both A and B are not invertible.

(c) The product AB is given by:

$$\begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & -4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ -5 & -9 \end{pmatrix}.$$

(d) AB is invertible iff its determinant is non-zero. We have

$$det(AB) = ((5)(-9) - (6)(-5)) = -15 \neq 0.$$

Thus AB is invertible.

Problem 5, 2016 Final Exam

Problem 5

Consider the real 3×3 matrix A given by

$$A = \begin{pmatrix} -1 & -2 & -1 \\ 2 & 1 & 5 \\ 4 & -3 & 7 \end{pmatrix}.$$

(a) **Determine invertibility of A by examining the column vectors of A.** [5 pts]

(b) **Determine invertibility of A by computing the determinant** $\det(A)$ **of A.** [5 pts]

(c) **Compute A^2 (i.e. AA).** [5 pts]

(d) **Compute** $\det(A^2)$. [5 pts]

This is a good problem because it directly makes use of the theorems stated above.

a) Theorem 31 above states "A is **invertible** if and only if the column vectors of A are linearly independent."
Write $\propto_1 A_{;1} + \propto_2 A_{;2} + \propto_3 A_{;3} = \mathbf{0}$

$$\propto_1 \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + \propto_2 \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} + \propto_3 \begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} = \mathbf{0}$$

i) $-\propto_1 - 2\propto_2 - \propto_3 = 0$

ii) $2\propto_1 + \propto_2 + 5\propto_3 = 0$

iii) $4\propto_1 - 3\propto_2 + 7\propto_3 = 0$

Last equation gives

$$3\propto_2 = 4\propto_1 + 7\propto_3$$
$$\propto_2 = \frac{7}{3}\propto_3 + \frac{4}{3}\propto_1$$

Substitute this into equation 2 to get

$$2\propto_1 + 4\propto_1 + \frac{7}{3}\propto_3 + 5\propto_3 = 0$$
$$\frac{10}{3}\propto_1 = -\frac{22}{3}\propto_3$$
$$\propto_1 = -\frac{11}{5}\propto_3$$

Substitute this into equation 1 to get $\propto_1 = \propto_2 = \propto_3 = 0$
Therefore, by Theorem 31, since the column vectors of A are linearly independent, A is invertible.

b)

$$\det(A) = -1\det\begin{pmatrix} 1 & 5 \\ -3 & 7 \end{pmatrix} + 2\det\begin{pmatrix} 2 & 5 \\ 4 & 7 \end{pmatrix} - 1\det\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$
$$= -22 - 12 + 10 = -24,$$

Since det(A)≠0, A is invertible (Theorem 32 above)

b) **AA =** $\begin{pmatrix} -1 & -2 & -1 \\ 2 & 1 & 5 \\ 4 & -3 & 7 \end{pmatrix}\begin{pmatrix} -1 & -2 & -1 \\ 2 & 1 & 5 \\ 4 & -3 & 7 \end{pmatrix} = \begin{pmatrix} -7 & 3 & -16 \\ 20 & -18 & 38 \\ 18 & -32 & 30 \end{pmatrix}$

c) $det(A^2) = (det(A))^2 = 576$ (Theorem 35 above)

Section 13: Eigenvalues & Eigenvectors

Definition 35: Let v ∈ V with v ≠ 0 (i.e. v is not the zero vector of V); v is said to be an **eigenvector** of the linear transformation L if there exists a real number λ such that:

L(v) = λv.

The real number λ in the above relation is called an **eigenvalue** of L. We then say that v is an eigenvector of L associated to the eigenvalue λ.

Theorem 41: Let A ∈ $M_n(\mathbb{R})$ be a real n × n matrix. Let λ ∈ ℝ. We have λ is an **eigenvalue** of A if and only if det(λI − A) = 0 (where again I denotes the n × n **identity matrix**). This theorem gives us a systematic way of computing eigenvalues.

Theorem 42: Let A ∈ $M_n(\mathbb{R})$ be a real n × n matrix. Then 0 is an eigenvalue of A if and only if A is not invertible.

Steps for Finding Eigenvalues/Vectors

1. Determine characteristic polynomial by calculating det(λI-A) for a matrix A. For example, if
A=$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, find det($\begin{bmatrix} \lambda-1 & -2 & -2 \\ 0 & \lambda-3 & -1 \\ 0 & 0 & \lambda-2 \end{bmatrix}$)
2. Once all of the eigenvalues are found (in this case they are 1,2,3), find the corresponding eigenvectors for each eigenvalue. In order to do this, find the ker(λI-A). Continuing the above example, choosing 2 as our eigenvalue, take (2I-A)v = **0** to find an eigenvector. This is equivalent to finding a vector in ker($\begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$). This gives the eigenvector [0,-1,1].

Problem 5, 2015 Final Exam

5. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$, and let

$$A = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix}.$$

(a) Say what it means for a vector $\mathbf{v}$ to be an eigenvector of A. (That is, give the definition of "$\mathbf{v}$ is an eigenvector of A".) (**3 pts**)

(b) Compute $A\mathbf{v}_1$, $A\mathbf{v}_2$, and $A\mathbf{v}_3$. (**3 pts**)

(c) Your computations in (b) should show that each of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of A. What are their eigenvalues? (**3 pts**)

(d) What is $A^4\mathbf{v}_3$? (i.e., the result of putting $\mathbf{v}_3$ through A four times.) (**4 pts**)

(e) Write the vector $\mathbf{w} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (**4 pts**)

(f) For a given integer $n \geq 1$, give a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of A. (**3 pts**)

Solution:

(a) We say that $\mathbf{v}$ is an eigenvector of A if $\mathbf{v}$ is not equal to the zero vector and there exists a real number λ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

In this case λ is called the eigenvalue of A associated with $\mathbf{v}$.

(b) We have

$$A\mathbf{v}_1 = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix},$$

$$A\mathbf{v}_2 = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

and

$$A\mathbf{v}_3 = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}.$$

(c) Indeed,

$$A\mathbf{v}_1 = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = (-1)\mathbf{v}_1$$

and hence the eigenvalue of $\mathbf{v}_1$ is $\lambda_1 = -1$. Also,

$$A\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = (1) \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = (1)\mathbf{v}_2$$

and hence the eigenvalue of $\mathbf{v}_2$ is $\lambda_2 = 1$. Finally,

$$A\mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix} = (2) \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = (2)\mathbf{v}_3$$

and hence the eigenvalue of $\mathbf{v}_3$ is $\lambda_3 = 2$.

(d) Using the fact that $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$ repeatedly, we have

$$\begin{aligned} A^4\mathbf{v}_3 &= A^3(A\mathbf{v}_3) = A^3(\lambda_3\mathbf{v}_3) = \lambda_3 A^2(A\mathbf{v}_3) \\ &= \lambda_3 A^2(\lambda_3\mathbf{v}_3) = \lambda_3^2 A(A\mathbf{v}_3) = \lambda_3^2 A(\lambda_3\mathbf{v}_3) \\ &= \lambda_3^3(A\mathbf{v}_3) = \lambda_3^3(\lambda_3\mathbf{v}_3) \\ &= \lambda_3^4\mathbf{v}_3 \end{aligned}$$

Thus

$$A^4\mathbf{v}_3 = \lambda_3^4\mathbf{v}_3 = (2)^4\begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 16 \\ -16 \\ -48 \end{pmatrix}.$$

(e) To write the vector $\mathbf{w} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we need to find scalars α, β and γ such that

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = w$$

or equivalently

$$\alpha\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}.$$

In other words, we have to solve the following system of linear equations:

$$\begin{cases} \alpha + \gamma = 5 \\ 2\alpha + \beta - \gamma = 4 \\ 2\alpha + 2\beta - 3\gamma = -1 \end{cases}$$

Solving the above system via the Gaussian elimination method, we obtain a unique solution given by:

$$(\alpha, \beta, \gamma) = (4, -3, 1).$$

Thus

$$w = 4\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

(f) Given integer $n \geq 1$, we deduce that for any eigenvector $\mathbf{v}$ with eigenvalue λ,

$$A^n\mathbf{v} = \lambda^n\mathbf{v}.$$

In other words, if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n. This can be shown iteratively on n using the same procedure as in (d). Thus, using the above fact and the results in (e) and (c), we have

$$\begin{aligned} A^n w &= A^n(4\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3) \\ &= 4(A^n\mathbf{v}_1) - 3(A^n\mathbf{v}_2) + (A^n\mathbf{v}_3) \\ &= 4(\lambda_1^n\mathbf{v}_1) - 3(\lambda_2^n\mathbf{v}_2) + (\lambda_3^n\mathbf{v}_3) \\ &= 4(-1)^n\mathbf{v}_1 - 3(1)^n\mathbf{v}_2 + (2)^n\mathbf{v}_3 \\ &= 4(-1)^n\mathbf{v}_1 - 3\mathbf{v}_2 + (2)^n\mathbf{v}_3. \end{aligned}$$

Problem 3, 2016 Final Exam

Problem 3

Consider the following real vector space $(\mathbf{W}_2, +, \cdot)$ with

$$\mathbf{W}_2 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x > 0, y > 0\}$$

under the following addition and scalar multiplication operations:

- **Addition:** For any (x_1, y_1) and (x_2, y_2) in $\mathbf{W}_2$,

$$(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2).$$

- **Scalar multiplication:** For any scalar $\alpha \in \mathbb{R}$ and $(x, y) \in \mathbf{W}_2$,

$$\alpha \cdot (x, y) = (x^\alpha, y^\alpha).$$

Let now $L : \mathbf{W}_2 \to \mathbf{W}_2$ be the mapping defined by:

$$L((x, y)) = (x^2, y^3), \ \forall (x, y) \in \mathbf{W}_2.$$

(a) **Show that L is a linear mapping.** [4 pts]

(b) **Show that 2 is an eigenvalue of L, and determine a corresponding eigenvector.** [3 pts]

(c) **Show that 3 is an eigenvalue of L, and determine a corresponding eigenvector.** [3 pts]

Solution:

a) To check linearity need to check:

i) $L(v_1+v_2) = L(v_1) + L(v_2)$

ii) $L(av) = aL(v)$

First, $L(x_1,y_1 + x_2,y_2) = L(x_1x_2, y_1y_2) = [(x_1x_2)^2,(y_1y_2)^3] = (x_1^2 x_2^2, y_1^3 y_2^3)$

$L(x_1,y_1) + L(x_2,y_2) = (x_1^2,y_1^3) + (x_2^2,y_2^3) = (x_1^2 x_2^2, y_1^3 y_2^3)$

Therefore $L(v_1+v_2) = L(v_1) + L(v_2)$

Second, $L(a\bullet(x,y)) = L(x^a,y^a) = (x^{2a},y^{3a}) = aL(x,y)$

L satisfies both conditions of a linear map, therefore it is a linear map

(b) We have: $L((2,1)) = (2^2, 1^3) = (4,1) = 2 \cdot (2,1)$, and since $(2,1)$ is not equal to the zero vector of $\mathbf{W}_2$ (which is equal to $(1,1)$), it follows that the real number 2 is an eigenvalue of L, and that $(2,1)$ is a corresponding eigenvector.

(c) We have: $L((1,2)) = (1^2, 2^3) = (1,8) = 3 \cdot (1,2)$, and since $(1,2)$ is not equal to the zero vector of $\mathbf{W}_2$ (which is equal to $(1,1)$), it follows that the real number 3 is an eigenvalue of L, and that $(1,2)$ is a corresponding eigenvector.

Alternative solution to b and c:

b) $M(L) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ since these are the scalar multiples we "multiply" x and y by respectively. Since this is a diagonal matrix, 2 is an eigenvalue.
Eigenvector satisfies Av = λv. ∴ we must have $(\lambda I - A)v = \mathbf{0}$, which is equivalent to finding a vector v ∈ ker($\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$).

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$(v_1)^0 \bullet (v_2)^0 = 1$
$(v_1)^0 \bullet (v_2)^{-1} = 1 \rightarrow (v_2)^{-1} = 1 \rightarrow v_2 = 1$.

Therefore v_1 = any real number > 0, ≠ 1 , $v_2 = 1$. An eigenvector is [2,1]. Note $v_1 \neq 1$ because otherwise $(v_1, v_2) = (1,1)$ which is not an eigenvector because eigenvectors must not be the zero vector. (Definition 35)

c) Again, since M(L) is a diagonal matrix, 3 is clearly an eigenvalue. Eigenvector satisfies Av = λv. ∴ we must have $(\lambda I - A)v = \mathbf{0}$, which is equivalent to finding a vector v ∈ ker($\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$).

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$(v_1)^1 \bullet (v_2)^0 = 1 \rightarrow (v_1)^1 = 1 \rightarrow v_1 = 1$
$(v_1)^0 \bullet (v_2)^0 = 1$

Therefore $v_1 = 1$, v_2 = any real number > 0, ≠ 1. An eigenvector is [1,2]. Note $v_2 \neq 1$ because this would mean $(v_1, v_2) = (1,1)$ which is not an eigenvector because eigenvectors must not be the zero vector. (Definition 35)