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Research Article

Asymptotic Stability of Distributed-Order Nonlinear Time-Varying Systems with the Prabhakar Fractional Derivatives

MohammadHossein Derakhshan and Azim Aminataei

Faculty of Mathematics, K. N. Toosi University of Technology, P.O. Box: 16765-3381, Tehran, Iran

Correspondence should be addressed to MohammadHossein Derakhshan; m.h.derakhshan.20@gmail.com

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In this article, we survey the Lyapunov direct method for distributed-order nonlinear time-varying systems with the Prabhakar fractional derivatives. We provide various ways to determine the stability or asymptotic stability for these types of fractional differential systems. Some examples are applied to determine the stability of certain distributed-order systems.

1. Introduction

In recent years, distributed-order fractional calculus has played a significant role in many areas of science, engineering, and mathematics [1–5]. For the first time in 1969, the distributed-order fractional calculus with the Caputo fractional derivatives was surveyed by Caputo [6]. Later, other research on the distributed-order fractional derivatives was presented. For example, Fernández-Anaya et al. [7] studied asymptotic stability of distributed-order nonlinear dynamical systems with the Caputo fractional derivative. Moreover, Duong et al. [3] studied the deterministic analysis of distributed-order systems using operational matrix. A new method for obtaining the numerical solution of distributed-order time-fractional-subdiffusion equations (DO-TFSDE) of the fourth order is studied in [8], and solving a two-dimensional distributed-order time-fractional fourth-order partial differential equation by using of the space-time Petrov-Galerkin spectral method is studied in [9]. The stability of distributed-order fractional differential systems with respect to the nonnegative density function has also been studied [10, 11]. We define fractional distributed-order nonautonomous systems of the form

$$ {}^{C}D^{\gamma}_{,c(\mu),\omega}x(t) = f(x(t), t), \tag{1}$$

where $c(\mu)$ is an absolutely integrable function in the interval $\mu \in [0, 1]$ and ${}^{C}D^{\gamma}_{\rho,c(\mu),\omega}$ is a distributed-order fractional differential/integral operator in the sense of a given fractional differential/integral operator of order $c(\mu)$ which discusses about the stability or asymptotic stability for these systems. Our interest in choosing this type of derivative is related to the three-parameter Mittag-Leffler function. One useful application of the three-parameter Mittag-Leffler function in mathematics has been related to their importance in fractional calculus as a model of complex susceptibility in the response of disordered materials and heterogeneous systems [12], in the response in anomalous dielectrics of Havriliak-Negami type [13], in fractional viscoelasticity [14], in the discussion of stochastic processes [15], in probability theory [16], in the description of dynamical models of spherical stellar systems [17], in the polarization processes in Havriliak-Negami models [13, 18], and in fractional or integral differential equations [19]. In this paper, we intend to survey the stability or asymptotic stability analysis of a distributed-order fractional differential/integral operator containing the Prabhakar fractional derivatives. This type of fractional derivative was introduced by Garra et al. [20] in that it is considered in terms of the generalized Mittag-Leffler function and can be considered as a generalization of the most popular definitions of fractional derivatives. In the field of stability and asymptotic stability, several papers have been published as follows: in [21], the Hyers-Ulam stability of the linear and nonlinear differential equations of fractional order with Prabhakar derivative by using the Laplace transform method is studied and the authors show that the fractional equation introduced is Hyers-Ulam stable, and in [22], the authors obtained the stability regions of differential systems of fractional order with the Prabhakar fractional derivatives. For this purpose, in Section 2, we recall some definitions and lemmas in generalized fractional calculus. In Section 3, we introduce the distributed-order nonlinear time-varying systems containing the Prabhakar fractional derivative and discuss about the stability analysis of these types of fractional differential systems. In Section 4, we plot two examples in order to show the performance and accuracy of the proposed method.

2. Preliminaries

In this section, we recall some definitions and lemmas which are used in the next sections.

$$\begin{aligned} {}_{0^+}I_t^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \quad t>0, 0<\alpha<1, \\ {}_{0^+}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(\tau)(t-\tau)^{-\alpha} d\tau, \quad t>0, 0<\alpha<1, \end{aligned} \tag{2}$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$, $\Re(z) > 0$ is the gamma function. Also, for the absolutely continuous function f, the Caputo fractional derivatives of order α is defined as follows:

$${}_{0^+}^{C}D_t^{\alpha} f(t) = {}_{0^+}I_t^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \tag{3}$$

$$\left(\mathbf{E}_{\rho,\mu,\omega,0^+}^{\gamma} f\right)(t) = \int_0^t (t-\tau)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(t-\tau)^{\rho}) f(\tau) d\tau, \tag{4}$$

where $E_{\rho,\mu}^{\gamma}$ is the generalized Mittag-Leffler function introduced by Prabhakar in 1971 [25]:

$$E_{\rho,\mu}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{n!\,\Gamma(\rho n+\mu)} z^n, \quad \Re(\rho), \Re(\mu) > 0. \tag{5}$$

$$\left(D_{\rho,\mu,\omega,0^+}^{\gamma} f\right)(t) = \frac{d^m}{dt^m} \mathbf{E}_{\rho,m-\mu,\omega,0^+}^{-\gamma} f(t), \quad t>0. \tag{6}$$

Definition 1. (see [23, 24]). Let $0<\alpha<1$ and $f \in L^1[0,b]$, $0<t<b\le\infty$. Then, the Riemann-Liouville fractional integral and derivative of order α are defined as

Definition 2. (see [20]). For $m-1<\mu\le m$ and function $f \in L^1[0,b]$, $0<t<b\le\infty$, the Prabhakar fractional integral is defined as follows:

Definition 3. (see [20]). For the function $f \in L^1[0,b]$, the Prabhakar fractional derivative is defined as

Also, analogous formulas for the Caputo-Prabhakar fractional derivative are given by

$$\left({}^{C}D_{\rho,\mu,\omega,0^+}^{\gamma} f\right)(t) = \mathbf{E}_{\rho,m-\mu,\omega,0^+}^{-\gamma} \frac{d^m}{dt^m} f(t), \quad t>0, \tag{7}$$

where $m-1<\Re(\mu)<m$ and $\rho,\mu,\omega,\gamma \in \mathbb{C}$.

$$\begin{aligned} \mathscr{L}\left\{D_{\rho,\mu,\omega,0^+}^{\gamma} f(t)\,;s\right\} &= s^{\mu}(1-\omega s^{-\rho})^{\gamma} F(s) \\ &\quad - \sum_{k=0}^{m-1} s^k \left(D_{\rho,\mu-k,\omega,0^+}^{\gamma} f\right)(0^+), \end{aligned} \tag{8}$$

where $F(s)$ is the Laplace transform of $f(t)$

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{9}$$

$$\mathscr{L}\left\{t^{\mu-1} E_{\rho,\mu}^{-\gamma}(\omega t^{\rho})\,;s\right\} = s^{-\mu}(1-\omega s^{-\rho})^{\gamma}, \quad \left|\frac{\omega}{s^{\rho}}\right| < 1, \quad \Re(\mu) > 0. \tag{10}$$

$$\left(\frac{d}{dt}\right)^n \left(t^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega t^{\rho})\right) = t^{\mu-n-1} E_{\rho,\mu-n}^{\gamma}(\omega t^{\rho}). \tag{11}$$

$$\begin{aligned} \mathscr{L}\left\{{}^{C}D_{\rho,\mu,\omega,0^+}^{\gamma} f(t)\,;s\right\} &= s^{\mu}(1-\omega s^{-\rho})^{\gamma} F(s) \\ &\quad - s^{\mu-m}(1-\omega s^{-\rho})^{\gamma} \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0^+), \end{aligned} \tag{12}$$

$$\begin{aligned} \mathscr{L}\left\{{}^{C}D_{\rho,\mu,\omega,0^+}^{\gamma} f(t)\,;s\right\} &= \mathscr{L}\left\{t^{m-\mu} E_{\rho,\mu}^{-\gamma}(\omega t^{\rho})\,;s\right\} \mathscr{L}\left\{\frac{d^m}{dt^m} f(t)\,;s\right\} \\ &= s^{\mu}(1-\omega s^{-\rho})^{\gamma} F(s) \\ &\quad - (1-\omega s^{-\rho})^{\gamma} \sum_{k=0}^{m-1} s^{\mu-k-1} f^{(k)}(0^+), \end{aligned} \tag{13}$$

and the proof is completed.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{14}$$

$${}^{C}D_{\rho,c(\mu),\omega,0^+}^{\gamma} x(t) = \int_{m-1}^{m} c(\mu)\, {}^{C}D_{\rho,\mu,\omega,0^+}^{\gamma} x(t) d\mu, \quad m-1<\mu<m. \tag{15}$$

Lemma 4. *(see [20]). The Laplace transform of the Prabhakar fractional derivative for $m-1<\Re(\mu)<m$ is given by*

Lemma 5. *The Laplace transform of the generalized Mittag-Leffler function $t^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega t^{\rho})$ is given by [20]*

Lemma 6. *(see [23]). Let $\rho, \mu, \gamma, \omega \in \mathbb{C}(\Re(\rho) > 0, \Re(\mu) > 0)$. Then, for any $n \in \mathbb{N}$, the generalized Mittag-Leffler function derivative is defined as*

Lemma 7. *The Laplace transform of (7) is given by*

Proof. Using the definition of $({}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}f)(t)$ and equation (10), we obtain

Lemma 8. *(see [26]). Let $F(s) = \mathscr{L}\{f(t)\,;s\}$. If all poles of s $F(s)$ are in the open left-half complex plane, then*

Definition 9. The distributed-order fractional integral operator in the Caputo-Prabhakar sense with respect to an order density function $c(\mu) \geq 0$ is defined by

The Laplace transform of the Caputo-Prabhakar distributed-order derivative is obtained as

$$\begin{aligned}\mathscr{L}\left\{{}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+}x(t)\,;s\right\} &= \int_{m-1}^{m} c(\mu)\Bigg[s^{\mu}(1-\omega s^{-\rho})^{\gamma}X(s) \\ &\quad - (1-\omega s^{-\rho})^{\gamma}\sum_{k=0}^{m-1} s^{\mu-k-1}x^{(k)}(0^+)\Bigg]d\mu \\ &= C(s)(1-\omega s^{-\rho})^{\gamma}X(s) \\ &\quad - (1-\omega s^{-\rho})^{\gamma}\sum_{k=0}^{m-1}\frac{1}{s^{k+1}}C(s)x^{(k)}(0^+),\end{aligned} \tag{16}$$

where $X(s)$ is the Laplace transform of $x(t)$ and $C(s) = \int_{m-1}^{m} c(\mu)s^{\mu}d\mu$.

$$|f(t,x) - f(t,y)| \leq L|x-y|, \tag{17}$$

for all $(t,x), (t,y) \in D$. The constant L is called a Lipschitz constant for $f(t,x)$ with respect to x on D.

Definition 10. A real-valued continuous function $f(t,x)$ is said to satisfy a Lipschitz condition with respect to x on $D = [0,\infty)$ provided there is a constant L such that

3. The Distributed-Order Fractional Integral Operator

In this section, we state the stability and asymptotic stability of the distributed-order nonlinear time-varying systems as

$${}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+}x(t) = f(x(t),t),\ t \in [0,\infty), \quad 0<\mu<1, \quad x(0) = x_0, \tag{18}$$

where ${}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+}x(t) < M, f(x(t),t) \in L^1[0,\infty]$ and f is a real-value continuous function. Also. in the above, $c(\mu)$ is an absolutely integrable function and it satisfies $\int_0^1 c(\mu)s^{\mu}d\mu \neq 0, \Re(s) > 0$. Assuming the above conditions are satisfied for the system (18), in this case, to prove the existence and uniqueness of system (18), we can perform a process similar to [4], and assuming that the system solution will be as follows, these solutions are obtained by taking the Laplace transform from both sides of system ((18)):

$$x(t) = x(0) + \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma}C(s)}\,;t-\xi\right\}f(\xi)d\xi. \tag{19}$$

$$\frac{1}{2}{}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}x^2(t) \leq x(t)\,{}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}x(t), \quad \mu \in (0,1). \tag{20}$$

$$x(t)\,{}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}x(t) - \frac{1}{2}{}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}x^2(t) \geq 0, \tag{21}$$

using equation (7) in Definition 3 for (21), it can be written as follows:

$${}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}x(t) = \int_0^t (t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho})\dot{x}(\tau)d\tau, \tag{22}$$

and in the same way,

$$\frac{1}{2}{}^{C}D^{\gamma}_{\rho,\mu,\omega,0^+}x^2(t) = \int_0^t (t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho})x(\tau)\dot{x}(\tau)d\tau. \tag{23}$$

Lemma 11. *Let $x \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq 0$, we have:*

Proof. Proving that expression (20) is true, to prove that

Relation (21) can be written as

$$\int_0^t (t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho})[x(t)-x(\tau)]\dot{x}(\tau)d\tau \geq 0. \tag{24}$$

Let us define the auxiliary variable $y(\tau) = x(t) - x(\tau)$; in this way, expression (24) can be written as

$$\int_0^t (t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho})y(\tau)\dot{y}(\tau)d\tau \leq 0, \tag{25}$$

defining

$$\begin{aligned} du &= y(\tau)\dot{y}(\tau)d\tau \Rightarrow u = \frac{1}{2}y^2(\tau), \\ v &= (t-\tau)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^{\rho}) \Rightarrow dv \\ &= -(t-\tau)^{-\mu-1}E^{-\gamma}_{\rho,-\mu}(\omega(t-\tau)^{\rho})d\tau,\end{aligned} \tag{26}$$

and taking the integration by parts of (25) turns it into

$$-\left[\frac{y^2(\tau)E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^\rho)}{2(t-\tau)^\mu}\right]\Bigg|_{\tau=t}+\left[\frac{y^2(0)E^{-\gamma}_{\rho,1-\mu}(\omega(t)^\rho)}{2t^\mu}\right]+\frac{1}{2}\int_0^t\frac{y^2(\tau)E^{-\gamma}_{\rho,2-\mu}(\omega(t-\tau)^\rho)}{(t-\tau)^{\mu+1}}d\tau\geq 0. \tag{27}$$

Let us check the first term of relation (27) which has an indetermination at $\tau=t$, so let us analyze the corresponding limit. Now, we show that there exists $\lim_{\tau\to t}-[y^2(\tau)E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^\rho)/2(t-\tau)^\mu]$ and its value is zero, then we have

$$\lim_{\tau\to t}-\left[\frac{y^2(\tau)E^{-\gamma}_{\rho,1-\mu}(\omega(t-\tau)^\rho)}{2(t-\tau)^\mu}\right]=-\frac{1}{2\Gamma(1-\mu)}\lim_{\tau\to t}\frac{[x(t)-x(\tau)]^2}{(t-\tau)^\mu}, \tag{28}$$

since it results in 0/0, by applying the L'Hôpital rule on (3-10), we obtain

$$\begin{aligned}&-\frac{1}{2\Gamma(1-\mu)}\lim_{\tau\to t}\frac{[-2\dot{x}(\tau)x(t)+2x(\tau)\dot{x}(\tau)]}{-\mu(t-\tau)^{\mu-1}}\\&=-\frac{1}{2\Gamma(1-\mu)}\lim_{\tau\to t}\frac{[-2\dot{x}(\tau)x(t)+2x(\tau)\dot{x}(\tau)](t-\tau)^{1-\mu}}{-\mu}=0.\end{aligned} \tag{29}$$

Thus, relation (27) is reduced to

$$\left[\frac{y^2(0)E^{-\gamma}_{\rho,1-\mu}(\omega(t)^\rho)}{2t^\mu}\right]+\frac{1}{2}\int_0^t\frac{y^2(\tau)E^{-\gamma}_{\rho,2-\mu}(\omega(t-\tau)^\rho)}{(t-\tau)^{\mu+1}}d\tau\geq 0, \tag{30}$$

$$\frac{y^2(0)}{2\Gamma(-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(-\gamma+n)\omega^n}{n!\Gamma(\rho n+1-\mu)}t^{\rho n-\mu}+\frac{1}{2\Gamma(-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(-\gamma+n)\omega^n}{n!\Gamma(\rho n+1-\mu)}\int_0^t y^2(\tau)(t-\tau)^{\rho n-\mu-1}\geq 0. \tag{31}$$

Due to $t\geq\tau, t\geq 0$ and features of the gamma function, equation (31) is clearly true, and this concludes the proof.

$$\frac{1}{2}{}^C D^\gamma_{\rho,\mu,\omega,0^+}x^T(t)x(t)\leq x^T(t)\,{}^C D^\gamma_{\rho,\mu,\omega,0^+}x(t),\quad \mu\in(0,1). \tag{32}$$

$$\frac{1}{2}{}^C D^\gamma_{\rho,c(\mu),\omega,0^+}x^T(t)x(t)\leq x^T(t)\,{}^C D^\gamma_{\rho,c(\mu),\omega,0^+}x(t),\quad \mu\in(0,1). \tag{33}$$

$${}^C D^\gamma_{\rho,c(\mu),\omega,0^+}x(t)={}^C D^\gamma_{\rho,c(\mu),\omega,0^+}y(t)+M(t). \tag{34}$$

Remark 12. Lemma 11 is valid for $x(t)\in\mathbb{R}^n$

Lemma 13. *Let $x(t)$ be defined as in Remark 12. Then, for any $t\geq t_0$, the following relationship holds:*

Proof. Multiplying both sides of (20) by $c(\mu)\geq 0$ and integrating with respect to μ in the interval (0,1), the desired result is obtained.

Lemma 14. *Let $\mu\in(0,1)$ and $c(\mu)$ is such that the operator $\int_0^t\mathscr{L}^{-1}\{1/(1-\omega s^{-\rho})^\gamma C(s)\,;t-\xi\}(.)d\xi$ takes nonnegative functions into nonnegative functions. If ${}^C D^\gamma_{\rho,c(\mu),\omega,0^+}x(t)\geq{}^C D^\gamma_{\rho,c(\mu),\omega,0^+}y(t)$ and $x(0)=y(0)$, then $x(t)\geq y(t)$.*

Proof. Adding up a nonnegative function $M(t)$ to the right-hand side of the inequality ${}^C D^\gamma_{\rho,c(\mu),\omega,0^+}x(t)\geq{}^C D^\gamma_{\rho,c(\mu),\omega,0^+}y(t)$, we have

Using formula (16) and taking the Laplace transform of (34), we have

$$\begin{aligned}&(1-\omega s^{-\rho})^\gamma C(s)x(s)-\frac{(1-\omega s^{-\rho})^\gamma C(s)}{s}x(0)\\&=(1-\omega s^{-\rho})^\gamma C(s)y(s)-\frac{(1-\omega s^{-\rho})^\gamma C(s)}{s}y(0)+M(s).\end{aligned} \tag{35}$$

Thus,

$$x(s)=y(s)+\frac{M(s)}{(1-\omega s^{-\rho})^\gamma C(s)}. \tag{36}$$

At this point, by applying the inverse of the Laplace transform on both sides of the above relation (36) and using the convolution theorem, we then obtain

$$x(t)=y(t)+\int_0^t\mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^\gamma C(s)}\,;t-\xi\right\}M(\xi)d\xi. \tag{37}$$

The second term of the right-hand side of (37) is nonnegative, because $\mathscr{L}^{-1}\{1/(1-\omega s^{-\rho})^\gamma C(s)\,;t-\xi\}, M(\xi)$ are nonnegative, then $x(t)\geq y(t)$.

According to Lemma 14, the following corollary is obtained.

$$\alpha_1\|x\|^a\leq V(x(t),t)\leq\alpha_2\|x\|^{ab}, \tag{38}$$

$${}^C D^\gamma_{\rho,c(\mu),\omega,0^+}V(x(t),t)\leq-\alpha_3\|x\|^{ab}, \tag{39}$$

where $\mu\in(0,1)$, $a,b,\alpha_1,\alpha_2,\alpha_3>0$. The distributed-order fractional system of (18) is asymptotically stable in $x=0$ if the roots of $(1-\omega s^{-\rho})^\gamma C(s)+\alpha_3/\alpha_2$ are in the open left-half complex plane, and $c(\mu)$ is such that $\mathscr{L}^{-1}\{1/((1-\omega s^{-\rho})^\gamma C(s)+\alpha_3/\alpha_2)\,;t-\xi\}\geq 0,\forall t\geq 0$.

$${}^C D^\gamma_{\rho,c(\mu),\omega,0^+}V(x(t),t)\leq-\frac{\alpha_3}{\alpha_2}V(x(t),t). \tag{40}$$

Corollary 15. *Let $\mu \in (0,1)$, and the features of Lemma 14 and Lemma 13 are established. Then, the origin of system (18) is stable in that the origin is the equilibrium point, if $x^T f(x(t),t) \leq 0$.*

Theorem 16. *Let $x = 0$ be an equilibrium point for system (18). Let that there exists a Lyapunov function $V(x(t),t)$ satisfying*

Proof. Using equations (38) and (39), we can get

Adding up a nonnegative function $M(t)$ to the left-hand side of the last inequality, we have

$$ {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} V(x(t),t) + M(t) = -\frac{\alpha_3}{\alpha_2} V(x(t),t). \tag{41}$$

By applying the Laplace transform on both sides of (41), we have

$$\begin{aligned}(1-\omega s^{-\rho})^{\gamma} C(s) V(x(s),s) - \frac{(1-\omega s^{-\rho})^{\gamma} C(s)}{s} V(0) + M(s) \\ = -\frac{\alpha_3}{\alpha_2} V(x(s),s),\end{aligned} \tag{42}$$

and solving for $V(s)$:

$$V(s) = \frac{C(s)(1-\omega s^{-\rho})^{\gamma} V(0)}{s((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)} - \frac{M(s)}{(1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2}. \tag{43}$$

By applying the inverse of the Laplace transform on both sides of the above relation (43), we obtain

$$\begin{aligned}V(t) = \mathscr{L}^{-1}\left\{\frac{C(s)(1-\omega s^{-\rho})^{\gamma} V(0)}{s((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)}\right\} \\ - \mathscr{L}^{-1}\left\{\frac{M(s)}{(1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2}\right\}.\end{aligned} \tag{44}$$

We can rewrite the second term of the right-hand side of (44) as

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{M(s)}{(1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2}\right\} &= M(t) * g(t) \\ &= \int_0^t M(\xi) g(t-\xi) d\xi,\end{aligned} \tag{45}$$

where $g(t) = \mathscr{L}^{-1}\{1/((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)\,;t\}$. Considering that $C(s)$ is such that $g(t) \geq 0$, $\forall t \geq 0$, and $M(t), \forall t \geq 0$, then

$$V(t) \leq \mathscr{L}^{-1}\left\{\frac{C(s)(1-\omega s^{-\rho})^{\gamma} V(0)}{s((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)}\right\}. \tag{46}$$

Using Lemma 8 and the hypothesis for the function, we get

$$\begin{aligned}\lim_{t\to\infty} V(t) &\leq \lim_{t\to\infty} \mathscr{L}^{-1}\left\{\frac{C(s)(1-\omega s^{-\rho})^{\gamma} V(0)}{s((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)}\right\} \\ &= \lim_{s\to 0} s\left\{\frac{C(s)(1-\omega s^{-\rho})^{\gamma} V(0)}{s((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)}\right\} = 0.\end{aligned} \tag{47}$$

Using equations (38) and (3-28) and considering that $V(t) \geq 0, \forall t \geq 0$, we can get

$$\lim_{t\to\infty} \alpha_1 \|x(t)\|^a \leq \lim_{t\to\infty} V(x(t),t) = 0, \tag{48}$$

since $\alpha_1, a > 0$, then we obtain $\lim_{t\to\infty}\|x(t)\| = 0$. The proof is complete.

The following lemma allows us to determine asymptotic stability by analyzing the integer order derivative of an appropriate Lyapunov function.

$$\begin{aligned}\left\|\int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma} C(s)}\,;t-\xi\right\} f(x(\xi),\xi) d\xi\right\| \leq \\ \leq \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma} C(s)}\,;t-\xi\right\} \|f(x(\xi),\xi)\| d\xi.\end{aligned} \tag{49}$$

$$x(t) = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma} C(s)}\,;t-\xi\right\} f(x(\xi),\xi) d\xi, \tag{50}$$

norming both sides of (50)

$$\|x(t)\| = \left\|\int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma} C(s)}\,;t-\xi\right\} f(x(\xi),\xi) d\xi\right\|, \tag{51}$$

and applying inequality properties, we get

$$\|x(t)\| \leq \int_0^t \left\|\mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma} C(s)}\,;t-\xi\right\}\right\| \|f(x(\xi),\xi)\| d\xi, \tag{52}$$

since $\mathscr{L}^{-1}\{1/(1-\omega s^{-\rho})^{\gamma} C(s)\,;t-\xi\} \geq 0$, then we have

$$\|x(t)\| \leq \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{(1-\omega s^{-\rho})^{\gamma} C(s)}\,;t-\xi\right\} \|f(x(\xi),\xi)\| d\xi, \tag{53}$$

setting (51) in (53) and we get inequality (49).

$$\gamma_1 \|x(t)\|^a \leq V(x(t),t) \leq \gamma_2 \|x(t)\|, \tag{54}$$

$$\frac{dV(x(t),t)}{dt} \leq -\gamma_3 \|x(t)\|, \quad t \geq 0, \tag{55}$$

where $a, \gamma_1, \gamma_2, \gamma_3 > 0$, $({}^C D^{\gamma}_{\rho,-(1-c(\mu)),\omega,0^+} x(t))_{t=0} = 0$. And the distribution function $1 - c(\mu)$ satisfies the conditions of Theorem 16, then system (18) is asymptotically stable.

$$ {}^C D^{\gamma}_{\rho,1-c(\mu),\omega,0^+} x(t) = {}^C D^{\gamma}_{\rho,-c(\mu),\omega,0^+} \frac{dV(x(t),t)}{dt}. \tag{56}$$

Lemma 17. *If $\mathscr{L}^{-1}\{1/(1-\omega s^{-\rho})^{\gamma} C(s)\,;t-\xi\} \geq 0$ then*

Proof. We define $x(t)$ as follows:

Theorem 18. *Assume that $f(x(t),t)$ satisfies a Lipschitz condition with respect to x on D. If $x(0)=0$, $\mathscr{L}^{-1}\{1/(1-\omega s^{-\rho})^{\gamma} C(s)\,;t-\xi\} \geq 0$ and there exists a Lyapunov function $V(x(t), t)$ that satisfies*

Proof. By properties of the distributed derivative

Setting (56) in (55), we have

$$ {}^C D^{\gamma}_{\rho,1-c(\mu),\omega,0^+} V(x(t),t) \leq -\gamma_3 {}^C D^{\gamma}_{\rho,-c(\mu),\omega,0^+} \|x(t)\|. \tag{57}$$

Let L be a Lipschitz constant for $f(t,x)$ with respect to x on D, and using the Lipschitz condition, then we have from (57)

$$ {}^C D^{\gamma}_{\rho,1-c(\mu),\omega,0^+} V(x(t),t) \leq \frac{-\gamma_3}{L} {}^C D^{\gamma}_{\rho,-c(\mu),\omega,0^+} \|f(x(t),t)\|, \tag{58}$$

using Lemma 17 and considering that $D^{\gamma}_{\rho,-c(\mu),\omega,0^+} f(x(t), t) = x(t)$ we have

$$\begin{aligned} {}^C D^{\gamma}_{\rho,1-c(\mu),\omega,0^+} V(x(t),t) &\leq \frac{-\gamma_3}{L} \|{}^C D^{\gamma}_{\rho,-c(\mu),\omega,0^+} f(x(t),t)\| \\ &\leq \frac{-\gamma_3}{L} \|x(t)\|. \end{aligned} \tag{59}$$

Considering $c(\mu) = 1 - c(\mu)$, $\alpha_3 = \gamma_3/L$, $b = a^{-1}$, it follows from Theorem 16 that the system is asymptotically stable.

4. Numerical Results

In this section, two numerical examples for the distributed-order linear and nonlinear systems are presented to verify the efficiency of the proposed method.

$$ {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_1(t) = -5(x_1(t) + \exp(t) x_2(t)), \tag{60}$$

$$ {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_2(t) = -5(x_2(t) - \exp(t) x_1(t)). \tag{61}$$

Example 19. Consider the following system of a fractional distributed order, when $\mu \in (0,1)$

To use Theorem 16, first we show that $\mathscr{L}^{-1}\{1/((1-\omega s^{-\rho})^{\gamma} C(s) + \alpha_3/\alpha_2)\} \geq 0$, for all $t \geq 0$ is hold, then letting $\alpha_1 = 1/4$, $\alpha_2 = 1$, and $\alpha_3 = 4$, we obtain

$$ (1-\omega s^{-\rho})^{\gamma} C(s) + \frac{\alpha_3}{\alpha_2} = 4 - s^{2/3}(1-\omega s^{-\rho})^{\gamma}, \tag{62}$$

$$ \frac{1}{4 - s^{2/3}(1-\omega s^{-\rho})^{\gamma}} = \frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{s^{2/3}(1-\omega s^{-\rho})^{\gamma}}{4}\right)^n, \quad |s^{2/3}(1-\omega s^{-\rho})^{\gamma}| < 1. \tag{63}$$

Using Lemma 5 on equation (63), we obtain

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{4 - s^{2/3}(1-\omega s^{-\rho})^{\gamma}}\right\} &= \frac{1}{4}\sum_{n=0}^{\infty} \mathscr{L}^{-1}\left\{\left(\frac{s^{2/3}(1-\omega s^{-\rho})^{\gamma}}{4}\right)^n\right\} \\ &= \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{4^n} t^{-(2/3)n-1} E^{-n\gamma}_{\rho,-(2/3)n}(\omega t^{\rho}), \quad t \geq 0. \end{aligned} \tag{64}$$

For each $t \geq 0$, inequality $(1/4)\sum_{n=0}^{\infty} (1/4^n) t^{-(2/3)n-1} E^{-n\gamma}_{\rho,-(2/3)n}(\omega t^{\rho}) \geq 0$ is hold. Then, the first part is established. Also, all the roots of this function $4 - s^{2/3}(1-\omega s^{-\rho})^{\gamma} = 0$ are located in the open left-half complex plane and this roots can be obtained by $s = re^{i\theta}$. Now, let us consider the following Lyapunov candidate function:

$$ \alpha_1 \|x(t)\|^2 \leq V(x_1(t), x_2(t)) = \frac{1}{2}x_1^2(t) + \frac{1}{2}x_2^2(t) \leq \|x(t)\|^2. \tag{65}$$

Using Lemma 13 for (65), we obtain

$$\begin{aligned} {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} V(x_1(t), x_2(t)) &= \frac{1}{2} {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_1^2(t) \\ &\quad + \frac{1}{2} {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_2^2(t) \\ &\leq x_1(t) {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_1(t) \\ &\quad + x_2(t) {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_2(t). \end{aligned} \tag{66}$$

Substituting system (60) in (66), then we obtain

$$ {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} V(x_1(t), x_2(t)) \leq -5(x_1(t)^2 + x_2(t)^2) \leq -\alpha_3 \|x(t)\|^2. \tag{67}$$

By Theorem 16, we can conclude that the origin of (60) is asymptotically stable. Figures 1 and 2 demonstrate the behavior of system (60) for a short time scale.

$$ {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_1(t) = -2(x_1(t) x_2^2(t) + x_1(t)), \tag{68}$$

$$ {}^C D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_2(t) = -2(-x_2(t) x_1^2(t) + x_2(t)). \tag{69}$$

Example 20. In this example, we consider the following nonlinear system of fractional distributed order when $\mu \in (0,1)$:

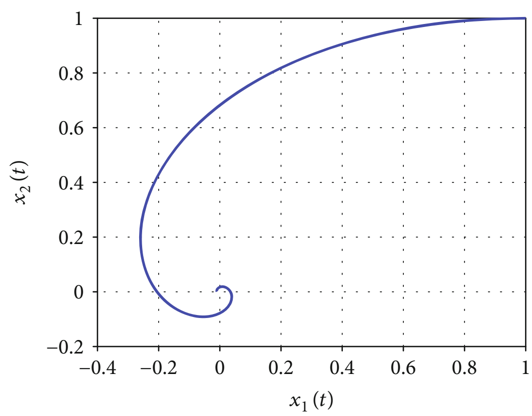

FIGURE 1: The numerical results for system (60) at $c(\mu) = -\delta(\mu - 2/3)$ for Example 19.

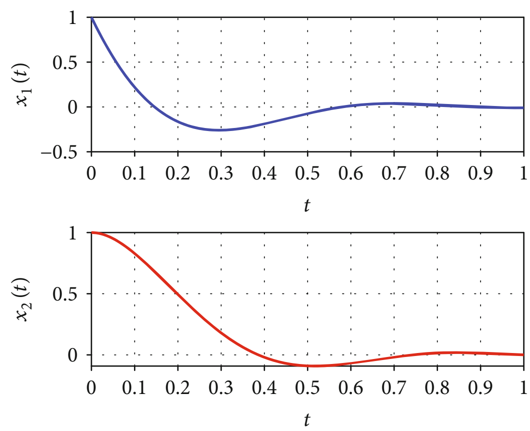

FIGURE 2: Stability diagram of Example 19 for $x_1(t)$ and $x_2(t)$.

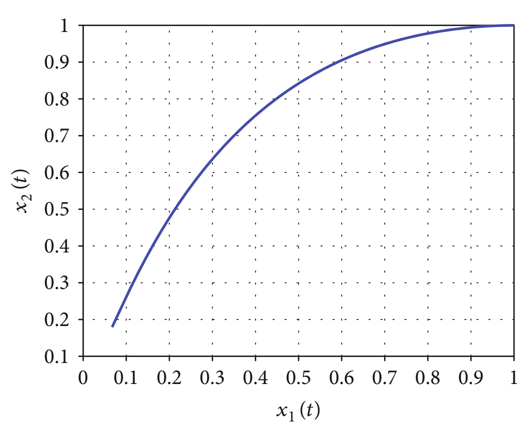

FIGURE 3: The numerical results for system (60) at $c(\mu) = -\delta(\mu - 2/3)$ for Example 20.

With the same process as Example 19, we consider the following Lyapunov candidate function:

$$\alpha_1 \|x(t)\|^2 \le V(x_1(t), x_2(t)) = \frac{1}{2} x_1^2(t) + \frac{1}{2} x_2^2(t) \le \|x(t)\|^2. \tag{70}$$

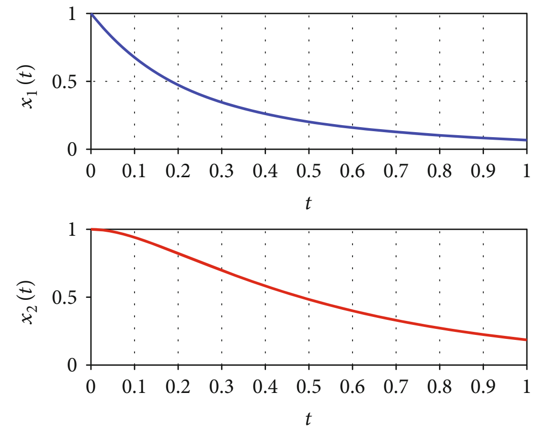

FIGURE 4: Stability simulation of Example 20 for $x_1(t)$ and $x_2(t)$.

Using Lemma 13 for (70), we obtain

$$\begin{aligned} {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} V(x_1(t), x_2(t)) &= \frac{1}{2} {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_1^2(t) \\ &\quad + \frac{1}{2} {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_2^2(t) \\ &\le x_1(t) {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_1(t) \\ &\quad + x_2(t) {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} x_2(t). \end{aligned} \tag{71}$$

Substituting system (68) in (71), we then have

$$\begin{aligned} {}^{C}D^{\gamma}_{\rho,c(\mu),\omega,0^+} V(x_1(t), x_2(t)) &\le -2x_1(t)\left(x_1(t)x_2^2(t) + x_1(t)\right) \\ &\quad - 2x_2(t)\left(-x_2(t)x_1^2(t) + x_2(t)\right) \\ &\le -\alpha_3 \|x(t)\|^2. \end{aligned} \tag{72}$$

By Theorem 16, we can conclude that the origin of (68) is asymptotically stable. Figures 3 and 4 demonstrate the behavior of system (68) for a short time scale.

5. Conclusion

In this paper, we focus on the distributed-order linear and nonlinear time-varying systems containing Caputo-Prabhakar fractional derivative of order $c(\mu)$. With the expansion the Lyapunov direct method to the distributed-order case, we state that stability and asymptotic stability results in this kind of systems. Also, in this paper, Lemma 11 is a generalization of Lemma 1 in [27], Theorem 16 is a generalization of Theorem 3 in [7], Lemma 17 is a generalization of Lemma 4 in [7], and Theorem 18 is a generalization of Theorem 4 in [7]. In order to demonstration the validity and applicability of the obtained results in this paper, two examples are shown.

Data Availability

The [MATLAB] data used to support the findings of this study are included within the article. The idea of this article is taken from the article [https://scholar.google.com.ua/scholar?hl=en~~~~~~~~~^~^~^~^~~~~~~~~~~~amp;as_sdt=0%2C5q=Asymptotic+stability+of+distributed+order+nonlinear+dynamical+systems~~~~~~~~~^~^~^~^~~~~~~~~~~~amp;btnG=], and it is similar to the data in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] T. M. Atanackovic, M. Budincevic, and S. Pilipovic, "On a fractional distributed-order oscillator," *Journal of Physics A: Mathematical and General*, vol. 38, no. 30, pp. 6703–6713, 2005.

[2] H. Aminikhah, A. Refahi Sheikhani, and H. Rezazadeh, "Stability analysis of distributed order fractional Chen system," *The Scientific World Journal*, vol. 2013, 13 pages, 2013.

[3] P. L. T. Duong, E. Kwok, and M. Lee, "Deterministic analysis of distributed order systems using operational matrix," *Applied Mathematical Modelling*, vol. 40, no. 3, pp. 1929–1940, 2016.

[4] N. J. Ford and M. L. Morgado, "Distributed order equations as boundary value problems," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 2973–2981, 2012.

[5] A. N. Kochubei, "Distributed order calculus and equations of ultraslow diffusion," 2007, http://arxiv.org/abs/math-ph/0703046.

[6] M. Caputo, *Elasticita e dissipazione*, Zanichelli, Italian, Bologna, 1969.

[7] G. Fernández-Anaya, G. Nava-Antonio, J. Jamous-Galante, R. Muñoz-Vega, and E. G. Hernández-Martínez, "Asymptotic stability of distributed order nonlinear dynamical systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 48, pp. 541–549, 2017.

[8] M. A. Abdelkawy, M. M. Babatin, and A. M. Lopes, "Highly accurate technique for solving distributed-order time-fractional-sub-diffusion equations of fourth order," *Computational and Applied Mathematics*, vol. 39, no. 2, pp. 1–22, 2020.

[9] M. Fei and C. Huang, "Galerkin-Legendre spectral method for the distributed-order time fractional fourth-order partial differential equation," *International Journal of Computer Mathematics*, vol. 97, no. 6, pp. 1183–1196, 2020.

[10] H. S. Najafi, A. R. Sheikhani, and A. Ansari, "Stability analysis of distributed order fractional differential equations," in *abstract and applied analysis (Vol. 2011)*, Hindawi, 2011.

[11] A. Refahi, A. Ansari, H. S. Najafi, and F. Merhdoust, "Analytic study on linear systems of distributed order fractional differential equations," *Le Matematiche*, vol. 67, no. 2, pp. 3–13, 2012.

[12] P. Miskinis, "The Havriliak-Negami susceptibility as a nonlinear and nonlocal process," *Physica Scripta*, vol. 2009, no. T136, article 014019, 2009.

[13] R. Garrappa, F. Mainardi, and M. Guido, "Models of dielectric relaxation based on completely monotone functions," *Fractional Calculus and Applied Analysis*, vol. 19, no. 5, pp. 1105–1160, 2016.

[14] A. Giusti and I. Colombaro, "Prabhakar-like fractional viscoelasticity," *Communications in Nonlinear Science and Numerical Simulation*, vol. 56, pp. 138–143, 2018.

[15] M. D'Ovidio and F. Polito, "Fractional diffusion-telegraph equations and their associated stochastic solutions," 2013, http://arxiv.org/abs/1307.1696.

[16] K. Górska, A. Horzela, L. Bratek, K. A. Penson, and G. Dattoli, "The probability density function for the Havriliak-Negami relaxation," 2016, http://arxiv.org/abs/1611.06433.

[17] J. An, E. Van Hese, and M. Baes, "Phase-space consistency of stellar dynamical models determined by separable augmented densities," *Monthly Notices of the Royal Astronomical Society*, vol. 422, no. 1, pp. 652–664, 2012.

[18] A. Stanislavsky and K. Weron, "Atypical case of the dielectric relaxation responses and its fractional kinetic equation," *Fractional Calculus and Applied Analysis*, vol. 19, no. 1, pp. 212–228, 2016.

[19] A. Liemert, T. Sandev, and H. Kantz, "Generalized Langevin equation with tempered memory kernel," *Physica A: Statistical Mechanics and its Applications*, vol. 466, pp. 356–369, 2017.

[20] R. Garra, R. Gorenflo, F. Polito, and Ž. Tomovski, "Hilfer-Prabhakar derivatives and some applications," *Applied Mathematics and Computation*, vol. 242, pp. 576–589, 2014.

[21] M. H. Derakhshan, M. Ahmadi Darani, A. Ansari, and R. Khoshsiar Ghaziani, "On asymptotic stability of Prabhakar fractional differential systems," *Computational Methods for Differential Equations*, vol. 4, no. 4, pp. 276–284, 2016.

[22] M. H. Derakhshan and A. Ansari, "On Hyers–Ulam stability of fractional differential equations with Prabhakar derivatives," *Analysis*, vol. 38, no. 1, pp. 37–46, 2018.

[23] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations (Vol. 204)*, Elsevier Science Limited, North-Holland Mathematics Studies, Amsterdam, London and New York, 2006.

[24] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of Their Applications*, Elsevier, Academic Press Inc, San Diego, CA, USA, 1998.

[25] T. R. Prabhakar, *A singular integral equation with a generalized Mittag Leffler function in the kernel*, 1971.

[26] D. G. Duffy, *Transform Methods for Solving Partial Differential Equations*, Chapman and Hall/CRC, New York, 2004.

[27] N. Aguila-Camacho, M. A. Duarte-Mermoud, and J. A. Gallegos, "Lyapunov functions for fractional order systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 9, pp. 2951–2957, 2014.