11.2 Areas of Regular Polygons

What you should learn

GOAL 1 Find the area of an equilateral triangle.

GOAL 2 Find the area of a regular polygon, such as the area of a dodecagon in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the area of a hexagonal mirror on the Hobby-Eberly Telescope in **Exs. 45 and 46**.

REAL LIFE

GOAL 1 FINDING THE AREA OF AN EQUILATERAL TRIANGLE

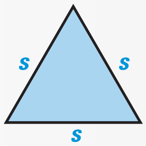
The area of *any* triangle with base length b and height h is given by $A = \frac{1}{2}bh$. The following formula for equilateral triangles, however, uses only the side length.

THEOREM

THEOREM 11.3 ***Area of an Equilateral Triangle***

The area of an equilateral triangle is one fourth the square of the length of the side times $\sqrt{3}$.

$$A = \frac{1}{4}\sqrt{3}s^2$$



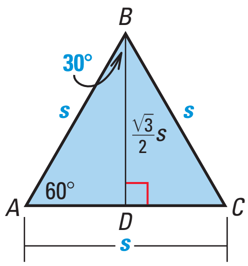
EXAMPLE 1 *Proof of Theorem 11.3*

Prove Theorem 11.3. Refer to the figure below.

SOLUTION

GIVEN ▶ $\triangle ABC$ is equilateral.

PROVE ▶ Area of $\triangle ABC$ is $A = \frac{1}{4}\sqrt{3}s^2$.



Paragraph Proof Draw the altitude from B to side $\overline{AC}$. Then $\triangle ABD$ is a 30°-60°-90° triangle. From Lesson 9.4, the length of $\overline{BD}$, the side opposite the 60° angle in $\triangle ABD$, is $\frac{\sqrt{3}}{2}s$. Using the formula for the area of a triangle,

$$A = \frac{1}{2}bh = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2}s\right) = \frac{1}{4}\sqrt{3}s^2.$$

EXAMPLE 2 *Finding the Area of an Equilateral Triangle*

Find the area of an equilateral triangle with 8 inch sides.

SOLUTION

Use $s = 8$ in the formula from Theorem 11.3.

$$A = \frac{1}{4}\sqrt{3}s^2 = \frac{1}{4}\sqrt{3}(8^2) = \frac{1}{4}\sqrt{3}(64) = \frac{1}{4}(64)\sqrt{3} = 16\sqrt{3} \text{ square inches}$$

▶ Using a calculator, the area is about 27.7 square inches.

STUDENT HELP

Study Tip

Be careful with radical signs. Notice in Example 1 that $\sqrt{3}s^2$ and $\sqrt{3s^2}$ do not mean the same thing.

GOAL 2 FINDING THE AREA OF A REGULAR POLYGON

You can use equilateral triangles to find the area of a regular hexagon.

ACTIVITY Developing Concepts

Investigating the Area of a Regular Hexagon

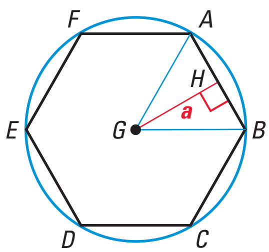
Use a protractor and ruler to draw a regular hexagon. Cut out your hexagon. Fold and draw the three lines through opposite vertices. The point where these lines intersect is the *center* of the hexagon.

1. How many triangles are formed? What kind of triangles are they?
2. Measure a side of the hexagon. Find the area of one of the triangles. What is the area of the entire hexagon? Explain your reasoning.

Think of the hexagon in the activity above, or another regular polygon, as inscribed in a circle.

The **center of the polygon** and **radius of the polygon** are the center and radius of its circumscribed circle, respectively.

The distance from the center to any side of the polygon is called the **apothem of the polygon**. The apothem is the height of a triangle between the center and two consecutive vertices of the polygon.



Hexagon *ABCDEF* with center *G*, radius *GA*, and apothem *GH*

STUDENT HELP

Study Tip
In a regular polygon, the length of each side is the same. If this length is s and there are n sides, then the perimeter P of the polygon is $n \cdot s$, or $P = ns$.

As in the activity, you can find the area of any regular n-gon by dividing the polygon into congruent triangles.

$$A = \boxed{\text{area of one triangle}} \cdot \boxed{\text{number of triangles}}$$

$$= \left(\frac{1}{2} \cdot \text{apothem} \cdot \text{side length } s\right) \cdot \text{number of sides}$$

$$= \frac{1}{2} \cdot \text{apothem} \cdot \text{number of sides} \cdot \text{side length } s$$

$$= \frac{1}{2} \cdot \text{apothem} \cdot \text{perimeter of polygon}$$

The number of congruent triangles formed will be the same as the number of sides of the polygon.

This approach can be used to find the area of any regular polygon.

THEOREM

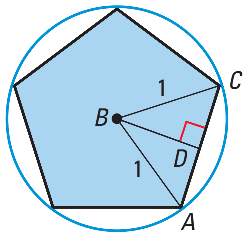
THEOREM 11.4 ***Area of a Regular Polygon***

The area of a regular n-gon with side length s is half the product of the apothem a and the perimeter P, so $A = \frac{1}{2}aP$, or $A = \frac{1}{2}a \cdot ns$.

A **central angle of a regular polygon** is an angle whose vertex is the center and whose sides contain two consecutive vertices of the polygon. You can divide 360° by the number of sides to find the measure of each central angle of the polygon.

EXAMPLE 3 *Finding the Area of a Regular Polygon*

A regular pentagon is inscribed in a circle with radius 1 unit. Find the area of the pentagon.



SOLUTION

To apply the formula for the area of a regular pentagon, you must find its apothem and perimeter.

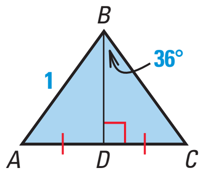
The measure of central $\angle ABC$ is $\frac{1}{5} \cdot 360°$, or $72°$.

In isosceles triangle $\triangle ABC$, the altitude to base $\overline{AC}$ also bisects $\angle ABC$ and side $\overline{AC}$. The measure of $\angle DBC$, then, is 36°. In right triangle $\triangle BDC$, you can use trigonometric ratios to find the lengths of the legs.

STUDENT HELP

Look Back
For help with trigonometric ratios, see p. 558.

$$\cos 36° = \frac{BD}{BC} = \frac{BD}{1} = BD \qquad \sin 36° = \frac{DC}{BC} = \frac{DC}{1} = DC$$



▶ So, the pentagon has an apothem of $a = BD = \cos 36°$ and a perimeter of $P = 5(AC) = 5(2 \cdot DC) = 10 \sin 36°$. The area of the pentagon is

$$A = \frac{1}{2}aP = \frac{1}{2}(\cos 36°)(10 \sin 36°) \approx 2.38 \text{ square units.}$$

EXAMPLE 4 *Finding the Area of a Regular Dodecagon*

FOCUS ON APPLICATIONS

FOUCAULT PENDULUMS swing continuously in a straight line. Watching the pendulum, though, you may think its path shifts. Instead, it is Earth and you that are turning. The floor under this pendulum in Texas rotates fully about every 48 hours.

APPLICATION LINK
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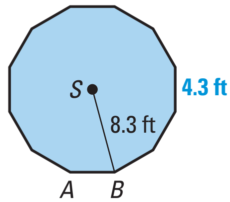
PENDULUMS The enclosure on the floor underneath the Foucault Pendulum at the Houston Museum of Natural Sciences in Houston, Texas, is a regular dodecagon with a side length of about 4.3 feet and a radius of about 8.3 feet. What is the floor area of the enclosure?



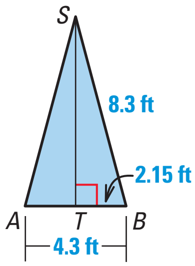
SOLUTION

A dodecagon has 12 sides. So, the perimeter of the enclosure is

$$P \approx 12(4.3) = 51.6 \text{ feet.}$$

In $\triangle SBT$, $BT = \frac{1}{2}(BA) = \frac{1}{2}(4.3) = 2.15$ feet. Use the Pythagorean Theorem to find the apothem ST.

$$a = \sqrt{8.3^2 - 2.15^2} \approx 8 \text{ feet}$$



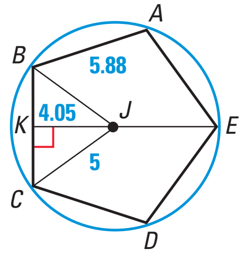
▶ So, the floor area of the enclosure is

$$A = \frac{1}{2}aP \approx \frac{1}{2}(8)(51.6) = 206.4 \text{ square feet.}$$

Guided Practice

Vocabulary Check ✓

In Exercises 1–4, use the diagram shown.



1. Identify the *center* of polygon *ABCDE*.

2. Identify the *radius* of the polygon.

3. Identify a *central angle* of the polygon.

4. Identify a segment whose length is the *apothem*.

Concept Check ✓

5. In a regular polygon, how do you find the measure of each central angle?

Skill Check ✓

6. What is the area of an equilateral triangle with 3 inch sides?

STOP SIGN **The stop sign shown is a regular octagon. Its perimeter is about 80 inches and its height is about 24 inches.**

7. What is the measure of each central angle?

8. Find the apothem, radius, and area of the stop sign.

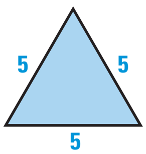
Practice and Applications

STUDENT HELP

Extra Practice to help you master skills is on p. 823.

FINDING AREA **Find the area of the triangle.**

9.



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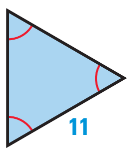
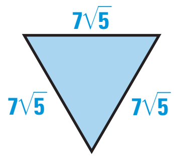


11.

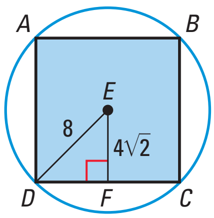


MEASURES OF CENTRAL ANGLES **Find the measure of a central angle of a regular polygon with the given number of sides.**

12. 9 sides **13.** 12 sides **14.** 15 sides **15.** 180 sides

FINDING AREA **Find the area of the inscribed regular polygon shown.**

16.



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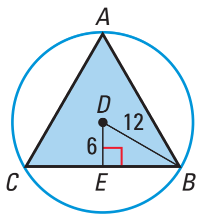

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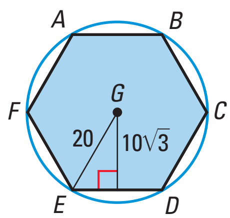


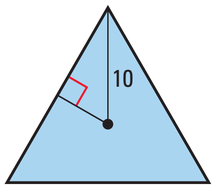
STUDENT HELP

HOMEWORK HELP

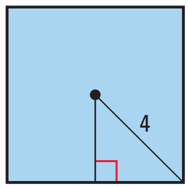
Example 1: Exs. 9–11, 17, 19, 25, 33
Example 2: Exs. 9–11, 17, 19, 25, 33
Example 3: Exs. 12–24, 26, 34
Example 4: Exs. 34, 45–49

PERIMETER AND AREA **Find the perimeter and area of the regular polygon.**

19.



20.



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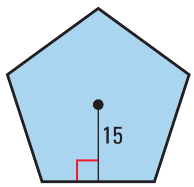
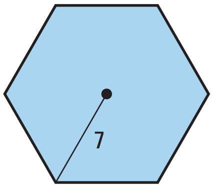
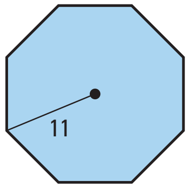


Perimeter and Area **In Exercises 22–24, find the perimeter and area of the regular polygon.**

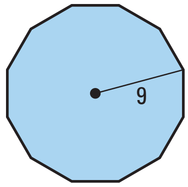
22.



23.



24.



25. **Area** Find the area of an equilateral triangle that has a height of 15 inches.

26. **Area** Find the area of a regular dodecagon (or 12-gon) that has 4 inch sides.

Logical Reasoning **Decide whether the statement is *true* or *false*. Explain your choice.**

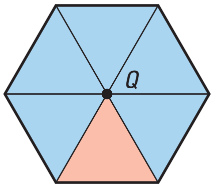
27. The area of a regular polygon of fixed radius r increases as the number of sides increases.

28. The apothem of a regular polygon is always less than the radius.

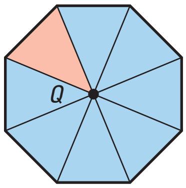
29. The radius of a regular polygon is always less than the side length.

Area **In Exercises 30–32, find the area of the regular polygon. The area of the portion shaded in red is given. Round answers to the nearest tenth.**

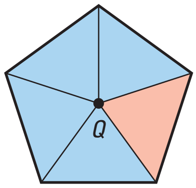
30. Area $= 16\sqrt{3}$



31. Area $= 4 \tan 67.5°$



32. Area $= \tan 54°$

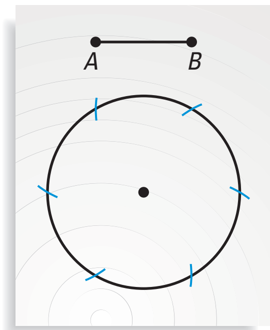


33. **Using the Area Formulas** Show that the area of a regular hexagon is six times the area of an equilateral triangle with the same side length.

$\left(\textit{Hint}\text{: Show that for a hexagon with side lengths } s,\ \frac{1}{2}aP = 6 \cdot \left(\frac{1}{4}\sqrt{3}\,s^2\right).\right)$

34. **Basaltic Columns** Suppose the top of one of the columns along the Giant's Causeway (see p. 659) is in the shape of a regular hexagon with a diameter of 18 inches. What is its apothem?

Construction **In Exercises 35–39, use a straightedge and a compass to construct a regular hexagon and an equilateral triangle.**



35. Draw $\overline{AB}$ with a length of 1 inch. Open the compass to 1 inch and draw a circle with that radius.

36. Using the same compass setting, mark off equal parts along the circle.

37. Connect the six points where the compass marks and circle intersect to draw a regular hexagon.

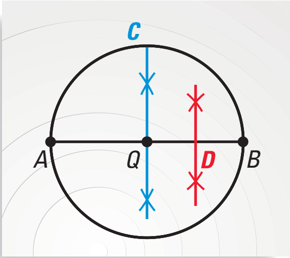
38. What is the area of the hexagon?

39. *Writing* Explain how you could use this construction to construct an equilateral triangle.

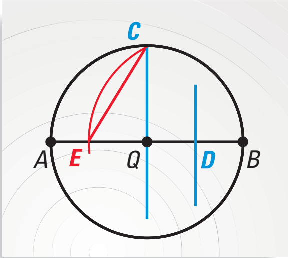
STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell for help with construction in Exs. 40–44.

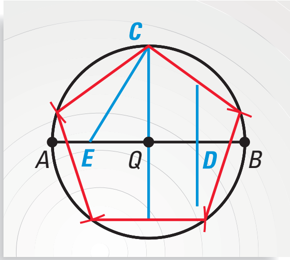
CONSTRUCTION **In Exercises 40–44, use a straightedge and a compass to construct a regular pentagon as shown in the diagrams below.**



Exs. 40, 41



Ex. 42



Exs. 43, 44

40. Draw a circle with center Q. Draw a diameter $\overline{AB}$. Construct the perpendicular bisector of $\overline{AB}$ and label its intersection with the circle as point C.

41. Construct point D, the midpoint of $\overline{QB}$.

42. Place the compass point at D. Open the compass to the length DC and draw an arc from C so it intersects $\overline{AB}$ at a point, E. Draw $\overline{CE}$.

43. Open the compass to the length CE. Starting at C, mark off equal parts along the circle.

44. Connect the five points where the compass marks and circle intersect to draw a regular pentagon. What is the area of your pentagon?

FOCUS ON CAREERS

REAL LIFE ASTRONOMERS use physics and mathematics to study the universe, including the sun, moon, planets, stars, and galaxies.

INTERNET CAREER LINK
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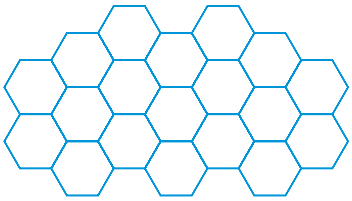
TELESCOPES **In Exercises 45 and 46, use the following information.**
The Hobby-Eberly Telescope in Fort Davis, Texas, is the largest optical telescope in North America. The primary mirror for the telescope consists of 91 smaller mirrors forming a hexagon shape. Each of the smaller mirror parts is itself a hexagon with side length 0.5 meter.

45. What is the apothem of one of the smaller mirrors?

46. Find the perimeter and area of one of the smaller mirrors.

TILING **In Exercises 47–49, use the following information.**
You are tiling a bathroom floor with tiles that are regular hexagons, as shown. Each tile has 6 inch sides. You want to choose different colors so that no two adjacent tiles are the same color.



47. What is the minimum number of colors that you can use?

48. What is the area of each tile?

49. The floor that you are tiling is rectangular. Its width is 6 feet and its length is 8 feet. At least how many tiles of each color will you need?

Test Preparation

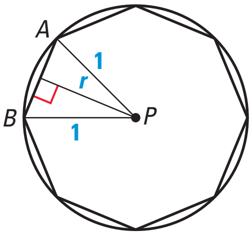
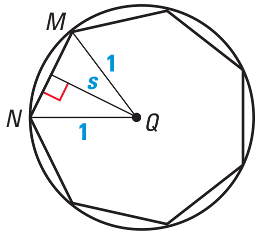
QUANTITATIVE COMPARISON **In Exercises 50–52, choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

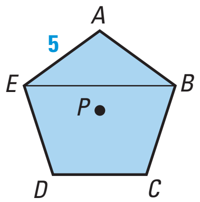
Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
		
50.	$m\angle APB$	$m\angle MQN$
51.	Apothem r	Apothem s
52.	Perimeter of octagon with center P	Perimeter of heptagon with center Q

★ Challenge

53. USING DIFFERENT METHODS Find the area of $ABCDE$ by using two methods. First, use the formula $A = \frac{1}{2}aP$, or $A = \frac{1}{2}a \cdot ns$. Second, add the areas of the smaller polygons. Check that both methods yield the same area.



EXTRA CHALLENGE

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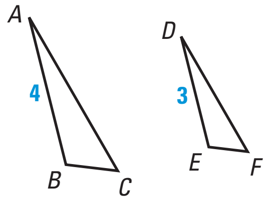
MIXED REVIEW

SOLVING PROPORTIONS **Solve the proportion.** (Review 8.1 for 11.3)

54. $\frac{x}{6} = \frac{11}{12}$ **55.** $\frac{20}{4} = \frac{15}{x}$ **56.** $\frac{12}{x+7} = \frac{13}{x}$ **57.** $\frac{x+6}{9} = \frac{x}{11}$

USING SIMILAR POLYGONS **In the diagram shown, $\triangle ABC \sim \triangle DEF$. Use the figures to determine whether the statement is true.** (Review 8.3 for 11.3)

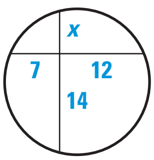


58. $\frac{AC}{BC} = \frac{DF}{EF}$ **59.** $\frac{DF}{AC} = \frac{EF + DE + DF}{BC + AB + AC}$

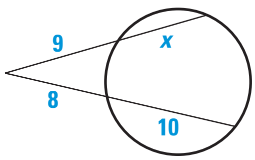
60. $\angle B \cong \angle E$ **61.** $\overline{BC} \cong \overline{EF}$

FINDING SEGMENT LENGTHS **Find the value of x.** (Review 10.5)

62.



63.



64.