B.Sc. COMPUTER SCIENCE, BCA

Allied Mathematics

ALLIED COURSE I (AC) - ALGEBRA AND CALCULUS

UNIT I

Theory of Equations: Relation between roots & coefficients – Transformations of Equations – Diminishing , Increasing & multiplying the roots by a constant- Forming equations with the given roots – Rolle's Theorem, Descarte's rule of Signs(statement only) –simple problems.

UNIT II

Matrices : Singular matrices – Inverse of a non-singular matrix using adjoint method - Rank of a Matrix –Consistency - Characteristic equation , Eigen values, Eigen vectors – Cayley Hamilton's Theorem (proof not needed) –Simple applications only

UNIT III

Differentiation: Maxima & Minima – Concavity , Convexity – Points of inflexion - Partial differentiation – Euler's Theorem - Total differential coefficients (proof not needed) –Simple problems only.

UNIT IV

Integration : Evaluation of integrals of types $\int \frac{px+q}{ax^2+bx+c}dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x}$ Evaluation using Integration by parts – Properties of definite integrals – Fourier Series in the range (0 , 2 π) – Odd & Even Functions – Fourier Half range Sine & Cosine Series

UNIT V

Differential Equations: Variables Separables – Linear equations – Second order of types (a D^2 + b D + c) y = F (x) where a,b,c are constants and F (x) is one of the following types (i) e^{Kx} (ii) sin (kx) or cos (kx) (iii) x n , n being an integer (iv) e^{Kx} F (x).

Unit – I
Theory of Equations

Let us consider

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

This a polynomial in 'x' of degree 'n' provided $a_0 \neq 0$.

The equation is obtained by putting f(x) = 0 is called an **algebraic equation** of degree n.

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS OF EQUATIONS

Let the given equation be $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$

Let α_1, α_2,…… , α_n be its roots.

$\sum \alpha_1$ = sum of the roots taken one at a time = $-\frac{a_1}{a_0}$

$\sum \alpha_1 \alpha_2$ = sum of the product of the roots taken two at a time = $\frac{a_2}{a_0}$

$\sum \alpha_1 \alpha_2 \alpha_3$ = sum of the product of the roots taken three at a time = $-\frac{a_3}{a_0}$

finally we get $\alpha_1.\alpha_2 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$.

Problem:

If α and β are the roots of $2x^2 + 3x + 5 = 0$, find α + β, αβ.

Solution:

Here $a_0 = 2, a_1 = 3, a_3 = 5.$

$\sum \alpha$ = α + β = $-\frac{a_1}{a_0} = -\frac{3}{2}$

αβ = $\frac{a_2}{a_0} = \frac{5}{2}$.

Problem:

Solve the equation $x^3 + 6x + 20 = 0$, one root being 1 + 3i.

Solution:

Given equation is cubic. Hence we have 3 roots. One root is (1+3i) = α (say) complex roots occur in pairs.

∴ β = 1 – 3i is another root.

To find third root γ (say)

Sum of the roots taken one at a time

$$\alpha + \beta + \gamma = \frac{0}{1} = 0.$$

i.e., $1 + 3i + 1 - 3i + \gamma = 0$

$$\gamma = -2$$

∴ The roots of the given equation are 1 + 3i, 1 – 3i, -2.

Problem:

Solve the equation $3x^3 - 23x^2 + 72x - 20 = 0$ having given that $3 + \sqrt{-5}$ is a root.

Solution:

Given equation is cubic. Hence we have three roots.

One root is $3 + i\sqrt{5} = \alpha$

Since complex roots occur in pairs, $3 - i\sqrt{5} = \beta$ is another root.

Sum of the roots is $\alpha + \beta + \gamma = \frac{23}{3}$

i.e., $3 + i\sqrt{5} + 3 - i\sqrt{5} + \gamma = \frac{23}{3}$

$$6 + \gamma = \frac{23}{3}$$

$$\gamma = \frac{23}{3} - 6$$

$$\gamma = \frac{5}{3}$$

Hence the roots of the given equation are $3 + i\sqrt{5}$, $3 - i\sqrt{5}$, $\frac{5}{3}$.

Problem:

Solve the equation $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$ which has a root $2 + \sqrt{3}$.

Solution:

Given $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$. ----- (1)

This equation is biquadratic, i.e., fourth degree equation.

∴ It has 4 roots. Given $2 + \sqrt{3}$ is a root which is clearly irrational. Since irrational roots occur in pairs, $2 - \sqrt{3}$ is also a root of the given equation.

$\therefore$ [x $-$ (2 + $\sqrt{3}$)] [x $-$ (2 $-$ $\sqrt{3}$)] is a factor of (1)

i.e., $x^2 - 4x + 1 = 0$ is a factor.

Dividing (1) by $x^2 - 4x + 1 = 0$, we get

	$x^2 + 6x + 7$
$x^2 - 4x + 1$	$x^4 + 2x^3 - 16x^2 - 22x + 7$
	$x^4 - 4x^3 + x^2$
	(-) $6x^3 - 17x^2 - 22x + 7$
	$6x^3 - 24x^2 + 6x$
	(-) $7x^2 - 28x + 7$
	$7x^2 - 28x + 7$
	0

Hence the quotient is $x^2 + 6x + 7$ = 0. Solving this quadratic equation, we get $= -3 \pm \sqrt{2}$.

Hence the roots of the given equation are 2 + $\sqrt{3}$, 2 $-$ $\sqrt{3}$, $-3 + \sqrt{2}$, $-3 - \sqrt{2}$.

Problem:
Form the equation, with rational coefficients one root of whose roots is $\sqrt{2} + \sqrt{3}$.
Solution:

One root is $\sqrt{2} + \sqrt{3}$

i.e., x = $\sqrt{2} + \sqrt{3}$

i.e., x $-$ $\sqrt{2}$ = $\sqrt{3}$

Squaring on both sides we get

$$(x - \sqrt{2})^2 = 3$$

$$x^2 - 2\sqrt{2}x + 2 = 3$$

$$x^2 - 1 = 2\sqrt{2}x$$

Again squaring, we get

$(x^2 - 1)^2 = \left(2\sqrt{2}x\right)^2$

$x^4 - 2x^2 + 1 = 4.2.x^2$

$x^4 - 10x^2 + 1 = 0$, which is the required equation.

Problem:

Form the equation with rational coefficients having 1 + $\sqrt{5}$ and 1 + $\sqrt{-5}$ as two of its roots.

Solution:

Given x = 1 + $\sqrt{5}$ and x = 1 + $i\sqrt{5}$

i.e., [x − (1 + $\sqrt{5}$)] [x − (1 + $i\sqrt{5}$)] are the factors of the required equation.

Since complex and irrational roots occur in pairs, we have x = 1 − $\sqrt{5}$, x = 1 − $i\sqrt{5}$ are also the roots of the required equation.

i.e., x − (1 − $\sqrt{5}$) and x − (1 − $i\sqrt{5}$) are also factors of the required equation.

Hence the required equation is,

[x − (1 + $\sqrt{5}$)] [x − (1 + $i\sqrt{5}$)] [x − (1 − $\sqrt{5}$)] [x − (1 − $i\sqrt{5}$)] = 0

i.e., $[(x-1)^2 - 5][(x-1)^2 + 5]$ = 0

$(x^2 - 2x - 4)(x^2 - 2x + 6) = 0$

Simplifying we get

$x^4 - 4x^3 + 6x^2 - 4x - 24 = 0$ which is the required equation.

Problem:

Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$ whose roots are in A.P.

Solution:

Let the roots be α – d, α, α + d.

Sum of the roots taken one at a time is,

α – d + α + α + d = $\frac{48}{32}$

3α = $\frac{48}{32}$

$\alpha = \frac{1}{2}$

$\therefore \frac{1}{2}$ is a root of the given equation. By division we have,

$\frac{1}{2}$	32	-48	22	-3
	0	16	-16	3
	32	-32	6	0

The reduced equation is $32x^2 - 32x + 6 = 0$

Solving this quadratic equation we get the remaining two roots $\frac{1}{4}, \frac{3}{4}$.

Hence the roots of the given equation are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Problem:

Find the value of k for which the roots of the equation $2x^3 + 6x^2 + 5x + k = 0$ are in A.P.

Solution:

Given $2x^3 + 6x^2 + 5x + k = 0$ --------- (1)

Let the roots be α – d, α, α + d.

Sum of the roots taken one at a time is,

$\alpha - d + \alpha + \alpha + d = \frac{-6}{2}$

$3\alpha = -3$

i.e., $\alpha = -1$

i.e., α = -1 is a root of (1).

∴ put x = - 1 in (1), we get k = 1.

Problem:

Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P.

Solution:

Given $27x^3 + 42x^2 - 28x - 8 = 0$ --------- (1)

Let the roots be $\frac{\alpha}{r}$, α, αr

Product of the roots taken three at a time is $\frac{\alpha}{r} \cdot \alpha \cdot \alpha r = \frac{8}{27}$

i.e., $\alpha^3 = \frac{8}{27}$

i.e., $\alpha = \frac{2}{3}$.

i.e., $\alpha = \frac{2}{3}$ is a root of the given equation (1)

i.e., $x = \frac{2}{3}$ is a root of the given equation (1)

i.e., $(x - \frac{2}{3})$ is a factor of (1).

$$27x^2 + 60x + 12$$

$$x - \frac{2}{3} \;\Big|\; 27x^3 + 42x^2 - 28x - 8$$

$$27x^3 - 18x^2$$

(-) $60x^2 - 28x - 8$

$$60x^2 - 40x$$

(-) $12x - 8$

$$12x - 8$$

0

Hence the quotient is $27x^2 + 60x + 12 = 0$

i.e., $9x^2 + 20x + 4 = 0$

Solving this quadratic equation we get x = $-2 \text{ or } -\frac{2}{9}$

Hence the roots of the given equation are $-2, -\frac{2}{9}, \frac{2}{3}$.

Problem:
Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in G.P.
Solution:

Given $x^3 - px^2 + qx - r = 0$ --------- (1)

Let the roots be $\frac{\alpha}{r}$, α, αr

Product of the roots taken three at a time $\frac{\alpha}{r} \cdot \alpha \cdot \alpha r = r$

i.e., $\alpha^3 = r$ -------- (2)

But α is a root of the equation (1). Put x = α in (1), we get,

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0 \quad \text{-------- (3)}$$

Substituting (2) in (3) we get

$$r - p\alpha^2 + q\alpha - r = 0$$

$$p\alpha^2 - q\alpha = 0$$

$$\alpha(p\alpha - q) = 0$$

$\alpha \neq 0$ $\therefore \quad p\alpha - q = 0$

i.e., $p\alpha = q$

i.e., $\alpha = \frac{q}{p}$

$$\therefore \alpha^3 = \frac{\alpha^3}{\beta^3}$$

$$r = \frac{q^3}{p^3}$$

Hence the required condition is $p^3 r = q^3$.

Transformation of Equations:

Problem:

If the roots of $x^3 - 12x^2 + 23x + 36 = 0$ are -1, 4, 9, find the equation whose roots are

1, -4, -9.

Solution:

Given $x^3 - 12x^2 + 23x + 36 = 0$ -------- (1)

The roots are -1, 4, 9.

Now we find an equation whose roots are 1, -4, -9 ie., to find an equation whose roots are the roots of (1) but the signs are changed. Hence in (1) we have to change the sign of odd powers of x.

Hence the required equation is

$$-x^3 - 12x^2 - 23x + 36 = 0$$

i.e., $x^3 + 12x^2 + 23x - 36 = 0$

This gives the required equation.

Problem:

Multiply the roots of the equation $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ by $\frac{1}{2}$.

Solution:

Given $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ -------- (1)

To multiply the roots of (1) by $\frac{1}{2}$, we have to multiply the successive coefficients beginning with the second by $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4$

i.e., $$x^4 + \frac{1}{2}\, 2x^3 + \left(\frac{1}{2}\right)^2 4x^2 + \left(\frac{1}{2}\right)^3 6x + \left(\frac{1}{2}\right)^4 8 = 0$$

$$x^4 + x^3 + x^2 + \frac{3}{4}x + \frac{1}{2} = 0$$

i.e., $$4x^4 + 4x^3 + 4x^2 + 3x + 2 = 0$$

which is the required equation.

Problem:

Remove the fractional coefficients from the equation $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$.

Solution:

Given $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$ --------- (1)

Multiply by the roots of (1) by m, we get

$$x^3 - \frac{m}{4}x^2 + \frac{m^2}{3}x - m^3 = 0 \qquad \text{--------- (2)}$$

If m = 12 (L.C.M. of 4 and 3), the fractions will be removed. Put m = 12 in (2), we get

i.e., $x^3 - 3x^2 + 48x - 1728 = 0$.

Problem:

Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$ given that its roots are in H.P.

Solution:

Given $6x^3 - 11x^2 - 3x + 2 = 0$ ---------- (1)

Its roots are in H.P. x to $\frac{1}{x}$ in (1), we get

$$6\left(\frac{1}{x}\right)^3 - 11\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 2 = 0$$

$$\Rightarrow 2x^3 - 3x^2 - 11x + 6 = 0 \qquad -----(2)$$

Now the roots of (2) are in A.P. (Since H.P. is a reciprocal of A.P.). Let the roots of (2) be $\alpha - d,\ \alpha,\ \alpha + d$.

Sum of the roots

$$\alpha - d + \alpha + \alpha + d = \frac{3}{2}$$

$$\Rightarrow 3\alpha = \frac{3}{2}$$

$$\alpha = \frac{1}{2} \qquad -----(3)$$

Product of the roots taken 3 at the time is $(\alpha - d) \times \alpha \times (\alpha + d) = \frac{-11}{2}$

d = $\pm\frac{5}{2}$.

Case(i) :

$$When\ d=\frac{5}{2}\ and\ \alpha=\frac{1}{2}, the\ roots\ of\ \phi(y)\ are\ \frac{1}{2}-\frac{5}{2}, \frac{1}{2}, \frac{1}{2}+\frac{5}{2}$$

$$i.e., -2, \frac{1}{2}, 3.$$

$$\therefore The\ roots\ of\ the\ given\ equation\ are\ the\ reciprocal\ of\ the\ roots\ of\ \phi(y)$$

$$i.e., -\frac{1}{2}, 2, \frac{1}{3}.\ are\ roots\ of\ \phi(x)$$

Case (ii) :

$$When\ d=\frac{-5}{2}\ and\ \alpha=\frac{1}{2}, the\ roots\ of\ \phi(y)\ are\ \frac{1}{2}+\frac{5}{2}, \frac{1}{2}, \frac{1}{2}-\frac{5}{2}$$

$$i.e., 3, \frac{1}{2}, -2.$$

$$\therefore The\ roots\ of\ the\ given\ equation\ are\ the\ reciprocal\ of\ the\ roots\ of\ \phi(y)$$

$$i.e., \frac{1}{3}, 2, -\frac{1}{2}.\ are\ roots\ of\ \phi(x)$$

Problem:

Diminish the roots of $x^4-5x^3+7x^2-4x+5=0$ *by* 2 and find the transformed equation.

Solution :

Diminishing the roots by 2, we get

2	1	-5	7	-4	5	
	0	2	-6	2	-4	
2	1	-3	1	-2	1	(constant term of the transformed equation)
	0	2	-2	-2		
2	1	-1	-1	-4	(coefficient of x)	
	0	2	2			
2	1	1	1	(coefficient of x^2)		
	0	2				
2	1	3	(coefficient of x^3)			
	0					
	1	(coefficient of x^4 in the transformed equation)				

The transformed equation whose roots are less by 2 of the given equation is

$$x^4+3x^3+x^2-4x+1=0$$

Problem:

Increase by 7 the roots of the equation $3x^4+7x^3-15x^2+x-2=0$ and find the transformed equation.

Solution :

Increasing by 7 the roots of the given equation is the same as diminishing the roots by -7.

-7	3	7	-15	1	-2	
	0	-21	98	-581	4060	
-7	3	-14	83	-580	4058	(constant term of the transformed equation)
	0	-21	245	-2296		
-7	3	-35	328	-2876	(coefficient of x)	
	0	-21	392			
-7	3	-56	720	(coefficient of x^2)		
	0	-21				
-7	3	-77	(coefficient of x^3)			
	0					
	3	(coefficient of x^4 in the transformed equation)				

The transformed equation is $3x^4-77x^3+720x^2-2876x+4058=0$.

Problem:

Find the equation whose roots are the roots of $x^4-x^3-10x^2+4x+24=0$ increased by 2.

Solution :

-2	1	-1	-10	4	24	
	0	-2	6	8	24	
-2	1	-3	-4	12	0	(constant term of the transformed equation)
	0	-2	10	-12		
-2	1	-5	6	0	(coefficient of x)	
	0	-2	14			
-2	1	-7	20	(coefficient of x^2)		
	0	-2				
-2	1	-9	(coefficient of x^3)			
	0					
	1	(coefficient of x^4 in the transformed equation)				

The transformed equation is $x^4-9x^3+20x=0$.

Problem:

If α, β, γ are the roots of the equation $x^3 - 6x^2 + 12x - 8 = 0$, find an equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$.

Solution :

2	1	-6	12	-8
	0	2	-8	8
2	1	-4	4	0
	0	2	-4	
2	1	-2	0	
	0	2		
2	1	0		
	0			
	1			

The transformed equation is $x^3 = 0$.

i.e., the roots are = 0, 0, 0.

i.e., $\alpha - 2 = 0,\ \beta - 2 = 0,\ \gamma - 2 = 0$

i.e., $\alpha = 2,\ \beta = 2,\ \gamma = 2$.

Problem:

Find the transformed equation with sign changed $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$.

Solution:

Given that $x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$

Given sign =	+	+	+	-	+	-
	+	-	+	-	+	-
	+	-	+	+	+	+

Now the transformed equation $x^5 - 6x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$ which is the required equation.

Nature of the Roots:

Problem:
Determine completely the nature of the roots of the equation $x^5 - 6x^2 - 4x + 5 = 0$.
Solution:

Given that $f(x) = x^5 - 6x^2 - 4x + 5$

There are 2 times sign changed.

∴ There exist 2 positive roots.

Put x = -x

$$f(-x) = (-x)^5 - 6(-x)^2 - 4(-x) + 5$$

$$= -x^5 - 6x^2 + 4x + 5$$

There is 1 time sign changed.

∴ There is only one positive root.

∴ There are 3 real roots.

The degree of the equation is 5.

Number of imaginary roots = degree of equation – number of real roots

= 5 – 3

= 2

∴ The number of imaginary roots = 2.

UNIT -2

MATRICES

A matrix is defined to be a rectangular array of numbers arranged into rows and columns. It is written as follows:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots\dots & a_{3n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} \end{bmatrix}$$

Special Types of Matrices:

(i) A **row matrix** is a matrix with only one row. E.g., [2 1 3].

(ii) A **column matrix** is a matrix with only one column. E.g., $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(iii) **Square matrix** is one in which the number of rows is equal to the number of columns.

If A is the square matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots\dots & a_{3n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} \end{bmatrix}$$

then the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots\dots & a_{3n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} \end{vmatrix}$$

is called the determinant of the matrix A and it is denoted by |A| or detA.

(iv) **Scalar matrix** is a diagonal matrix in which all the elements along the main diagonal are equal.

E.g., $\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$

(v) **Unit matrix** is a scalar matrix in which all the elements along the main diagonal are unity.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(vi) **Null or Zero matrix**. If all the elements in a matrix are zeros, it is called a null or zero matrix and is denoted by 0.

(vii) **Transpose matrix**. If the rows and columns are interchanged in matrix A, we obtain a second matrix that is called the transpose of the original matrix and is denoted by A^t.

(viii) **Addition of matrices.** Matrices are added, by adding together corresponding elements of the matrices. Hence only matrices of the same order may be added together. The result of addition of two matrices is a matrix of the same order whose elements are the sum of the same elements of the corresponding positions in the original matrices.

E.g., $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \\ a_5 + b_5 & a_6 + b_6 \end{bmatrix}$

Problem:

Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$; compute 3A-4B

Solution :

$$3A-4B=3\begin{bmatrix}1&0&2\\3&1&4\\5&0&6\end{bmatrix}-4\begin{bmatrix}2&1&-1\\3&0&-2\\0&1&1\end{bmatrix}$$

$$=\begin{bmatrix}3&0&6\\9&3&12\\15&0&18\end{bmatrix}-\begin{bmatrix}8&4&-4\\12&0&-8\\0&4&4\end{bmatrix}$$

$$=\begin{bmatrix}-5&-4&10\\-3&3&20\\15&-4&14\end{bmatrix}$$

Problem: Find values of $x, y, z\ and\ \omega$ that satisfy the matrix relationship

$$\begin{bmatrix}x+3&2y+5\\z+4&4x+5\\\omega-2&3\omega+1\end{bmatrix}=\begin{bmatrix}1&-5\\-4&2x+1\\2\omega+5&-20\end{bmatrix}$$

Solution :

From the equality of these two matices we get the equations

$x+3=1 \qquad 4x+5=2x+1$

$2y+5=-5 \qquad \omega-2=2\omega+5$

$z+4=-4 \qquad 3\omega+1=-20$

Solving these equations we get

$x=-2, y=-5,\ z=-8,\ \omega=-7$

Multiplication of Matrices.

If A is a m × n matrix with rows A_1, A_2, ……, A_m and B is a n × p matrix with columns B_1, B_2, ….., B_p, then the prodduct AB is a m × p matrix C whose elements are given by the formula $C_{ij} = A_i \,.\, B_j$.

$$\text{Hence } C = AB = \begin{bmatrix}A_1.B_1 & A_1.B_2 & \cdots & A_1.B_p\\ A_2.B_1 & A_2.B_2 & \cdots & A_2.B_p\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ A_m.B_1 & A_m.B_2 & \cdots & A_m.B_p\end{bmatrix}$$

Inverse of a Matrix

Problem: Find the inverse of the matrix $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$.

Solution:

$$\det\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = 2\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix}$$

= 2(1 + 3) – 1(-6) – 1(-2)

= 8 + 6 + 2

= 16.

Form the matrix of minor determinants:

$$\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -6 & -2 \\ 0 & 4 & -4 \\ 4 & 6 & 2 \end{pmatrix}.$$

Adjust the signs of every other element (starting with the second entry):

$$\begin{pmatrix} 4 & 6 & -2 \\ 0 & 4 & 4 \\ 4 & -6 & 2 \end{pmatrix}$$

Take the transpose and divide by the determinant:

$$\frac{1}{16}\begin{pmatrix} 4 & 0 & 4 \\ 6 & 4 & -6 \\ -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

So the inverse matrix is $\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$.

Problem: Show that A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$. Hence determine its inverse.

Solution: $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $A^2 - 4A - 5I = 0$.

Multiplying by A^{-1}, we have

$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$

i.e., $A - 4I - 5A^{-1} = 0$

Therefore $5A^{-1} = A - 4I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Therefore $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$.

Rank of a Matrix

A sub-matrix of a given matrix A is defined to be either A itself or an array remaining after certain rows and columns are deleted from A.

The determinants of the square sub-matrices are called the minors of A.

The rank of an $m \times n$ matrix A is r iff every minor in A of order r + 1 vanishes while there is at least one minor of order r which does not vanish.

Problem: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$.

Solution:

Minor of third order = $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{vmatrix}$

= 0.

The minors of order 2 are obtained by deleting any one row and any one column.

One of the minors of orders 2 is $\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}$

Its value is 8.

Hence the rank of the given matrix is 2.

Rank of a Matrix by Elementary Transformations:

Problem: Find the rank of the matrix A = $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

Solution: The given matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -6 \\ 0 & -1 & -8 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & -1 & -8 \end{bmatrix} R_2 \to R_2(-1)$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 5C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} C_3 \to C_3 - 6C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \frac{R_3}{-2}$$

Hence A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is a unit matrix of order 3.

Hence the rank of the given matrix is 3.

Procedure for finding the solutions of a system of equations:

Let the given system of linear equations be

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n = b_2$

……………………………………………………

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n = b_m$

Step 1: Construct the coefficient matrix which is denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$$

Step 2: Construct the augmented matrix which is denoted by [A, B]

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & \ldots\ldots & a_{3n} & b_3 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & b_m \end{bmatrix}$$

Step 3: Find the ranks of both the coefficient matrix and augmented matrix which are denoted by R(A) and R(A, B).

Step 4: Compare the ranks of R (A) and R(A, B) we have the following results.

(a) If R(A) = R(A, B) = n (number of unknowns) then the given system of equations are consistent and have unique solutions.

(b) If R(A) = R(A, B) < n (number of unknowns) then the given system of equations are consistent and have infinite number of solutions.

(c) If R(A) ≠ R(A, B) then the given system of equations are inconsistent (that is the given system of equations have no solution).

Problem: Test for consistency and hence solve $x-2y+3z=2, 2x-3z=3, x+y+z=0.$

Solution: The coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

The augmented matrix

$$[A, B] \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -9 & -1 \\ 0 & -3 & -2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & -3 & -2 & -2 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & -3 & -2 & -2 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-5}{19} \end{bmatrix} R_3 \to \frac{4R_3}{19}$$

Here rank of coefficient matrix is 3.

Rank of augmented matrix is 3.

Hence the given system of equations are consistent and have unique solution.

Problem: Test the consistency of the following system of equations and if consistent solve

$$2x - y - z = 2, x + 2y + z = 2, 4x - 7y - 5z = 2.$$

Solution:

The coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$$

The augmented matrix

$$[A, B] \sim \begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix} R_1 \sim R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 5 & 3 & 2 \end{bmatrix} R_3 \to R_3 + R_2$$

Here rank of coefficient matrix is R(A) = 2.

Rank of augmented matrix is R(A, B) = 2.

i.e., R(A) = R(A, B) < 3 (the number of unknowns)

Hence the given system of equations are consistent but have infinite number of solutions. Here the reduced system is

5y +3z = 2

x + 2y + z = 2

i.e., $y = \frac{2-3z}{5}$

$x = 2 - z - 2\left(\frac{2-3z}{5}\right)$

$= \frac{6+z}{5}$

i.e., $x = \frac{6+k}{5}$, $y = \frac{2-3z}{5}$, z = k where z = k is the parameter.

Solution of Simultaneous Equations

Problem: Solve the system of equations $\begin{aligned} 2x+y+z&=6 \\ x+2y+3z&=6.5 \\ 4x-2y-5z&=2 \end{aligned}$

Solution:

It can be represented as: $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 6.5 \\ 2 \end{pmatrix}$.

To see whether a solution exists we need to find $\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}$.

This determinant is $2\begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}$ = 2(-4) – (-17) + (-10) = -1

Therefore we know that the equations do have a unique solution.

To find the solution we need to find the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}$.

Find the determinant: we have already found that this is -1.

Form the matrix of minor determinants (which, for a particular entry in the matrix, is the determinant of the 2 by 2 matrix that is left when the row and column containing the entry are deleted):

$$\begin{pmatrix} -4 & -17 & -10 \\ -3 & -14 & -8 \\ 1 & 5 & 3 \end{pmatrix}$$

Adjust the signs of every other element (starting with the second entry):

$$\begin{pmatrix} -4 & 17 & -10 \\ 3 & -14 & 8 \\ 1 & -5 & 3 \end{pmatrix}$$

Take the transpose and divide by the determinant:

$$\frac{1}{-1}\begin{pmatrix} -4 & 3 & 1 \\ 17 & -14 & -5 \\ -10 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}$$

So the inverse matrix is $\begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}$.

Hence the solutions to the equations are found by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ 6.5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -1 \\ 2 \end{pmatrix}.$$

Therefore x = 2.5, y = -1 and z = 2.

Cayley – Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

Problem: *Verify Cayley – Hamilton theorem for the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ *and hence find the inverse of A.*

Solution : The characteristic equation of matrix A is

$\lambda^3 - \lambda^2(1+4+6) + \lambda(-1-3+0) - [1(-1)-2(-3)+3(-2)] = 0$

$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$, which is the characteristic equation.

By Cayley – Hamilton theorem , we have to prove

$A^3 - 11A^2 - 4A + 1 = 0$

$$A^2 = A \times . A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix}$$

$$A^3 - 11A^2 - 4A + I = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix} - 11 \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} - 4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence the theorem is verified.

To find A^{-1}

We have $A^3 - 11A^2 - 4A + I = 0$

$I = -A^3 + 11A^2 + 4A$

$A^{-1} = -A^2 - 11A + 4I$

$$= - \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} - 11 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{A^{-1}} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

Problem: Find all the eigen values and eigen vectors of the matrix A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+1+1) + \lambda(-3+1+1) - [2(-3)-1(-1)-1(-1)] = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$, which is the characteristic equation.

1	1	-4	-1	4
	0	1	-3	-4
	1	-3	-4	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4 , -1$

Hence **λ = 1 , 4 , -4.**

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + 0x_3 = 0$

Take first and second equation,

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$\Rightarrow \dfrac{x_1}{-2+0} = \dfrac{-x_2}{-2+1} = \dfrac{x_3}{0-1}$

$\Rightarrow \dfrac{x_1}{-2} = \dfrac{x_2}{1} = \dfrac{x_3}{-1}$

$\therefore$ **x_1 =** $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

When λ = -1 , equation (1) becomes

$3x_1+x_2-x_3 = 0$

$x_1+2x_2-2x_3 = 0$

$\Rightarrow \dfrac{x_1}{-2+2} = \dfrac{-x_2}{-6+1} = \dfrac{x_3}{6-1}$

$\Rightarrow \dfrac{x_1}{0} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$

$\therefore$ **x_2=** $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

When λ = 4 , equation (1) becomes

$-2x_1+x_2-x_3 = 0$

$x_1-3x_2-2x_3 = 0$

$\Rightarrow \dfrac{x_1}{-2-3} = \dfrac{-x_2}{4+1} = \dfrac{x_3}{6-1}$

$\Rightarrow \dfrac{x_1}{-1} = \dfrac{x_2}{-1} = \dfrac{x_3}{1}$

$\therefore$ **x_3=** $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Hence **Eigen vector =** $\begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

Problem: Find all the eigen values and eigen vectors of $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+3+2) + \lambda(4+3+4) - [2(4)-2(1)+1(-1)] = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$, which is the characteristic equation.

1	1	-7	11	-5
	0	1	-6	5
	1	-6	5	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 5) = 0$

$\Rightarrow \lambda = 1, 5$

Hence λ = 1 , 1 , 5.

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + x_3 &= 0 \\ x_1 + (3-\lambda)x_2 + x_3 &= 0 \\ x_1 + 2x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

Here all the equations are same.

Put $x_3 = 0$, we get $x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$

$\therefore$ **x_1 =** $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda = 1$, put $x_2 = 0$,we get

$x_1+x_3 = 0$

$x_1= -x_3$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$

$\therefore$ **x_2 =** $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

When $\lambda = 5$, equation(1) becomes

$-3x_1+2x_2+x_3 = 0$

$x_1-2x_2+x_3 = 0$ (taking first and second equation)

$\Rightarrow \frac{x_1}{2+2} = \frac{-x_2}{-3-1} = \frac{x_3}{6-2}$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$

$\therefore$ **x_3=** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. **Hence Eigen vector =** $\begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Problem: ***Find the eigen values and eigen vectors of*** $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Solution : The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(1+2-1) + \lambda(-3-1+3) - [1(-3)-1(1)-2(-1)] = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

2	1	-2	-1	2
	0	2	0	-2
	1	0	-1	0

λ = 2 is a root.
The other roots are

$$\lambda^2 - 1 = 0$$

$$(\lambda-1)(\lambda+1) = 0$$

$$\lambda = 1, -1$$

Hence **λ = 2 , 1 , -1**

The eigen vectors of matrix A is given by

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (-1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{..............................(1)}$$

When λ = 1 ,Equation (1) becomes

$$0x_1 + x_2 - 2x_3 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

$$\Rightarrow \frac{x_1}{1+2} = \frac{-x_2}{0-2} = \frac{x_3}{0+1}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

When λ = -1 ,Equation (1) becomes

$x_2=0$

$2x_1-2x_3 = 0$

$x_1 = x_3$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

When λ = 2,

Equation (1) becomes

$-x_1+x_2-2x_3=0$

$-x_1+0x_2+x_3=0$(taking first and second equation)

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{-1-2} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Hence **Eigen vector =** $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

Unit – III

Maxima And Minima

If a continuous function increases up to a certain value and then decreases, that value is called a **maximum** value of the function.

If a continuous function decreases up to a certain value and then increases, that value is called a **minimum** value of the function.

Theorem: If $f'(a) = 0 \; and \; f''(a) \neq 0$, then f(x) has a maximum if $f''(a) < 0$ and a minimum if $f''(a) > 0$.

Problem:

Find the maxima and minima of the function $2x^3 - 3x^2 - 36x + 10$.

Solution:

Let f(x) be $2x^3 - 3x^2 - 36x + 10$.

At the maximum or minimum point $f'(x) = 0$

Here $f'(x) = 6x^2 - 6x - 36$

$= 6(x-3)(x+2)$

∴ x=3 and x=-2 give maximum or minimum.

To distinguish between the maximum and minimum,we need $f''(x) = 6\,(2x-1)$.

When x=3 , $f''(x) = 6\,(6-1) = 30 \quad i.e. f'' \; is \; positive.$

When x=-2, $f''(x) = 6\,(-4-1) = -30 \quad i.e. f'' \; is \; negative.$

∴ x=-2 gives the maximum and x=3 gives the minimum.

Hence Maximum value = f(-2) = 54 and Minimum value = f(3) = -71.

Problem: Find the maximum value of $\frac{\log x}{x}$ for positive values of x.

Solution : Let f(x) be $\frac{\log x}{x}$

$\therefore \; f'(x) = \frac{1-\log x}{x^2}$

$f''(x) = \frac{-3+2\log x}{x^3}$

At a maximum or a minimum, $f'(x) = 0$.

$\therefore 1 - \log x = 0$. $\therefore x = e$.

$f''(e) = \frac{-3+2\log e}{e^3} = \frac{-1}{e^3}$, i.e., - ve.

$\therefore x = e$ gives a maximum.

Maximum value of the function $f(e) = \frac{1}{e}$.

Concavity and Convexity, Points of inflexion:

If the neighbourhood of a point P on a curve is above the tangent at P, it is said to be Concave upwards; if the curve is below the tangent at P, it is said to be concave downwards or convex upwards.

If at a point P, a curve changes its concavity from upwards to downwards or vice versa, P is called a point of inflexion.

Problem:

For what values of x is the curve $y = 3x^2 - 2x^3$ concave upwards and when is it convex upwards?

Solution:

$$y = 3x^2 - 2x^3$$

Then $\frac{dy}{dx} = 6x - 6x^2$,

$$\frac{d^2y}{dx^2} = 6 - 12x = -6(2x - 1).$$

If $x > \frac{1}{2}$, $\frac{d^2y}{dx^2}$ is negative and so convex upwards.

If $x < \frac{1}{2}$, $\frac{d^2y}{dx^2}$ is positive and so concave upwards.

If $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} = -12$ and so there is a point of inflexion at $x = \frac{1}{2}$. i.e., at the point $(\frac{1}{2}, \frac{1}{2})$

Partial Differentiation

Let u = f(x, y) be a function of two independent variables. Differentiating u w.r.t. 'x' keeping 'y' constant is known as the partial differential coefficient of 'u' w.r.t. 'x'.

It is denoted by $\frac{\partial u}{\partial x}$.

$\therefore \frac{\partial u}{\partial x}$ means differentiate u w.r.t. 'x' keeping 'y' constant.

Similarly if we differentiate u w.r.t. 'y' keeping 'x' constant is known as the partial differential coefficient of 'u' w.r.t. 'y'.

It is denoted by $\frac{\partial u}{\partial y}$.

$\therefore \frac{\partial u}{\partial y}$ means differentiate u w.r.t. 'y' keeping 'x' constant.

Symbolically, if u = f(x, y), then

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

Problem:

If $u = \log(x^2 + y^2 + z^2)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2+y^2+z^2}$.

Solution:

Given $u = \log(x^2 + y^2 + z^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x)$$

$$= \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\frac{(x^2+y^2+z^2)(1)-(x)(2x)}{(x^2+y^2+z^2)^2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}\right] \quad \text{------ (1)}$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = 2\left[\frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}\right] \quad \text{------ (2)}$$

$$\frac{\partial^2 u}{\partial z^2} = 2\left[\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}\right] \qquad \text{------ (3)}$$

Adding (1), (2) and (3) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 2\left[\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\right]$$

$$= \frac{2}{x^2+y^2+z^2}\,.$$

Problem:

If u = log(tanx + tany + tanz), show that $sin2x.\frac{\partial u}{\partial x} + sin2y.\frac{\partial u}{\partial y} + sin2z.\frac{\partial u}{\partial z}$ = 2.

Solution:

Given u = log(tanx + tany + tanz)

$$\frac{\partial u}{\partial x} = \frac{sec^2 x}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{sin2x\ .\ sec^2 x}{\tan x + \tan y + \tan z} \qquad \text{------ (1)}$$

Similarly,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{sin2y\ .\ sec^2 y}{\tan x + \tan y + \tan z} \qquad \text{------ (2)}$$

$$\sin 2z \frac{\partial u}{\partial z} = \frac{sin2z\ .\ sec^2 z}{\tan x + \tan y + \tan z} \qquad \text{------ (3)}$$

Adding (1), (2) and (3) we get

$$sin2x.\frac{\partial u}{\partial x} + sin2y.\frac{\partial u}{\partial y} + sin2z.\frac{\partial u}{\partial z} = \frac{sin2x.sec^2 x + sin2y\ .\ sec^2 y + sin2z\ .\ sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\ sinx\ cosx.\frac{1}{cos^2 x} + 2\ siny\ cosy.\frac{1}{cos^2 y} + 2\ siny\ cosy.\frac{1}{cos^2 y}}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$$

= 2

Euler's Theorem on Homogeneous Function

Theorem: If u is a homogeneous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Problem: If u = $\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = tanu$.

Solution: Given u = $\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

i.e., sin u = $\frac{x^2+y^2}{x+y} = \frac{x^2\left[1+\frac{y^2}{x^2}\right]}{x\left[1+\frac{y}{x}\right]}$

sin u = x f$\left(\frac{y}{x}\right)$, where f$\left(\frac{y}{x}\right) = \frac{1+\frac{y^2}{x^2}}{1+\frac{y}{x}}$

∴ sin u is a homogeneous function of degree 1. By Euler's theorem

$$x\frac{\partial(sinu)}{\partial x} + y\frac{\partial(sinu)}{\partial y} = 1 .\sin u$$

$$x\ cosu\ \frac{\partial u}{\partial x} + y\ cosu\ \frac{\partial u}{\partial y} = sinu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{sinu}{cosu} = \tan u .$$

Problem:

Verify Euler's Theorem when u = $x^3 + y^3 + z^3 + 3xyz$.

Solution:

$\frac{\partial u}{\partial x} = 3x^2 + 3yz.$

$\frac{\partial u}{\partial y} = 3y^2 + 3zx.$

$\frac{\partial u}{\partial z} = 3z^2 + 3xy.$

∴ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3zx) + z(3z^2 + 3xy)$

$= 3(x^3 + y^3 + z^3 + 3xyz)$

$= 3u.$

Total Differential Coefficient:

Problem:

Find $\frac{du}{dt}$ where u = $x^2 + y^2 + z^2$, x = e^t, y = e^t sint and z = e^t cost.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$= 2\,x\,e^t + 2\,y\,(e^t \sin t + e^t \cos t) + 2\,z\,(e^t \cos t - e^t \sin t)$

$= 2\,e^t\,(x + y \sin t + y \cos t + z \cos t – z \sin t)$

$= 2\,e^t\,(e^t + e^t \sin^2 t + e^t \sin t \cos t + e^t \cos^2 t - e^t \sin t \cos t)$

$= 2\,e^t \,.\, 2\,e^t$

$= 4\,e^t$.

Problem:

If $x^3 + y^3 + 3axy$, find $\frac{dy}{dx}$.

Solution:

$x^3 + y^3 + 3axy = 0$, i.e., f(x, y) = 0.

$\frac{\partial u}{\partial x} = 3x^2 – 3ay$

$\frac{\partial f}{\partial y} = 3y^2 – 3ax$

$\therefore \quad \frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$

$= -\frac{x^2 - ay}{y^2 - ax}$.

UNIT – IV

Evaluation of Integrals & Fourier Series

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$

Problem: Evaluate $\int \frac{dx}{x^2+2x+5}$.

Solution:

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+2^2}$$

Put x +1 = y; ∴ dx = dy

$$\int \frac{dx}{(x+1)^2+2^2} = \int \frac{dy}{y^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right).$$

Problem: Evaluate $\int \frac{dx}{4+5x-x^2}$.

Solution:

$$\int \frac{dx}{4+5x-x^2} = \int \frac{dx}{-(x^2-5x-4)} = -\int \frac{dx}{x^2-5x-4}$$

$$= -\int \frac{dx}{\left(x-\frac{5}{2}\right)^2-\left(\frac{41}{4}\right)}$$

$$= -\int \frac{dx}{\left(x-\frac{5}{2}\right)^2-\left(\frac{\sqrt{41}}{2}\right)^2}$$

Put $x - \frac{5}{2} = y$; ∴ dx = dy

$$= -\int \frac{dy}{(y)^2-\left(\frac{\sqrt{41}}{2}\right)^2}$$

$$= -\frac{1}{2\times\frac{\sqrt{41}}{2}}\log\left\{\frac{\frac{\sqrt{41}}{2}+y}{\frac{\sqrt{41}}{2}-y}\right\}$$

$$= -\frac{1}{\sqrt{41}}\log\left\{\frac{\sqrt{41}+2x-5}{\sqrt{41}-2x+5}\right\}$$

Integral of the form $\int \frac{px+q}{ax^2+bx+c}\,dx$

Problem: Evaluate $\int \frac{3x+1}{2x^2-x+5}\,dx$

Solution:

Let $3x + 1 = A\frac{d}{dx}(2x^2 - x + 5) + B$

$3x + 1 = A(4x - 1) + B$

Equating coefficient of 'x' on both sides we get

$3 = 4A \rightarrow A = \frac{3}{4}$

Equating constant coefficients we get,

$1 = -A + B$

$B = A + 1 = \frac{3}{4} + 1 = \frac{3+4}{4} = \frac{7}{4}$.

$3x + 1 = \frac{3}{4}(4x - 1) + \frac{7}{4}$.

$$\int \frac{3x+1}{2x^2-x+5}\,dx = \int \frac{\frac{3}{4}(4x-1)+\frac{7}{4}}{2x^2-x+5}\,dx$$

$$= \frac{3}{4}\int \frac{4x-1}{2x^2-x+5}\,dx + \frac{7}{4}\int \frac{dx}{2x^2-x+5}$$

$$= \frac{3}{4}\log(2x^2 - x + 5) + \frac{7}{4}\int \frac{dx}{2(x^2-\frac{x}{2}+\frac{5}{2})}$$

$$= \frac{3}{4}\log(2x^2 - x + 5) + \frac{7}{8}\int \frac{dx}{\left(x-\frac{1}{4}\right)^2+\left(\frac{39}{16}\right)}$$

$$= \frac{3}{4}\log(2x^2 - x + 5) + \frac{7}{8}\,\frac{1}{2\times\frac{\sqrt{39}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{4}}{\frac{\sqrt{39}}{2}}\right)$$

$$= \frac{3}{4}\log(2x^2 - x + 5) + \frac{7}{4\sqrt{39}}\tan^{-1}\left(\frac{4x-1}{\sqrt{39}}\right).$$

Integrals of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Problem: Evaluate $\int \frac{dx}{\sqrt{3x^2+4x+2}}$

Solution:

$$\int \frac{dx}{\sqrt{3x^2+4x+2}} = \int \frac{dx}{\sqrt{3}\left(\sqrt{x^2+\frac{4}{3}x+\frac{2}{3}}\right)}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\left(x+\frac{2}{3}\right)^2+\left(\frac{2}{9}\right)} = \frac{1}{\sqrt{3}} \int \frac{dx}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= \frac{1}{\sqrt{3}} \sinh^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{2}}{3}}\right)$$

$$= \frac{1}{\sqrt{3}} \sinh^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)$$

Integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

Problem: Evaluate $\int \frac{2x+1}{\sqrt{3+4x-x^2}}\,dx$.

Solution:

Let $2x + 1 = A \frac{d}{dx}(3+4x-x^2) + B$

$$2x + 1 = A(-2x+4) + B$$

Equating coefficient of ‘x’ on both sides we get

$$2 = -2A \Rightarrow A = -1$$

Equating constant coefficients we get,

$$1 = 4A + B$$

$$B = 1 - 4A = 1 + 4 = 5 .$$

$$2x + 1 = -1\,(2x + 1) + 5 .$$

$$\therefore \int \frac{2x+1}{\sqrt{3+4x-x^2}}\,dx = \int \frac{-(-2x+4)+5}{\sqrt{3+4x-x^2}}\,dx$$

$$= \int \frac{2x-4}{\sqrt{3+4x-x^2}}\,dx + \int \frac{5}{\sqrt{3+4x-x^2}}\,dx$$

$$= 2\sqrt{3+4x-x^2} + 5 \int \frac{dx}{\sqrt{-(x-2)^2+7}}$$

$$= 2\sqrt{3+4x-x^2} + 5\int \frac{dx}{\sqrt{(\sqrt{7})^2-(x-2)^2}}$$

$$= 2\sqrt{3+4x-x^2} + 5\ \sinh^{-1}\left(\frac{x-2}{\sqrt{7}}\right).$$

Properties of Definite Integrals:

$\int_a^b f(x)dx = F(b) - F(a)$ where $\int f(x)dx = F(x) + c$.

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^b f(x)dx = \int_a^b f(y)dy$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
5. $\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx$
6. If f(x) is an odd function i.e., f(-x) = - f(x) then
   $$\int_{-a}^{a} f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx$$
7. If f(x) is an even function i.e., f(-x) = f(x) then
   $$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx \ = 2\int_0^a f(x)dx$$

Problem: Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^n x\, dx}{\sin^n x+\cos^n x}\, dx$.

Solution:

Let $\quad I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x\, dx}{\sin^n x+\cos^n x}\, dx \qquad$ ------ (1)

Also $I = \int_0^{\frac{\pi}{2}} \frac{\sin^n(\frac{\pi}{2}-x)\, dx}{\sin^n(\frac{\pi}{2}-x)+\cos^n(\frac{\pi}{2}-x)}\, dx \qquad [\because \int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x\, dx}{\cos^n x+\sin^n x}\, dx \qquad$ ------ (2)

Adding (1) and (2) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x\, dx}{\sin^n x+\cos^n x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x\, dx}{\cos^n x+\sin^n x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^n x+\cos^n x}{\sin^n x+\cos^n x}\, dx$$

$= \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$

$2I = \frac{\pi}{2}$

$\therefore \ I = \frac{\pi}{4}$.

Problem: Evaluate $\int_0^{\frac{\pi}{2}} \log(sinx)dx$.

Solution:

Let $\quad I = \int_0^{\frac{\pi}{2}} \log(sinx)dx \qquad$ ------ (1)

Also $I = \int_0^{\frac{\pi}{2}} \log[sin\left(\frac{\pi}{2} - x\right)]dx \qquad [\because using\ property\ 4]$

$= \int_0^{\frac{\pi}{2}} \log(\cos x)dx \qquad$ ------ (2)

Adding (1) and (2) we get

$2I = \int_0^{\frac{\pi}{2}} \log(sinx)dx + \int_0^{\frac{\pi}{2}} \log(\cos x)dx$

$= \int_0^{\frac{\pi}{2}} [\log(sinx) + \log(\cos x)]dx$

$= \int_0^{\frac{\pi}{2}} \log(sinx\ .\cos x)dx$

$= \int_0^{\frac{\pi}{2}} \log\left(\frac{sin2x}{2}\right) dx$

$= \int_0^{\frac{\pi}{2}} [\log(sin2\,x) - log2]dx$

$= \int_0^{\frac{\pi}{2}} \log(\sin 2\,x)dx - \int_0^{\frac{\pi}{2}} \log 2\ dx$

$= \int_0^{\frac{\pi}{2}} \log(sin2x)\ dx - log2\ [x]_0^{\frac{\pi}{2}}$

$2I = \int_0^{\frac{\pi}{2}} \log(sin2x)\ dx - \frac{\pi}{2}\ log2 \qquad$ ------ (3)

To evaluate $\int_0^{\frac{\pi}{2}} \log(sin2x)\ dx$

Put 2x = y, 2dx = dy. When x = 0, y = 0; x = $\frac{\pi}{2}$, y = π

$\therefore \int_0^{\frac{\pi}{2}} \log(sin2x)\, dx = \frac{1}{2}\int_0^{\pi} \log(siny)\, dy$

$= \frac{1}{2} . 2 \int_0^{\frac{\pi}{2}} \log(siny)\, dy$

$= \int_0^{\frac{\pi}{2}} \log(sinx)\, dx = \text{I}$

i.e., $\int_0^{\frac{\pi}{2}} \log(sin2x)\, dx = I$ ------ (4)

Substituting (4) in (3), we get

$2\text{I} = \text{I} + \frac{\pi}{2}\, log2$

$\text{I} = -\frac{\pi}{2}\, log2$

i.e., $\int_0^{\frac{\pi}{2}} \log(sinx) dx = -\frac{\pi}{2}\, log2 = \frac{\pi}{2}\, log(2)^{-1}$

$= \frac{\pi}{2}\, log\left(\frac{1}{2}\right).$

FOURIER SERIES

Particular Cases

Case (i)

If f(x) is defined over the interval (0,2*l*).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,.....\infty$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

If f(x) is defined over the interval (0,2π).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$, n=1,2,..... ∞

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$ n=1,2,..... ∞

Case (ii)

If f(x) is defined over the interval (-*l* , *l*).

f(x) = $$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx$$

n =1,2,...... ∞

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

n=1,2,...... ∞

If f(x) is defined over the interval (-π, π).

f(x) = $$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

a_0 = $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$, n=1,2,..... ∞

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$ n=1,2,..... ∞

Problem: Obtain the Fourier expansion of

$$f(x) = \frac{1}{2}(\pi - x) \text{ in } -\pi < x < \pi$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

Problem: Obtain the Fourier expansion of f(x)=e^{-ax} in the interval ($-\pi$, π). Deduce that

$$\cos ech\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi}-e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

Here, $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

b_n = $$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

= $$\frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

= $$\frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Thus,

f(x) = $$\frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For x=0, a=1, the series reduces to

f(0)=1 = $$\frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

or

1 = $$\frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

or 1 = $$\frac{2\sinh\pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Thus,

$$\pi \cos ech\pi = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

This is the desired deduction.

Problem: Obtain the Fourier expansion of f(x) = x^2 over the interval (-π, π). Deduce that

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+......+\infty$$

Solution:

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

or $$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx,$$ since f(x)cosnx is even

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ since f(x)sinnx is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

Problem: Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Solution:

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} xdx = \pi$$

since f(x)cosnx is even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nxdx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0,$$ since f(x)sinnx is odd

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos n\pi$$

or
$$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus,

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

This is the series as required.

Problem: Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Solution:

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx dx + \int_{0}^{\pi} x\cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx dx + \int_{0}^{\pi} x\sin nx dx\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is

$$f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1 - 2(-1)^n\right]}{n}\sin nx$$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x=0. Hence

$$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or\ \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get, $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

Problem: Obtain the Fourier series of $f(x) = 1-x^2$ over the interval (-1,1).

Solution:

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$= 2\int_{0}^{1}(1 - x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1} f(x)\cos(n\pi x)dx$$ as $f(x)\cos(n\pi x)$ is even

$$= 2\int_{0}^{1}(1 - x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$$ =0, since $f(x)\sin(n\pi x)$ is odd.

The Fourier series of f(x) is $f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$

Problem: Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1+\dfrac{4x}{3}, & -\dfrac{3}{2} < x \le 0 \\ 1-\dfrac{4x}{3}, & 0 \le x < \dfrac{3}{2} \end{cases}$$

Deduce that $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots$

Solution:

The period of f(x) is $\dfrac{3}{2} - \left(\dfrac{-3}{2}\right) = 3$

Also f(-x) = f(x). Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_0^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - \left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right]_0^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also,

$$b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$$

Thus

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$$

putting x=0, we get

$$f(0) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

or $$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$$

Thus, $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period 2*l* may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0,*l*) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose f(x) = φ(x) is given in the interval (0,*l*). Then we define f(x) = -φ(-x) in (-*l*,0). Hence f(x) becomes an odd function in (-*l* , *l*). The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad (11)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over (0,*l*).

Putting l=π in (11), we obtain the half-range sine series of f(x) over (0,π) given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0,l) \quad .....given \\ \phi(-x) & \text{in } (-l,0) \quad .....\text{in order to make the function even.} \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad (12)$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over (0,*l*)

Putting l = π in (12), we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

where

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx dx \quad \text{n}=1,2,3,\ldots..$$

Problem: Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).

Solution: We have,

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n=\frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

The sine series of f(x) is

$$f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1-(-1)^n\right]\sin nx$$

Problem: Obtain the cosine series of

$$f(x)=\begin{cases}x, 0<x<\frac{\pi}{2}\\ \pi-x, \frac{\pi}{2}<x<\pi\end{cases} \qquad over(0,\pi)$$

Solution:

$$a_0=\frac{2}{\pi}\left[\int_0^{\pi/2}xdx+\int_{\pi/2}^{\pi}(\pi-x)dx\right]=\frac{\pi}{2}$$

Here

$$a_n=\frac{2}{\pi}\left[\int_0^{\pi/2}x\cos nx dx+\int_{\pi/2}^{\pi}(\pi-x)\cos nx dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1+(-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

Thus, the Fourier cosine series is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

Problem: Obtain the half-range cosine series of f(x) = c-x in $0<x<c$

Solution:

Here

$$a_0 = \frac{2}{c}\int_0^c (c-x)dx = c$$

$$a_n = \frac{2}{c}\int_0^c (c-x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

$$a_n = \frac{2c}{n^2\pi^2}\left[1-(-1)^n\right]$$

The cosine series is given by

$$f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$$

UNIT-V

DIFFERENTIAL EQUATIONS

Definition:

A differential equation is an equation in which differential coefficients occur.

Differential equations are of two types(i) Ordinary and (ii) Partial.

An ordinary differential equation is one which a single independent variable enters, either explicity or implicity. For example,

$$\frac{dy}{dx} = 2\sin x, \frac{d^2x}{dr^2} + m^2x = 0$$

$$x2\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + y = \sin x$$

are ordinary differential equations.

Variable separable.

Suppose an equation is of the form $f(x)dx + F(y)dy = 0$.

We can directly integrate this equation and the solution is $\int f(x)dx + \int F(y)dy = c,$ where c is an arbitrary constant.

Problem: Solve $\frac{dy}{dx} + \left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}} = 0$

Solution:

We have $\frac{dy}{\sqrt{1-y^2}} + \frac{dy}{\sqrt{1-x^2}} = 0.$

Integrating, $\sin^{-1}y + \sin^{-1}x = c$.

Problem: Solve tany $\frac{dy}{dx}$ = cotx.

Solution: tany $\frac{dy}{dx}$ = cotx

tany dy = cotx dx

$$\int \tan y \, dy = \int \cot x \, dx$$

log secy = log sinx + logc

log secy – log sinx = logc

$$\log \left(\frac{\sec y}{\sin x} \right) = \log c$$

$$\frac{\sec y}{\sin x} = c .$$

Problem: Solve tanx $\sec^2$y dy + tany $\sec^2$x dx = 0

Solution:

tanx $\sec^2$y dy = - tany $\sec^2$x dx

$$\frac{\sec^2 y}{\tan y} dy = \frac{\sec^2 x}{\tan x} dx$$

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{\sec^2 x}{\tan x} dx$$

put t = tany put u = tanx

dt = $\sec^2$y dy du = $\sec^2$x (- dx)

log t = - log u + log c

log t + log u = log c

log (tu) = log c

tu = c

tan y tan x = c.

Problem: Solve secx dy + secy dx = 0

Solution: secx dy = - secy dx

$$\frac{dy}{\sec y} = \frac{dx}{\sec x}$$

$$\int \cos y \, dy = \int \cos x \, dx$$

sin y = - sin x + c

sin x + sin y = c.

Linear Equation:

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no products of these occur.

The linear equation of the first order is of the form $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x only.

Problem: Solve $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$.

Solution:

Divided by $1 + x^2$

$$\frac{(1+x^2)}{(1+x^2)}\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{4x^2}{1+x^2}$$

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{4x^2}{1+x^2}$$

This is of the form $\frac{dy}{dx}+Py=Q$.

P = $\frac{2x}{1+x^2}$ and Q = $\frac{4x^2}{1+x^2}$

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y e^{\int \frac{2x}{1+x^2} dx} = \int \frac{4x^2}{1+x^2} e^{\int \frac{2x}{1+x^2} dx} dx + c \qquad \rightarrow (1)$$

$e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx}$

put $t = 1 + x^2$

$dt = 2x\, dx$

$e^{\int \frac{2x}{1+x^2} dx} = e^{\int \frac{dt}{t}}$

$= e^{\log t}$

$= t$

$e^{\int \frac{2x}{1+x^2} dx}$ = 1 + x^2. → (2)

Using (2) in (1),

y (1 + x^2) = $\int \frac{4x^2}{1+x^2}(1+x^2)dx + c$

y (1 + x^2) = $\int 4x^2 dx + c$

y (1 + x^2) = $\frac{4x^3}{3} + c$.

Problem: Solve $\frac{dy}{dx}$ + y sec x = tan x.

Solution:

This is of the form $\frac{dy}{dx} + Py = Q$.

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

P = sec x & Q = tan x

$$y e^{\int \sec x\, dx} = \int \tan x\, e^{\int \sec x\, dx} dx + c \qquad -->(1)$$

$$Now\, e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)}$$

$$= \sec x + \tan x$$

(1) → y (secx + tanx) = $\int \tan x$ (secx + tanx) $dx + c$

$$= \int \tan x \sec x\, dx + \int \tan^2 x\, dx + c$$

$$= \sec x + \int (1 - \sec^2 x) dx$$

$$= \sec x + \int dx - \int \sec^2 x dx$$

$$= \sec x + x - \tan x + c.$$

Problem: Solve $\frac{dy}{dx}$ - tan xy = -2 sin x.

Solution:

This is of the form $\frac{dy}{dx} + Py = Q$.

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

P = - tan x & Q = -2 sin x

$$y e^{\int -\tan x\, dx} = \int -2 \sin x\, e^{\int -\tan x\, dx} dx + c \qquad -->(1)$$

$$Now\, e^{-\int \tan x\, dx} = e^{-\log \sec x}$$

$$= -\sec x$$

- y sec x = $\int -2 \sin x (-\sec x) dx + c$

$= \int 2 \sin x \sec x\, dx + c$

$= 2 \int \frac{\sin x}{\cos x} dx + c$

$= 2 \int \tan x\, dx + c$

- y sec x = 2 log secx + c.

Problem: Solve $\cos^2 x \frac{dy}{dx}$ + y = tan x.

Solution:

Divided by $\cos^2 x$.

$$\frac{\cos^2 x}{\cos^2 x}\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x$$

$$P = \sec^2 x \;\&\; Q = \tan x \sec^2 x$$

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y e^{\int \sec^2 x\,dx} = \int \tan x \sec^2 x\, e^{\int \sec^2 x\,dx} dx + c \qquad -->(1)$$

$$Now\, e^{\int \sec^2 x\,dx} = e^{\tan x}$$

y $e^{\tan x}$ = $\int \tan x \sec^2 x\, e^{\tan x} dx + c$

put t = tanx

dt = $\sec^2$x dx

y e^t = $\int t\, e^t dt + c$

= t . e^t - e^t

= e^t (t – 1) + c

y $e^{\tan x}$ = $e^{\tan x}$ (tanx – 1) + c

Problem: Solve (1 + x^2) $\frac{dy}{dx}$ + 2xy = cosx.

Solution:

Divided by 1 + x^2

$$\frac{(1+x^2)}{(1+x^2)}\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cos x}{1+x^2}$$

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cos x}{1+x^2}$$

This is of the form $\frac{dy}{dx} + Py = Q$.

P = $\frac{2x}{1+x^2}$ and Q = $\frac{\cos x}{1+x^2}$

The solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$$y e^{\int \frac{2x}{1+x^2} dx} = \int \frac{\cos x}{1+x^2} e^{\int \frac{2x}{1+x^2} dx} dx + c \qquad \rightarrow (1)$$

$$e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx}$$

put t = 1 + x^2

dt = 2x dx

$e^{\int \frac{2x}{1+x^2}dx} = e^{\int \frac{dt}{t}}$

$= e^{\log t}$

$= t$

$e^{\int \frac{2x}{1+x^2}dx} = 1 + x^2.$ → (2)

Using (2) in (1),

$y(1 + x^2) = \int \frac{\cos x}{1+x^2}(1+x^2)dx + c$

$y(1 + x^2) = \int \cos x\, dx + c$

$y(1 + x^2) = \sin x + c$.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Problem: Solve $(D^2 + 5D + 6)\, y = e^x$.

Solution:

To find the C.F. solve $(D^2 + 5D + 6)\, y = 0$.

The auxiliary equation is $m^2 + 5m + 6 = 0$.

Solving, m = -2 and -3.

C.F. = $A e^{-2x} + B e^{-3x}$.

P.I. = $\frac{1}{D^2 + 5D + 6} e^x$

$= \frac{1}{12} e^x$ on replacing D by 1.

$y = A e^{-2x} + B e^{-3x} + \frac{1}{12} e^x$.

Problem: Solve $(D^2 - 2mD + m^2)\, y = e^{mx}$.

Solution:

To find the C.F. solve $(D^2 - 2mD + m^2)\, y = 0$.

The auxiliary equation is $k^2 - 2mk + m^2 = 0$.

i.e., $(k - m)^2 = 0$, ∴ k = m twice.

C.F. = $e^{mx}(A + Bx)$.

$$\text{P.I.} = \frac{1}{(k-m)^2} e^{mx}$$

$$= \frac{x^2}{2} e^{mx}$$

∴ $y = e^{mx}\left(A + Bx + \frac{x^2}{2}\right)$.

Problem: Solve $(D^2 - 3D + 2)\, y = \sin 3x$.

Solution:

To find the C.F. solve $(D^2 + 5D + 6)\, y = 0$.

The auxiliary equation is $m^2 - 3m + 2 = 0$.

Solving, m = 2 and 1.

C.F. = $A e^{2x} + B e^{x}$.

$$\text{P.I.} = \frac{\sin 3x}{D^2 - 3D + 2}$$

$$= \frac{\sin 3x}{-9 - 3D + 2} \text{ , put } D^2 = -a^2 = -9$$

$$= \frac{\sin 3x}{-7 - 3D} \times \frac{7 - 3D}{7 - 3D}$$

$$= \frac{7\sin 3x - 3D(\sin 3x)}{-49 + 9D^2}$$

$$= \frac{7\sin 3x - 3(3\cos 3x)}{-49 + 9(-9)}$$

$$= \frac{7\sin 3x - 9\cos 3x}{-49-81}$$

$$= \frac{7\sin 3x - 9\cos 3x}{-130}$$

$$= -\left[\frac{7\sin 3x - 9\cos 3x}{130}\right]$$

y = C.F. + P.I.

$$= A\,e^{2x} + B\,e^{x} - \left[\frac{7\sin 3x - 9\cos 3x}{130}\right].$$

Problem: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 5x^2$.

Solution:

$(D^2 + 2D + 3)\,y = 5x^2$

To find the C.F. solve $(D^2 + 2D + 3)\,y = 0$.

The auxiliary equation is $m^2 + 2m + 3 = 0$.

$$m = \frac{-2 \pm \sqrt{2^2 - 4.1.3}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4-12}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

$$= \frac{-2 \pm 2i\sqrt{2}}{2}$$

$$= -1 \pm i\sqrt{2}$$

$\alpha = -1$, $\beta = \sqrt{2}$

C.F. = $e^{-x}(A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x)$

$$\text{P.I.} = \frac{5x^2}{D^2 + 2D + 3}$$

$$= \frac{5x^2}{3+2D+D^2}$$

$$= \frac{5x^2}{3\left[1+\frac{2D+D^2}{3}\right]}$$

$$= \frac{5}{3}\left[1+\frac{2D+D^2}{3}\right]^{-1} x^2$$

$$= \frac{5}{3}\left[1-\left(\frac{2D+D^2}{3}\right)+\left(\frac{2D+D^2}{3}\right)^2 - \ldots\ldots\ldots\right] x^2$$

$$= \frac{5}{3}\left[1-\left(\frac{2D+D^2}{3}\right)+\left(\frac{4D^2+4D^3+D^4}{9}\right) - \ldots\ldots\ldots\right] x^2$$

$$= \frac{5}{3}\left[1-\left(\frac{2D+D^2}{3}\right)+\left(\frac{4D^2}{9}\right)\right] x^2 \quad \text{(Neglecting Higher Powers)}$$

$$= \frac{5}{3}\left[x^2-\left(\frac{2D(x^2)+D^2(x^2)}{3}\right)+\left(\frac{4D^2(x^2)}{9}\right)\right]$$

$$= \frac{5}{3}\left[x^2-\left(\frac{2(2x)+2}{3}\right)+\left(\frac{4(2)}{9}\right)\right]$$

$$= \frac{5}{3}\left[x^2-\left(\frac{4x+2}{3}\right)+\left(\frac{8}{9}\right)\right]$$

$$= \frac{5}{3}\left[x^2-\frac{4x}{3}-\frac{2}{3}+\frac{8}{9}\right]$$

$$= \frac{5}{3}\left[x^2-\frac{4x}{3}+\frac{2}{9}\right]$$

y = C.F. + P.I.

$$= e^{-x}(A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x) + \frac{5}{3}\left[x^2-\frac{4x}{3}+\frac{2}{9}\right].$$

Problem: Solve $(D^2 + 4)\, y = e^{2x} \sin 2x$.

Solution:

The auxiliary equation $m^2 + 4 = 0$.

$m^2 = -4$

$m = \sqrt{-4}$

$m = \pm 2i$.

C.F. $= e^{0x}(A \cos 2x + B \sin 2x)$

$= A \cos 2x + B \sin 2x$

$$\text{P.I.} = \frac{e^{2x} \sin 2x}{D^2 + 4}$$

$$= \frac{e^{2x} \sin 2x}{(D+2)^2 + 4} \quad \text{, replace D by D+2}$$

$$= \frac{e^{2x} \sin 2x}{D^2 + 4D + 8}$$

$$= \frac{e^{2x} \sin 2x}{-4 + 4D + 8} \quad , \quad \text{replace } D^2 \text{ by -4}$$

$$= \frac{e^{2x} \sin 2x}{4D + 4} \times \frac{4D - 4}{4D - 4}$$

$$= \frac{e^{2x}[4D(\sin 2x) - 4\sin 2x]}{16D^2 - 16} = \frac{e^{2x}[4D(\sin 2x) - 4\sin 2x]}{16(-4) - 16}$$

$$= \frac{4e^{2x}[2\cos 2x - \sin 2x]}{-80}$$

y = C.F. + P.I.

$$= A \cos 2x + B \sin 2x - \frac{4e^{2x}[2\cos 2x - \sin 2x]}{80}.$$