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Volume 25

# Asymptotic Behavior of Dissipative Systems

Jack K. Hale

American Mathematical Society

http://dx.doi.org/10.1090/surv/025

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# Asymptotic Behavior of Dissipative Systems

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2000 *Mathematics Subject Classification.* Primary 58–XX;
Secondary 34–XX, 35–XX.

**Library of Congress Cataloging-in-Publication Data**

Hale, Jack K.
Asymptotic behavior of dissipative systems.
(Mathematical surveys and monographs, ISSN 0076-5376; no. 25)
Bibliography: p.
1. Differentiable dynamical systems. 2. Differential equations, Partial–Asymptotic theory. 3. Stability. 4. Differential equations–Asymptotic theory. I. Title. II. Series.

QA614.8.H35 1988 003 87-33495
ISBN 0-8218-1527-X (alk. paper)

AMS softcover ISBN: 978-0-8218-4934-7



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10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

# Contents

# Acknowledgment

The author is indebted to many students and colleagues who have helped to develop the ideas in these notes over the years. In particular, I owe a special acknowledgment to Genevieve Raugel who read and constructively criticized almost all of the original manuscript. Thanks to Ezoura Fonseca, Kate MacDougall, Jeri Murgo, and the professional staff of the American Mathematical Society, especially Janet Scappini, for their assistance in the preparation of the text. Finally, the author gratefully appreciates financial support from the National Science Foundation, the Air Force Office of Scientific Research, the Army Research Office, and a Carnegie Fellowship from Heriot-Watt University.

## APPENDIX

### Stable and Unstable Manifolds.

In this section, we state the basic properties of the stable and unstable manifolds of an hyperbolic equilibrium point of an abstract evolutionary equation. Indications of the proofs also will be given. To simplify the presentation and in order not to obscure the fundamental ideas, we concentrate on ordinary differential equations in finite dimensions and then give references for the appropriate modifications in infinite dimensional cases.

Consider the system of differential equations

$$\dot{x} = Ax + f(x), \tag{A.1}$$

where $x \in R^n$, $A$ is an $n \times n$ constant matrix whose eigenvalues have nonzero real parts, $f: R^n \to R^n$ is a Lipschitz continuous function satisfying

$$\begin{aligned} &f(0) = 0, \\ &|f(x) - f(y)| \le \eta(\sigma)(x - y) \quad \text{if } |x|,\ |y| \le \sigma, \end{aligned} \tag{A.2}$$

where $\eta: [0, \infty) \to [0, \infty)$ is a continuous function with $\eta(0) = 0$.

For any $x_0 \in R^n$, let $\phi(t, x_0)$ be the solution of (A.1) through $x_0$. The unstable set $W^u(0)$ and the stable set $W^s(0)$ of 0 are defined as

$$\begin{aligned} W^u(0) = \{x_0 \in R^n :\ &\phi(t, x_0) \text{ is defined for } t \le 0 \\ &\text{and } \phi(t, x_0) \to 0 \text{ as } t \to -\infty\}, \end{aligned}$$

$$\begin{aligned} W^s(0) = \{x_0 \in R^n :\ &\phi(t, x_0) \text{ is defined for } t \ge 0 \\ &\text{and } \phi(t, x_0) \to 0 \text{ as } t \to +\infty\}. \end{aligned}$$

For a given neighborhood $U$ of 0, we can also define

$$\begin{aligned} W^u(0, U) &= \{x_0 \in W^u(0):\ \phi(t, x_0) \in U,\ t \le 0\}, \\ W^s(0, U) &= \{x_0 \in W^s(0):\ \phi(t, x_0) \in U,\ t \ge 0\}. \end{aligned}$$

These latter sets also are called local unstable and stable sets and are designated by $W^u_{\text{loc}}(0), W^s_{\text{loc}}(0)$.

Since the eigenvalues of $A$ have nonzero real parts, there is a projection operator $P: R^n \to R^n$ such that $PR^n$ and $QR^n$, $Q = I - P$, are invariant under $A$ and the spectrum $\sigma(AP)$ of $AP$ has positive real parts and $\sigma(AQ)$ has negative real parts.

A basic lemma is the following.

LEMMA A.1. *If $\phi(t, x_0)$, $t \leq 0$, is a bounded solution of* (A.1), *then $\phi(t, x_0)$ satisfies the integral equation*

$$(\text{A.3}) \qquad y(t) = e^{At}Px_0 + \int_0^t e^{A(t-s)}Pf(y(s))\,ds + \int_{-\infty}^t e^{A(t-s)}Qf(y(s))\,ds.$$

*If $\phi(t, x_0)$, $t \geq 0$, is a bounded solution of* (A.1), *then $\phi(t, x_0)$ satisfies the integral equation*

$$(\text{A.4}) \qquad z(t) = e^{At}Qx_0 + \int_0^t e^{A(t-s)}Qf(z(s))\,ds - \int_t^\infty e^{A(t-s)}Pf(z(s))\,ds.$$

*Conversely, if $y(t)$, $t \leq 0$ [or $t \geq 0$], is a bounded solution of* (A.3) [*or* (A.4)], *then $y(t)$ satisfies* (A.1).

PROOF: Let $y(t) = \phi(t, x_0)$, $t \leq 0$, be a bounded solution of (A.1). Then, for any $\tau$ in $(-\infty, 0]$,

$$(\text{A.5}) \qquad Qy(t) = e^{A(t-\tau)}Qy(\tau) + \int_\tau^t e^{A(t-s)}Qf(y(s))\,ds.$$

There are positive constants $k$, $\alpha$ such that

$$(\text{A.6}) \qquad |e^{A(t-\tau)}Q| \leq ke^{-\alpha(t-\tau)}, \qquad t \geq \tau.$$

If we let $\tau \to -\infty$ in (A.5) using the fact that $y(s)$ is bounded in $s$, we obtain

$$Qy(t) = \int_{-\infty}^t e^{A(t-s)}Qf(y(s))\,ds.$$

Since

$$Py(t) = e^{At}Px_0 + \int_0^t e^{A(t-s)}Pf(y(s))\,ds,$$

we see that $y(t)$ must satisfy (A.3). The proof for the case when $x_0 \in W^s(0)$ is similar and therefore omitted.

The converse statement is proved by direct computation.

We say that $W^u(0, U)$ is a *Lipschitz graph over $PR^n$* if there is a neighborhood $V$ of 0 in $PR^n$ such that $W^u(0, U) = \{y \in R^n : y = g(x),\ x \in V$, where $g$ is a Lipschitz continuous function$\}$. The set $W^u(0, U)$ is said to be *tangent to $PR^n$ at* 0 if $|Qx|/|Px| \to 0$ as $x \to 0$ in $W^u(0, U)$. Similar definitions hold for $W^s(0, U)$.

A classical theorem on local unstable and stable sets which can be traced initially to the work of Lyapunov and Poincaré is contained in the following result.

THEOREM A.2. *Suppose $f$ satisfies* (A.2) *and* $\operatorname{Re}\sigma(A) \neq 0$. *There is a neighborhood $U$ of* 0 *in $R^n$ such that $W^u(0, U)$ [or $W^s(0, U)$] is a Lipschitz graph over $PR^n$ [or $QR^n$] which is tangent to $PR^n$ [or $QR^n$] at* 0.

PROOF: The proof is a standard application of the contraction mapping principle (for example, see Hale [1969] or [1980]) and gives exponential decay rates of the solutions to the origin. Suppose $k$, $\alpha$ are chosen so that (A.6) is satisfied and also so that

$$(\text{A.7}) \qquad |e^{At}P| \leq ke^{\alpha t}, \qquad t \leq 0.$$

Choose $\delta > 0$ so that $4k\eta(\delta) < \alpha$, $8k^2\eta(\delta) < \alpha$. For $x_0 \in PR^n$ with $|x_0| \leq \delta/2k$, define $S(x_0, \delta)$ as the set of continuous functions $x: (-\infty, 0] \to R^n$ such that $|x| = \sup_{-\infty \leq t \leq 0} |x(t)| \leq \delta$ and $Px(0) = x_0$. The set $S(x_0, \delta)$ is a closed bounded subset of the Banach space of all continuous functions taking $(-\infty, 0]$ into $R^n$ with the uniform topology. For any $x$ in $S(x_0, \delta)$, define

$$(A.8) \qquad (Tx)(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Pf(x(s))\,ds + \int_{-\infty}^t e^{A(t-s)}Qf(x(s))\,ds$$

for $t \leq 0$.

It is easy to show that $T: S(x_0, \delta) \to S(x_0, \delta)$ is a contraction mapping with contraction constant $x_0$ and therefore has a unique fixed point $x^*(\cdot, x_0)$. This fixed point satisfies (A.3) and thus is a solution of (A.1) from Lemma A.1.

The function $x^*(\cdot, x_0)$ is continuous in $x_0$. Also,

$$|x^*(t, x_0)| \leq ke^{\alpha t}|x_0| + k\eta(\delta) \int_t^0 e^{\alpha(t-s)}|x^*(s, x_0)|\,ds$$
$$+ k\eta(\delta) \int_{-\infty}^t e^{-\alpha(t-s)}|x^*(s, x_0)|\,ds.$$

From this inequality, one can prove that (see, for example, Lemma 6.2, p. 110 of Hale [1980]) that

$$(A.9) \qquad |x^*(t, x_0)| \leq 2ke^{\alpha t/2}|x_0|, \qquad t \leq 0.$$

This estimate shows that $x^*(\cdot, 0) = 0$ and also that $x^*(0, x_0) \in W^u(0)$.

The same type of computations show that

$$(A.10) \qquad |x^*(t, x_0) - x^*(t, \overline{x}_0)| \leq 2ke^{\alpha t/2}|x_0 - \overline{x}_0|, \qquad t \leq 0.$$

In particular, $x^*(\cdot, x_0)$ is Lipschitzian in $x_0$.

One next observes that

$$|x^*(0, x_0) - x^*(0, \overline{x}_0)| \geq |x_0 - \overline{x}_0| - \int_{-\infty}^0 k\eta(\delta)e^{\alpha s}|x^*(s, x_0) - x^*(s, \overline{x}_0)|\,ds$$
$$\geq |x_0 - \overline{x}_0| \left[1 - \frac{4k^2\eta(\delta)}{3\alpha}\right] \geq \frac{1}{2}|x_0 - \overline{x}_0|.$$

Thus, the mapping $x_0 \mapsto x^*(0, x_0)$ is one-to-one with a continuous inverse.

These estimates together with the fact that $\phi(t, x_0)$, $t \leq 0$, $x_0 \in W^u(0)$ must satisfy (A.3) imply that $W^u(0, U)$ for some $U$ satisfies the conclusions of the theorem. The same type of argument applied to (A.4) will yield a complete proof of the theorem.

It is more difficult to obtain more regularity of the manifolds $W^u_{\text{loc}}(0), W^s_{\text{loc}}(0)$. If we assume that the vector field is $C^k$, then a standard application of the contraction mapping principle to (A.8) will not show that the above manifolds are $C^k$. However, with considerable effort, one can show that they are $C^{k-1,1}$; that is, they are represented by a function which is $C^{k-1}$ with the $k-1$ first derivatives being Lipschitz (see, for example, Carr [1981], Sijbrand [1985]). One must then use some other method to show that the manifolds actually are $C^k$. A convenient way is to use the following lemma of Henry [1983] based on a remark in Hirsch, Pugh, and Shub [1977, p. 35].

LEMMA A.3. *Let $X, Y$ be Banach spaces, $Q \subset X$ an open set, and $g: Q \to Y$ locally Lipschitzian. Then $g$ is continuously differentiable if and only if, for each $x_0 \in Q$,*

$$|g(x+h) - g(x) - g(x_0+h) + g(x_0)| = o(|h|_X) \tag{A.11}$$

*as $(x, h) \to (x_0, 0)$.*

PROOF. It is easy to see that (A.11) holds if $g$ is a $C^1$-function. Without loss in generality, we take $Q$ to be a ball and $g$ to be Lipschitzian in $Q$. If the derivative $g'$ of $g$ exists at each point of $Q$ and (A.11) is satisfied, then $g'$ is continuous. Thus, it is enough to prove that $g'$ exists at each point of $Q$.

*Case* 1. Let us first suppose that $X = Y = R$. Since $g$ is absolutely continuous, it is differentiable almost everywhere. For any $x_0 \in Q$, $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|g(x+h) - g(x) - g(x_0+h) + g(x_0)| \le \varepsilon|h| \quad \text{if} \quad |x - x_0| + |h| < \delta.$$

There is an $x^*$ in $(x_0 - \delta, x_0 + \delta)$ such that $g'(x^*)$ exists. Thus, for $h \neq 0$ sufficiently small,

$$\left|\frac{g(x_0+h) - g(x_0)}{h} - g'(x^*)\right| \le 2\varepsilon,$$

$$0 \le \{\overline{\lim_{h\to 0}} - \underline{\lim_{h\to 0}}\}\frac{g(x_0+h) - g(x_0)}{h} \le 4\varepsilon.$$

Since $\varepsilon$ is arbitrary, this implies that $g'(x_0)$ exists.

*Case* 2. Now suppose that $X = R$ and $Y$ is a Banach space with $Y^*$ being the dual space. If $\eta \in Y^*$, then Case 1 implies that $\eta g$ is a $C^1$-function and $|(\eta g)'(x)| \le |\eta| \cdot \operatorname{Lip} g$. If $D(x): \eta \to (\eta g)'(x)$, then $D(x) \in Y^{**}$ is continuous in $x$ from (A.11).

For $x_0 \in Q$, $\eta \in Y^*$, $|\eta| \le 1$, we have

$$\eta\left(\frac{g(x_0+h) - g(x_0)}{h}\right) - D(x_0)\eta = \frac{1}{h}\int_{x_0}^{x_0+h} (D(x) - D(x_0))\eta \to 0$$

as $h \to 0$ uniformly for $|\eta| \le 1$. Let $\tau: Y \to Y^*$ be the canonical inclusion. Then $\tau[g(x_0+h) - g(x_0)]/h \to D(x_0)$ as $h \to 0$. Since $\tau$ is an isometry, this implies that $[g(x_0+h) - g(x_0)]/h \to$ a limit in $Y$ as $h \to 0$; that is, $g(x_0)$ exists.

*Case* 3. Finally, let $X, Y$ be arbitrary Banach spaces. From Case 2, for any $x \in Q$, $h \in X$, the map $t \mapsto g(x+th)$ taking $R$ to $Y$ is $C^1$ if $t$ is small. Thus, the Gateaux derivative $dg(x,h) = dg(x+th)/dt|_{t=0}$ exists. Condition (A.11) implies that $dg(x,h) \to dg(x_0,h)$ in $Y$ as $x \to x_0 \in Q$, uniformly for $|\eta| \le 1$. This implies that $h \mapsto dg(x,h)$ is linear and continuous. Thus, $dg(x,h)$ is the Fréchet derivative at $h$ of $g$ and the proof is complete.

THEOREM A.4. *Suppose* $\operatorname{Re}\sigma(A) \neq 0$, *$f$ satisfies* (A.2) *and is also a $C^1$-function. Then the sets $W^u(0,U)$, $W^s(0,U)$ of Theorem* A.2 *are $C^1$-manifolds.*

PROOF. Let $x^*(t, x_0)$ be the fixed point of the map $T$ in (A.8) which was used to define $W^u_{\mathrm{loc}}(0) = W^u(0,U)$. Define $y(x,x_0,h)(t) = x^*(t,x+h) - x^*(t,x) - x^*(t,x_0+h) + x^*(t,x_0)$. From Lemma (A.3), it is sufficient to show that

$$\overline{\lim_{(x,h)\to(x_0,0)}} \frac{1}{h} y^*(x,x_0,h)(t) = 0 \tag{A.12}$$

uniformly for $-\infty < t \le 0$.

From the definition of $x^*(t,x_0)$ and the fact that $f$ is a $C^1$-function, we have, for $t \le 0$,

$$\begin{aligned}
&y^*(x,x_0,h)(t)\\
&\quad = \int_0^t e^{A(t-s)} P f_x(x^*(s,x_0)) y^*(x,x_0,h)(s)\,ds\\
&\quad + \int_{-\infty}^t e^{A(t-s)} Q f_x(x^*(s,x_0)) y^*(x,x_0,h)(s)\,ds\\
&\quad + \int_0^t e^{A(t-s)} P[f_x(x^*(s,x)) - f_x(x^*(s,x_0))][x^*(s,x_0+h) - x^*(s,x_0)]\,ds\\
&\quad + \int_{-\infty}^t e^{A(t-s)} Q[f_x(x^*(s,x)) - f_x(x^*(s,x_0))][x^*(s,x_0+h) - x^*(s,x_0)]\,ds\\
&\quad + o(h)
\end{aligned}$$

as $|h| \to 0$. Using the estimates (A.6), (A.7), (A.2), and (A.10), we have

$$\begin{aligned}
|y^*(x,x_0,h)(t)| \le k\eta(\delta) &\int_t^0 e^{\alpha(t-s)} |y^*(x,x_0,h)(s)|\,ds\\
&+ k\eta(\delta) \int_{-\infty}^t e^{-\alpha(t-s)} |y^*(x,x_0,h)(s)|\,ds\\
&+ \frac{8k^2}{\alpha} |h| e^{\alpha t/2} \sup_{s\le 0} |f_x(x^*(s,x)) - f_x(x^*(s,x_0))|\\
&+ o(h)
\end{aligned}$$

as $h \to 0$.

One can now show that there is a constant $K > 0$ such that (see, for example, Lemma 6.2, p. 110 of Hale [1980])

$$\frac{1}{|h|} |y^*(x,x_0,h)(t)| \le K \sup_{s\le 0} |f_x(x^*(s,x)) - f_x(x^*(s,x_0))|$$

for $t \le 0$. Since the function $x^*(\cdot,x)$ is continuous in $x$ and $f$ is a $C^1$-function, we have (A.12) is satisfied and the theorem is proved.

Let us now suppose that $f = f(x,\lambda)$ depends on a parameter $\lambda$ varying in an open subset $V$ of a Banach space $\Lambda$, $0 \in V$. Also, suppose that $f(x,\lambda)$ is continuous in $(x,\lambda)$ and there is a continuous function $\eta\colon [0,\infty)\times\Lambda \to [0,\infty)$, $\eta(0,0) = 0$, such that

$$\begin{aligned}
&|f(x,\lambda) - f(y,\lambda)| \le \eta(\sigma,\lambda)|x-y| \quad \text{if } |x|,|y| \le \sigma, \qquad \lambda \in \Lambda\\
&f(0,0) = 0.
\end{aligned}$$

The equilibrium points of the equation are the solutions of $x = -A^{-1}f(x,\lambda)$. An application of the uniform contraction principle shows that there is a unique fixed point $\phi_\lambda$ of the map $-A^{-1}f(x,\lambda)$ for $(x,\lambda)$ in a neighborhood of zero. Furthermore, $\phi_0 = 0$ and $\phi_\lambda$ is continuous in $\lambda$. If $x \mapsto x + \phi_\lambda$, then one can use the proof of Theorem A.2 and the uniform contraction principle to see that there are neighborhoods $U$ of zero in $R^n$ and $V_1$ of zero in $\Lambda$ such that the local unstable manifold $W^u_\lambda(\phi_\lambda, U)$ of $\phi_\lambda$ and the local stable manifold $W^s_\lambda(\phi_\lambda, U)$ of $\phi_\lambda, \lambda \in V_1$, are continuous in $\lambda$.

If we suppose in addition that $f(x,\lambda)$ is a $C^1$-function in $(x,\lambda)$, then the Implicit Function Theorem implies that $\phi_\lambda$ is $C^1$ in $\lambda$. From Theorem A.4, we also know that $W^u_\lambda(\phi_\lambda, U), W^s_\lambda(\phi_\lambda, U)$ are $C^1$-manifolds for each $\lambda \in V_1$. To show that these manifolds are $C^1$ in $\lambda$, let $x^*(t, x_0, \lambda)$ be the fixed point of the operator $T = T_\lambda$ in (A.8) [we have made the transformation above: $x \mapsto \phi_\lambda + x$]. The same type of argument used in the proof of Theorem A.2 shows thatf $x^*(t, x_0, \lambda)$ is Lipschitz continuous in $\lambda$. One now may apply Lemma A.3 and the argument in the proof of Theorem A.4 to obtain the $C^1$-dependence on $\lambda$.

We have proved

THEOREM A.5. *Suppose* $\operatorname{Re}\sigma(A) \neq 0$, $f = f_\lambda$ *depends on a parameter* $\lambda$ *in an open subset* $V$ *in a Banach space* $\Lambda$, $\sigma \in V$, *and* $f_\lambda(x)$ *is a* $C^1$*-function of* $\lambda, x$ *with* $f(0,0) = 0$, $D_x f_\lambda(0,0) = 0$. *Then the manifolds* $W^u_\lambda(\phi_x, U), W^s_\lambda(\phi_\lambda, U)$ *are represented by a* $C^1$*-function of* $x, \lambda$.

One can extend Theorems A.4 and A.5 to the following

THEOREM A.6. *Suppose* $\operatorname{Re}\sigma(A) \neq 0$, $f$ *satisfies* (A.2) *and is a* $C^k$*-function,* $k \geq 1$. *Then the sets* $W^u(0,U)$, $W^s(0,U)$ *of Theorem* A.2 *are* $C^k$*-manifolds. In addition, if* $f = f_\lambda$ *depends on a parameter* $\lambda$ *in an open subset* $V$ *of a Banach space* $\Lambda, 0 \in V$ *and* $f_\lambda(x)$ *is a* $C^k$*-function of* $x, \lambda$ *with* $f_0(0) = 0, D_x f_0(0) = 0$, *then the manifolds* $W^u_\lambda(\phi_\lambda, U)$, $W^s_\lambda(\phi_\lambda, U)$ *are represented by a* $C^k$*-function of* $x, \lambda$.

We do not prove this theorem. A complete proof can be found in Hirsch, Pugh,and Shub [1977] where they used the graph transform (in contrast to fixed points of $T$ in Formula (A.8)) to show the existence of the unstable and stable sets and they used a fiber contraction theorem to obtain the smoothness. Another proof is contained in Henry [1983] Chow and Lu [1988a] where they use (A.8) for existence, Lemma A.3, and a $C^r$-section theorem similar to the one in Hirsch, Pugh, and Shub [1977, p. 31] for the smoothness. Another proof has been given by Vanderbauwhede and van Gils [1987] using (A.8) and a fixed point theorem in weighted Banach spaces. Recently, Chow and Lu [1988b] have given a proof using (A.8), weighted Banach spaces and the contraction theorem. We should remark that the results in Hirsch, Pugh, and Shub [1977] deal with the more general problem of stable and unstable sets (as well as the persistence under perturbations) of normally hyperbolic invariant sets.

For infinite dimension problems, there are analogues of Theorem A.6 for some situations. These include all of the examples discussed in this book except the nonlinear diffusion problem of §4.9.6. The analogue of Theorem A.2 can be found in Ball [1973] and the presentation in Henry [1977], as well as in Vanderbauwhede and van Gils [1987], Chow and Lu [1988a],[1988b], is given for infinite dimensional problems.

We state the result in the infinite dimensional case. Let $X \subset Y$ be Banach spaces with the embedding being continuous. Let $S(t): Y \to Y, t \geq 0$, and the spaces $X, Y$ satisfy the following properties:

($H_1$) $S(t)$ is a strongly continuous linear semigroup.

($H_2$) There is a decomposition $Y = Y_1 \oplus Y_2$ with continuous projections $P_i: Y \to Y_i$ such that

$$P_i S(t) = S(t) P_i, \qquad t \geq 0.$$

($H_3$) $P_i X \subset X$ and $S(t) Y \subset X$ for $t > 0$.

($H_4$) $S(t)$ can be extended to a group on $Y_1$.

($H_5$) There exist constants $M \geq 0$, $N \geq 0$, $\alpha > 0$, $\beta > 0$, $0 \leq \gamma < 1$ such that

$$\begin{aligned} |S(t)P_1 x|_X &\leq M e^{\alpha t} |x|_X \quad \text{for } t \leq 0, \qquad x \in X, \\ |S(t)P_2 x|_X &\leq M e^{-\beta t} |x|_X \quad \text{for } t \geq 0, \qquad x \in X, \\ |S(t)P_2 y|_X &\leq (M t^{-\gamma} + N) e^{-\beta t} |y|_Y \quad \text{for } t > 0, \qquad y \in Y. \end{aligned}$$

Let $F: X \to Y$ be a given function and consider the integral equation in $X$:

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau) F(x(\tau))\, d\tau. \tag{A.13}$$

For the linear semigroup $S(t)$, the subspace $X_1$ of $X$ is the unstable manifold of zero and the subspace $X_2$ is the stable manifold of zero. If we suppose that $F \in C^k(X, Y)$, $k \geq 0$, and

$$F(0) = 0, \qquad DF(0) = 0, \tag{A.14}$$

then one can use the contraction mapping principle as in the proof of Theorem A.2 to obtain the Lipschitz local unstable manifold $W^u_{\text{loc}}(0)$ and local stable manifold $W^s_{\text{loc}}(0)$ of the zero solution of the integral equation (A.13). These manifolds are actually $C^k$ as stated in the following result.

THEOREM A.7. *Suppose* (H1)-(H5) *and* (A.14) *are satisfied. If* $f \in C^k(X, Y), k \geq 1$, *then* $W^u_{\text{loc}}(0), W^s_{\text{loc}}(0)$ *are* $C^k$*-manifolds. If, in addition,* $F = F_\lambda$ *depends on a parameter* $\lambda$ *in an open subset* $V$ *of a Banach space* $\Lambda$ *and* $F_\lambda(x)$ *is* $C^k$ *in* $x, \lambda$ *with* $F_\lambda(0) = 0$, $D_x F_\lambda(0) = 0$, *then the manifolds* $W^u_{\text{loc},\lambda}(0)$, $W^s_{\text{loc},\lambda}(0)$ *are represented by a* $C^k$*-function of* $x, \lambda$.

As mentioned earlier, Theorem A.6 can be applied to the equations considered in this book (except for §4.9.6). For example, for the case in which $S(t)$ is an analytic semigroup with generator $-A$ (see §4.2), the two Banach spaces $X, Y$ are respectively $X^\alpha, X$ in the notation of §4.2. For the damped hyperbolic equations considered in §§4.7 and 4.8, $X = Y = H^1_0(\Omega) \times L^2(\Omega)$.

For applications, one must also consider center manifolds and the manifolds near an equilibrium point which have a specified exponential behavior either as $t \to +\infty$ or as $t \to -\infty$. Such results are obtained by splitting the spectrum of the linear semigroup $e^{At}$ by the circle $|z| = e^{\eta t}$ and then using weighted supremum norms so that the equilibrium point appears to be a saddle point for the linear operator in this norm (see Ball [1973a], Henry [1983], Vanderbauwhede and van Gils [1987], and Chow and Lu [1987b]).

As a final remark, we mention that results similar to the above ones are valid for fixed points of maps. In this case, the integrals in (A.8) are replaced by sums.

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# Index

*This monograph reports the advances that have been made in the area by the author and many other mathematicians; it is an important source of ideas for the researchers interested in the subject.*

***—Zentralblatt MATH***

*Although advanced, this book is a very good introduction to the subject, and the reading of the abstract part, which is elegant, is pleasant. ... this monograph will be of valuable interest for those who aim to learn in the very rapidly growing subject of infinite-dimensional dissipative dynamical systems.*

***—Mathematical Reviews***

This book is directed at researchers in nonlinear ordinary and partial differential equations and at those who apply these topics to other fields of science. About one third of the book focuses on the existence and properties of the flow on the global attractor for a discrete or continuous dynamical system. The author presents a detailed discussion of abstract properties and examples of asymptotically smooth maps and semigroups. He also covers some of the continuity properties of the global attractor under perturbation, its capacity and Hausdorff dimension, and the stability of the flow on the global attractor under perturbation. The remainder of the book deals with particular equations occurring in applications and especially emphasizes delay equations, reaction-diffusion equations, and the damped wave equations. In each of the examples presented, the author shows how to verify the existence of a global attractor, and, for several examples, he discusses some properties of the flow on the global attractor.

ISBN 978-0-8218-4934-7

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