M2T2

PATTERNS AND POLYGONS

Algebra

STATE GOAL 8: Use algebraic and analytical methods to identify and describe patterns and relationships in data, solve problems and predict results.

Statement of Purpose:

As the NCTM Principals and Standards document states, "students in the middle grades should learn algebra both as a set of competencies tied to the representation of quantitative relationships and as a style of mathematical thinking for formalizing patterns, functions, and generalizations." (NCTM, p. 223). In this section we have built on the notions of patterns and their expression both graphically and algebraically. We try to show the connections between algebra and geometry. We explore discrete data and graphing in the context of linear and non-linear relationships.

The introduction of appropriate tools at appropriate times (from computers to angle rulers) is an important part of exploring and developing mathematical understanding.

Highlights

Algebra is an important tool for discovering, describing, and exploring patterns and relations in a variety of contexts. In this unit, we use algebra to help us make and talk about important discoveries in *geometry*—in particular, to study the interior angles of polygons.

Connections to the Illinois Learning Standards.

Standard 8.A.—Describe numerical relationships using variables and patterns. Throughout this module, participants will be looking for patterns and relationships. They will use tables, graphs, manipulatives, measurements, formulas, reasoning find the patterns and express the relationships.

Standard 8.B.—Interpret and describe numerical relationships using tables, graphs, and symbols. Tables are used to find patterns in sets of numbers and measurements of interior angles in polygons. The data from the table is graphed and formulas are derived from the data.

Standard 8.D.—Use algebraic concepts and procedures to represent and solve problems. In this module participants use formulas and revise those formulas to find their applicability with various polygons. They apply the search for patterns to the "Handshake Problem" and the "Swimming Pool" problem. Both of these lead to patterns involving quadratic relationships.

Table of Contents

Materials

Minimal Materials List

- Angle rulers (or protractors)
- Ruler (or other straight edge)
- Graph paper (Grid paper is available in the Appendix)
- Scissors and tape (for cutting out angles)
- Color tiles or squares (for swimming pool problem)
- Colored pencils

Optimal list will include

- Graphing calculator
- Distance sensor for graphing calculator
- Internet access
- Geometer's Sketchpad®, Cabri®, or other geometry software
- Microsoft Excel®, Claris Works®, or other spread-sheet software

List of Internet Files Used in This Module

- <www.mste.uiuc.edu/users/mmckelve/applets/mlmtoothpick2/default.html> Toothpick applet
- <www.mste.uiuc.edu/activity/boxperim/> The box-perimeter applet.
- <www.mste.uiuc.edu/m2t2/algebra/tableForPolygons.xls> is an Excel® spreadsheet with a blank table at for polygon interior angle values.
- <www.mste.uiuc.edu/m2t2/algebra/Nonagon.gsp> is a Geometer's Sketchpad file of regular nonagon.
- <www.mste.uiuc.edu/m2t2/algebra/Nonascpt.gss> is a script for constructing the nonagon automatically.
- <www.mste.uiuc.edu/m2t2/algebra/table2Polygons.xls> is a spreadsheet with the values of the interior angles to plot.
- <www.mste.uiuc.edu/m2t2/algebra/20gon.gsp> is a Sketchpad file of a regular 20-gon.
- <www.mste.uiuc.edu/activity/angleobject/> This applet will show the angle sum of a polygon in circle units. Participants may use scissors and tape to cut out the angles and tape them together into circle units and half-units.
- <www.mste.uiuc.edu/m2t2/algebra/nonregs2.gsp> is a Geometer's Sketchpad® file of irregular polygons.

M2T2

Instructor Page

Looking for Patterns.

We begin with some activities in pattern recognition and expression.

Real and “virtual” manipulatives can be used to reinforce the effort to find patterns. The graphics at right use Java applets available at <www.mste.uiuc.edu/java/>

The progression of the unit involves seeing patterns in different ways: with manipulatives and technology, with formulas and with graphs, with charts, tables, and drawing tools.

The activities on the participant page can be done with real toothpicks and with square tiles as well as with the Java applets as pictured.

X	Y
1	4
2	7
3	10
4	13
5	16
6	19
7	22
10	31

At left is the table for the problem with toothpicks. Notice that the difference between each of the values in the right-hand column is 3.

The rule for this table is the following:
English version: In every row, the Y-value is always 3 times the corresponding X value, plus 1.
Algebra version: $Y = 3x + 1$

Point out that the algebra version is more succinct than the English version.

X	Y
1	4
2	6
3	8
4	10
5	12
6	14
10	22
49	100

At left is the table for the problem with boxes (the banquet table problem) . Notice that the difference between each of the successive values in the right-hand column is 2.

The rule for this table is
English version: In every row, the Y-value is always 2 times the corresponding X value, plus 2.
Algebraic version: $Y = 2x + 2$

Internet sites:

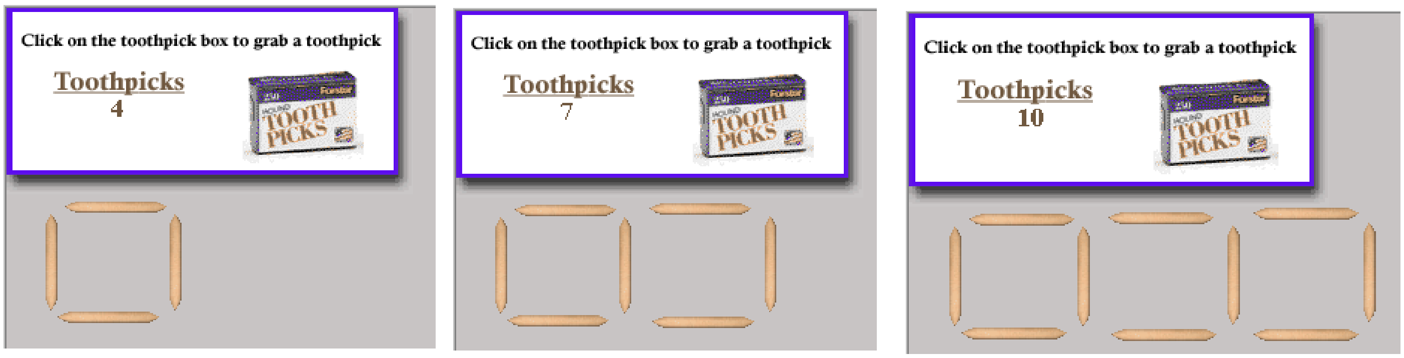
<www.mste.uiuc.edu/users/mmckelve/applets/mlmtoothpick2/default.html> Toothpick applet
<www.mste.uiuc.edu/activity/boxperim/> The box-perimeter applet.

M2T2

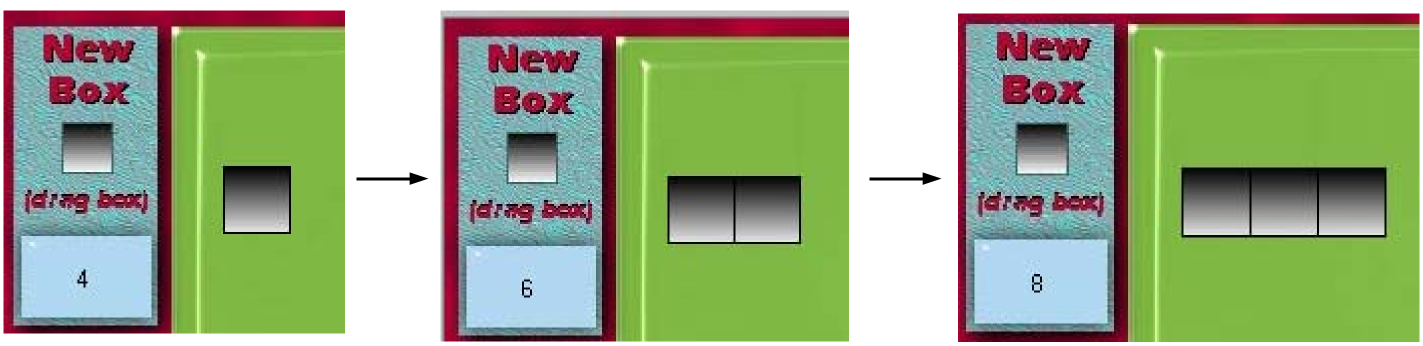
Participant Page

Looking for Patterns.

We begin with some activities in pattern recognition and expression.



If 4 toothpicks make one box, and 7 toothpicks make two boxes, and 10 toothpicks make three boxes, how many toothpicks make 10 boxes in a row? How many make 100 boxes in a row?



Let's look at the "banquet table problem". Each table can seat one person on a side. But when the tables are put end-to-end, one of the sides on each table cannot be used. How many a tables need to be set end-to-end to have enough seats for 8 people? For 100 people?

M2T2

Instructor Page

Examining Patterns

This activity should help participants get used to the idea of looking for patterns and expressing them in different ways.

X	Y
0	3
1	4
2	5
3	6
4	7
8	11
9	12
12	15

Description: Add three to each number on the left.

Algebraic expression: $y = x + 3$

X	Y
0	1
1	3
2	5
3	7
4	9
5	11
8	17
10	21

Description: Each number on the left is multiplied by 2 and then 1 is added.

Algebraic expression: $y = 2x + 1$

X	Y
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49

Description: Take each number on the left, multiply it by itself.

Algebraic expression: $y = x^2$

X	Y
1	4
2	6
3	8
4	10
5	12
6	14
10	22
49	100

Description: Take each value in the left-hand column, multiply it by 2 and add two.

Algebraic expression: $y = 2x + 2$

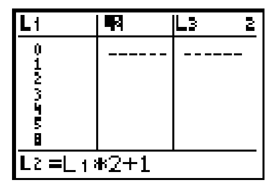
.

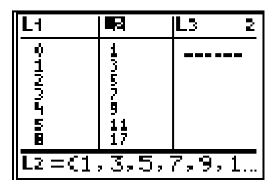
Calculator activity:

Participants can enter and graph these tables with calculators. Below are instructions for the TI-73 and second table of the worksheet.

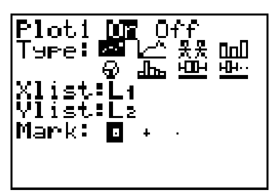
Turn the calculator on and press the [LIST] button. Then use the arrow keys to go to L1 and press [CLEAR] and [ENTER].

Next enter the values 0, 1, 2, 3,4, 5, 8, 10. Clear L2 if it is not already clear. There are two ways to enter the second column. You may enter the numbers individually, OR... you can enter the formula. To use the formula, highlight L2 and press [2nd] [STAT] [1]. Then [x] [2] [+] [1]. Your screen should look like this one. Next, press [ENTER]. The values of the second list should appear. Point out to participants that they just generated the second list from the formula. Have them try the same thing with the other tables.

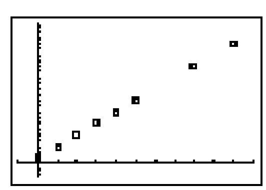




To graph these data, press [2nd] [PLOT] [1]. Your screen should look like the one at right. Be sure that "Plot 1" is turned on. Also, be sure Xlist is L1 and Ylist is L2. (To set them, [▼][▼][▼], then [2nd] [STAT] [1], [▼] [2nd] [STAT] [2]).



Finally, press [ZOOM] [7]. Press [TRACE] and use the arrows to view the coordinates of the points.



M2T2

Participant Page

Examining Patterns

For each table of numbers, fill in the missing numbers following the same rule as the other number pairs in the table. Then describe the rule using words. Next, use an algebraic expression to describe the rule. Consider the left column to be **X** and the right column to be **Y**.

X	Y
0	3
1	4
2	5
3	6
4	7
8	11
9	
12	

1) ____

Algebraic expression: ____________________

X	Y
0	1
1	3
2	5
3	7
4	9
5	
8	
	21

2) ____

Algebraic expression: ____________________

X	Y
0	0
1	1
2	4
3	9
4	16
5	
6	
7	

3) ____

Algebraic expression: ____________________

X	Y
1	4
2	6
3	8
4	10
5	12
6	
10	
	100

4) ____

Algebraic expression: ____________________

M2T2

Instructor Page

An n-gon is a polygon with n sides.

A regular polygon is a polygon with all sides equal to one another and all angles equal to one another.

Patterns and Regular Polygons

- ⇒ Give everyone a worksheet with several regular polygons like the one at the right. Larger versions of the polygons are included in Appendix A.
- ⇒ Allow participants time to measure the angles in each polygon and complete the table.
- ⇒ Ask questions about n-gons not provided. What would be the angle for a "15-gon?"
- ⇒ Ask participants to describe the patterns in words. Every time you add a side, you add 180 degrees! Why?

Participants should know that the measures of the angles are all equal. Have them look for patterns in the measures of the interior angles.

NOTE: Many participants (teachers especially) know the trick of dividing the polygons into triangles. If so, use that as a starting point, but have them step back and examine the patterns that come from measuring angles.

It is hard to see a pattern in the measure of the single interior angle. Have participants use the last column for the sum of the interior angles. See if they can find patterns that relate the number of sides to the to the interior angle sum. Discuss with them looking for patterns in tables of numbers. Do they look vertically and horizontally for patterns?

Polygon	Name of Polygon	Number of Sides	Number of Angles	Measurement of an Interior angle	Sum of Interior Angles
A	Triangle	3	3	60°	180°
B	Square	4	4	90°	360°
C	Pentagon	5	5	108°	540°
D	Hexagon	6	6	120°	720°
E	Heptagon	7	7	Approximately 129°	900°

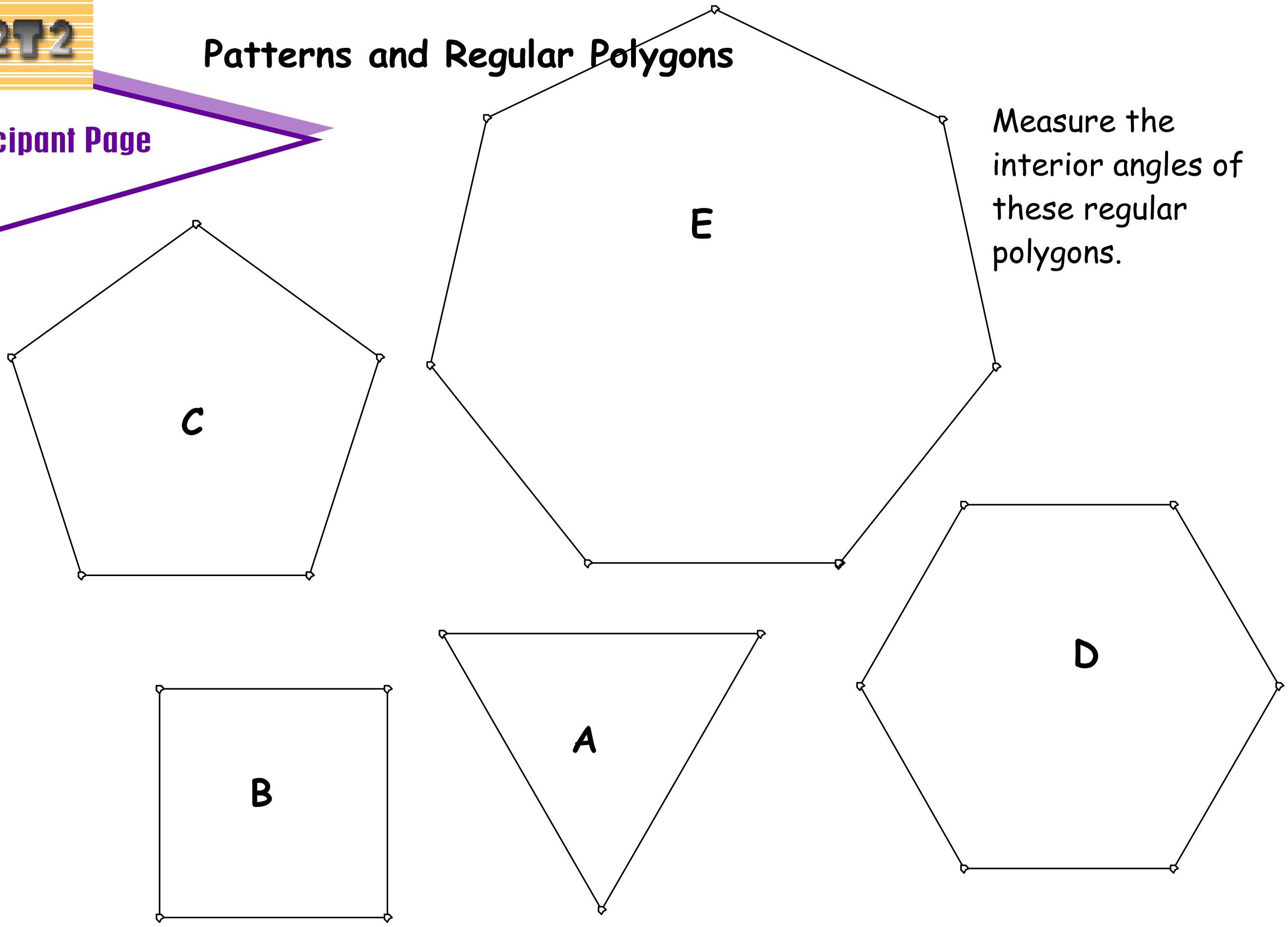
Internet Resources

<www.mste.uiuc.edu/m2t2/algebra/tableForPolygons.xls> is an Excel® spreadsheet with the table at the left. Participants can see the values, find the patterns and let the spreadsheet do the multiplication.

Participant Page

Patterns and Regular Polygons

Measure the interior angles of these regular polygons.



Polygon	Name of Polygon	Number of Sides	Number of Angles	Measurement of an interior angle	Sum of Interior Angles
A		3			
B		4			
C		5			
D		6			
E		7			
F*		8			
G*		9			
H*		n			

*Try to fill in these rows without the polygons.

What patterns do you see? Write them on the next page.

M2T2

Instructor Page

Applying the Pattern to Draw a Regular Polygon

The main idea here is to move from the figures and the table to the formula, and thus to show multiple representations of the same information.

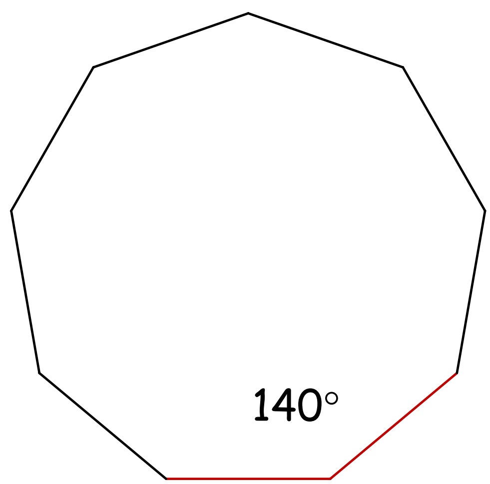
Participants may come up with different answers. Some may arrive at the formula before others. Some may divide the polygon into triangles (as we will do later) to find the formula.

- ⇒ Using a word version of the pattern, ask the participants to convert the pattern into the language of algebra. For example, multiply the number of sides times 180 and then subtract 360 degrees from the result. So, one algebraic expression is 180n—360 = Sum.
- ⇒ Discuss various algebraic representations of the verbal form of the pattern. Another expression of the pattern is (n-2)180 = Sum
- ⇒ Agree on a formula to represent the pattern.
- ⇒ Ask each participant to draw a nonagon using their paper, pencil, ruler, protractor, and/or angle ruler. Agree on a procedure for drawing the nonagon.
- ⇒ Discuss how the formula suggests a procedure for drawing any regular polygon.

$$Sum = (n-2)180$$

or

$$Sum = 180n - 360$$



A regular nonagon. Each interior angle is 140°

To be able to draw the regular nonagon, participants must recognize that the sum will be divided by 9 to get the value of each interior angle, 140° in the case of the regular nonagon. Next, they use that angle and the length of each side to draw the nonagon.

If Geometer's Sketchpad® is available, participants can use the sketch and script below to see how to apply the algorithm with a script and let the computer make the polygon.

Internet Resources:

<www.mste.uiuc.edu/m2t2/algebra/Nonagon.gsp> is a Geometer's Sketchpad file of a regular nonagon.

<www.mste.uiuc.edu/m2t2/algebra/Nonascpt.gss> is a script for constructing the nonagon automatically.

<www.mste.uiuc.edu/m2t2/algebra/Fastnona.gsp> is a Sketchpad file that explains how to draw a regular nonagon more quickly using the central angle.

M2T2

Participant Page

Applying the Pattern to Draw a Regular Polygon

Use this space to explain the pattern that you find.

In the box below, write a formula for the pattern that everyone agrees will relate the sum of the interior angles of a regular polygon to the number of sides.

Sum =

So now that you have found a formula, use it to draw a nonagon with a ruler and protractor (or angle ruler). Describe your step-by-step procedure.

Instructor Page

We want to explore different ways to get at the pattern. That is, look at it visually and with variables.

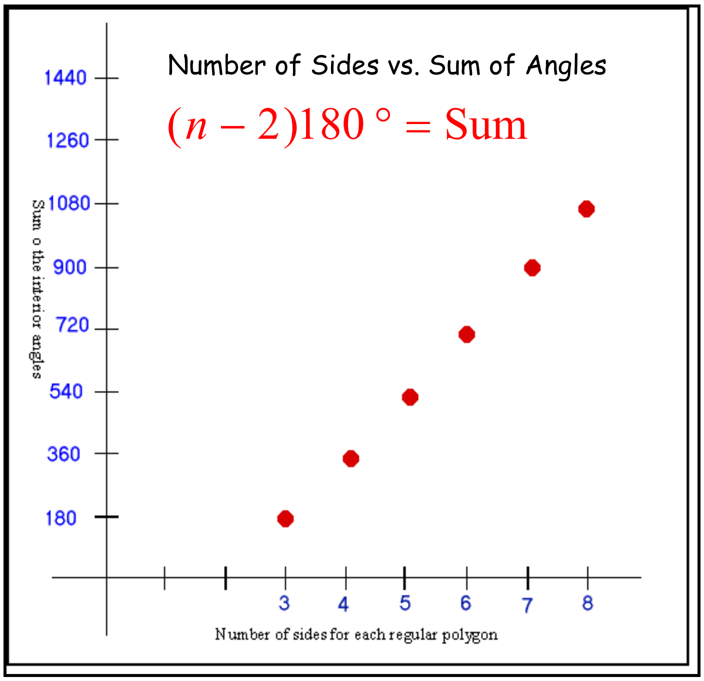
Spend time talking about discrete vs. continuous graphs. Why would it not make sense to connect the points on the graph when we are talking about polygons? On the other hand, it is useful to discuss the "linear relationship" depicted in the graph. (What would the slope be?!

NOTE: The dependent variable could be radians or even "number of circles".

Graphing the Pattern

⇒ Have the participants label the axes. Make the horizontal axis the number of sides and the vertical axis the angle sum. Be sure to note that the scales are different on the two axes.

⇒ Note that there should only be points. It is misleading to draw lines connecting the points because the values between the points do not have meaning.

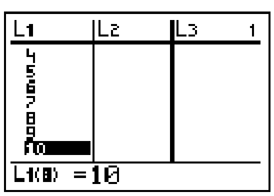


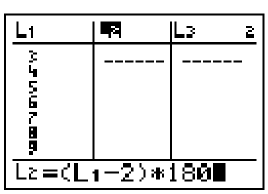
This is the completed graph. Note that the first point has the coordinates (3,180)

Notice too that this is a graph of a linear relationship.

Plotting the points with the TI-73.

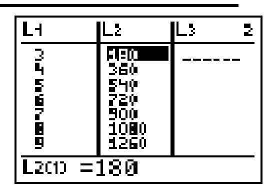
Hit [LIST]. Arrow to L1 and hit [CLEAR] [ENTER], Then enter 3, 4, 5, 6, 7, 8, 9, 10 in L1. Then use the arrows to highlight L2. Hit [CLEAR] [ENTER], arrow to L2 again and then [(] [2nd] [STAT] [1], then [−] [2] [)] [×] [1] [8] [0] [ENTER].

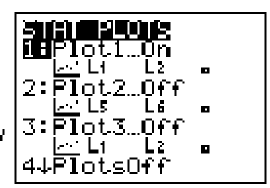


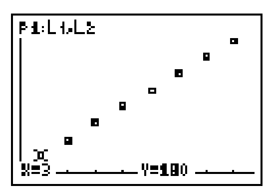


You screen should look like the one at right. Next, hit [2nd] [PLOT] and be sure that Plot1 is on and that L1 and L2 are the selected lists.

Finally, hit [ZOOM] [7] to get the graph, and [TRACE] to view the coordinates. Use the arrow keys to see the X and Y values.







Internet Resources:

Graphing points is very easy with a spreadsheet. An Excel® chart with these values is available at <www.mste.uiuc.edu/m2t2/algebra/table2Polygons.xls

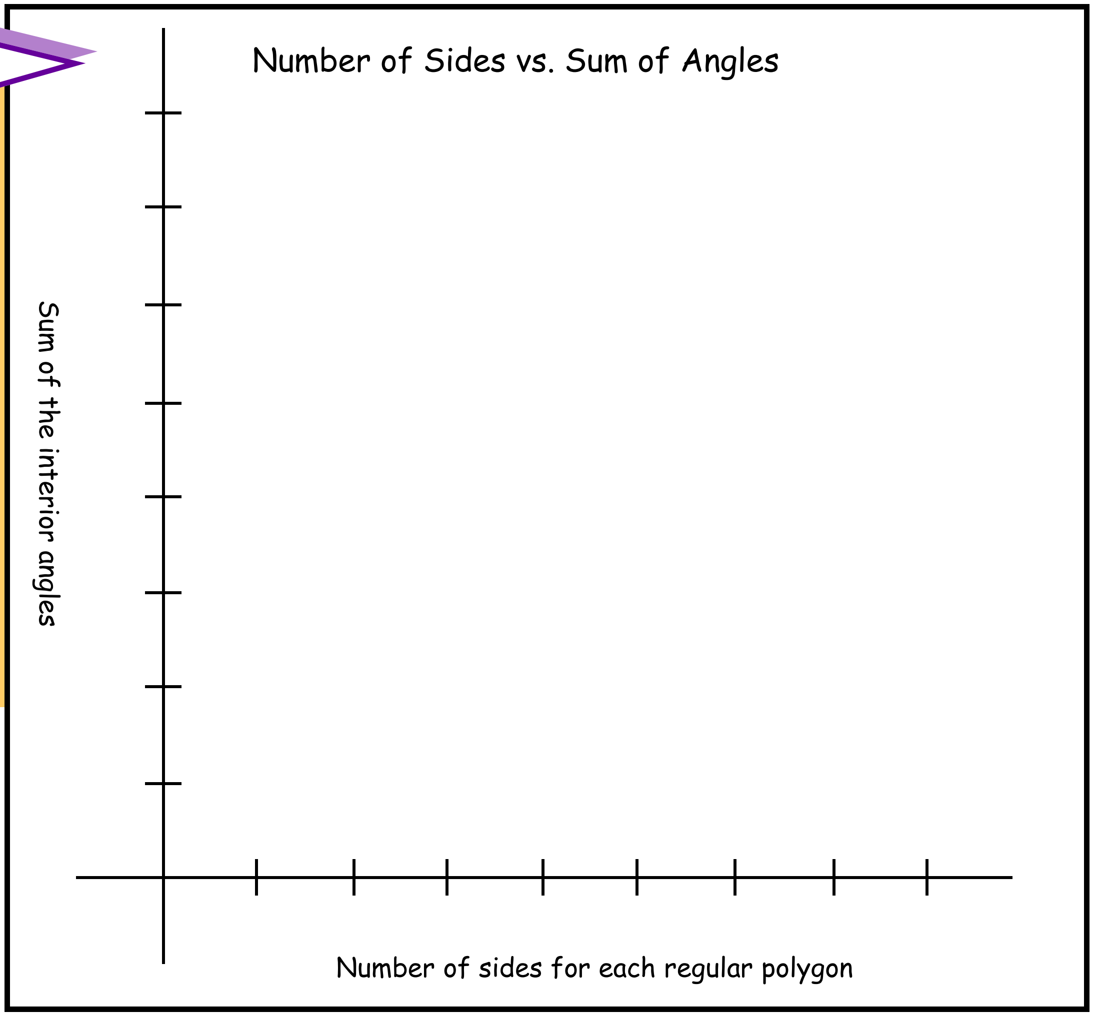
M2T2

Participant Page

Try making a graph with the sides as one axis and the angle sum as another axis.

Use the axes at the right. Be sure to label them.

Graphing the Pattern



Describe the graph.

____

The graph you made above should have many points on it. Should you connect those points? What would it mean if those points were connected?

____

____

____

M2T2

Instructor Page

We want to explore different ways to get at the pattern. That is, explore it with tables, formulas, and graphs. And look at different aspects of each.

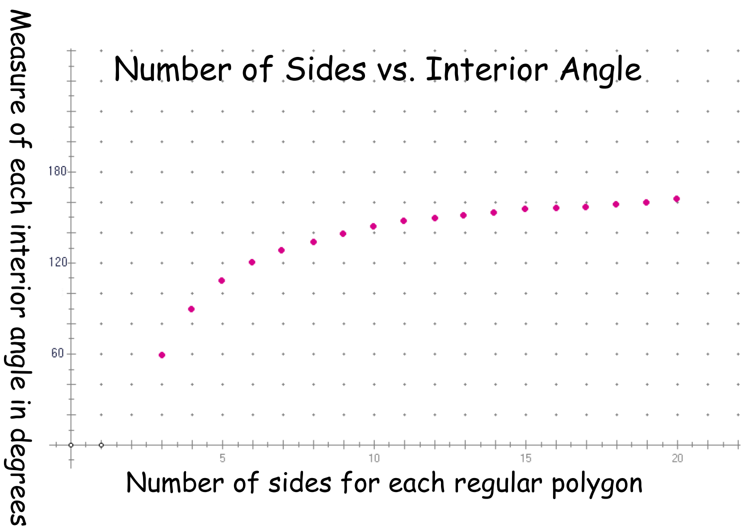
In mathematics a limit is a value that a function or a sequence approaches with increasing closeness, but does not reach.

The domain is the set of values assumed by the independent variable.

Graphing the Pattern: Part 2

Try the same activity again. However, this time graph the value of one interior angle against the number of sides in a polygon. That is, for the hexagon plot the point (6, 120) since it has six sides and each angle is 120°.

NOTE: A black-line master for the graph below is available in Appendix D. Instructions for plotting these points with the TI-73 are available in Appendix E.



This is what the plot on the side of the page should look like.

Notice that the dots do not (and will not) reach 180°. 180° is the *LIMIT*!

$$\frac{(n-2)180°}{n} = \text{Interior Angle Of Regular Polygon}$$

One way to make sense of this graph is to manipulate the expression (n-2)180/n. This is the same as (180n/n) - 360/n, which simplifies to 180 - (360/n). As n gets larger, 360/n gets smaller. An extension would be to fill in the table below. Ask participants to write what 360/n means? How does it relate to the graph above?

Notice that, as with the previous graph, the domain is restricted to whole numbers of 3 or larger. More details on this exploration can be found in Appendices D and E.

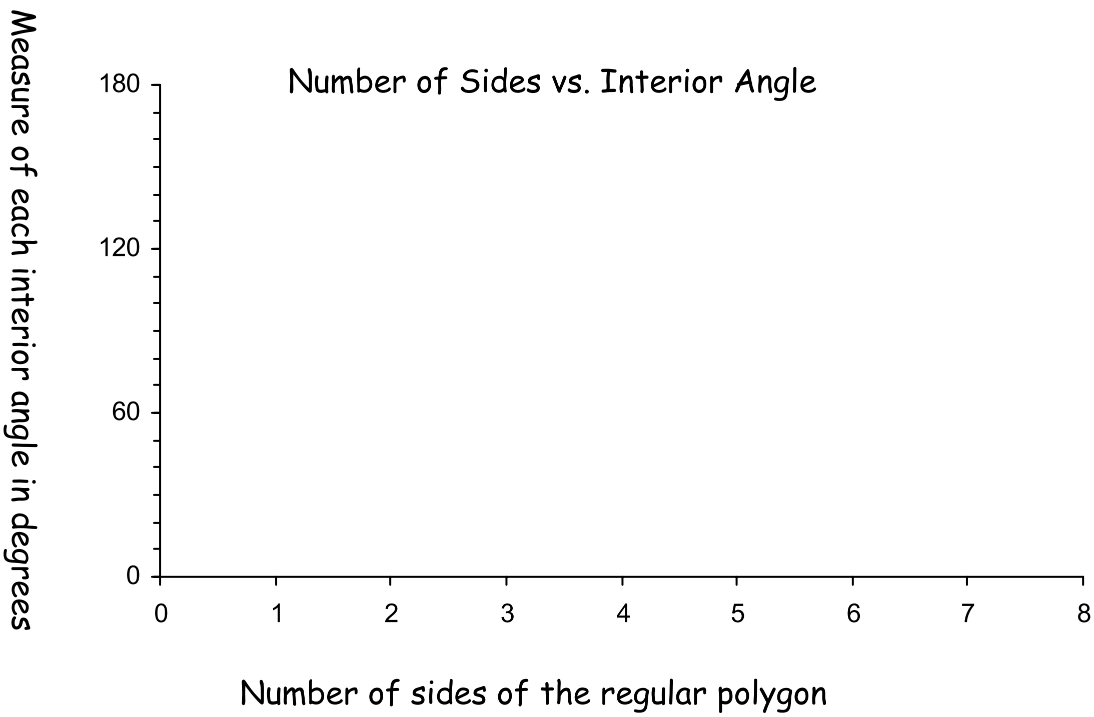
Number of Sides (n)	3	4	5	10	20
(n-2)180/n	60	90	108		162
360/n	120	90			

Participant Page

Graphing the Pattern: Part 2

Try plotting the value of an interior angle against the number of sides.

What do you notice?



Write the formula for the measure of an interior angle of a regular n-gon below:

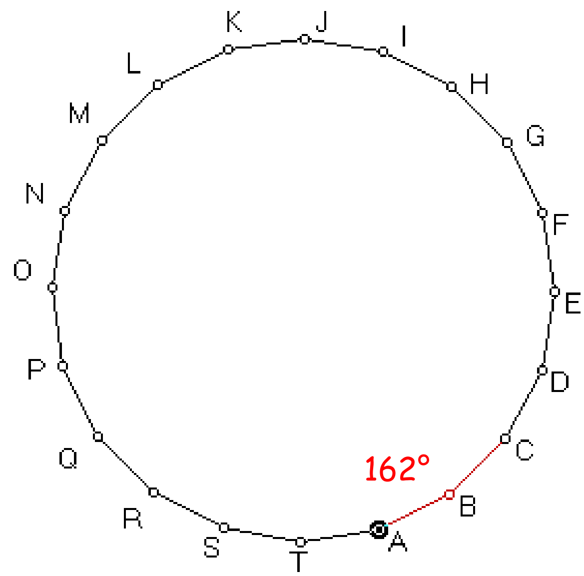
Interior Angle of a Regular Polygon =

Describe what happens to the value of each interior angle as n (the number of sides) gets larger.

Instructor Page

Turning the Problem Around

We just saw that if we know the number of sides of a regular polygon, then we can find its interior angles. So first we try an unusual polygon, like one with 20 sides to confirm the formula. That is the figure below.



Next, let's turn the problem around. If we start with an interior angle of a regular polygon, can we find the number of sides? We know that 60° goes with a triangle (3 sides), and 90° corresponds to 4 sides,

Every interesting solution leads to more problems.

Allow the participants plenty of time to work on the problem on finding the number of sides in a regular polygon with each interior angle = 179°. There may be several ways to come up with the answer.

Can we create a regular polygon with interior angle = 179°? __YES__

If we could, how many sides would it need to have? _____360 sides_

How do you know? Answers will vary, but one way is illustrated with the algebraic manipulations below.

$$\frac{(n-2)180}{n} = InteriorAngle \longrightarrow 180 - \frac{360}{n} = InteriorAngle \longrightarrow n = \frac{360}{180 - InteriorAngle} = 360$$

Use the formula, but this time, start with the angle and find the number of sides. Try different values. What happens? Write your observations below. Use the hand-outs in the appendices E and F to further explore these ideas.

Internet Resources:

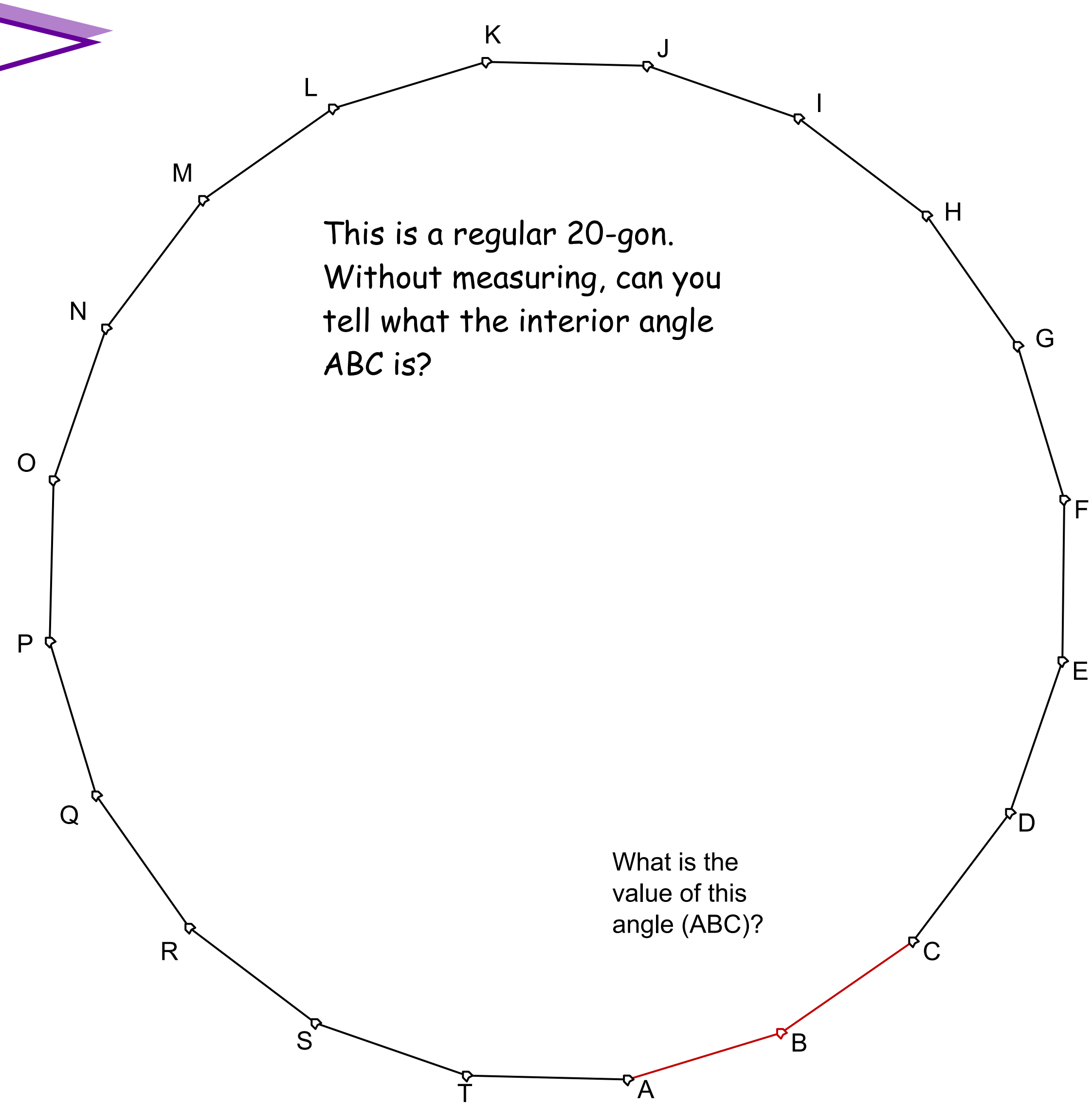
There is a Sketchpad file of a regular 20-gon at www.mste.uiuc.edu/m2t2/algebra/20gon.gsp

Have participants write scripts to build regular polygons.

M2T2

Participant Page

Turning the Problem Around



Can we create a regular polygon with interior angle = 179°? _________

If we could, how many sides would it need to have?

____

Explain

____

____

____

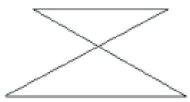
M2T2

Instructor Page

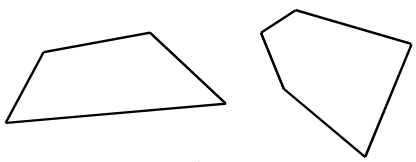
Irregular Polygons: Do the Patterns Still Hold True?

We saw that there was a pattern to the sum of the interior angles of a regular polygon. Next we found a formula for the pattern. We graphed the pattern; we used the pattern with a large number of sides (20), and then turned the problem around to find the number of sides of the regular polygon. Here is a new, but related, question. Will the pattern hold for irregular polygons?

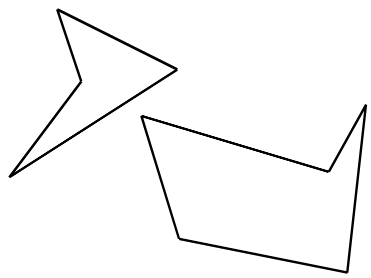
Give participants plenty of time to think and talk about the formulas



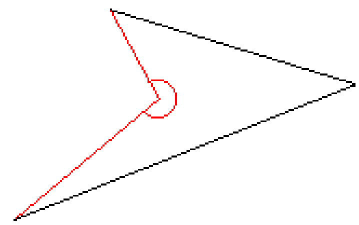
Not a polygon



Convex Polygons



Concave Polygons



The angle greater than 180° is the reflex angle.

$$(n-2)180° = \text{Sum}$$

This is the formula for the sum of the interior angles of a regular polygon. Will it work for irregular polygons?

Yes, it will. When participants complete the table, they should see that the formula holds. The next section on partitions will illustrate why the formula holds.

$$\frac{(n-2)180°}{n} = InteriorAngle$$

This is the formula for finding a single interior angle of a regular polygon. Will it work for irregular polygons?

This formula will only work when all the angle are congruent. So, it will not work for the polygons above. You might, however, think of it as the "average" angle size..

$$n = \frac{360}{180 - InteriorAngle}$$

This is the formula for finding the number of sides of a regular polygon given an interior angle. Will it work for irregular polygons?

This formula will not work at all. Without the polygon being regular, there is no way to move backwards from a single angle measure to a certain number of sides.

Technology Tip:

The program, Geometer's Sketchpad®, will not measure angles greater than 180° If you have Sketchpad, ask participants how this will effect the angle sum if measured in Sketchpad?

Internet Resources:

There is a polygon drawing tool at <http://www.mste.uiuc.edu/java/java/angleobject/>
This applet will show the angle sum of a polygon in circle units. Participants may use scissors and tape to cut out the angles and tape them together into circle units and half-units.

M2T2

Participant Page

Convex Polygons: Polygons with all interior angles less than 180°.

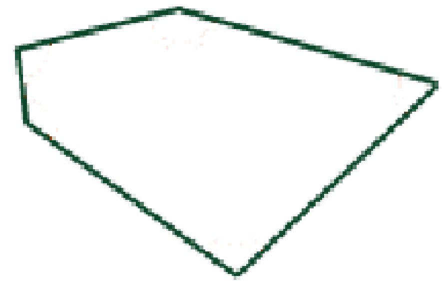
Concave Polygons are polygons with at least one interior angle greater than 180°

Irregular Polygons: Do the Patterns Still Hold True?

Find the following information for the polygon below.

Angle Measures: _____, _____, _____, _____, _____

Sum of Angle Measures =



Draw your own irregular polygons on another sheet of paper and measure their interior angles. Enter the values in the table below

Irregular Polygon	Name of Polygon	Number of	Concave or	List of Interior Angle	Sum of Interior
A					
B					
C					
D					
E					

$$(n-2)180^\circ = \text{Sum}$$

This is the formula for the sum of the interior angles of a regular polygon. Will it work for irregular polygons? Why or why not?

$$\frac{(n-2)180^\circ}{n} = InteriorAngle$$

This is the formula for finding a single interior angle of a regular polygon. Will it work for irregular polygons? Why or why not?

$$n = \frac{360}{180 - InteriorAngle}$$

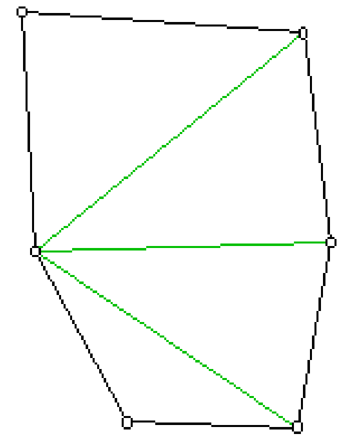
This is the formula for finding the number of sides of a regular polygon given an interior angle. Will it work for irregular polygons? Why or why not?

M2T2

Instructor Page

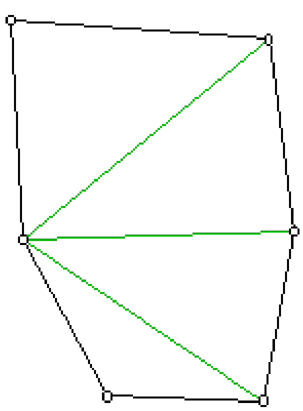
Partitioning into Triangles to See the Pattern

Have participants draw triangles by using line segments from vertex to vertex to partition their polygons. The interior (green) segments in this hexagon show how this is done. The sum of *all* the angles of *all* the triangles equals the sum of the interior angles. The number of triangles is two less than the number of sides

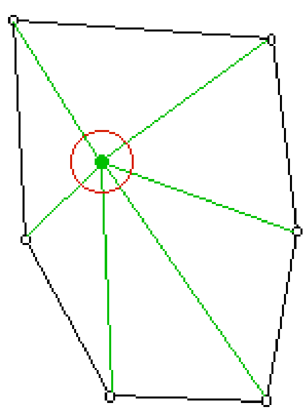


Questions:

1. What is the relationship between the number of triangles and the number of sides?
   Number triangles = Number of Sides — 2.
2. What is the relationship between the number of triangles and the sum of the interior angles?
   Sum of Interior Angles = Number of Triangles times 180°.
3. Does it matter to this relationship if the polygon is regular or irregular?
   No, the relationships are hold true for any polygon, regular or irregular, convex or concave.



(n-2)180



n180 - 360

Here is another way to cut the polygon into triangles. Pick any interior point and draw segments to the vertices. Notice that this piece of geometry confirms the algebra. That is, we have a geometric example of the distributive property as seen in the figures above.

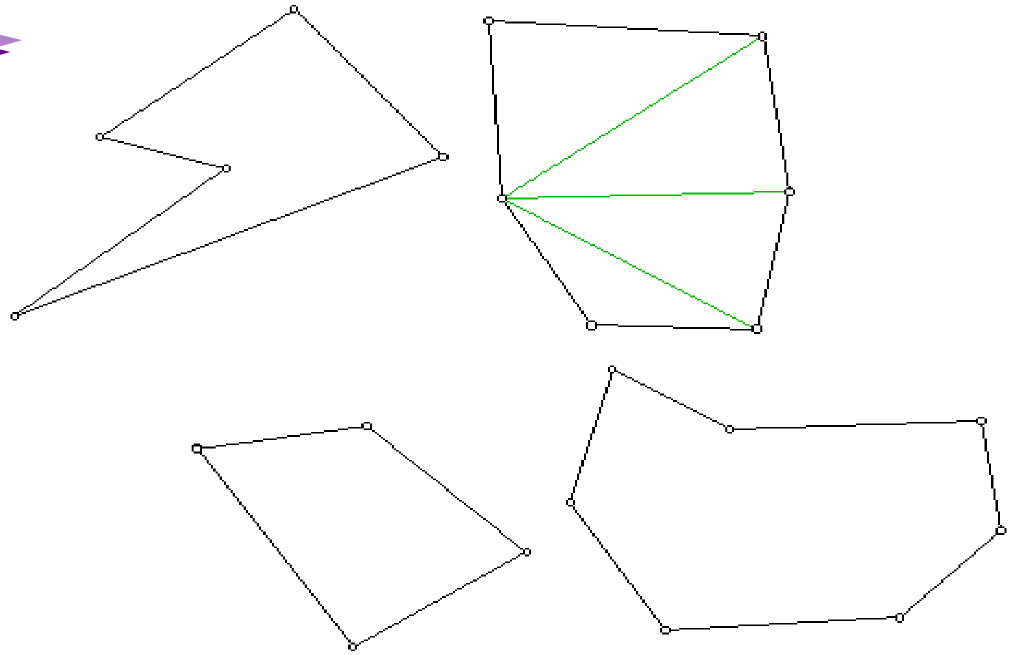
Notice that with the partitioning, we no longer even have to measure the angles to get the sum! We can just add up the triangles and multiply by 180°.

Internet Resources:

<www.mste.uiuc.edu/m2t2/algebra/nonregs2.gsp> is a Geometer's Sketchpad® file of the irregular polygons at the right.

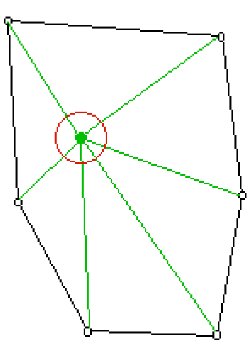
Participant Page

Partitioning into Triangles to See the Pattern



One of the hexagons above has been partitioned into triangles. How can you use this partition to find the sum of the measures of the interior angles of the hexagon without measuring? (Hint: The sum of the measures of the interior angles of each triangle = 180°).
Partition the other polygons to see that this is true

In the space below, try drawing three different irregular polygons and use the partition method to find the sums of the measures of the interior angles.



You can also partition the polygon into triangles from a point inside the polygon. How would you state the relationship between the number of triangles and the sum of the interior angles in this case?

M2T2

Instructor Page

So What's the "Big Idea"!?

The authors of this section where interested in showing connections between algebra and geometry. On the team of developers was a teacher who happened to be teaching his students the rule for the sum of the interior angles of a polygon that week. We then took that geometry problem and attempted to look closely at it to see where the algebra was. That is, where could we find the interesting problems for which the tools of a traditional algebra class (tables, formulas, variables, equations, graphing, looking for patterns, and expressing relationships) could be used to get a deeper understanding of polygons and relationships? This module is the result of that effort.

Let's take stock of what we have done so far.

- We have used **tables** to find relationships between interior angle sums and numbers of sides.
- We developed a **formula** for that relationship expressed as an **equation** with the **variable** "n" being the number of sides.
- We have **graphed** that **linear relationship**.
- We found a formula and graphed the **non-linear relationship** between a single interior angle of a regular polygon and the number of sides. We saw that the value of that interior angle had a **limit** of 180.
- We turned the formula around and saw that we could find the number of sides of a regular polygon if we had an interior angle.
- We tested the formula for angle sum on irregular polygons and saw that it still held true. However, we saw that the formula for a single interior angle was not applicable to irregular polygons and neither was the formula to calculate the number of sides from an interior angle. That is, we saw the constraints on those formulae.
- Then we explored the partitioning of regular and irregular polygons into triangles and observed that the geometry of this partitioning corresponded to the algebraic relation expressed in the application of the **distributive property**.

  That is **$(n - 2)180 = 180n - 360$**.

So, the "big idea" is that in mathematics we are often searching for patterns, and algebra is essential for the recognition and description of these patterns. We can find lots of algebra in geometry problems. And moreover, we can use the tools of algebra to get a deeper understanding of the relationships and patterns that emerge through our exploration of geometry. In the process of looking carefully at these problems, we encountered important notions for more advanced mathematics, such as discrete vs. continuous data, and the limit of a function, and restrictions on a domain. And there is much more that we can explore. In this final section we will look at the quadratic relationships that can be found in polygons.

M2T2

Participant Page

So What's the "Big Idea"!?

Let's take stock of some of the concepts we have explored. Looking at relationships between polygons may not seem like algebra at first, but let's list some of the tools that you have used.

- What variables have you used in this section, and how have you used them?
- Where in these sections have you used formulas? How have you used them?
- Where in this section have you graphed equations? What did the graphs illustrate?
- After this section, are you able to use terms like the following in a mathematical context?:
  - Linear relationship
  - Variable
  - Limit
  - Discrete data
  - Partition
  - Formula

We are not done exploring polygons yet. We have looked at sides and angles. Now we will look at sides, vertices, and diagonals. The context will be a question known as "the handshake problem."

M2T2

Instructor Page

Connect the Dots

There are many other problems which involve a search for patterns. This connect-the-dots-problem is an example. Here the pattern is somewhat more sophisticated.

Be sure to give participants time to explore and discuss.

The solution to the problem is: n(n-1)/2. Some participants may be able to see this from looking at the table. Others may look at the drawing and think of it this way; you can draw a line from each of the n dots to the n-1 remaining dots. This gives you n(n-1) lines. BUT, the line PQ is the same as QP so we must divide by 2 giving us: n(n-1)/2

This problem is also known as "**the handshake problem**." If a room of 6 people all shake hands how many handshakes take place? 25 people? N people? The dots represent people; the line segments represent handshakes.

Polygon Constructed	Number of Points	Number of Line Segments
Line (not a polygon)	2	1
Triangle	3	3
Quadrilateral	4	6
Pentagon	5	10
Hexagon	6	15
Heptagon	7	21
Octagon	8	28
Nonagon	9	36
Decagon	10	45
50-gon	50	50*49/2
n-gon	n	n(n-1)/2

First Differences: 2, 3, 4, 5, 6, 7, 8, 9

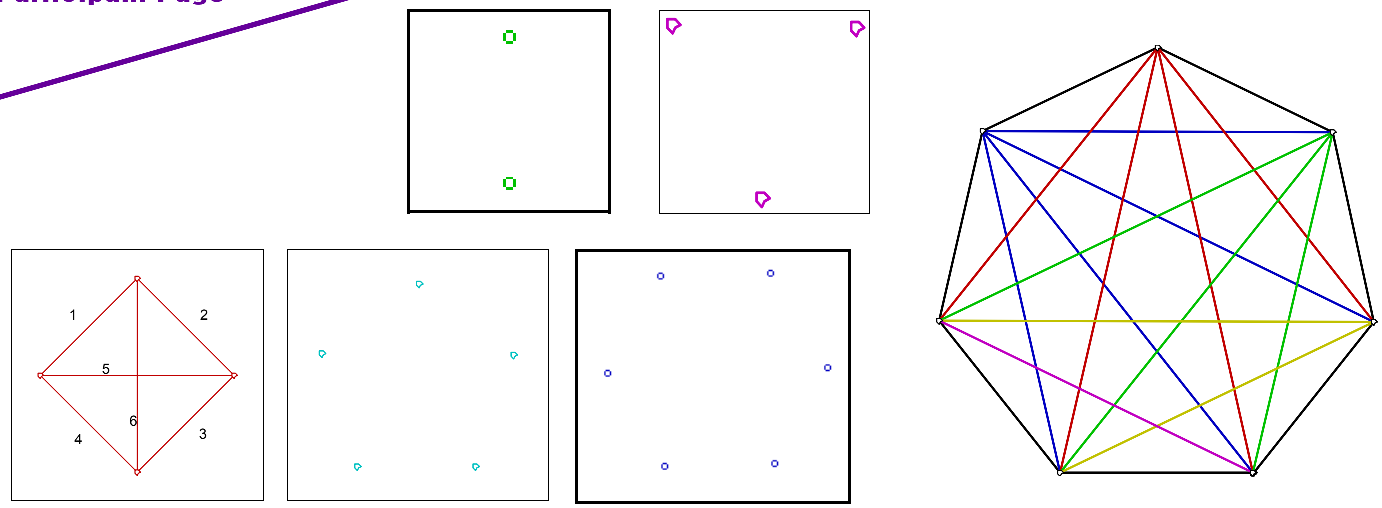
Second Differences: 1, 1, 1, 1, 1, 1, 1

If participants notice these difference patterns, they can use the method of "finite differences to find the formula. For more detail on this method, see Appendix H

M2T2

Participant Page

Connect the Dots



Using the sets of points above, connect all possible line segments between the points and record the total number of line segments for each set below:

Polygon Constructed	Number of Points	Number of Line Segments
	2	
	3	
Quadrilateral	4	6
	5	
	6	
	7	
	8	
	9	
Decagon	10	
50-gon	50	
n-gon	n	

Try graphing the pattern with a graphing calculator. Use the number of points as the independent variable (X) and the number of line segments as the dependent variable (Y). What patterns do you see? Describe them. Can you relate the number of points to the number of line segments using a formula?

The Handshake Problem

Another way to think of this problem is as “the handshake problem.” If 6 people all shake hands with each other, how many handshakes take place in all? If 10 people shake hands? n people?

If you used the dots to model people, what would represent the handshakes?

Instructor Page

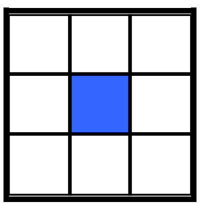
Extension: The Diagonal Problem

How many diagonals are there in each polygon? This problem is almost the connect-the-dots problem again. But you must subtract the n-sides of the polygon. .

The formula for the diagonals is: [n (n-1)/2] – n. Some participants may find these formulas by analyzing a table.

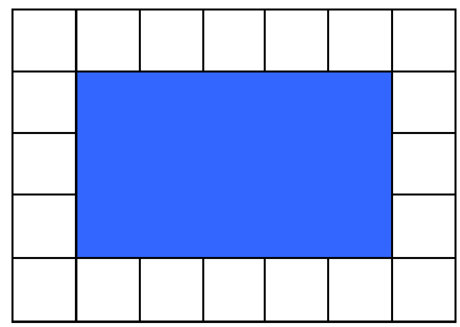
Other possible formulas are n(n-3)/2 or [(n-1)(n-2)/2] - 1
Be sure that participants understand that all of these expressions are equivalent.

Homework: The Swimming Pool Problem



Side B

Side A



Side B

Side A

Table of Relationships for Square Swimming Pool

Tiles on	Tiles on	Layers	Number
3	3	1	8
5	5	2	24
7	7	3	48
9	9	4	80
11	11	5	120
$2n + 1$	$2n + 1$	n	$4n^2 + 4n$
OR			
x	x	$(x - 1)/2$	$X^2 - 1$

There are at least two ways to approach this. One is to let the number of layers be the independent variable. Another is to let the tiles on one of the sides be the independent variable.

Table of Relationships for Rectangular Swimming Pool

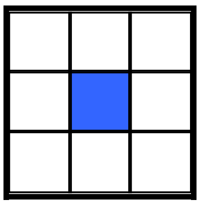
Tiles on	Tiles on	Layers	Number
7	5	1	20
9	7	2	48
11	9	3	84
13	11	4	128
15	13	5	180
$2n + 5$	$2n + 3$	n	$4n^2 + 16n$
OR			
x	$X - 2$	$(x - 5)/2$	$X^2 - 2x - 15$

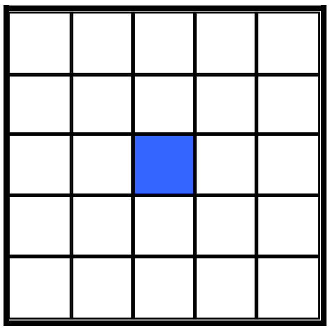
M2T2

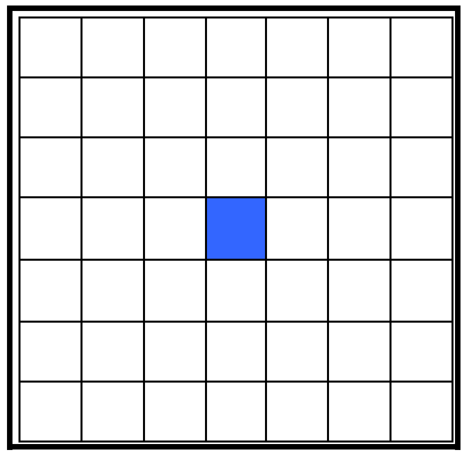
Participant Page

Homework: **The Swimming Pool Problem**

Suppose that you are building a deck around a swimming pool and need to tile the deck. As you increase the size of the rectangular frame of tiles, how does the number of tiles increase?







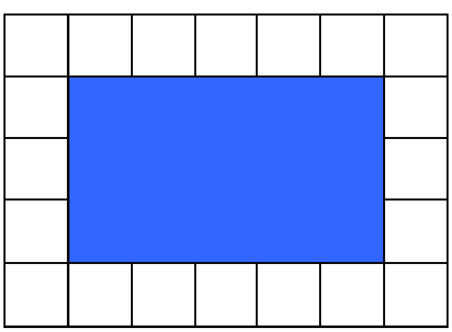
ETC...

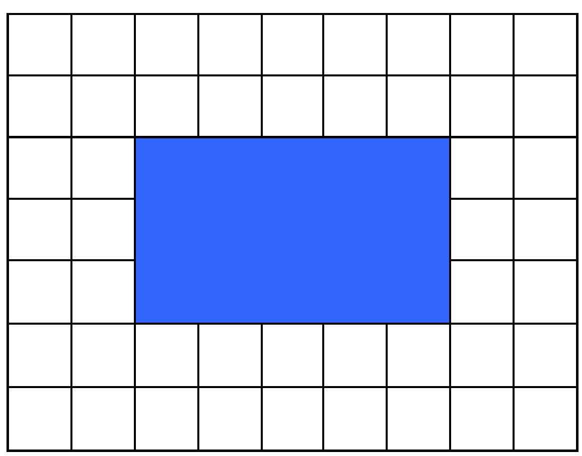
This pattern continues to grow in this manner. Make a table for the number of tiles needed in the first few designs in an attempt to make a generalization about how many tiles are needed for a certain size deck. Be prepared to discuss the patterns that you find.

1. What is the relationship between the number of tiles on a side and the total number of tiles?
2. What is the relationship between the number of layers and the number of tiles?

CHALLENGE:

What will the patterns be for a swimming pool like the one below?

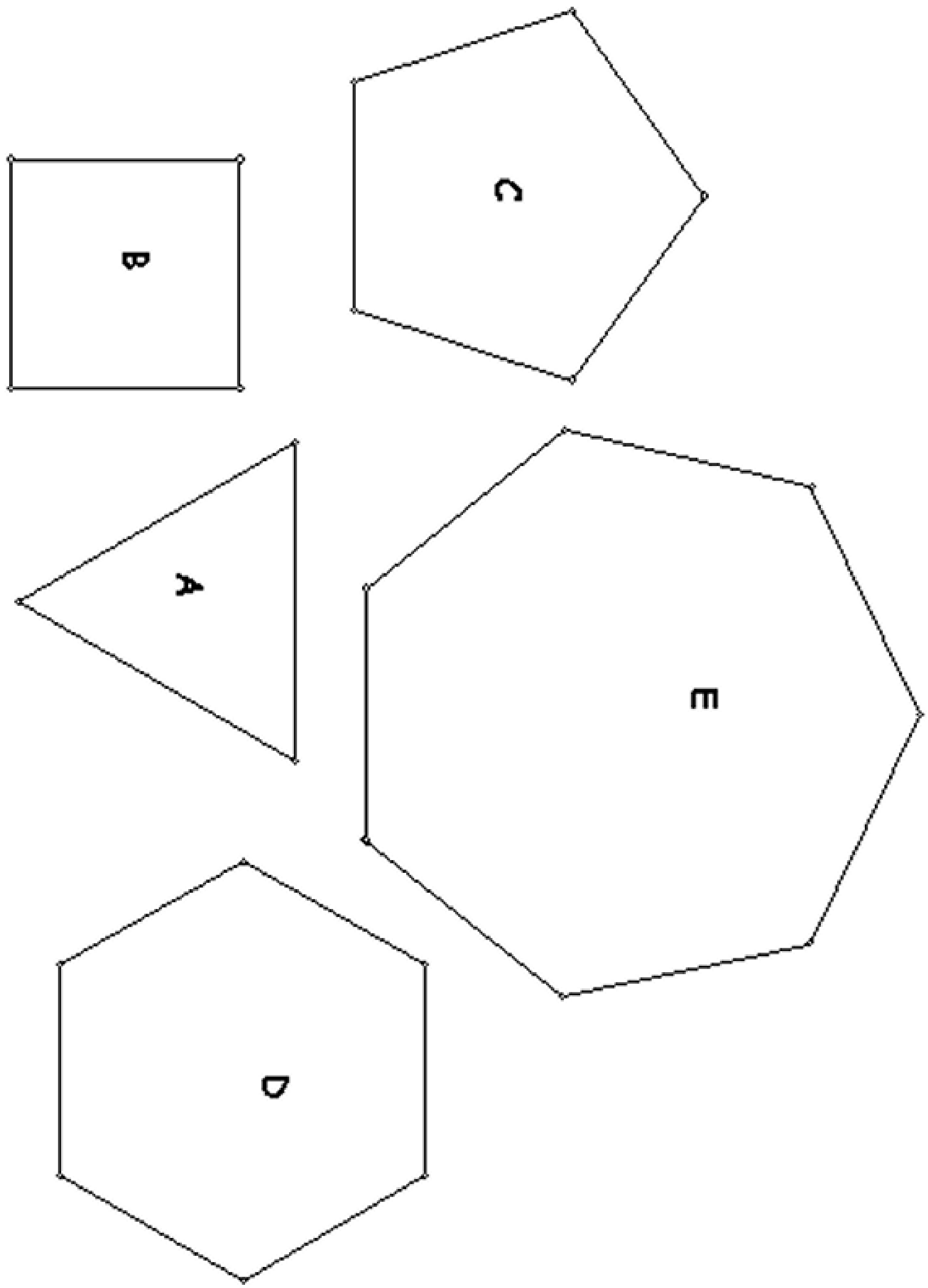




1. What is the relationship between the number of *tiles on one side* and the *total number of tiles*?
2. What is the relationship between the number of *layers of tiles* and the *number of tiles*?

This page intentionally blank

Regular Polygons to Measure



This page intentionally blank

Appendix B

Grid Paper

This page intentionally blank

Constructing a Regular Nonagon with Geometer's Sketchpad®

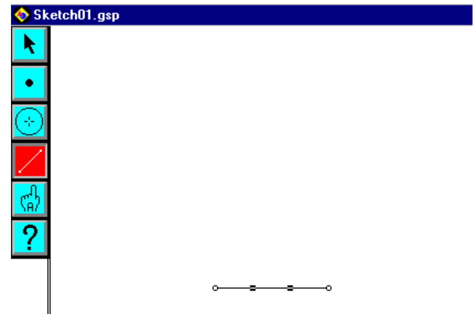


Figure 1: line segment

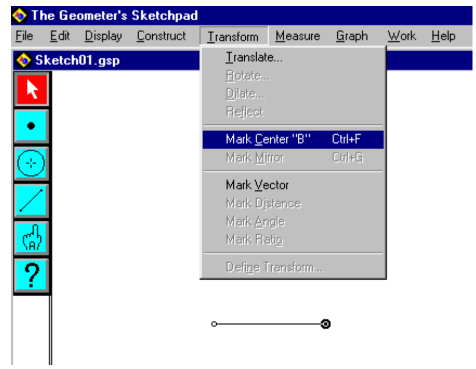


Figure 2: Mark the Center

- Open Sketchpad
- Click on the segment tool.
- Draw a segment as shown in Figure 1.
- Click on the selection arrow and select the right-hand point.
- Go to the Transform menu and choose "Mark Center 'B'" In this drawing the center is "B". It is possible that it is a different letter in your drawing.
- Hold down the Shift key and select the left-hand point and the line segment itself.
- Go to the Transform menu and choose "Rotate…" A box appears with an angle value in it.
- Change the value to -140. The rotation direction is counter-clockwise for positive values and clockwise for negative values. 140 is the angle sum for a nonagon (1260°) divided by 9 (the number of angles). The result is the rotation of line segment and the point by 140 degrees clockwise.
- Repeat this step with the rotated segment. Select the new point, mark it as a Center and rotate the segment and point around it. On the ninth segment, rotate the segment only.

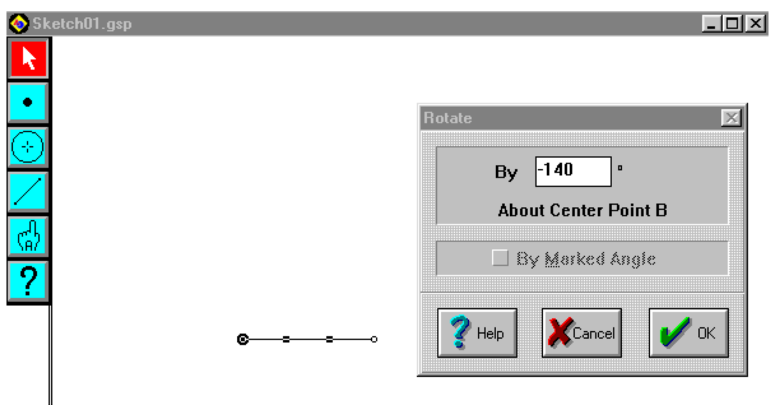


Figure 3: Enter -140° in the Rotate box.

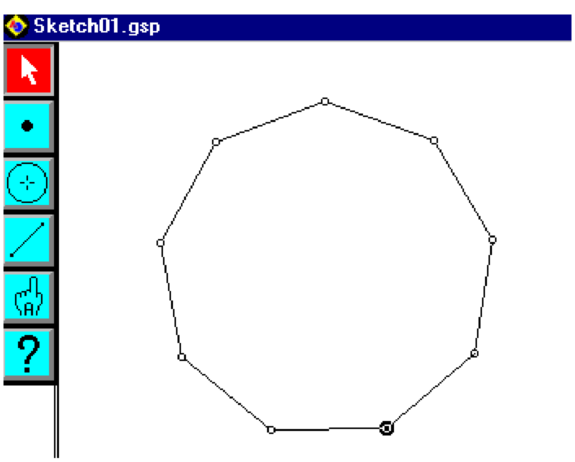
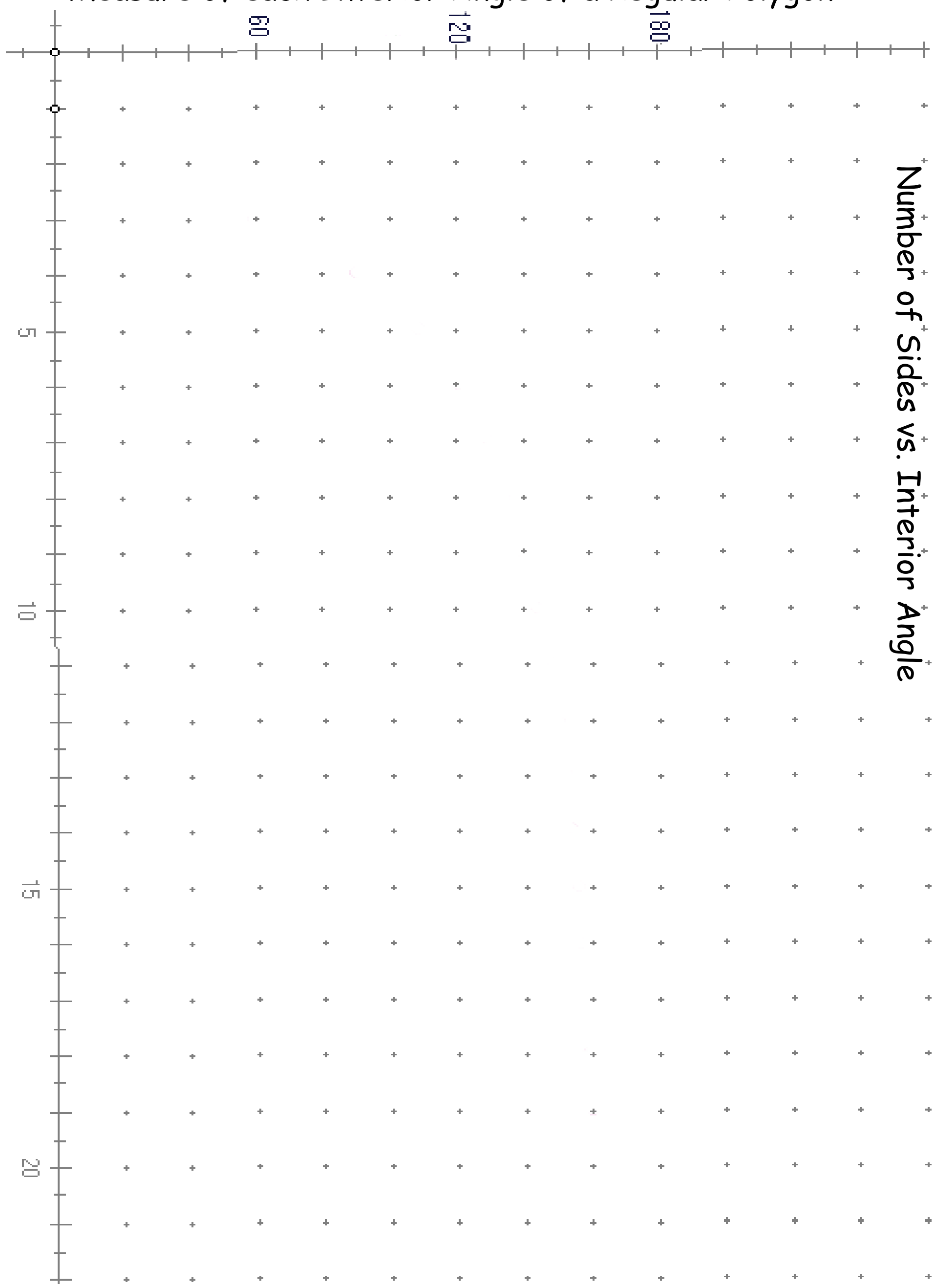


Figure 4: The completed nonagon.

A sketchpad file with the nonagon already made is available at
<www.mste.uiuc.edu/m2t2/algebra/nonagon.gsp>
A script to make the nonagon automatically is at
<www.mste.uiuc.edu/m2t2/algebra/nonascpt.gss>

This page intentionally blank

Graphing the Interior Angle Values of Regular Polygons.



Write the formula for the measure of an interior angle of a regular n-gon below:

Interior Angle of a Regular Polygon =

Number of Sides	Measure of Interior Angle

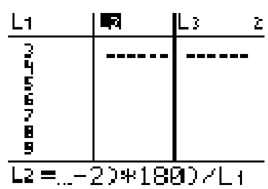
This page intentionally blank

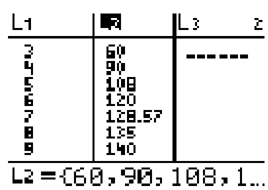
Appendix E

Exploring Interior Angle Patterns with the Graphing Calculator.

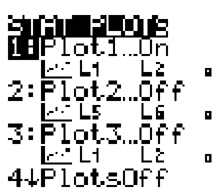
Graphing the interior angle values with the TI-73.

- Clear the Lists: [LIST] [▲] to L1, then [CLEAR] [ENTER], then [▶] [▲] [CLEAR] [ENTER] to clear L2.
- Enter the list of sides in L1: [3] [ENTER] [4] [ENTER], etc., up to 9 sides.
- Next, put the formula for calculating the interior angle (n-2)/180: [▶] [▲] to highlight L2, then [(] [(] [2nd] [STAT] [1] [−] [2] [)] [×] [1] [8] [0] [)] [÷] [2nd] [STAT] [1] [ENTER]. The data should look like the second picture at right.





- [2nd] [PLOT]. Check to be sure Plot1 is On and L1 and L2 are selected.

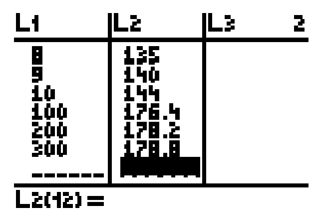


- To see the graph, press [ZOOM] [7] and then [TRACE] to see the X and Y values for the points

Using the TI-73 to extend the interior angle pattern.

What happens to the interior angle of a regular polygon when the number of sides increases?

- Clear the Lists: [LIST] [▲] to L1, then [CLEAR] [ENTER], then [▶] [▲] [CLEAR] [ENTER] to clear L2.
- Enter the list of sides in L1. This time, after 9, enter 10, 100, 200, and 300 for the last values on the list.
- Now repeat the steps from the previous section to enter the formula (n-2)180/n into the second list. You should get two lists like the ones at right.



What will the graph of these coordinates look like? Draw your own graph before you press the [ZOOM] [7] buttons to see the graph.

What will the interior angle be for a regular polygon with 1000 sides?

This page intentionally blank

Appendix F

Exploring the Limit of the Size the Interior Angles of a Regular Polygon

If we know the number of sides, we can use the formula (n-2)180/n to find the value of a single interior angle.

Another way to write this formula is (180n/n) + (360/n). Convince yourself that these two formulas are the same by doing algebraic manipulations on one to get the other.

Now, notice that (180n/n) is really just 180.
So the formula for a single interior angle becomes 180 - (360/n).
What happens to this formula as n (the number of sides) gets very big? Fill in the table below.

Number of Sides	3	4	5	10	100	200	300	1000
(n-2)180°/n	60°	90°	108°		176.4°			
360°/n	120°	90°						

Turning the problem around.
If we have the number of sides, we can find the interior angle, but if we have the interior angle, can we find the number of sides?

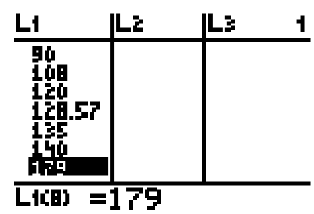
With algebraic operations, we can take the equation and solve for the value of n (the number of sides).

$180-\frac{360}{n}=InteriorAngle$ becomes $n=\frac{360}{180-InteriorArgle}$

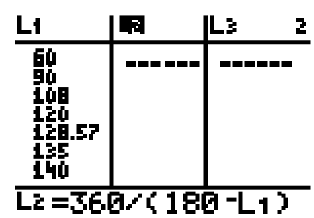
What if the interior angle is 179°? How many sides are there?

What if the value of the interior angle is 180° or more? In that case, the formula would not work, because the denominator would be zero or less. Geometrically, we can see that it is also impossible to have an interior angle greater than or equal to 180°. Mathematicians would say that the "domain" of this function is limited to certain values between 60° and 180°.

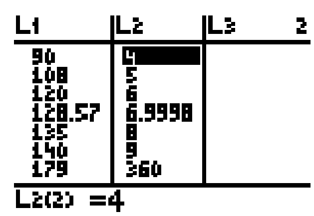
Finding the number of sides on the TI-73.



- Clear the L1 and L2.
- In L1, enter the values for the angles that you know for the interior angles of a regular triangle (60), quadrilateral (90), pentagon (108), hexagon (120), heptagon (128.57), octagon (135) and nonagon (140). Also enter a value of 179.



- For L2, enter the formula with [3][6][0] [÷] [(][1][8][0] [-] [2nd] [STAT][1][)], then [ENTER].



What does each value in L2 mean?
What are some values in L1 that would produce problematic results in L2?

This page intentionally blank

Using the Graphing Calculator and Distance Sensor

At the right is a picture of a TI-73 graphing calculator connected to a Calculator-Based Ranger (CBR). The CBR sends out sound waves and uses the reflection of those waves to determine distances. It sends the data to the calculator, which then generates a graph.

On these pages, we will explore the use of this tool to visualize patterns

1. To use the distance sensor with the TI-73, connect the cable to both the calculator and the CBR. Press [APPS] [2] [ENTER] [3] [ENTER] [1]. You should see a screen with options for the Ranger. Set the "Smoothing" to medium and "Units" to meters. Do this by using the [▼] key to get to the field, and press [ENTER] to change the field. After you have the settings as you want them, [▲] to get to the "START NOW" at the top.

And press [ENTER]. You should be ready to go! Aim your CBR at a wall or other target, press [ENTER] on more time, and watch it generate the graph.
To repeat the sample, press [ENTER] [3].

2. See if you can create a graph like **Figure 1** at the right. Describe in words how you generate this graph.

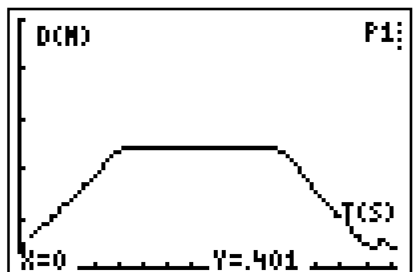


Figure 1

3. Try making a graph like **Figure 2**. Describe the motion you made in order to make this graph.

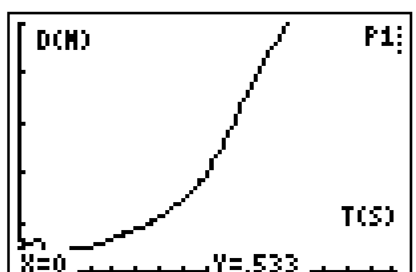


Figure 2

Continued on the next page

This page intentionally blank

Appendix G (continued)

Using the Graphing Calculator and Distance Sensor (continued)

4. Change the Units on the sensor to Feet and see if you can generate a graph that will go through the points described in the Distance Vs. Time Table at the right (Note to instructor: this is the data from the handshake problem.)

Distance Vs. Time Table

Seconds	Distance (in feet)
2	1
3	3
4	6
5	10
6	15
7	21

5. At the right is a blank set of axes. Draw a graph on these axes for your partner to walk. Write the description for this walk below. Be prepared to share your graph and description.

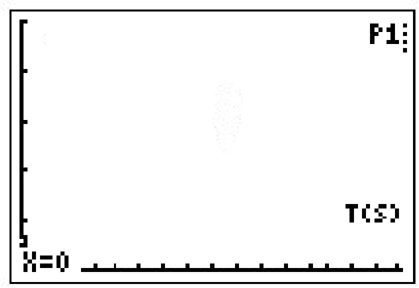


Figure 3: Draw your own graph above and have your teammates walk it.

6. Explain why the graph at the right is impossible to make with the distance sensor. (Note to instructor: This graph will fail the "vertical line test" for functions.)

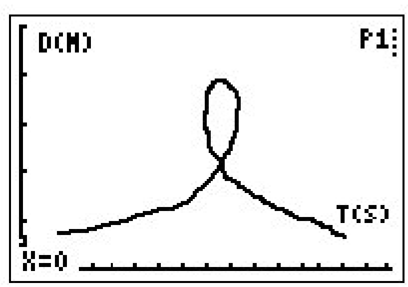


Figure 4: Impossible distance vs. time graph.

This page intentionally blank

Appendix H

Finite Differences

A handy method for finding the equation for a relationship that is quadratic is the method of finite differences. Like the quadratic formula, it is a tool to make computation easier.

Consider the general form of a quadratic equation: $y = ax^2 + bx + c$. We want to find the values of a, b, and c from a table of values of points on the quadratic function. The table below illustrates the general method.

The General Form of the Finite Differences Method of Finding Coefficients

X	Y
0	$a(0)^2 + b(0) + c = c$
1	$a + b + c$
2	$4a + 2b + c$
3	$9a + 3b + c$
4	$16a + 4b + c$

First Differences

$(a + b + c) - c = a + b$

$(4a + 2b + c) - (a + b + c) = 3a + b$

$5a + b$

$7a + b$

Second Differences

$2a$

$2a$

$2a$

We can use this method to find the equation of a function from a table of values if the function that fits the points is quadratic. Take for example the *Connect the Dots* problem (*Handshake Problem*) from this module. Since the second differences are all the same number, that is our clue that the function is quadratic, and the coefficient of the x^2 term (the quadratic term) is 1/2 (because 2a = 1). To find the coefficient of the linear term, we look at the first differences. Using the table, we see that we need to use the first difference of the terms where X = 3 and X = 2. In this case, 5a + b = 2. Since we know that a = 1/2, we can solve for b. 5/2 + b = 2 implies that b = -1/2. In this particular case, we don't worry about the value of c.

Applying the Finite Differences Method to the Connect the Dots/Handshake Problem

Polygon Constructed	Number of Points	Number of Line Segments
Line (not a polygon)	2	1
Triangle	3	3
Quadrilateral	4	6
Pentagon	5	10
Hexagon	6	15
Heptagon	7	21

First Differences

2

3

4

5

6

Second Differences

1

1

1

1

For more detail on this method, see Herr, T., & Johnson, K. (1994). Problem solving strategies: Crossing the river with dogs and other mathematical adventures. Berkeley, CA: Key Curriculum Press. ISBN: 1559530685 pp. 379-412.

This page intentionally blank

List of resources

American Association for the Advancement of Science. (2000). Middle grade mathematics textbooks: A benchmarks-based evaluation. Washington, DC.

Burns, M., & Humphreys, C. (1990). A collection of math lessons from grades 6 through 8. White Plains, NY: Math Solutions Publications.

Fennell, F., Bamberger, H. J., Rowan, T. E., Sammons, K. B., & Suarez, A. R. (2000). Connect to NCTM Standards 2000: Making the Standards work at grades 8. Chicago, IL: Creative Publications.

Greenes, C., & Findell, C. (1999). Groundworks. Chicago, IL: Creative Publications.
Algebraic thinking activities for grades 1 through 7.

Herr, T., & Johnson, K. (1994). Problem solving strategies: Crossing the river with dogs and other mathematical adventures. Berkeley, CA: Key Curriculum Press.

Herrera, T. (2001). An interview with Marilyn Burns: Meeting the Standards--Don't try to do it all by yourself. ENC focus, 8(2), 16-19.
Marilyn Burns talks about what a reform-based classroom looks like. See this at www.enc.org

Illinois State Board of Education (ISBE). (undated). Illinois Standards Achievement Test Sample Math Materials. Springfield, IL: Illinois State Board of Education.

Illinois State Board of Education, A. S. (1995). Effective scoring rubrics: A guide to their development and use. Springfield, IL: Illinois State Board of Education,.
ISBE documents can be obtained by calling 217-782-4823

Jackiw, N. (1991-2000). Geometer's Sketchpad. Emeryville, CA: Key Curriculum Press.
Geometer's Sketchpad is dynamic geometry software available at <www.keypress.com>.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (1996). Getting to know Connected Mathematics. White Plains, NY: Dale Seymour.
Connected Mathematics is the curriculum that ranked highest on the AAAS benchmarks-base evaluation. The CMP can be reached at: 517-336-2870. Dale Seymour Publications can be reached at: 800-872-1100

Murdock, J., Kamischke, E., & Kamischke, E. (2000). Discovering algebra: An investigative approach (Vol. 1). Emeryville, CA: Key Curriculum Press.

Office for Mathematics, S., and Technology Education,. (2001). [World Wide Web]. MSTE Office at the University of Illinois at Urbana-Champaign,. Available: <www.mste.uiuc.edu/>.
A set of java activities in mathematics can be found at <www.mste.uiuc.edu/java/>. The main M2T2 page is <www.mste.uiuc.edu/m2t2/>

School and Student Assessment Section of the Illinois State Board of Education (ISBE). (undated). Assessment handbook: A guide for developing assessment programs in Illinois Schools. Springfield, IL: Illinois State Board of Education.

Texas Instruments. (1997a). Cabri Geometry II. Austin, TX: Texas Instruments.
Cabri is another geometry program. It is available on some TI calculators as well as a free standing application.

Texas Instruments. (1997b). Calculator-based ranger (CBR) [Distance sensor]. Dallas, TX: Texas Instruments.

Texas Instruments. (1998). TI-73 Graphing Calculator [Graphing calculator]. Dallas, TX: Texas Instruments.
CBRs and calculators can be purchased through office supply stores or stores like Klaus Radio (1 800 545 5287).

Algebra

Send questions on these modules to m2t2@mail.mste.uiuc.edu