Asset Prices in An Exchange Economy

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SUFE

Math review

Reference: Stokey, Lucas with Prescott (1989)

Definition

A **metric space** is a set S, together with a metric (distance function) $\rho : S \times S \to R$, such that for all x, y, $z \in S$:

a. $\rho(x, y) \geq 0$, with equality if and only if $x = y$;

b. $\rho(x, y) = \rho(y, x)$; and

c. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

A metric abstracts the four basic properties of Euclidean distance:

- the distance between distinct points is strictly positive;
- the distance from a point to itself is zero;
- distance is symmetric;
- and the triangle inequality holds.

Math review - A complete metric space

Definition

A metric space $(S,\ \rho)$ is **complete** if every Cauchy sequence in S converges to an element in S.

A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence (satisfies the Cauchy criterion) if for each $\varepsilon > 0$, there exists N_ε such that $\rho(x_n, x_m) < \varepsilon$, all $n,\ m \geq N_\varepsilon$.

Math review -contraction mapping

Definition

Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself. T is a contraction mapping (with modulus β) if for some $\beta \in (0,1)$, $\rho(Tx, Ty) \leq \beta\rho(x,y)$, for all $x, y \in S$.

T is a function mapping: $Tx = 0.3x + 9x^2 + \log(x)$, $Tx = 0.3x$

The fixed points of T : $Tx = x$

Example

On a closed interval $S = [a, b]$, with $\rho(x,y) = |x - y|$. Then $T : S \to S$ is a contraction if for some $\beta \in (0,1)$:

$$\frac{|Tx - Ty|}{|x - y|} \leq \beta < 1, \text{ all } x, y \in S \text{ with } x \neq y.$$

Math review - Contraction Mapping Theorem

Theorem

If (S, ρ) is a complete metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β, then

a. T has exactly one fixed point v in S, and

b. for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2, ...$

Math review - Contraction Mapping Theorem

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- Choose $v_0 \in S$, and define $\{v_n\}_{n=0}^{\infty}$ by $v_{n+1} = Tv_n$, so that $v_n = T^n v_0$

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- Choose $v_0 \in S$, and define $\{v_n\}_{n=0}^{\infty}$ by $v_{n+1} = Tv_n$, so that $v_n = T^n v_0$
- By the contraction property of T,

$$\rho(v_2, v_1) = \rho(Tv_1, Tv_0) \leq \beta\rho(v_1, v_0).$$

Continuing by induction, we get

$$\rho(v_{n+1}, v_n) \leq \beta^n \rho(v_1, v_0), \ \ n = 1, 2, \ldots$$

Math review - Contraction Mapping Theorem

Proof.

Hence, for any $m > n$,

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + ... + \rho(v_{n+2}, v_{n+1}) + \rho(v_{n+1}, v_n) \\
&\quad \text{triangle inequality}
\end{aligned}$$

$$\begin{aligned}
\rho(v_m, v_n) &\leq \left[\beta^{m-1} + ... + \beta^{n+1} + \beta^n\right] \rho(v_1, v_0) \\
&\leq \frac{\beta^n}{1-\beta}\rho(v_1, v_0).
\end{aligned}$$

$\{v_n\}$ is a Cauchy sequence. Since S is complete, it follows that $v_n \to v \in S$. □

Proof.

To show that $Tv = v$, note that for all n and all $v_0 \in S$,

$$\begin{array}{rcl} \rho(Tv, v) & \leq & \rho(Tv, T^n v_0) + \rho(T^n v_0, v) \\ & \leq & \beta\rho(v, T^{n-1} v_0) + \rho(T^n v_0, v). \\ & \vdots & \quad\quad \to 0 \quad\quad\quad\quad \to 0 \text{ as }_{n\to\infty} \end{array}$$

Hence, $\rho(Tv, v) = 0$, or $Tv = v$. □

Note that $T^{n-1} v_0$ converges to v

Proof.

Finally, we must show that there is no other function $\hat{v} \in S$ satisfying $T\ \hat{v} = \hat{v}$. Suppose to the contrary that $\hat{v} \neq v$ is another solution. Then

$$0 < a = \rho(\hat{v}, v) = \rho(T\hat{v}, Tv) \leq \beta\rho(\hat{v}, v) = \beta a,$$

which cannot hold, since $\beta < 1$. This proves part (a). □

Math review - Contraction Mapping Theorem

Proof.

To prove part (b), observe that for any $n \geq 1$,

$$\rho(T^n v_0, v) = \rho(T(T^{n-1} v_0), Tv) \leq \beta \rho(T^{n-1} v_0, v),$$

So that (b) follows by induction, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$. □

$$
\begin{aligned}
(Tv)(x) &= \sup_y \left[F(x,y) + \beta v(y)\right] \\
& \quad s.t.\ y \text{ feasible given } x
\end{aligned}
$$

- $v(x)$, x is the beginning-of-period state variable
- y is the end-of-period state to be chosen
- $x \in X \subseteq R^l$, $y \in X$
- $F(x,y)$ current period return
- Operator T, fixed point $Tv = v$

Blackwell's sufficient conditions for a contraction

(Stokey, Lucas and Prescott 1989, page 54)

Theorem

Let $X \subseteq R^l$ and let $B(X)$ be a space of bounded functions $f : X \to R$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying

- *(monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;*
- *(discounting) there exists some $\beta \in (0, 1)$ such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ all } f \in B(X), \ a \geq 0, \ x \in X$$

[Here $(f + a)(x)$ is the function defined by $(f + a)(x) = f(x) + a$.] Then T is a contraction with modulus β.

Blackwell's sufficient conditions for a contraction

Proof.

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- For any $f, g \in B(X)$, $f \leq g + \|f - g\|$.

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- If $f(x) \leq g(x)$ for all $x \in X$, we write $f \leq g$.
- For any $f, g \in B(X)$, $f \leq g + \|f - g\|$.
- Then properties (a) and (b) imply that

$$Tf \leq T(g + \|f - g\|) \leq Tg + \beta \|f - g\|$$

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- Reversing the roles of f and g gives by the same logic

$$Tg \leq Tf + \beta \|f - g\|$$

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$$Tg \leq Tf + \beta \|f - g\|$$

- Combining these two inequalities, we find that $\|Tf - Tg\| \leq \beta \|f - g\|$, as was to be shown.

□

Theorem of the Maximum

(Stokey, Lucas and Prescott 1989, page 62)

Theorem

Let $X \subseteq R^l$ *and* $Y \subseteq R^m$, *let* $f : X \times Y \to R$ *be a continuous function, and let* $\Gamma : X \to Y$ *be a compact-valued and continuous correspondence.*

- *Then the function* $h : X \to R$ *defined in (1) is continuous,*
- *and the correspondence* $G : X \to Y$ *defined in (2) is nonempty, compact-valued, and upper hemi-continuous (u.h.c.).*

$$h(x) = \max_{y \in \Gamma(x)} f(x, y) \tag{1}$$

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\} \tag{2}$$

Theorecital examination of stochastic behavior of equilibrium asset prices

- Environment:
  - A single good
  - Pure exchange
  - Identical consumers
  - A number of different productive units
  - An assest is a claim to all or part of the output of one of these units
- Shock: productivity in each unit fluctuates stochastically through time
  - Equilibrium asset prices will fluctuate as well

Rationality of expectations

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  - Contraction Mapping Operator —- learn and converge

Consumption and Utility

- Representative consumer:

$$E\left\{\sum_{t=0}^{\infty}\beta^{t}U(c_{t})\right\}$$

c_t a stochastic process representing consumption of a single good

- n distinct productive units, output is perishable

$$0 \leq c_t \leq \sum_{i=1}^{n} y_{it}$$

Output

- Production is entirely "exogenous": no resources are utilized
- Output y_t follows a Markov process defined by its transition funciton

$$F(y', y) = \Pr\left\{y_{t+1} \leq y' | y_t = y\right\}$$

Asset holdings

- Ownership in this productive units is determined each period in a competitive stock market
  - Each unit has outstanding one perfectly divisible equity share
  - A share entitles its owner as of the beginning of t to all of the unit's output in period t
  - Shares are traded, after payment of real dividends, at a competitively determined price vector $p_t = (p_{1t}, ...p_{nt})$
  - Let $z_t = (z_{1t}, ..., z_{nt})$ denote a consumer's beginning-of-period share holdings

Quantities of consumption and asset holdings

- All output will be consumed ($c_t = \sum_{i=0}^{n} y_{it}$)
- All shares will be held ($z_t = (1, ..., 1) = \underline{1}$)

States and price function

- The current output vector y_t: summarizes all relevant information on the current and future physical state of the economy

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- Knowledge of the transition function $F(y', y)$ and this function $p(y)$ will suffice to determine the stochastic character of the price process $\{p_t\}$

Policy funcitons

- A consumer's current consumption and portfolio decisions, c_t and z_{t+1}, depend on his beginning of period portfolio, z_t, the prices he faces, p_t, and the relevant information he possesses on current and future states of the economy, y_t.

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- His behavior can be described by fixed decision rules $c_t = c(z_{t,} y_t, p_t)$ and $z_{t+1} = z(z_{t,} y_t, p_t)$

Definition of equilibrium

Definition

An equilibrium is a continuous function $p(y) : E^{n+} \to E^{n+}$ and a continuous, bounded function $v(z, y) : E^{n+} \times E^{n+} \to R^{+}$ such that
(i)

$$v(z, y) = \max_{c,\ x} \left\{ U(c) + \beta \int v(x, y') dF(y', y) \right\}$$

subject to

$$c + p(y) \cdot x \leq y \cdot z + p(y) \cdot z, \;\; c \geq 0, \;\; 0 \leq x \leq \bar{z},$$

where $\bar{z}$ is a vector with components exceeding one;
(ii) for each y, $v(\underline{1}, y)$ is attained by $c = \sum_i y_i$ and $x = \underline{1}$.

Definition of equilibrium

- Condition (i) says that, given the behavior of prices, a consumer allocates his resources $y \cdot z + p(y) \cdot z$ optimally among current consumption c and end-of-period share holdings x

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- Condition (i) says that, given the behavior of prices, a consumer allocates his resources $y \cdot z + p(y) \cdot z$ optimally among current consumption c and end-of-period share holdings x
- Condition (ii) requires that these consumption and portfolio decisions be market clearing

Value function

- Since the market is always cleared, the consumer will never be observed except in the state $z = \underline{1}$. On the other hand, the consumer has the option to choose security holdings $x \neq 1$
- To evaluate these options, he needs to know $v(z, y)$ for all z
- The value $v(z, y)$ will be interpreted as the value of the objective for a consumer who begins in state y with holdings z, and follows an optimum consumption-portfolio policy thereafter

Lucas's Tree

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- Output of fruit is subject to uncertainty
- Our interest:
  The movement of the asset prices

Construction of the equilibrium - propositions

Proposition 1: *For each continuous price function $p(\cdot)$ there is a unique, bounded, continuous, nonnegative function $v(z,\ y : p)$ satisfying (i). For each y, $v(z,\ y : p)$ is an increasing, concave function of z.*

Proof.

- Define the operator T on functions $v(z, y)$ such that (i) is equivalent to $Tv = v$.

$$(Tv)(z, y) = \max_{c,\ x} \left\{ U(c) + \beta \int v(x, y') dF(y', y) \right\} = v(z, y)$$

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- The domain of T is the nonnegative orthant L^{2n+} of the space L^{2n} of continuous, bounded functions $u : E^{n+} \times E^{n+} \to R$, normed (Norm assigns length to a vector) by

$$\|u\| = \sup_{z,\ y} |u(z,\ y)|$$

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- Since $U(c)$ is bounded, Tu is bounded, and by **the Theorem of Maximum** Tu is consitnuous

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Proof.

- Since applying T involves maximizing a continuous function over a compact set, Tu is well defined for any $u \in L^{2n+}$
- Since $U(c)$ is bounded, Tu is bounded, and by **the Theorem of Maximum** Tu is consitnuous
- Hence, T is monotone ($u \geq v$ implies $Tu \geq Tv$) and for any constant A, $T(u+a) = Tu + \beta a$.

$$(Tv)(z,y) = \max_{c,\ x} \left\{ U(c) + \beta \int v(x,y')dF(y',y) \right\}$$

$$\begin{aligned} c + p(y)\cdot x &\leq y\cdot z + p(y)\cdot z,\ c \geq 0,\ 0 \leq x \leq \bar{z}, \\ c + p(y)\cdot x &\leq A \end{aligned}$$

Proposition 1

$$(Tv)(z,y) = \max_x \left\{ U\left[A - p(y)x\right] + \beta \int v(x,y')dF(y',y) \right\}$$

Proof.

- Monotonicity: If $u \geq v$ everywhere, the maximum $Tu > Tv$

□

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- Monotonicity: If $u \geq v$ everywhere, the maximum $Tu > Tv$
- Discounting: $T(u+a) = Tu + \beta a$

$$\begin{aligned} & (T(v+a))(z,y) \\ = \quad & \max_x \left\{ U\left[A - p(y)x\right] + \beta \int v(x,y')dF(y',y) + \beta a \right\} \end{aligned}$$

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- Then according to the **Blackwell's condition for a contraction mapping**, T is a contraction mapping. It follows that $Tv = v$ has a unique solution v in L^{2n+}. Further, $\lim_{n\to\infty} T^n u = v$ for any $u \in L^{2n+}$ $(\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v))$.

□

Proposition 1

Proof.

- To prove that v is increasing in z, observe that Tu is an increasing function of z for any u.

$$(Tv)(z,y) = \max_x \left\{ U(z(y+p(y)) - p(y)x) + \beta \int v(x,y')dF(y',y) \right\}$$

Since $v = Tv$, this implies that v is increasing in z.

$\square$

Proposition 1

Proof.

To prove that v is concave in z, we first show that if $u(z, y)$ is concave in z, so is $(Tu)(z, y)$.

- Let z_0, z_1 be chosen, let $0 \leq \theta \leq 1$

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$$z^{\theta} = \theta z^{0} + (1 - \theta) z^{1}$$

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- Let (c_i, x_i) attain
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- Now
$$(c^\theta, x^\theta) = (\theta c^0 + (1-\theta)c^1, \theta x^0 + (1-\theta)x^1)$$
satisfies
$$c^\theta + p(y)x^\theta \leq y z^\theta + p(y) z^\theta$$

Proof.

- so that

$$
\begin{aligned}
(Tu)(z^{\theta}, y) &\geq U(c^{\theta}) + \beta \int u(x^{\theta}, y')dF(y', y) \\
&\geq \theta(Tu)(z^{0}, y) + (1-\theta)(Tu)(z^{1}, y)
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using the concavity of U and u.

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  using the concavity of U and u.
- Hence $(Tu)(z, y)$ is concave in z for all $n = 1, 2, ...$
- Then, since $\lim_{n\to\infty} T^n u = v$, v is concave

□

Proposition 2 (Envelope theorem)

Proposition 2: If $v(z, y; p)$ is attained at (c, x) with $c > 0$, then v is differentiable with respect to z at (z, y) and

$$\frac{\partial v(z, y; p)}{\partial z_i} = U'(c)\left[y_i + p_i(y)\right], \quad i = 1, ..., n$$

Proof.

- Define $f : R^+ \to R^+$ by

$$f(A) = \max_{c,x} \left\{ U(c) + \beta \int v(x, y')dF(y', y) \right\}$$

  subject to

$$c + p(y)x \leq A, \ \ c, x \geq 0$$

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- For each A, $f(A)$ is attained at $c(A)$, $x(A)$ say, and since the maximand is strictly concave in c, $c(A)$ is unique and varies continuously with A.

Proposition 2

Proof.

- If $c(A) > 0$ and if h is sufficiently small, $c(A) + h$ and $x(A)$ are feasible at 'income' $A + h$

$$
\begin{aligned}
f(A+h) &\geq U(c(A)+h) + \beta \int v(x(A), y')dF(y', y) \\
&= U(c(A)+h) - U(c(A)) + f(A)
\end{aligned}
$$

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- $c(A+h) - h$ and $x(A+h)$ are feasible at income A. Thus

$$\begin{aligned} f(A) &\geq U(c(A+h)-h) + \beta \int v(x(A+h), y') dF(y', y) \\ &= U(c(A+h)-h) - U(c(A+h)) + f(A+h) \end{aligned}$$

□

Proposition 2

Proof.

- Combining these inequalities gives

$$\begin{aligned} U(c(A)+h) - U(c(A)) &\leq f(A+h) - f(A) \\ &\leq U(c(A+h)) - U(c(A+h)-h) \end{aligned}$$

□

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Proof.

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$$\begin{aligned} U(c(A)+h) - U(c(A)) &\leq f(A+h) - f(A) \\ &\leq U(c(A+h)) - U(c(A+h) - h) \end{aligned}$$

- Dividing by h, letting $h \to 0$, and utilizing the continuity of $c(\cdot)$ gives

$$f'(A) = U'(c(A))$$

□

Proposition 2

Proof.

- Combining these inequalities gives

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- Dividing by h, letting $h \to 0$, and utilizing the continuity of $c(\cdot)$ gives

$$f'(A) = U'(c(A))$$

- Now letting $A = yz + p(y)z$, so that $v(z, y; p) = f(A)$, we obtain $(\partial v/\partial z_i) = f'(A)(\partial A/\partial z_i)$, as was to be shown.

□

Solution of the price function

- First order condition

$$U'(c)p_i(y) = \beta \int \frac{\partial v(x, y')}{\partial x_i} dF(y', y)$$

$$c + p(y)x = yz + p(y)z$$

provided $c, x > 0$

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provided $c, x > 0$

- If next period's optimum consumption $c\prime$ is also positive, Proposition 2 implies

$$\frac{\partial v(x, y')}{\partial x_i} = U'(c') \left[y_i' + p_i(y') \right]$$

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- One may think this equation, loosely, as equating the marginal rate of substitution of current for future consumption to the market rate of transformation, as given in the market rate of return on security i
- Mathematically, it is a stochastic Euler equation

Solution of the price function

$$U'\left(\sum_j y_j\right) p_i(y) = \beta \int U'\left(\sum_j y'_j\right) \left(y'_i + p_i(y')\right) dF(y', y) \qquad (*)$$

- Since this equation does not involve the particular value function $v(z, y; p)$ used in its derivation, it must hold for any equilibrium price function
- Conversely, if $p^*(y)$ solves this equation and $v(z, y; p^*)$ is as constructed in Proposition 1, then the pair $(p^*(y), v(z, y; p^*))$ is an equilibrium
- Thus solutions to this equation and equilibrium price functions are coincident

Solution of the price function

- Define

$$g_i(y) = \beta \int U'\left(\sum_j y'_j\right) y'_i dF(y', y)$$

$$f_i(y) = U'\left(\sum_j y_j\right) p_i(y)$$

We have n independent functional equations

$$f_i(y) = g_i(y) + \beta \int f_i(y') dF(y', y)$$

Solution of the price function

- If

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have solutions $(f_1(y), ..., f_n(y))$

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- the price functions
$$p_i(y) = \frac{f_i(y)}{U'\left(\sum_j y_j\right)}$$
will solve (*), and $p(y) = (p_1(y), ..., p_n(y))$ will be the equilibrium price function

Solution by contraction mapping

- If f is any continuous, bounded, nonnegative function on E^{n+}, the function $T_i f : E^{n+} \longrightarrow R^+$ given by

$$(T_i f)(y) = g_i(y) + \beta \int f(y')dF(y', y)$$

is well-defined and continuous in y

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- Since U is concave and bounded (by B, say) we have for any c :

$$0 = U(0) \leq U(c) + U'(c)(-c) \leq B - cU'(c)$$

So that $cU'(c) < B$ for all c.

Solution by contraction mapping

- $cU'(c) < B$, it follows that the functions $g_i(y)$ are bounded,

$$g_i(y) = \beta \int U' \left(\sum_j y'_j \right) y'_i dF(y', y)$$

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Solution by contraction mapping

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  since they are nonnegative and their sum is bounded by βB
- Evidently, solutions to $T_i f = f$ are solutions to $f(y) = g_i(y) + \beta \int f(y')dF(y', y)$

Characterize the price function – one asset

- The crucial issues are the information content of the current state y (that is, the way $F(y', y)$ varies with y) and the degree of "risk aversion" (the curvature of U)

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- The crucial issues are the information content of the current state y (that is, the way $F(y', y)$ varies with y) and the degree of "risk aversion" (the curvature of U)
- Suppose, as first case, that $\{y_t\}$ is a sequence of independent random variables: $F(y', y) = \phi(y')$
- Then $g(y)$ is the constant

$$\bar{g} = \beta \int y' U'(y') d\phi(y') = \beta E\left[yU'(y)\right]$$

Characterize the price function – one asset

- Calculating f from

$$(Tf)(y) = \bar{g} + \beta \int f(y')d\phi(y')$$

$$(T^2 f)(y) = \bar{g} + \beta \left[\bar{g} + \beta \int f(y')d\phi(y') \right]$$

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- We get

$$f(y) = \frac{\bar{g}}{1-\beta}, \; f'(y) = 0$$

Characterize the price function – one asset

- Differentiating

$$p(y) = \frac{f(y)}{U'(y)}$$

gives

$$p'(y) = -\frac{\beta E\left[yU'(y)\right] U''(y)}{(1-\beta)\left[U'(y)\right]^2} = p(y)\frac{-U''(y)}{U'(y)} > 0$$

Characterize the price function – one asset

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$$\frac{yp'(y)}{p(y)} = \frac{-yU''(y)}{U'(y)}$$

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Characterize the price function – one asset

- Rearranging

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- This is the elasticity of price with respect to income is equal to the Arrow-Pratt measure of relative risk aversion
- In a period of high transitory income, then, agents attempt to distribute part of the windfall over future periods (marginal utility decreases), via securities purchases. This attempt is frustrated (since storage is precluded) by an increase in asset prices

Autocorrelated production disturbances

- Restrict the stochastic difference equation governing y_t to have its root between zero and one

$$y_{t+1} = \rho y_t + \varepsilon_{t+1} \qquad \rho \in (0, 1)$$

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- CDF $F(y', y) = \Pr\{y_{t+1} \leq y' | y_t = y\}$
  $F_1 > 0$
  $F_2 < 0$: the higher the y_t, the more likely the higher y_{t+1}

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- CDF $F(y', y) = \Pr\{y_{t+1} \le y' | y_t = y\}$
  $F_1 > 0$
  $F_2 < 0$: the higher the y_t, the more likely the higher y_{t+1}
- Use the change of variable $u = F(y', y)$, and invert to get
  $y' = G(u, y)$, $G_2 = \partial y' / \partial y$
  By substitution we take into account that y affects $y\prime$,
  $u = F(G(u, y), y)$, completely differentiation gives

$$F_1 G_2 + F_2 = 0, \quad G_2 = -F_2 / F_1$$

Lemma 1

Lemma

Let F satisfy $0 < -F_2 < F_1$, and let $h(y)$ have a derivative bounded between 0 and $h'_M > 0$. Then

$$0 \leq \frac{d}{dy} \int h(y') dF(y', y) \leq h'_M$$

Proof.

$\frac{d}{dy} \int_0^1 h(y') dF(y', y) = \frac{d}{dy} \int_0^1 h(G(u, y)) du = \int_0^1 h'(G) G_2(u, y) du$, the result follows. □

Bounds

- $$\begin{aligned} (Tf)(y) &= g(y) + \beta \int f(y') dF(y', y) \\ \frac{d}{dy}(Tf)(y) &= g'(y) + \beta \frac{d}{dy} \int f(y') dF(y', y) \\ g'(y) &= \beta \frac{d}{dy} \int U'(y') y' dF(y', y) \end{aligned}$$

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- Let $f(y)$ be the solution to $f(y) = g(y) + \beta \int f(y')dF(y', y)$
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- Let $f(y)$ be the solution to $f(y) = g(y) + \beta \int f(y')dF(y', y)$
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- Take 0 and $\bar{a}$ as lower and upper bounds on $U''(y)y + U'(y)$, then apply Lemma 1
$$0 \leq g'(y) \leq \beta \bar{a}$$

Bounds

$$
\begin{aligned}
f'(y) &= g'(y) + \beta \frac{d}{dy} \int f(y') dF(y', y) \\
&= g'(y) + \beta \int f'(y') G_2(u, y) dF(y', y) \\
&= g'(y) + \beta \int \left[g'(y') + \beta \frac{d}{dy'} \int f(y'') dF(y'', y') \right] G_2(u, y) dF(y', \\
&= g'(y) + \beta \int g'(y') G_2(u, y) dF(y', y) + \ldots \\
&\geq 0
\end{aligned}
$$

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f'(y) &= g'(y) + \beta \frac{d}{dy} \int f(y') dF(y', y) \\
&= g'(y) + \beta \int f'(y') G_2(u, y) dF(y', y) \\
&\leq g'(y) + \beta \int f'(y') dF(y', y) \qquad \text{given } G_2 = \frac{-F_2}{F_1} < 1 \\
&\leq g'(y) + \beta \int \left[g'(y') + \beta \frac{d}{dy'} \int f(y'') dF(y'', y') \right] dF(y', y) \\
&\leq g'(y) + \beta \int g'(y') dF(y', y) + \beta^2 \int \int g'(y'') dF(y'', y') dF(y', y \\
&\leq \bar{a} + \beta \bar{a} + \beta^2 \bar{a} + \ldots \\
&\leq \frac{\beta \bar{a}}{1 - \beta}
\end{aligned}
$$

The elasticity of the equilibrium price function

- $$\begin{aligned} p(y) &= \frac{f(y)}{U'(y)} \\ p'(y) &= \frac{U'(y)f'(y) - f(y)U''(y)}{[U'(y)]^2} \end{aligned}$$

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- $$\frac{yp'(y)}{p(y)} = \frac{yf'(y)}{f(y)} - \frac{yU''(y)}{U'(y)}$$

income effect $(+)$

"information effect"

sign of $f'(y)$

$f(y)$ information about future dividends

The elasticity of the equilibrium price function

$$\frac{yp'(y)}{p(y)} = \frac{yf'(y)}{f(y)} - \frac{yU''(y)}{U'(y)}$$

- Depends on our knowledge of the curvature of U
  It shows how to translate such knowledge into knowledge about asset prices

The elasticity of the equilibrium price function

$$\frac{yp'(y)}{p(y)} = \frac{yf'(y)}{f(y)} - \frac{yU''(y)}{U'(y)}$$

- Depends on our knowledge of the curvature of U
  It shows how to translate such knowledge into knowledge about asset prices
- Relative risk aversion less than 1, $f'(y) > 0$, so that the information effect is positive
  Thus, new optimistic information on future dividends leads to increased asset prices