BSCPH-103

B. Sc. I Year

OSCILLATIONS AND WAVES

DEPARTMENT OF PHYSICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY

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Prof. H. M. Agrawal
Professor and Head, Department of Physics, CBSH, G.B.P.U.A.&T. Pantnagar, India

Dr. Kamal Devlal
Department of Physics
School of Sciences, Uttarakhand Open University

Programme Coordinator

Dr. Kamal Devlal
Department of Physics
School of Sciences, Uttarakhand Open University
Haldwani, Nainital

Unit writing and Editing

Editing

Dr Sudip Ranjan Jha
Professor of Physics
School of Sciences
Indira Gandhi National Open university, New Delhi

Writing

1.**Dr. H M Agrawal**
Professor of Physics, Department Physics
CBSH, GB Pant University of Ag. Tech. Pantnagar, US Nagar, Uttarakhand

2. **Dr. L. P. Verma**
Department Physics
MBPG Haldwani, Nainital, Uttarakhand

3. **Dr. Santosh K. Joshi**
Department Physics
University of Petroleum and Energy Studies,
Prem Nagar, Dehradun-248007 Uttarakhand

4. **Dr. Kailash Pandey**
Department Physics
University of Petroleum and Energy Studies,
Prem Nagar, Dehradun-248007 Uttarakhand

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Oscillations and Waves

DEPARTMENT OF PHYSICS
SCHOOL OF SCIENCES
UTTARAKHAND OPEN UNIVERSITY

Phone No. 05946-261122, 261123
Toll free No. 18001804025
Fax No. 05946-264232, E. mail info@uou.ac.in
htpp://uou.ac.in

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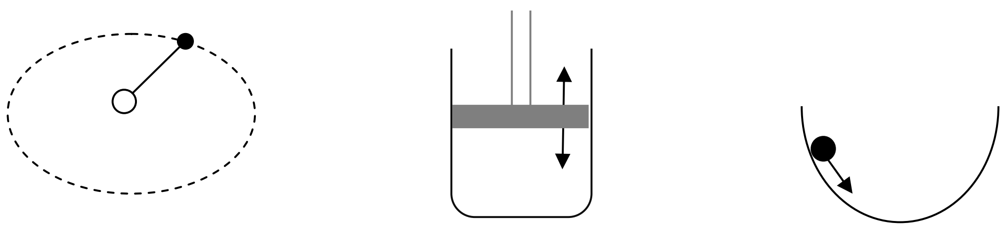
UNIT 1 SIMPLE HARMONIC MOTION

Structure

1.1 INTRODUCTION

Any motion which repeats itself after regular interval is called *periodic or harmonic motion* and the time interval after which the motion is repeated (i.e. the position and the velocity of the moving body is the same) is called its time period. Some examples of periodic motion include (see Fig. 1)

- motion of planets around the sun,
- motion of a piston inside a cylinder, used in automobile engines, or
- motion of a ball in a bowl.



(a) (b) (c)

Figure 1: Some examples of periodic motion: (a) motion of the earth around the sun, or moon around the earth; (b) motion of a piston in a cylinder which is used in automobile engines; (c) motion of a ball in a bowl.

If in case of periodic motion, the body moves back and forth repeatedly about a fixed position (called equilibrium or mean position), the motion is said to be *oscillatory or vibratory*. For instance, the motion of the earth around the sun or the motion of the hands of the clock, are examples of periodic motion, but they are not oscillatory in nature. The motion of piston in an automobile engine, motion of a ball in a bowl, motion of needle of sewing machine or the bob of a pendulum clock are all examples of oscillatory motion.

An oscillating body is said to execute *simple harmonic motion* (SHM) if the magnitude of the forces acting on it is directly proportional to the magnitude of its displacement from the mean position and the force (called restoring force) is always directed towards the mean position. Thus, we can see that simple harmonic motion or SHM is actually a special case of oscillatory or vibratory motion. We will study SHM in detail in this unit. Some examples of simple harmonic motion include (see Fig. 2)

- motion of a simple pendulum,
- a vibrating tuning fork, or
- a spring-mass system.

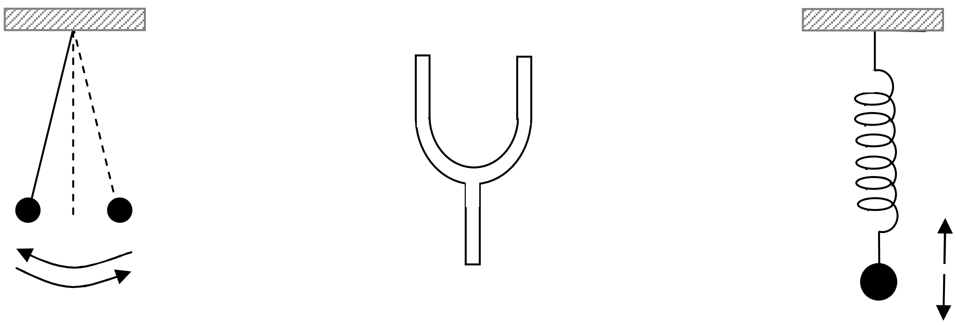


Figure 2: Some examples of SHM: (a) A simple pendulum; (b) a vibrating tuning fork; (c) an oscillating spring-mass system.

1.2 OBJECTIVES

After studying this unit, you should be able to

- describe examples of oscillating systems
- explain what is meant by simple harmonic motion
- explain what is meant by the amplitude and the time period of an oscillating system
- write down the general equation of simple harmonic motion and solve it
- describe how the acceleration, velocity and displacement of an oscillating system change with time
- define angular frequency
- differentiate between vertical and horizontal spring-mass systems
- calculate the time period for composite spring-mass systems

1.3 OSCILLATORY MOTION

Any oscillating system moves to and fro (back and forth) repeatedly. Oscillations may be very complex such as those of a piano string or those of the earth during an earthquake or beating of the heart. There are also oscillations which are not very evident to our senses like the oscillations of the air molecules that transmit the sensation of sound, the oscillations of the atoms in a solid that convey the sensation of temperature or the oscillations of the electrons in the antennas of radio and TV transmitters. It would not be an exaggeration to say that we are indeed surrounded by oscillations all the time because oscillations are not just confined to material objects such as musical instruments but visible light, micro waves, radio waves and X-rays are also the outcome of oscillatory phenomena. Thus, the study of oscillations is essential for the understanding of various systems, be it mechanical, acoustical, electrical or atomic.

The oscillatory motion in a physical system results from two properties – the property of *inertia*, and the property of *elasticity*. We will begin with two illustrative physical systems which are described in the following sections. Studying such simple systems will help us in understanding the motion of more complicated oscillating systems.

1.3.1 Simple Pendulum

Do you remember that in your senior secondary class, you performed an experiment with a simple pendulum in your physics laboratory, where you measured the change in time period with the length of the string?

A simple pendulum consists of a heavy point mass, suspended from a fixed support through a weightless inextensible string. Here, we must understand that a simple pendulum is an idealized model. In practice, a simple pendulum is realized by suspending a small metallic sphere by a thread hanging from a fixed support like a stand. Fig. 3 shows a simple pendulum in which a bob of mass *m* is suspended from the fixed support *P* through a light string of length *l*. Left to itself, the bob occupies the position *PO*, with the angle $\theta = 0$, which is known as the mean or equilibrium position. From this mean position, the pendulum is drawn towards one extreme A such that the angle θ remains small. In doing so, the bob gains some finite potential energy. When the bob is released from *A*, it begins to move downward towards the mean position *O*. As a result, its potential energy begins to decrease as the bob approaches *O*. As the potential energy decreases, the bob gains kinetic energy. At the mean position, the bob's kinetic energy is maximum and its potential energy is minimum. Further. Due to having gained kinetic energy, the bob does not stop at the mean position; it overshoots the mean position and reaches the other extreme *B*. At position B, bob's potential energy becomes maximum and the kinetic energy is zero because its velocity becomes zero momentarily. After a momentary rest, the bob once again retraces its path from *B* to *O* to *A*. Thus, we see that the bob oscillates in a circular arc with the center at the point of suspension *P*.

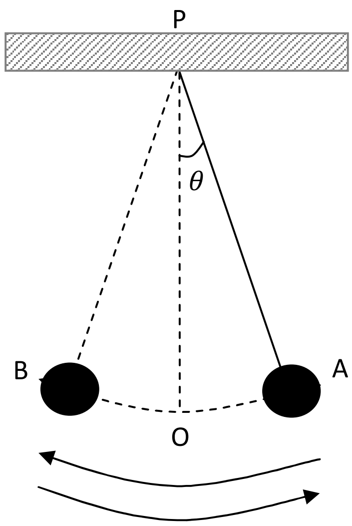


Figure 3: A simple pendulum.

Under ideal conditions, if there is no air resistance, losses due to friction do not affect the oscillatory motion. In such a situation, the pendulum should, in principle, oscillate forever! Each complete cycle of its oscillating motion takes it from one side of equilibrium to the other side and then back again.

1.3.2 Spring-Mass System

Just like simple pendulum, you must also be familiar with the spring pendulum.

Spring mass-system or spring pendulum consists of a weightless spring of constant *k*, one end of which is fixed rigidly to a wall and the other end is attached to a body of mass *m*, which is free to move horizontally or vertically depending on the system. If it is a loaded spring, it can move to and fro vertically. In the case of horizontal spring-mass system, the body is free to move on a frictionless horizontal surface, as shown in Fig. 4. When the spring is stretched, the elasticity of the spring tries to bring back the mass to its mean position. As the mass reaches the mean position, it has attained some velocity. As a result, the mass continues to move in the same direction and eventually compresses the spring until it reaches the other extreme position. The compressed spring pushes the mass back towards its mean position and the mass retraces its path. Thus, each cycle of oscillation takes the mass *m* from one extreme position to the extreme position on the other side of the mean position.

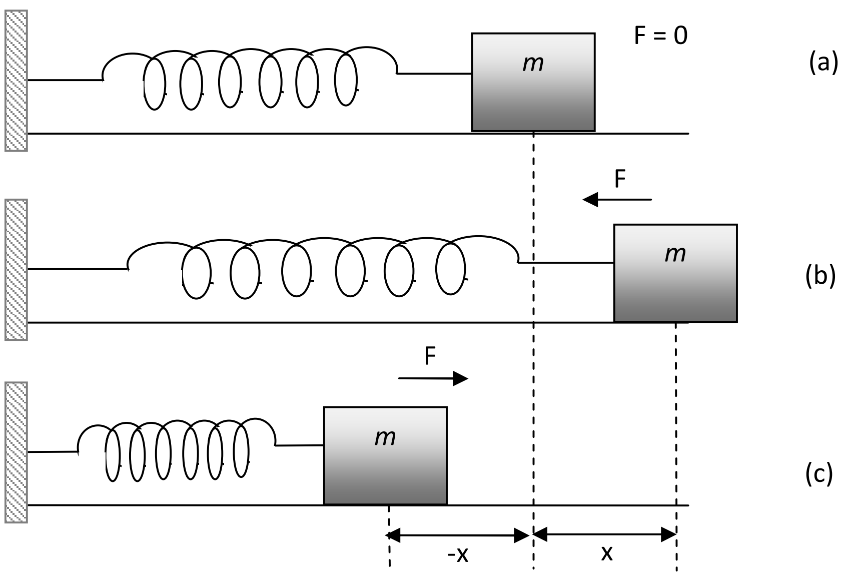


Figure 4: (a) Normal, (b) stretched, (c) compressed configurations of a horizontal spring-mass system.

Under ideal conditions, that is, if there is no air resistance and if the horizontal surface on which the mass is moving is frictionless, the spring – mass system should oscillate forever!

We will use spring-mass system, described above, to discuss the characteristics of SHM in the next section. You will also learn to calculate the force F shown in Fig. 4. But, before you proceed further, you should try to answer some questions based on what you have studied until now.

Self Assessment Question (SAQ) 1: In practice, the oscillations in a simple harmonic motion or a spring-mass system die away gradually and the mass *m* stops moving. What do you think is the reason for that?

Self Assessment Question (SAQ) 2: What are the two properties that are responsible for the oscillations?

Self Assessment Question (SAQ) 3: Do you think that the minute hand of the clock moves periodically? If so, can we also infer that its motion is oscillatory? Explain.

Self Assessment Question (SAQ) 4: Choose the correct option:

The motion of Halley's Comet around the sun is

(a) Periodic (b) Oscillatory (c) Simple harmonic (d) Translatory.

1.4 SIMPLE HARMONIC MOTION

In the previous Section, we discussed two examples of oscillatory motion. Let us now use the spring-mass system to understand simple harmonic motion (SHM). What is SHM? Let us first answer this question.

1.4.1 Definition of SHM

SHM can be defined in a number of ways:

1. If the force acting on the oscillating body is always in the direction opposite to the displacement of the body from the equilibrium or the mean position and its magnitude is proportional to the magnitude of displacement, the body is said to be executing SHM.
2. If the displacement vs. time curve of the oscillating body is sinusoidal in nature, the body is said to be executing SHM. This is another definition of SHM.
3. If the potential energy of the oscillating body is proportional to the square of its displacement with reference to the mean position, the body is said to be executing SHM. This is yet another definition of SHM.

Let us consider the first definition for now. The spring-mass system shown in Fig. 4 (a) is in the position of static equilibrium: the spring is relaxed (neither stretched nor compressed) and there is no force acting on the body. When the body is pulled to the right through a small distance x, the spring starts behaving like an elastic system under stress. You may recall the Young's modulus experiment that you did in your school. We know that if a wire of length L is stretched through a distance x by a force of magnitude F, the Young's modulus Y of the material of the wire is given by

$$Y = \frac{F/\alpha}{x/L} \quad (1.1)$$

Here α is the cross-sectional area of the wire. By rearranging the terms in equation (1.1), we can easily get the following form

$$F = \left(\frac{Y\alpha}{L}\right)x \quad (1.2)$$

We already know that elasticity is the property by virtue of which a body offers resistance to any change in its size or shape or both and makes the body regain its original condition when the deforming force, applied within a certain maximum limit, is removed. In other words, one can say that in the deformed condition, the body develops a restoring force and according to the Newton's third law of motion, this force is equal in magnitude but opposite in direction to the deforming force. Equation (1.2) implies that the restoring force is proportional to the elongation and is directed towards the equilibrium position (relaxed position when there is no restoring force acting).

Similarly, for the spring-mass system, Hooke's Law states that the restoring force is proportional to the displacement of the spring in case of stretched as well as compressed configurations. In our case, the restoring force exerted by the spring on the body is directed to the left [see Fig. 4 (b)] and is given by the following relation:

$$F = -kx \quad (1.3)$$

Since, the restoring force, F is proportional to the displacement[1] and is opposite in sign to the displacement, the resulting motion is simple harmonic. Here k is called the spring constant or stiffness constant. The SI unit of k is Nm^{-1}.

Example 1: If, in a spring-mass system as shown in Fig. 4, the spring constant is 50 Nm^{-1} and the block of mass 1 kg is displaced by 0.01 m to the right before being released, calculate the

(a) restoring force at t = 0,
(b) restoring force when the block travels to the other extreme, and
(c) The restoring force in the static equilibrium position.

Solution:

(a) If x is taken as positive to the right of the mean position, then the restoring force is given by

$$F = -kx = -(50\ Nm^{-1})(0.01\ \text{m}) = -0.5\ N$$

[1] The relationship (Eq. (1.3)) is linear only for small values of displacement, x and the elastic force produced in the linear spring is given by $F = -kx$, where x is the change in the length of the spring.

(b) Similarly, the restoring force is given by

$$F = -kx = -(50\ Nm^{-1})(-0.01\ \text{m}) = +0.5\ N$$

(c) At the mean position, $x = 0$

$$F = -(50\ Nm^{-1})(0) = 0$$

Self Assessment Question (SAQ) 5: If the displacement x in the above example is halved, how will be the restoring force change in all the three cases.

Self Assessment Question (SAQ) 6: What will happen if instead the initial displacement is doubled?

Self Assessment Question (SAQ) 7: Will the answer change if the mass of the block in the above example is changed to 5 kg?

Self Assessment Question (SAQ) 8: Challenge Question:

In the spring-mass system shown in Fig. 5, the spring constants (k) of both the springs is 50 Nm^{-1} and the block of mass 1 kg is displaced by 0.01 m to the right from its mean position before being released. In the equilibrium position of the system, the springs are stretched by 0.02 m each. Calculate the

(a) restoring force at t = 0,
(b) restoring force when the block travels to the other extreme, and
(c) the restoring force in the static equilibrium position.

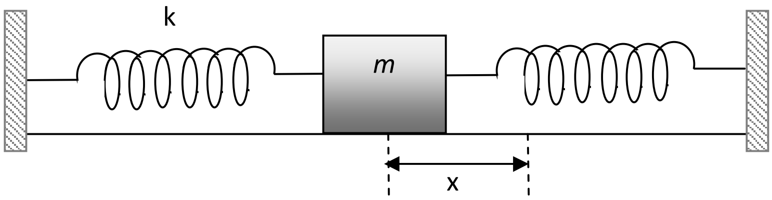


Figure 5: A spring-mass system where a body of mass m is connected by two identical weightless springs which are attached to rigid walls.

[Hint: The total restoring force on the block will be given by the elastic force exerted by each of two springs. We also know that the elastic force for a spring is proportional to the total elongation or compression.]

1.4.2 Basic Characteristics of SHM

Since we now know what SHM is, let us define some of the basic characteristics of SHM. What comes to your mind? The first important characteristic in SHM is the initial displacement that actually results in oscillations in the first place. The magnitude of the initial displacement, which is also the maximum displacement, is called the ***amplitude (A)*** of oscillations. As we mentioned before, the energy of the system executing SHM alternates between kinetic and potential forms. At the extremities of the oscillations, the kinetic energy is zero as the velocity is zero and the potential energy is the maximum.

Another characteristic of SHM is the ***time period (T)*** which is the time taken for one complete cycle of oscillation. This is the least time taken by an oscillating object to move from a certain position and velocity back to the same position and velocity. Generally, for convenience, we measure the time period from either the mean position or the extreme ends.

Instead of time period, many a times we talk in terms of the ***frequency (ν)*** to characterize SHM. Frequency is the number of complete oscillations executed per second and is the inverse of the time period, i.e.

$$\nu = \frac{1}{T} \tag{1.4}$$

It is expressed in cycles per second or simply s^{-1} or hertz (Hz). We also define a term called ***angular frequency***, denoted by $\boldsymbol{\omega}$, which is given by

$$\omega = 2\pi\nu \tag{1.5}$$

It is expressed in radian per second or simply$rad\ s^{-1}$, since 2π is the angle around a circle in radians and *T* is in seconds.

Example 2: A mass on a spring oscillates along a vertical line, taking 12 s to complete 10 oscillations. Calculate the

(a) time period, and
(b) the angular frequency.

Solution:

(a) Time period is the time taken for one complete cycle of oscillation; therefore, to complete one oscillation, time needed will be

$$T = \frac{(12\ s)}{(10\ oscillations)} = 1.2\ s$$

(b) The frequency is given by

$$\nu = \frac{1}{T} = \frac{1}{1.2}\ Hz$$

Therefore, the angular frequency is

$$\omega = 2\pi\nu = \frac{2\pi}{1.2} = 5.23\ rad\ s^{-1}$$

Example 3: The motion of a vibrating blade is frozen (when the frequency of the vibrating blade becomes equal to the stroboscope frequency) by illuminating it with a stroboscope (a flashing light). The least stroboscope frequency at which this occurs is 40 Hz. Calculate the

(a) time period, and
(b) the angular frequency of the vibrations.

Solution:

(a) Time period is the inverse of frequency

$$T = \frac{1}{\nu} = \frac{1}{40} = 0.025\ s$$

(b) The angular frequency is given by

$$\omega = 2\pi\nu = 2\pi(40) = 251.2\ rad\ s^{-1}$$

Self Assessment Question (SAQ) 9: An object executes simple harmonic motion with an angular frequency of $1.26 rad\ s^{-1}$. Calculate its time period.

Self Assessment Question (SAQ) 10: If the angular frequency ω is one revolution per minute. Calculate its time period. [Hint: One revolution = (2π) radians]

1.5 DIFFERENTIAL EQUATION OF SHM

Let us now express equation (1.3) in the differential form by using Newton's second law of motion. From Newton's second law of motion, we know that force experienced by a body of mass *m* can be expressed as a function of acceleration,

$$F = ma = m\ddot{x}$$

Therefore, in a spring-mass system, the force can be written as

$$F = m\ddot{x} = -kx$$

Or we can say that

$$m\ddot{x} + kx = 0$$

$$or, \qquad \ddot{x} + \frac{k}{m}x = 0 \qquad (1.6)$$

(**Comment**: Either follow the double dot notation or d^2x/dt^2 notation for double differentiation. In later Units, d^2x/dt^2 notation has been used. For students, it will be better if we follow d^2x/dt^2 notation.)

The above equation is the differential equation of SHM. k is the force constant (for our case of spring-mass system, it is called the spring constant) and has dimensions (MLT^{-2}/L) = MT^{-2}.

Therefore, the dimension of k/m is T^{-2}, i.e. square of reciprocal of time. We can replace k/m by ω^2. Thus, the equation (1.6) takes the form

$$\ddot{x} + \omega^2 x = 0 \tag{1.7}$$

We will find the physical meaning of ω, that it is actually the angular frequency that we already defined earlier, when we solve the differential equation (1.7).

1.5.1 Solution of the Differential Equation of SHM

The second time derivative of displacement ($\ddot{x}$) can be written as

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

Multiplying and dividing by dx in the numerator and the denominator, we get

$$\ddot{x} = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right)$$

We already know that $\dot{x}$ or dx/dt actually define the velocity$v$. Therefore, the above expression can take the following form

$$\ddot{x} = v\frac{d}{dx}(v)$$

Since,

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) = v\frac{dv}{dx}$$

We get

$$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right) \tag{1.8}$$

From (1.7) and (1.8), we get

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) + \omega^2 x = 0$$

$$or \;\; \frac{d}{dx}\left(\frac{v^2}{2} + \omega^2\frac{x^2}{2}\right) = 0$$

$$\therefore d(v^2 + \omega^2 x^2) = 0 \tag{1.9}$$

On integrating both the sides, we get

$$v^2 + \omega^2 x^2 = constant\ (C_1) \tag{1.10}$$

We already know that on the two extremes, when the magnitude of the displacement is equal to the amplitude ($x = \pm A$), the kinetic energy or the velocity is zero ($v = 0$). Using this boundary condition in equation (1.10), we can calculate the constant (C_1). Thus, C_1 is given by

$$(0)^2 + \omega^2(\pm A)^2 = C_1$$

$$or\ \ C_1 = \omega^2 A^2$$

Using this value in equation (1.10) and rearranging the terms, we get

$$v^2 = \omega^2(A^2 - x^2)$$

$$or\ \ \ v = \pm\omega\sqrt{(A^2 - x^2)} \tag{1.11}$$

The above relation is the expression for velocity of a particle executing SHM. We can see how the velocity has a maximum magnitude at $x = 0$ or in other words, the mean position. From (1.11), the maximum velocity is given by

$$|v|_{max} = \omega A \tag{1.12}$$

Example 4: A 50 g mass vibrates in SHM at the end of a spring. The amplitude of the motion is 12 cm and the period is 0.1 minutes. Find the maximum speed of the mass. What will be the speed at$x = A/2$?

Solution:

$$\omega = 2\pi\nu = 2\pi\left(\frac{1}{0.1 \times 60\ s}\right) = 1.047\ rad\ s^{-1}$$

$$\therefore |v|_{max} = \omega A = (1.047\ rad\ s^{-1})(12 \times 10^{-2}\ m)$$

$$= 0.1256\ m/s$$

From equation (1.11), we get

$$|v| = \omega\sqrt{A^2 - \left(\frac{A}{2}\right)^2} = \frac{3}{4}\omega A$$

$$= \frac{3}{4}(1.047\ rad\ s^{-1})(12 \times 10^{-2}\ m) = 0.0942\ m/s$$

Self Assessment Question (SAQ) 11: In the above question, calculate the speed at $x = 1\ cm$.

Self Assessment Question (SAQ) 12: In the above question, at what location will the speed of the vibrating mass be 5 cm/s?

Now, we will determine the expression for the displacement of a particle executing SHM. From (1.11), we get

$$\frac{dx}{dt} = \pm\omega\sqrt{(A^2 - x^2)}$$

Rearranging the terms, we get

$$\pm\frac{dx}{\sqrt{(A^2 - x^2)}} = \omega dt$$

On integrating both the sides, we get corresponding to the $(+)$ sign

$$\sin^{-1}\frac{x}{A} = \omega t + \delta_1$$

And, corresponding to the $(-)$ sign

$$\cos^{-1}\frac{x}{A} = \omega t + \delta_2$$

where δ_1 and δ_2 are dimensionless constants.

Therefore, we can see that the SHM is defined by a sinusoidal curve

$$x(t) = A\sin(\omega t + \delta) \tag{1.13}$$

Depending on the value of constant δ and ωt the displacement from the equilibrium position and velocity of the SHM at any instant can be determined.

1.5.2 Angular Frequency of SHM

We know that the displacement $x(t)$ should return to its initial value after one time period T of the motion. Or

$$x(t) = x(t + T)$$

We also know from trigonometry that the sine or cosine function repeats itself when its argument has increased by $2\pi\ rad$. Thus,

$$\omega(t + T) = \omega t + 2\pi$$

Or, we get

$$\omega = \frac{2\pi}{T} = 2\pi\nu \tag{1.14}$$

The quantity ω is therefore, the angular frequency that we defined earlier. Its SI unit is $rad\ s^{-1}$.

From equation (1.6), we know that

$$\omega^2 = \frac{k}{m}$$

$$\therefore \ \omega = \sqrt{\frac{k}{m}} \tag{1.15}$$

Example 5: A particle of mass 0.2 kg undergoes SHM according to the equation: $x(t) = 3\sin(\pi t + \pi/4)$. [*t* is in s and *x* in m]

(a) What is the amplitude of oscillation?
(b) What is the time period of oscillation?
(c) What is the initial value of *x*?
(d) What is the initial velocity when the SHM starts?
(e) At what instants is the particle's energy purely kinetic?

Solution:

(a) Comparing the given equation with $x(t) = A\sin(\omega t + \delta)$, we get the amplitude, $A = 3\ m$.

(b) On comparing, we get $\omega = \pi\ rad\ s^{-1}$. Therefore, from (1.14), we get the time period as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\ s$$

(c) Initial conditions are at $t = 0$

$$x(0) = 3\sin(\pi/4) = 1.5\sqrt{2}\ m$$

(d)

$$\frac{dx}{dt} = v(t) = 3\pi\sin(\pi t + \pi/4)$$

$$v(0) = 3\pi\sin(\pi/4) = \frac{3\pi}{\sqrt{2}}\ m/s$$

(e) The energy is purely kinetic when the particle is at the mean position, i.e. when $x(t) = 0$. Or

$$0 = 3\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$\therefore \pi t + \frac{\pi}{4} = 0, \pi, 2\pi, 3\pi, \ldots$$

$$i.e.\ \ t = -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \ldots$$

Rejecting the negative value of *t*, we get *t* = 3/4, 7/4, 11/4… At these instants, the particle crosses origin and hence its energy is purely kinetic.

Self Assessment Question (SAQ) 13: How are the following characteristics of SHM affected by doubling the amplitude? Explain.

(a) Time period, and (b) maximum velocity.

Self Assessment Question (SAQ) 14: Choose the correct option:

Which of the following functions represent SHM?

(a) $\sin(2\omega t)$ (b) $\sin^{-1}\omega t$ (c) $\sin(\omega t) + 2\cos(\omega t)$ (d) $\sin(\omega t) + \cos(2\omega t)$

1.6 DIFFERENT KINDS OF SPRING-MASS SYSTEM

We have already talked about spring-mass system in the previous sections. Now, we are going to discuss the different spring-mass systems.

1.6.1 Horizontal Oscillations

We have already talked about the horizontal oscillations in section 1.3.2. Here a weightless spring is fixed rigidly to a wall on one end and the other end is attached to a body which is free to move on a frictionless horizontal surface. When the body is pulled in one direction through a small distance, the spring is stretched from its earlier relaxed configuration. As a result a restoring force proportional to the magnitude of displacement is exerted on the body and it starts executing SHM.

We have also determined that the angular frequency of SHM is given by

$$\omega = \sqrt{\frac{k}{m}}$$

And the time period is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The displacement x as a function of time is of the form

$$x(t) = A\sin(\omega t + \delta)$$

1.6.2 Vertical Oscillations

Let us now consider the oscillations of a vertical spring-mass system. Consider a weightless spring having spring constant, k suspended vertically from a fixed point, supporting an object of mass *m*, at its lower end, as shown in Fig. 6.

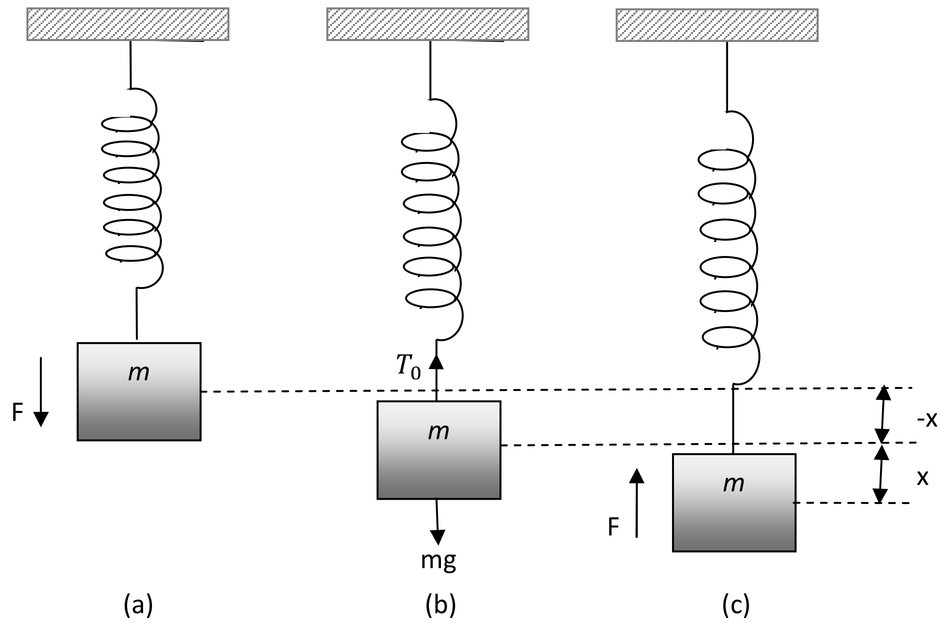


Figure 6: The oscillations of a loaded spring; (a) compressed, (b) equilibrium, (c) stretched configurations.

When the object is at rest initially at its equilibrium position (Fig. 6b), the tension in the spring at equilibrium, T_0, is equal and opposite to the weight, $F = mg$, of the object, where g is the acceleration due to gravity ($= 9.8\ ms^{-2}$). Hence, no net force is acting on the object and by force balance, we can write

$$T_0 = mg \tag{1.16}$$

When the object is oscillating, the tension in the spring changes as the length of the spring changes. When the object is at small displacement x from the equilibrium position, the restoring force is provided by the change of tension from equilibrium, ΔT. Applying Hooke's law to this change of tension gives

$$\Delta T = -kx \tag{1.17}$$

where the $(-)$ sign signifies that the change of tension is in the opposite direction to the displacement and hence, a loaded spring is also said to oscillate with simple harmonic motion.

The angular frequency is given by

$$\omega = \sqrt{\frac{k}{m}}$$

And the time period by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

which is the same as that for horizontal oscillations?

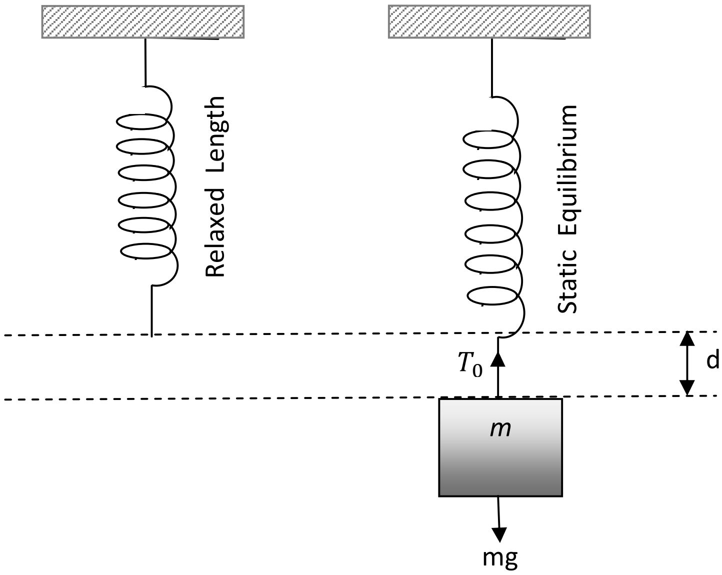


Figure 7: Relaxed spring hanging from the ceiling and a loaded spring.

The spring constant k in the vertical oscillations is determined with the help of equation (1.16)

$$T_0 = mg$$

Here, the tension T_0 stretches the spring and is given by

$$T_0 = -\mathrm{kd}$$

where d is the difference of the relaxed length and the equilibrium length of the spring as shown in Fig. 7. Therefore, we get

$$k = \frac{F}{d} \tag{1.18}$$

where F is the weight of the object, which in equal to mg,

Example 6: A copper spring suspended from a fixed point supports a scale pan of mass 0.05 kg at equilibrium. The scale pan descends 40 mm to a new equilibrium position when a 1 N weight is placed on it. Calculate the

(a) spring constant,
(b) the total mass of the scale pan and the 1 N weight. [g can be taken as 10 ms^{-2}]
(c) The scale pan, with 1 N weight on it, is pulled a distance of 15 mm downwards from equilibrium and then released. Calculate the time period of the oscillations, and
(d) the maximum speed of the scale pan.

Solution:

(a) From equation (1.18),

$$k = \frac{F}{d} = \frac{1}{0.04} = 25\ N/m$$

(b) Mass of 1 N weight = weight/g = 0.1 kg. Therefore, the total mass = 0.1 + 0.05 = 0.15 kg.

(c) The time period is given as

$$T = 2\pi\sqrt{\frac{0.15}{25}} = 0.49\ s$$

(d) The amplitude of the oscillations = 15 mm = 0.015 m. Therefore,

$$|v|_{max} = \omega A = \left(\frac{2\pi}{0.49\ s}\right)(0.015\ m) = 0.195\ m/s$$

Example 7: A block of mass 0.2 kg which slides without friction on a $\theta = 30^0$ incline is connected to the top of the incline by a mass-less spring of relaxed length of 23.75 cm and spring constant 80 N/m as shown in the following figure.

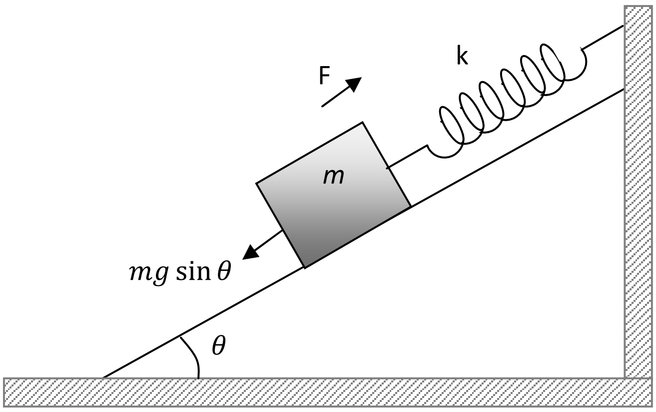


Figure 8: Spring pendulum on an incline.

(a) How far from the top of the incline does the block stop?

(b) If the block is pulled slightly down the incline and released, what is the period of the ensuing oscillations? [g can be taken as 10 ms^{-2}]

Solution:

(a) At static equilibrium, using force balance along the incline, where *d* is the difference of the relaxed length and the equilibrium length of the spring

$$mg\sin\theta = -\mathrm{kd}$$

$$\therefore d = -\frac{mg\sin\theta}{k} = -\frac{2\left(\frac{1}{2}\right)}{80} \times 100\ cm = -1.25\ cm$$

The negative sign is just an indicative of the direction. Now, the position of the mass from the top is the sum of the relaxed length of the spring and d, which is equal to

$$= 23.75 + 1.25 = 25\ cm$$

(b) The time period for a spring pendulum is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2\ kg}{80\ N/m}} = \frac{\pi}{10}\ s$$

Self Assessment Question (SAQ) 15: A steel spring, suspended from a fixed point, supports a 0.2 kg stone hung from its lower end. The stone is displaced downwards from its equilibrium position by a distance of 25 mm and then released. The time for 20 oscillations is measured as 22 s. Calculate (a) its time period, (b) its angular frequency, (c) its maximum speed, (d) the maximum tension in the spring.

1.6.3 Composite Spring-Mass System

Now we shall study some important composite spring-mass system. The first one we consider is the one that was given to you as a challenge question (SAQ 8). Let's try to determine the general solution to the problem. Here, as shown in Fig. 9, there are two springs with spring constants k_1 and k_2, which are attached to the rigid wall on one end and to the block of mass *m* on the other

.

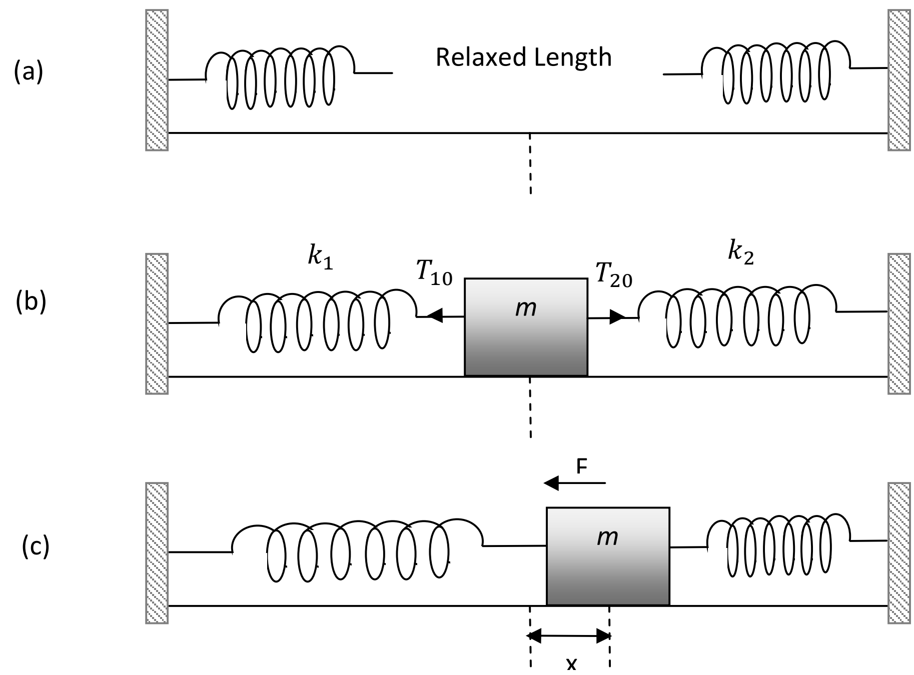


Figure 9: (a) Relaxed springs attached to the rigid wall on one side, (b) the spring-mass system at static equilibrium, (c) oscillating spring-mass system.

If the value of the spring constant of one of the springs is not known, it can be determined by balancing the forces. Let d_1 and d_2 be the lengths by which the two springs are stretched out when the composite spring-mass system is in static equilibrium (Fig.9). In this condition, the tensions in the two springs will balance each other. Thus, we can write

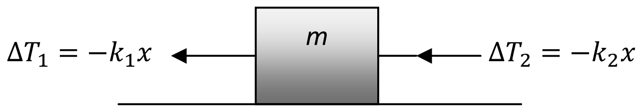
$$T_{10} = T_{20}$$

$$or \;\; k_1 d_1 = k_2 d_2 \tag{1.19}$$

When the block is pulled in the right direction by a small distance *x* and then released, the system executes SHM. The restoring force *F*, that the block experiences is given by the sum of the change in tension in the two springs, because the spring on the left is stretched and so it tries to pull the block in the left direction, and the spring on the right is compressed so it also tries to push the block toward left. Hence, we see that both the springs try to force the block in the same direction (see Fig. 10).

Thus, we can write, using Newton's second law,

$$F = \Delta T_1 + \Delta T_2$$



$$\therefore F = -k_1 x + -k_2 x$$

$$= -(k_1 + k_2)x$$

Thus, we can say that this composite spring-mass system too behaves like the simple horizontal spring-mass system. The net spring constant of the composite system is given by the sum of the two individual spring constants:

$$k = k_1 + k_2 \qquad (1.20)$$

Now, you can again try to solve SAQ 8, if you could not solve it earlier.

Example 8: The spring constants (k_1 and k_2) of both the springs is 50 Nm^{-1} and the block of mass 1 kg is displaced by 0.01 m to the right from its mean position before being released. In the equilibrium position of the system, the springs are stretched by 0.02 m each. Calculate the

(a) restoring force at t = 0,
(b) restoring force when the block travels to the other extreme, and
(c) the restoring force in the static equilibrium position.

Solution:

(a) The restoring force $F = -(k_1 + k_2)x$
$= -(50 + 50\ Nm^{-1})(0.01\ m) = -1\ N$
(b) When the block travels to the left extreme, $x = -0.01\ m$.
Therefore, $F = -(50 + 50\ Nm^{-1})(-0.01\ m) = 1\ N$
(c) In the static equilibrium position, there is no restoring force!

Now let us look at another composite system shown in Fig. 10 which is similar to the one discussed above. You may like to know how will the treatment change if the two springs are attached to the rigid wall on the same side.

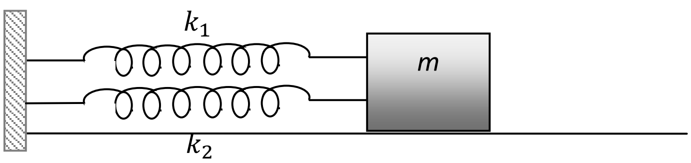


Figure 10: Composite spring – mass system with the two springs attached to the same wall.

Let us see what happens if we displace the mass m by a small distance x towards the right. This will cause both the springs to get stretched from their equilibrium lengths and hence try to pull the mass back toward left. Therefore, the total restoring force experienced by the block is given by

$$F = (-k_1 x) + (-k_2 x)$$

$$\therefore F = -(k_1 + k_2)x$$

Hence, just like the previous case, the net spring constant of the composite system is given by the sum of the two individual spring constants. i.e.

$$k = k_1 + k_2 \quad (1.21)$$

Example 9: A weightless spring whose spring constant is 100 Nm^{-1} is cut into two halves.

(a) What is the spring constant of each half?
(b) The two halves suspended separately support a block of mass *m*. If the system vibrates at a frequency of $(10/\pi)$ Hz, find the value of mass m.

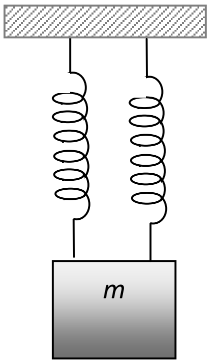


Figure 11: A mass m suspended by two springs which are attached to the ceiling.

Solution:

(a) From equation (1.2), we know that

$$F = \left(\frac{Y\alpha}{L}\right)x$$

Therefore, we get that the spring constant is inversely proportional to the length of the spring.

$$k\ (spring\ constant) \propto \frac{1}{L}$$

Thus, when the length is reduced to half, the value of the spring constant must double. Hence, the spring constant of each of the half spring is$2 \times 100\ Nm^{-1} = 200\ Nm^{-1}$.

(b) The net spring constant of the composite spring-mass system shown above will be $k = k_1 + k_2 = 200 + 200 = 400\ Nm^{-1}$. The frequency is given by

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Therefore,

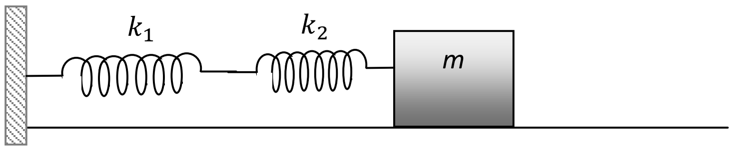
$$\frac{10}{\pi} = \frac{1}{2\pi}\sqrt{\frac{400}{m}}$$

Or,

$$m = 1\ kg$$

Self Assessment Question (SAQ) 16: In the above example, what will be the net spring constant if, instead of cutting the spring into two half, it is cut into three pieces? Also calculate the mass *m* if the system vibrates with the same frequency.

There is yet another case of composite spring-mass system, where the springs are joined in series as shown in Fig.12 below.



If the block is displaced to the right by *x*, the two springs will be stretched differently depending upon their respective spring constants. However, the force exerted by the two springs on each other will be same. Thus, if the two springs are stretched by x_1 and x_2, we can write

$$k_2 x_2 = k_1 x_1$$

Or,

$$x_1 = \frac{k_2}{k_1} x_2$$

Since the sum of the stretch in each of the two springs should be equal to the distance, *x* by which the block has been displaced, we have

$$x = x_1 + x_2 = x_2\left(1 + \frac{k_2}{k_1}\right)$$

or,

$$x_2 = \frac{x}{\left(1 + \frac{k_2}{k_1}\right)}$$

Now, the restoring force experienced by the block will be equal to the tension in spring 2,

$$F = -k_2 x_2 = -k_2 \frac{x}{\left(1 + \frac{k_2}{k_1}\right)}$$

$$= -\frac{1}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)} x$$

Therefore, the net spring constant when the springs are in series is

$$k = \frac{1}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}$$

Or,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \qquad (1.22)$$

Self Assessment Question (SAQ) 17: Calculate the time period of composite spring-mass system when the springs are joined in series?

1.7 SUMMARY

In this unit, we have studied about what is meant by the periodic motion, the oscillatory motion and what are the conditions and basic characteristics of SHM. We studied about the restoring force that comes in to play due to the displacement from the mean or the equilibrium position and how the restoring force is proportional to the magnitude of the displacement in case of SHM. We studied the two simple systems, simple pendulum and spring-mass system, which are both examples of SHM. We are also aware now that the two properties of inertia and elasticity are responsible for oscillation of a physical system.

Using the knowledge of Newton's second law of motion, we wrote the equation of motion for SHM and derived the solution of the differential equation used to describe SHM. We also calculated the time period and angular frequency of spring-mass system, including for some composite spring-mass systems.

1.8 GLOSSARY

Displacement – net change in location of a moving body; in case of SHM, it is measured from the equilibrium position.

Elasticity – ability of a material to regain its shape after being distorted.

Elastic Limit – the maximum limit up to which a solid material can be stretched without a permanently altering its length, shape and size.

Force – anything that can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a vibrating object.

Hooke's Law – the extension of a spring is proportional to the tension in the spring.

Inertia – the tendency of a physical object to remain still or to continue moving, unless a force is applied to it.

Microwaves – electromagnetic waves of wavelength between about 0.1 mm and 10 mm.

Radio waves – electromagnetic waves of wavelength longer than about a millimeter.

Sound – vibrations in a substance that travel through the substance.

Stiffness – a measure of the force needed to change the shape of an object.

Tension – the force in an object that has been stretched.

Velocity – speed in a given direction.

Wavelength – the distance between two adjacent wave-crests.

Weight – the force of gravity on an object. For a mass, m, its weight = mg.

X-rays – electromagnetic waves of wavelength less than about 1 nm.

Young's Modulus of elasticity – Stress divided by strain.

1.9 TERMINAL QUESTIONS

1. A horizontal spring-mass system of spring constant k and mass M executes SHM with frequency ν. When the block is passing through its equilibrium position, an object of mass m is put on it and the two move together. Find the new frequency of vibration.

2. A particle executes SHM with amplitude of 0.5 cm and frequency of 100 s^{-1}. What is the maximum speed of the particle?

3. A weight suspended from a spring oscillates up and down. The restoring force in the weight is zero at (a) highest point, (b) lowest point, (c) middle point, (d) none of these.

4. A person goes to bed at sharp 10:00 pm every day. Is it an example of periodic motion? If yes, what is the time period? If no, why?

5. In the above question, is it an example of SHM? If yes, why?

6. A particle moves on the x-axis according to the equation $x = A + B\sin\omega t$. Is the motion SHM? If yes, what is the amplitude?

7. The displacement of a particle in SHM in one time period is

(a) A, (b) 2A, (c) 4A, (d) zero.

8. The distance moved by a particle in SHM in one time period is

(a) A, (b) 2A, (c) 4A, (d) zero.

9. The distance moved by a particle in SHM in half time period is

(a) A, (b) 2A, (c) 4A, (d) zero.

10. Mention the differences among periodic motion, oscillatory motion and SHM.

11. Select the correct statement(s). More than one choice may be correct.

(a) A simple harmonic motion is necessarily periodic.
(b) A simple harmonic motion is necessarily oscillatory.
(c) An oscillatory motion is necessarily periodic.
(d) A periodic motion is necessarily oscillatory.

12. Write notes on:

(i) SHM (ii) Spring-Mass System (iii) Time period (iv) Angular Frequency

1.10 ANSWERS

Selected Self Assessment Questions (SAQs):

4. (a)

7. No.

10. $\omega = 2\pi/60\ rad/s$ and $T = 1/30\ s$

13. Period remains unchanged. Maximum velocity is doubled.

14. (a)

15. Hint: The maximum tension in the spring will be when it is stretched to the extreme, which is equal to the sum of the difference of the relaxed length and the equilibrium length of the spring, and the amplitude of the oscillations; i.e. $d + A = (T_0 + \Delta T)/k$.

16. $k = 300\ Nm^{-1}$

17. The time period for such a system is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{m(\frac{1}{k_1} + \frac{1}{k_2})}$$

Selected Terminal Questions:

1. Original frequency of SHM,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

The new frequency of SHM,

$$\nu_{new} = \frac{1}{2\pi}\sqrt{\frac{k}{m+M}}$$

Therefore,

$$\nu_{new} = \nu\sqrt{\frac{M}{m+M}}$$

2. $|v|_{max} = \omega A = (2\pi \times 100)(0.5 \times 10^{-2}) = \pi\ m/s$

3. (c) because at the equilibrium or mean position the restoring force is zero.

4. Yes. Time period = 24 hours.

5. No. SHM is a special case of oscillatory motion, where a body moves back and forth repeatedly about a fixed position. Here nothing like that happens!

6. Yes. Amplitude = A + B.

7. (d)

8. (c)

9. (b)

11. (a), (b)

1.11 REFERENCES

1. Concepts of Physics, Part I, H C Verma – Bharati Bhawan, Patna

2. The Physics of Waves and Oscillations, N K Bajaj – Tata McGraw-Hill, New Delhi
3. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
4. Physics, Jim Breithaupt – Palgrave

1.12 SUGGESTED READINGS

1. Concepts of Physics, Part I, H C Verma – Bharati Bhawan, Patna
2. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
3. Berkeley Physics Course Vol 3, Waves, C Kittel et al, McGraw- Hill Company

UNIT 2 ENERGY IN SIMPLE HARMONIC MOTION

Structure

2.1 INTRODUCTION

Let us start with revising what we learned in Unit 1. We defined SHM as oscillatory motion of an object in which the restoring force is proportional to the magnitude of the displacement and is always in a direction opposite to the displacement from the equilibrium or the mean position. We also know that the net force at the mean position is zero.

We then learned how to calculate the time period for spring-mass system executing simple harmonic motion. The time interval after which the periodic motion is repeated (i.e. the position and the velocity of the moving body is the same) is called its time period. It is worth noting here that the time period, T, is independent of the amplitude of SHM.

We also solved the differential equation for SHM and obtained its solution. The displacement of an object executing SHM is given by the expression: $x(t) = A\sin(\omega t + \delta)$. We shall explore the implications of this in detail, in this unit. We shall also obtain the expression for the energy of the system executing SHM.

2.2 OBJECTIVES

After studying this unit, you should be able to

- understand that the oscillatory motion described by a sinusoidal curve is called SHM
- explain what is meant by phase of an oscillator
- explain what is the phase constant or the phase angle
- write down the expressions for velocity and acceleration for SHM
- describe how the acceleration, velocity and displacement of an oscillating system change with time and how they are related to each other
- calculate the potential energy and kinetic energy of an oscillator executing SHM

2.3 PHASE OF AN OSCILLATOR EXECUTING SHM

The word phase is synonymous to what we call "state;". For example, in thermodynamics, phase of a substance refers to the solid or liquid or vapor or plasma state of the substance. You may recall from Unit 1 that the equation of motion of a system executing SHM is a second order differential equation. To obtain the expression for the displacement of the oscillator, we integrate the differential equation twice and obtain the value of the integration constant by making use of the initial and the boundary conditions. The first integration gives us the expression for velocity and the second integration gives us the expression for displacement:

$$x(t) = A\sin(\omega t + \delta)$$

In the above expression, as the argument of the sine function - time-varying quantity $(\omega t + \delta)$ - changes, the position also changes. In other words, this quantity actually specify the 'state' of the oscillator at that instant. This quantity is called the phase of the SHM.

If the phase $\varphi = (\omega t + \delta)$ is zero at a certain instant, we have

$$x(t) = A\sin(0) = 0$$

And

$$v(t) = \frac{dx}{dt} = A\omega\cos(0) = A\omega$$

This means that the oscillating body is crossing the mean position and is going towards the positive direction. If the phase is $\pi/2$, we get $x(t) = A$ and $v(t) = 0$ so that the oscillating body is at the positive extreme position. The following figure shows the state of the oscillating body at different phases.

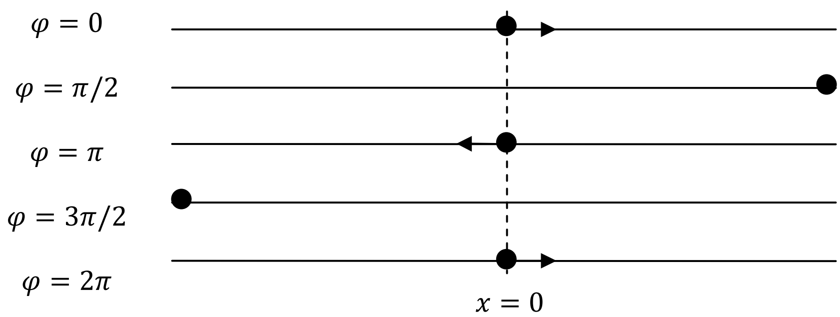


Figure 12: State of an oscillating body at different phases.

We can see that as the time increases, the phase increases. An increase of 2π brings the body to the same state of motion. Thus, a phase of $(\omega t + \delta)$ is equivalent to $(\omega t + \delta + 2\pi)$. Similarly, a phase change of $4\pi, 6\pi, 8\pi$ ... are equivalent to no phase change.

2.3.1 Phase Constant

The constant of integration δ appearing in the expression for displacement $x(t) = A\sin(\omega t + \delta)$ is called the phase constant or the phase angle. The value of δ depends on the initial condition, i.e. the displacement and the velocity of the oscillating body at $t = 0$.

In order to describe SHM quantitatively, a particular instant has to be assigned $t = 0$ and measurement of time should begin from this instant. This instant can be chosen according to the convenience. Suppose we choose $t = 0$ at an instant when the oscillating body is passing through its mean position and is going towards the positive direction. So, at this instant, the displacement is zero and hence the phase $\varphi = (\omega t + \delta)$ should be zero if we represent the SHM by the sine function. As $t = 0$, this means the phase constant δ is zero. The equation for displacement can then be written as

$$x(t) = A\sin(\omega t)$$

If we choose $t = 0$ at an instant when the oscillating body is at its positive extreme position, the phase angle is $\pi/2$ at this instant. Thus, the phase $\varphi = (\omega t + \delta)$ is $\pi/2$ and hence, $\delta = \pi/2$. The equation for the displacement is therefore given by

$$x(t) = A\sin(\omega t + \pi/2\,) = A\cos(\omega t)$$

As any instant can be chosen as $t = 0$, the phase constant δ can be chosen arbitrarily. Sometimes, we may have to consider two or more simple harmonic motions together. The phase constant δ of any one of them can be chosen as zero. The phase constants of the rest of them will be determined by the actual situation.

Example 1: Show that the sine and the cosine functions describing the displacement of the oscillating body executing SHM are equivalent.

Solution: The general expression for displacement is given by

$$x(t) = A\sin(\omega t + \delta)$$

Defining another arbitrary constant δ_1 such that $(\pi/2 + \delta_1) = \delta$, the above expression may be written as

$$x(t) = A\sin(\omega t + \pi/2 + \delta_1) = A\cos(\omega t + \delta_1)$$

Therefore, we can say that the sine and the cosine forms are equivalent. The value of phase constant, however, depends on the form chosen.

Example 2: A particle starts at $t = 0$ from the mean position with a velocity $v = 3\pi\ m/s$ in the positive direction. If the time period of the oscillation is 2 sec., write the expression for the displacement of the particle.

(a) What minimum time does the particle take to go from mean position to a point P, which lies midway between the mean position and the right extreme position?
(b) What minimum time does the particle take to reach the right extreme position from the mean position?

Solution: From equation (1.14), we know that

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\,s} = \pi\ rad\ s^{-1}$$

Let $x(t) = A\sin(\omega t + \delta)$ be the expression for the displacement of the particle executing SHMon. Therefore, the velocity is given by

$$\frac{dx}{dt} = v(t) = A\omega\cos(\omega t + \delta)$$

Applying the initial conditions, at $t = 0$

$$x(0) = 0\,m;\;\; v(0) = 3\pi\,m/s$$

On the above expressions for displacement and velocity, we get

$$0 = A\sin(\delta)$$

$$\therefore \delta = 0, \pi$$

And

$$3\pi = A\omega\cos(\delta)$$

Hence, $\delta = 0$ is possible but $\delta = \pi$ is not possible. Therefore, $\delta = 0$ is a possible solution. Substituting it in the above equation, we get

$$3\pi = A\omega\cos(0) = A\omega$$

$$A = \frac{3\pi}{(\pi\,rad\,s^{-1})} = 3\,m$$

Therefore, the equation of motion is $x(t) = 3\sin(\pi t)$

(a) The particle is at the mean position at $t = 0$. Let us assume that the particle reaches the point P (midway between the mean position and the right extreme) from its mean position in time t. Thus, we have, $x(t) = A/2 = 1.5\,m.$
Thus, $1.5 = 3\sin(\pi t)$ or t = 1/6 s.
(b) $x(t) = A = 3\,m$. Thus, $\sin(\pi t) = 1$ or t = 0.5 s.

Self Assessment Question (SAQ) 1: If a particle executing SHM starts at $t = 0$ from the right extreme, what is its equation?

Self Assessment Question (SAQ) 2: If a particle executing SHM starts at $t = 0$ from the left extreme, what is its equation?

Self Assessment Question (SAQ) 3: If a particle executing SHM starts at $t = 0$ from the mean position and the initial velocity is positive, what is its equation?

Self Assessment Question (SAQ) 4: If a particle executing SHM starts at $t = 0$ from the mean position and the initial velocity is negative, what is its equation?

2.3.2 SHM as Projection of Circular Motion

Consider a particle P moving on a circular path of radius r as shown in Fig. 2. We can write the coordinates of point P as $x = r\cos\theta$ and $y = r\sin\theta$, where r is the radius of the circle and θ is the angle between the line OP and the x-axis.

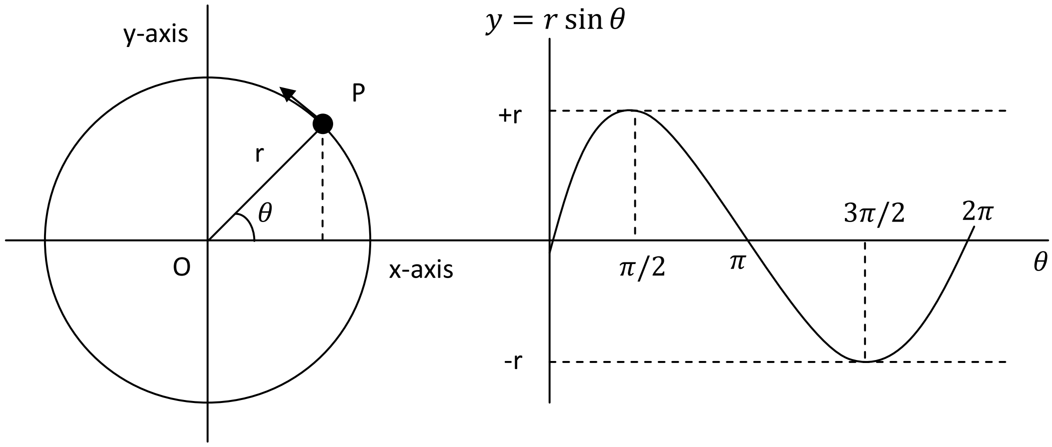


Figure 13: A particle P moving on a circular path. Its projection on the diameter generates a sinusoidal curve.

$x = r\cos\theta\, y = r\sin\theta\, r\theta$. From Fig. 2, you may note that, as the particle P moves along the circular path,, its y-coordinate $(= r\sin\theta)$ changes because θ changes from 0 to 2π. Thus, we can see that the y-coordinate of the particle P executes SHM. Similarly, you should convince yourself that the x-coordinate $(= r\cos\theta)$ of the particle P will also execute SHM. However, the phases of the two harmonic motions differ by $\pi/2$ as $\cos\theta = \sin(\theta + \pi/2)$.

Therefore, the projection of a uniform circular motion on a diameter of the circle is a simple harmonic motion. This representation **of SHM** is known as the **rotating vector representation** or **phasor model**.

2.4 VELOCITY AND ACCELERATION IN SHM

From your school mathematics, you may recall that, if we know the expression for the displacement of a particle, we can obtain expressions for its velocity and acceleration using differential calculus. In the previous Section, we obtained the expression for the velocity of the particle executing SHM by differentiating the expression for dixplacement, x(t):

$$x(t) = A\sin(\omega t + \delta)$$

$$\frac{dx}{dt} = v(t) = A\omega\cos(\omega t + \delta)$$

From the above expression, you may note that the amplitude of velocity or maximum velocity is given by

$$|v|_{max} = \omega A$$

In Unit 1, we obtained an expression for the velocity in terms of displacement and other parameters of SHM:

$$v(x) = \pm\omega\sqrt{(A^2 - x^2)}$$

To show that the above two expressions for the velocity are equivqlent, we write $x(t) = A\sin(\omega t + \delta)$:

$$v = \pm\omega\sqrt{A^2\{1 - \sin^2(\omega t + \delta)\}}$$

From basic trigonometry, we know that $1 - \sin^2\theta = \cos^2\theta$ is an identity. Therefore, we get $v = \pm A\omega\cos(\omega t + \delta)$.

Further, to obtain the expression for the acceleration of the particle executing SHM, we shall differentiate the expression for displacement twice, i.e.

$$\frac{d^2x(t)}{dt^2} = a(t)$$

$$or \quad \frac{d^2\{A\sin(\omega t + \delta)\}}{dt^2} = a(t)$$

$$\therefore a(t) = -A\omega^2\sin(\omega t + \delta) \tag{2.1}$$

The acceleration can also be expressed in terms of the displacement of the particle, i.e.

$$a(t) = -A\omega^2 x(t) \tag{2.2}$$

From equation (2.2), you may note that the acceleration in SHM is always directed towards the mean position. The magnitude of acceleration is minimum at the mean position and maximum at the extremes.

$$\begin{aligned} |a|_{min} &= 0 \quad at\ mean\ position \\ |a|_{max} &= \omega^2 A \quad at\ the\ two\ extremes \end{aligned} \tag{2.3}$$

Using the expression for acceleration, we can determine the restoring force acting on the oscillating object:

$$F(t) = ma(t) = -mA\omega^2\sin(\omega t + \delta) \tag{2.4}$$

At any position x, it is given by

$$F(x) = -m\omega^2 x \qquad (2.5)$$

We also know that $F = -kx$ in case of spring-mass system. Comparing it with equation (2.5) gives us the familiar expression for angular velocity

$$\omega = \sqrt{\frac{k}{m}}$$

Example 3: A horizontal platform executes SHM in a vertical line with time period of $\pi\ s$ and an amplitude of 0.5 m. A book of mass 2 kg is placed on the platform and oscillates with it. Calculate the greatest and least values of the force exerted by the book on the platform.

Solution: The angular frequency of oscillation of the horizontal platform is given by

$$\omega = \frac{2\pi}{T} = 2\ rad/s$$

For any particle executing SHM, the net force acting at a distance x from the mean position must be

$$F(x) = -m\omega^2 x$$

There are only two forces acting on the book: its weight and normal reaction. The net effect of these two forces must be towards the mean position.

At the mean position, there is no net force and hence, the normal reaction equals the weight (mg). Above the mean position, the normal reaction is less than the weight (mg) and below the mean position, the normal reaction is greater than the weight (mg). See figure below.

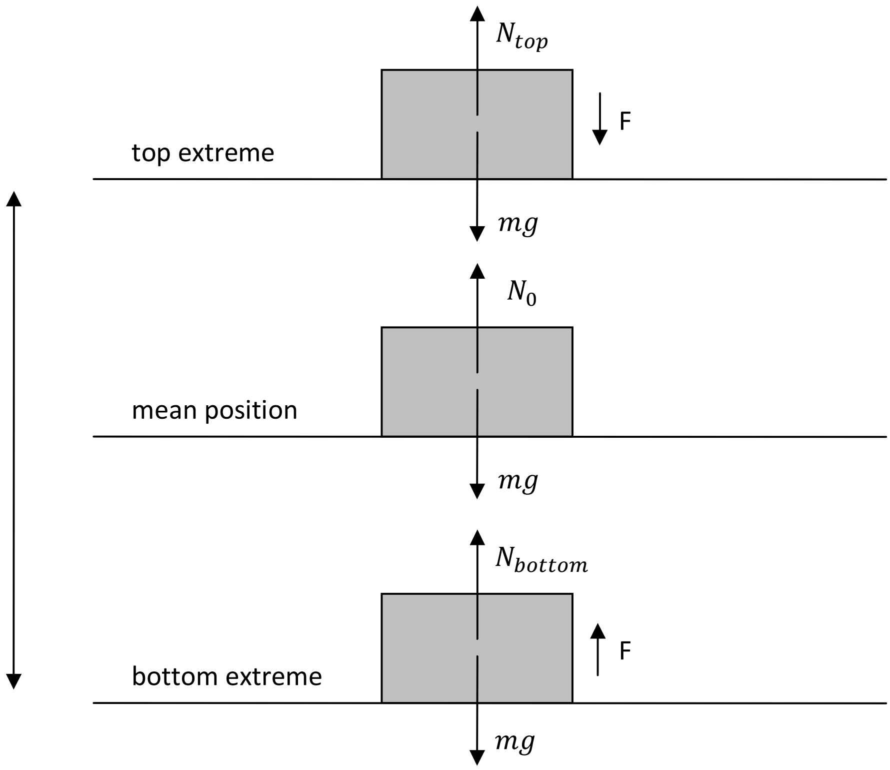


Figure 14: Force acting on the book placed on an oscillating platform.

At the top extreme: N is minimum at the top extreme.

The net force towards mean position is

$$F(x) = -m\omega^2 A$$

By balancing the forces, we get

$$N_{top} = N_{min} = mg - m\omega^2 A$$

$$= 2 \times 9.8 - 2(2)^2(0.5) = 15.6\ N$$

At the bottom extreme: N is maximum at the bottom.

The net force towards mean position is

$$F(x) = -m\omega^2(-A) = m\omega^2 A$$

By balancing the forces, we get

$$N_{bottom} = N_{max} = mg + m\omega^2 A$$

$$= 2 \times 9.8 + 2(2)^2(0.5) = 23.6\ N$$

Self Assessment Question (SAQ) 5: In the above example, what do you think is the condition for the book to maintain contact with the platform?

2.5 TRANSFORMATION OF ENERGIES IN SHM

While discussing the motion of simple oscillatory systems, we discovered that the energy of the oscillation alternates between potential and kinetic forms; the potential energy being minimum at the mean position and maximum at the extremities. On the other hand, the kinetic energy is maximum at the mean position and minimum at the extremities. While the sum of potential energy (U) and kinetic energy (K), which is the total mechanical energy (E) of the oscillator, remains constant.

Let us now derive an expressions for the potential, kinetic and total mechanical energy in SHM.

2.5.1 Potential Energy

We shall derive the elastic potential energy of the simple spring – mass system that we studied in Unit 1. The value of the elastic potential energy of the spring-mass system depends entirely on how much the spring is stretched or compressed, i.e. the displacement $x(t)$ of the mass from its equilibrium position$x(t)$Further, the elastic potential energy dU gained by the system is equal to the work done against the force in moving it through a distance *dx*. In other words,

$$dU = -F(x)dx \qquad (2.6)$$

Replacing $F(x) = -m\omega^2 x$ in the above equation, we get

$$dU = m\omega^2 x\ dx$$

Thus, the total elastic potential energy at a point x will be equal to the total work done in moving the oscillator from the mean position ($x = 0$). Therefore, integrating the above expression from 0 to *x*, we get

$$U = m\omega^2 \int_0^x x\ dx$$

$$or\ \ U = \frac{1}{2} m\omega^2 x^2$$

$$\therefore U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \sin^2(\omega t + \delta) \qquad (2.7)$$

Let us also calculate the average potential energy of the spring-mass system over one complete cycle. This can be determined by integrating it over time from 0 to T, i.e. one time period. Thus,

$$\langle U\rangle = \frac{1}{2}kA^2\left[\frac{\int_0^T \sin^2(\omega t+\delta)dt}{\int_0^T dt}\right]$$

$$= \frac{1}{2}kA^2\left[\frac{1}{2}\right]$$

$$\therefore \langle U\rangle = \frac{1}{4}kA^2 \qquad (2.8)$$

2.5.2 Kinetic Energy

The kinetic energy of the spring-mass system is entirely associated with the moving object. Its value depends on how fast the object is moving, that is, on $v(t)$. Hence,

$$K = \frac{1}{2}mv^2$$

$$\therefore K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\delta) = \frac{1}{2}kA^2\cos^2(\omega t+\delta) \qquad (2.9)$$

Therefore, the average kinetic energy, which can be calculated by integrating it over time from 0 to T, i.e. one time period, will be

$$\langle K\rangle = \frac{1}{2}kA^2\left[\frac{\int_0^T \cos^2(\omega t+\delta)\,dt}{\int_0^T dt}\right]$$

$$= \frac{1}{2}kA^2\left[\frac{1}{2}\right]$$

$$\therefore \langle K\rangle = \frac{1}{4}kA^2 \qquad (2.10)$$

Thus, we find that the average potential energy of the spring-mass system is equal to its average kinetic energy.

2.5.3 Total Mechanical Energy

Using equations (2.7) and (2.9), we can determine the total mechanical energy at a particular instant, by summing the potential and the kinetic energies,

$$E = U + K$$

$$= \frac{1}{2}kA^2\sin^2(\omega t+\delta) + \frac{1}{2}kA^2\cos^2(\omega t+\delta)$$

$$= \frac{1}{2}kA^2[\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)]$$

From trigonometry, we know that $\sin^2\theta + \cos^2\theta = 1$ is an identity. Thus,

$$E = \frac{1}{2}kA^2 \qquad (2.11)$$

The total mechanical energy of the oscillator (spring-mass system) is indeed a constant and is independent of time or position.

The potential energy and kinetic energy of a linear oscillator are shown as the function of time in the figure below. Note that all the energies are positive and that the potential energy and the kinetic energy peak twice during every period.

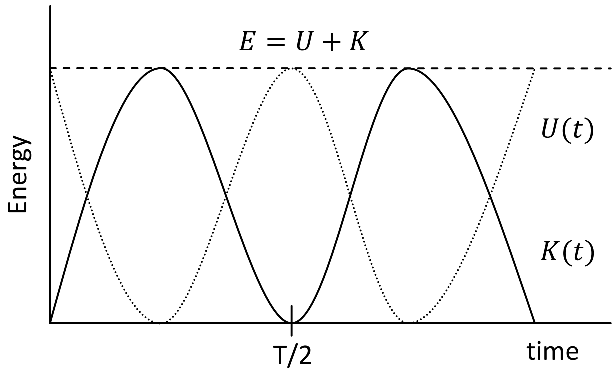


Figure 15: Potential energy, kinetic energy and total energy as functions of time, for SHM.

Next, in Fig. 5, we show the variation of potential energy and kinetic energy of a linear oscillator as the function of displacement. Note that, at $x = 0$, that is, at the mean position, the energy is all kinetic while at the extremities, i.e.at $x = \pm A$, it is all potential.

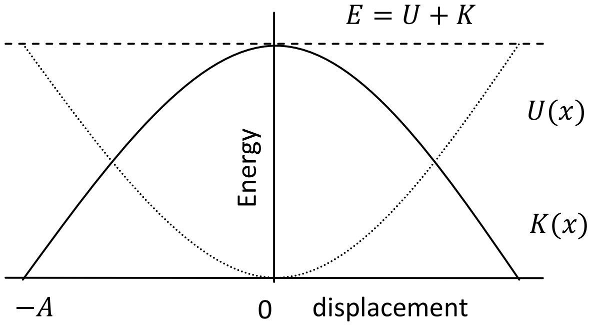


Figure 16: Potential energy, kinetic energy and total energy as functions of position, for SHM.

Example 4: A block, whose mass is *680 g*, is fastened to a spring whose spring constant *k* is *65 N/m*. The block is pulled a distance *x = 11 cm* from its equilibrium position at *x = 0* on a frictionless horizontal surface and released from rest at *t = 0*.

(a) What force does the spring exert on the block just before the block is released?
(b) What are the angular frequency, the frequency, and the period of the resulting oscillation?
(c) What is the amplitude of the oscillation?
(d) What is the maximum speed of the oscillating block?
(e) What is the magnitude of the maximum acceleration of the block?
(f) What is the phase angle for the motion?
(g) What is the total mechanical energy of the oscillator?
(h) What is the potential energy of this oscillator when the block is halfway to its end-point?
(i) What is the kinetic energy of the oscillator when the block is halfway to its end-point?

Solution:

(a) From Hooke's law

$$F = -kx = -(65)(0.11) = -7.2\,N$$

(b) For the givem spring-mass system, the angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65}{0.68}} = 9.78\,rad/s$$

Thus, the frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{9.78}{2\pi} = 1.56\,Hz$$

And the time period is

$$T = \frac{1}{\nu} = \frac{1}{1.56} = 0.64\,s$$

(c) Since the block is released from rest at 11 cm distance from its equilibrium point, the kinetic energy it possesses at this point is zero. We already know that at the position of maximum displacement, the energy is all potential and the kinetic energy is zero. Hence, the amplitude A should be equal to 11 cm or 0.11 m.

(d) The maximum speed is given by

$$|v|_{max} = \omega A = (9.78)(0.11) = 1.1\,m/s$$

(e) The maximum acceleration is when the block is at the ends of its path. At those points the force acting on the block has its maximum magnitude.

$$|a|_{max} = \omega^2 A = (9.78)^2(0.11) = 11\,ms^{-2}$$

(f) At $t = 0$, when the block is released, the displacement of the block has maximum value equal to the amplitude and the velocity of the block is zero. Using these initial conditions, we get

$$1 = \sin\delta$$

And

$$0 = \cos\delta$$

The smallest angle that satisfies both these conditions is $\delta = \pi/2$.

Note: Any angle $(2n\pi + \pi/2)$ *rad, where n is an integer, will also satisfy these conditions.*

(g) We already know that the total energy will be constant.

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(65)(0.11) = 0.393\,J$$

(**Comment**: correct the above equation: it should be $(0.11)^2$)

(h) The potential energy is given by

$$E = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{2}\right)^2$$

$$= \frac{1}{8}kA^2 = \frac{1}{4}E = 0.098\,J$$

(**Comment**: check the arithmetic above; it will change in view of correction in (g) above)

(i) The kinetic energy can be determined by subtracting the potential energy component from the total energy

$$K = E - U$$

$$= 0.393 - 0.098 = 0.295\,J$$

Thus, we see at this position during the oscillation, about 25% of the energy is in the potential form and the rest 75% is in kinetic form.

Self Assessment Question (SAQ) 6: Two simple harmonic motions are represented by the equations $x_1 = 10\sin(3t + \pi/4)$ and $x_2 = 5\cos(9t + \pi/3)$. Their amplitudes are of the ratio ______________.

Self Assessment Question (SAQ) 7: Choose the correct option:

A body is in SHM. The motion is represented graphically. The valid representation of the position will be

(a) A square wave
(b) A straight line
(c) A sinusoidal curve
(d) A $(y = x^2)$ curve
(e) A curve of the form $y = 5|\sin\varphi|$

Self Assessment Question (SAQ) 8: In the previous question, what will be the valid representation of the velocity?

Self Assessment Question (SAQ) 9: In the previous question, what will be the valid representation of the acceleration?

Self Assessment Question (SAQ) 10: Choose the correct option:

For a particle executing SHM, which of the following statements does not hold good?

(b) The total energy of the particle always remains the same.
(c) The restoring force is always directed towards a fixed poin.t
(d) The restoring force is maximum at the extreme positions.
(e) The acceleration of the particle is minimum at the mean position.
(f) The velocity of the particle is minimum at the mean position.

2.6 SUMMARY

In this unit, we studied the concept of phase of the oscillatory motion and what is meant by the phase angle or phase constant. We also learned that the the phase and phase constant for an oscillatory motion, which represents the state of the SHM, can be determined by using the boundary and the initial conditions. We also learned how SHM can be represented as a projection of circular motion on diameter and that the sine and the cosine forms are interchangeable.

Thereafter, we learned how to calculate the velocity and acceleration of a particle executing SHM. We saw that the acceleration in SHM is always opposite to the direction of displacement and that it always points towards the mean position. We then calculated the potential energy, kinetic energy and the total energy of the system. We discussed the variation of these energies with time as well as position and learned that the total energy remains constant throughout and only the kinetic energy and the potential energy of the system varies. At the mean position, the total energy is composed fully of the kinetic energy and at the extremities it is fully formed by the potential energy and the kinetic energy at these two ends is zero, because the oscillating body is at rest in that instant. Until now, we have considered the spring-mass system as our model oscillator. Now, after having understood the different characteristics of SHM, we are in a position to move forward and discuss some other physical systems executing SHM.

2.7 GLOSSARY

Acceleration – the change of velocity of an object per second. The unit of acceleration is m/s.

Amplitude – the maximum displacement of an oscillating system from its mean position.

Displacement – net change in location of a moving body.

Force – anything that can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a vibrating object.

Kinetic energy – energy of a moving object.

Total mechanical energy – it is the sum of the kinetic energy and the potential energy.

Phase difference – the fraction of a cycle between the motion of two objects vibrating at the same frequency.

Potential energy – energy due to position.

Tension – the force in an object that has been stretched.

Velocity – speed in a given direction.

Weight – the force of gravity on an object. For a mass, m, its weight = mg.

Work – energy transferred by a force.

2.8 TERMINAL QUESTIONS

1. For a particle executing SHM along x-axis, the restoring force is given by
   (a) $-Akx$ (b) $A\cos kx$ (c) $A\exp(-kx)$ (d) Akx

2. The potential energy of a particle executing SHM is given by
   (a) $U = k/2(x-a)^2$ (b) $U = kx + kx^2 + kx^3$ (c) $U = A\exp(-bx)$ (d) $U = constant$

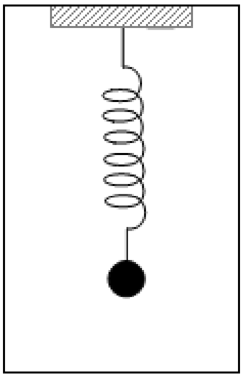
3. A particle executes simple harmonic motion of amplitude A along the x-axis. At $t = 0$, the position of the particle is $x = A/2$ and it moves along the positive x-direction. Find the phase constant if the equation is written as $x(t) = A\sin(\omega t + \delta)$.

4. A body of mass 2 kg, suspended through a vertical spring, executes SHM of period 4 s. If the oscillations are stopped and the body hangs in equilibrium, find the potential energy stored in the spring. [$g = 10\ ms^{-2}$]

5. In the previous question, if the system, instead of being on the earth's surface, is transported to the moon, how will your answer change? The acceleration due to gravity on the moon's surface is 1/6th of that on the earth.

6. If the system described in question 4 above is kept in an elevator which is moving downward with an acceleration of $5\ ms^{-2}$, how will your answer change? How about the condition

when the elevator is accelerating upwards with the same acceleration? And, what will happen if the elevator is experiencing a free fall?



7. A spring stores 5 J of energy when stretched by 25 cm. It is kept vertical with the lower end fixed. A block fastened to its other end is made to undergo small oscillations. If the block makes 5 oscillations each second, what is the mass of the block?

8. A mass M, attached to a spring, oscillates with a period of 2 s. If the mass is increased by 2 kg, the period increases by 1 s. Assuming Hooke's law is obeyed, the initial mass M was __________.

9. The work done by the spring-mass system during one complete oscillation is equal to

(a) The total energy of the system
(b) Kinetic energy of the system
(c) Potential energy of the system
(d) Zero

10. A particle of mass m is hanging vertically by an ideal spring of force constant k. If the mass is made to oscillate vertically, its total energy is

(a) maximum at the extreme position
(b) maximum at the mean position
(c) minimum at the mean position
(d) none of the above

11. Write short notes on:

(i) Acceleration in SHM (ii) Energy Variation in SHM
(iii) Phasor model of SHM

12. Four mass-less springs, whose force constants are 2k, 2k, k and 2k respectively, are attached to a mass M kept on a frictionless plane as shown in the following figure. If the mass M is displaced in the horizontal direction, what will be the frequency of the oscillation?

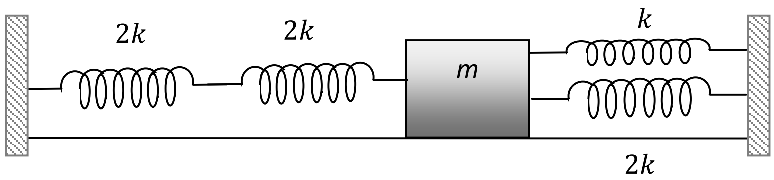


Figure 17: Composite spring-mass system.

13. A particle is executing SHM. Its velocity has values of 3 m/s and 2 m/s when its distance from the mean position is 1 m and 2 m, respectively. calculate the length of its path and period of its motion.

2.9 ANSWERS

Selected Self Assessment Questions (SAQs):

1. $x(t) = A\cos\omega t$

2. $x(t) = -A\cos\omega t$

3. $x(t) = A\sin\omega t$

4. $x(t) = -A\sin\omega t$

5. To maintain contact between the book and the platform, N_{min} must be positive. If N_{min} becomes less than zero, the book will leave contact with the platform.

Hence to maintain contact,

$$mg - m\omega^2 A \geq 0$$

$$or \quad \omega^2 A \leq g$$

6. Ratio of amplitudes = 10:5 or 2:1

7. (c)

8. (c)

9. (c)

10. (e)

Selected Terminal Questions:

1. (a)

2. (a)

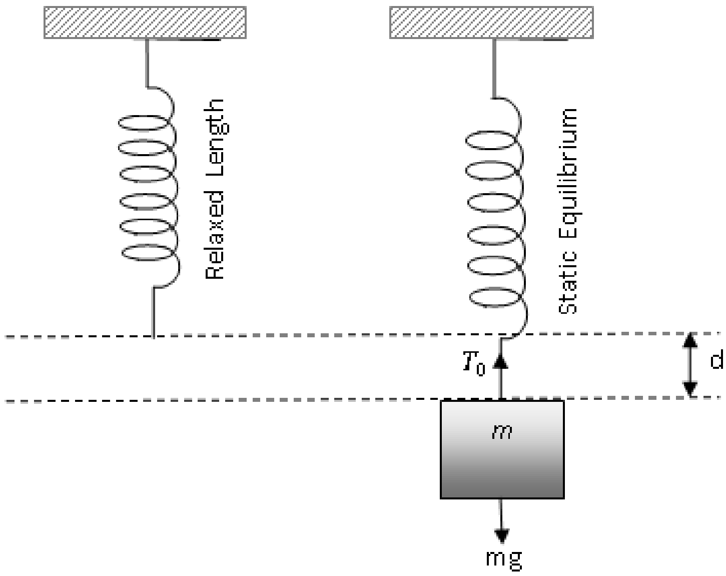
3. At $t = 0, x = A/2$. Therefore, $A/2 = A\sin\delta$ or $\delta = \pi/6$ or $5\pi/6$. The velocity is given by

$$\frac{dx}{dt} = v(t) = A\omega\cos(\omega t + \delta)$$

At $t = 0, v = A\omega\cos\delta$. Now, $\cos\pi/6 = \sqrt{3}/2$ and $\cos 5\pi/6 = -\sqrt{3}/2$. We are given that the velocity is positive at $t = 0$, therefore the phase constant cannot be $5\pi/6$.

$$\therefore \delta = \pi/6$$

4. The body hangs in equilibrium at the end of a spring as shown below.



Now the potential energy stored in the equilibrium position in the spring will be because of the elongation in the spring, i.e.

$$U = \frac{1}{2}m\omega^2 d^2$$

So, now our task is to calculate *d*. We know the frequency of the oscillation and the mass m of the block. By applying force balance, we have

$$T_0 = mg$$

Also, we know that the tension provides the restoring force,

$$T_0 = -kd$$

From the two equations, we get

$$d = \frac{mg}{k} = \frac{mg}{m\omega^2} = \frac{g}{\omega^2}$$

Therefore, the potential energy will be

$$U = \frac{1}{2}m\omega^2\left(\frac{g}{\omega^2}\right)^2 = \frac{1}{2}m\frac{g^2}{\omega^2}$$

$$= \frac{1}{2}m\frac{g^2}{\left(\frac{2\pi}{T}\right)^2} = \frac{1}{2}(2)\frac{(10)^2}{\left(\frac{2\pi}{4}\right)^2}$$

$$\therefore U = 40.5\,J$$

5. The only thing that will change in this question is the value of *g*. We will have to use the *g* on the moon. Therefore,

$$U = \frac{1}{2}m\frac{g^2}{\left(\frac{2\pi}{T}\right)^2} = \frac{1}{2}(2)\frac{\left(\frac{10}{6}\right)^2}{\left(\frac{2\pi}{4}\right)^2} = 1.12\,J$$

6. Again, what will change is the value of the acceleration due to gravity experienced by the system.

The elevator is accelerating down, hence the net acceleration experienced by the spring-mass system will be equal to the difference between the two accelerations. i.e.

$$g_{experienced} = g - a_{elevator}$$

$$\therefore g_{experienced} = 10 - 5 = 5\,ms^{-2}$$

Therefore, the potential energy stored in the spring will be

$$U = \frac{1}{2}m\frac{g^2_{experienced}}{\left(\frac{2\pi}{T}\right)^2} = \frac{1}{2}(2)\frac{(5)^2}{\left(\frac{2\pi}{4}\right)^2} = 10.1\,J$$

If instead the elevator is accelerating up, the acceleration experienced by the system will increase

$$g_{experienced} = g - a_{elevator}$$

$$\therefore g_{experienced} = 10 - (-5) = 15\ ms^{-2}$$

Therefore, the potential energy stored in the spring will be

$$U = \frac{1}{2} m \frac{g_{experienced}^2}{\left(\frac{2\pi}{T}\right)^2} = \frac{1}{2}(2)\frac{(15)^2}{\left(\frac{2\pi}{4}\right)^2} = 91\ J$$

Now, in case of free fall, the system will experience weightlessness, i.e. the acceleration experienced by the system will be zero.

$$g_{experienced} = g - a_{elevator}$$

$$\therefore g_{experienced} = 10 - 10 = 0$$

Therefore, the potential energy stored in the spring in the equilibrium position will be

$$U = \frac{1}{2} m \frac{g_{experienced}^2}{\left(\frac{2\pi}{T}\right)^2} = \frac{1}{2}(2)\frac{(0)^2}{\left(\frac{2\pi}{4}\right)^2} = 0$$

In other words, the spring will be relaxed.

7. The potential energy is given by

$$U = \frac{1}{2} m\omega^2 x^2$$

$$= \frac{1}{2} m(2\pi\nu)^2 x^2$$

Therefore,

$$m = \frac{2U}{(2\pi\nu)^2 x^2} = \frac{2(5)}{(2\pi \times 5)^2 (0.25)^2} = 0.16\ kg$$

8. We know that the time period is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The spring constant remains constant. Therefore, we can say that

$$\frac{T}{\sqrt{m}} = constant$$

Or,

$$\frac{2}{\sqrt{M}} = \frac{3}{\sqrt{M+2}}$$

Squaring both the sides, we get

$$4(M+2) = 9M$$

$$\therefore M = \frac{8}{5} = 1.6\ kg$$

9. (d)

10. (d) The total energy remains constant.

12. In the composite spring-mass system, the two springs on the left are in series. Hence, we find the resultant spring constant for the left side springs.

$$\frac{1}{k_{left}} = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$$

Now all other springs are in parallel and hence they all contribute to the restoring force equally (recall what we did in Unit 1). Therefore, the resultant spring constant is given by

$$k_{resultant} = k + k + 2k = 4k$$

Thus, the time period is given by

$$T = 2\pi\sqrt{\frac{M}{4k}}$$

And the frequency is inverse of time period,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{4k}{M}}$$

13. Using the relation for velocity as a function of position and squaring it on both the sides,

$$v^2 = \omega^2(A^2 - x^2)$$

Plugging in the known values of speed and time, we get

$$3^2 = \omega^2(A^2 - 1^2)$$

$$or\ \ 9 = \omega^2 A^2 - \omega^2$$

And

$$2^2 = \omega^2(A^2 - 2^2)$$

$$or\ \ 4 = \omega^2 A^2 - 4\omega^2$$

Subtracting the first equation from the second, we get

$$5 = 3\omega^2$$

$$\omega = \sqrt{\frac{5}{3}} = 1.29\ rad/s$$

Putting its value in one of the equations, we can get the value of the amplitude,

$$A = 2.53\ m$$

Hence, the length of the path = 2A = 5.06 m.

The time period is given by

$$T = \frac{2\pi}{\omega} = 4.87\ s$$

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UNIT 3 PHYSICAL SYSTEMS EXECUTING SHM

Structure

3.1 INTRODUCTION

After studying the previous two units, you now have a fairly good understanding of SHM and its basic characteristics like amplitude, time period and frequency. You also know that the phase of oscillator gives us information about the state of motion of the oscillator and the phase angle or the phase constant can be determined on the basis of the initial conditions of the oscillator. You also leared how to calculate the potential energy, kinetic energy and total energy for an oscillator. All these aspects of SHM have been discussed by considering the spring-mass system as our model oscillator.

The spring-mass system is the simplest physical system executing SHM. There are several other physical systems, such as simple pendulum and compound pendulum, which are of practical interest and are good examples of SHM. In this unit you will study about some of them. While doing so, you will also learn to calculate the characteristics parameters and energy associated with the oscillatory motion of such systems.

3.2 OBJECTIVES

After studying this unit, you should be able to

- give examples of physical systems exhibiting SHM,
- explain the origin of the restoring force in case of simple pendulum,
- explain why the angular displacement is taken small in case of simple pendulum,
- explain what is angular simple harmonic motion,
- write down the equations for angular velocity and angular acceleration for SHM,
- calculate the tension in the string in case of simple pendulum,
- calculate the angular frequency and the time period for compound pendulum, and
- calculate the angular frequency and the time period for torsional pendulum.

3.3 EXAMPLES OF PHYSICAL SYSTEMS EXECUTING SHM

You may recall from the previous unit that in the spring-mass system, the potential energy was stored in the stretched/compressed spring and the kinetic energy was stored the moving mass. . There are many oscillatory systems such as the simple pendulum, in which the potential energy is associated with the gravitational force rather than with the elastic properties of a compressed or stretched spring. Similarly, in the torsional pendulum, the potential energy of the system is associated with the elastic properties of a twisted wire. In the following, we will discuss the oscillatory motion of three such physical systems (pendulums(:

- Simple Pendulum
- Compound or Physical Pendulum
- Torsional Pendulum

In the first two pendulums, the gravitational force is associated with the springiness. And, in the torsional pendulum, springiness is associated with the elastic properties of a twisted wire. Yet another important feature of these oscillatory systems is that all three of them are examples of angular simple harmonic motion, That is, as the name suggests, these oscillatory systems are the angular version of the linear simple harmonic motion (that we studied by considering the spring-mass system). However, the motion of the simple pendulum and compound pendulum can also be analysed as linear simple harmonic oscillators.

Before we study each of these individual pemdulum, you may like to know some general features of the angular simple harmonic motion.

3.4 ANGULAR SIMPLE HARMONIC MOTION

A body free to rotate about a given axis can execute angular oscillations. For example, oscillations of a simple pendulum (Fig. 1) swinging about the point where one end of the string is attached, or the oscillations of a torsional pendulum (Fig. 3), where a body suspended by a thread is rotated through an angle are angular oscillations.

You may recall from Unit 1 that the linear simple harmonic motion is characterised by a restoring force generated in the system which is always directed towards the mean position and which is proportional to the displacement. Similarly, angular oscillations are considered angular simple harmonic motion if

- The restoring torque, τ is proportional to the angular displacement, θ and is always directed towards the mean or equilibrium angular position, i.e.

$$\tau = -K\theta \tag{3.1}$$

where K is a constant. The position of the system when the restoring torque is zero is called its mean or equilibrium position. At the mean position, the angular displacement is zero.

Analogous to the linear SHM, the moment of inertia, I takes the place of mass, m and the angular acceleration, α takes the place of linear acceleration, a. The angular acceleration is given as

$$\alpha = \frac{\tau}{I} = -\frac{K}{I}\theta$$

We can write the above expression as

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

Therefore, the angular frequency is given by

$$\omega = \sqrt{\frac{K}{I}} \tag{3.2}$$

Further, in case of angular simple harmonic motion, we talk in terms of angular displacement and integrating the above differential equation, we get

$$\theta = \theta_0 \sin(\omega t + \delta) \tag{3.3}$$

where θ_0 is the maximum angular displacement on either side or the amplitude.

Analogous to the linear SHM, the angular velocity at time t is given by

$$\Omega = \frac{d\theta}{dt} = \theta_0 \omega \cos(\omega t + \delta)$$

And at an angle θ

$$\Omega = \pm\omega\sqrt{\theta_0^2 - \theta^2}$$

With this background, let's now study each of the physical systems listed earlier.

3.5 SIMPLE PENDULUM

Fig. 1 shows a simple pendulum in which a bob of mass *m* is suspended from the fixed support *P* through a string of length *l*. Left to itself, the bob hangs along the line *PO*, and this alignment of the simple pendulum is called its mean or equilibrium position with the angle $\theta = 0$. If the bob is drawn towards one extreme A from this mean position, such that the angle θ remains small, and then released, it oscillates in a circular arc with the center at the point of suspension *P*.

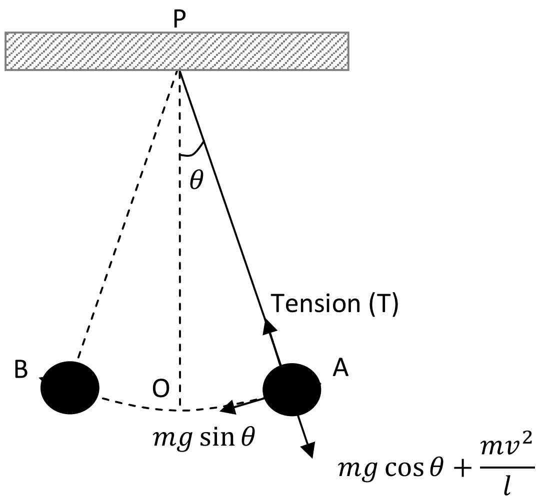


Figure 1: A simple pendulum.

Let us now examine whether or not the motion of the bob is simple harmonic and determine its oscillation time period.

When the bob is at point A, the forces acting on it are

- The tension in the string (T)
- The weight of the bob (mg)
- The centrifugal force because of the speed of the bob as it moves along the circular arc. However, at the end points such as A and B, its contribution is zero as the velocity at these points is zero.

The weight (mg) of the bob is resolved into a radial component $mg\cos\theta$ and a tangential component $mg\sin\theta$. The tangential component, which is the component along the tangent to the path traced by the bob, provides the the restoring force, because it always acts opposite to the displacement of the bob so as to bring it back toward the mean or equilibrium position ($\theta = 0$). Therefore, for the simple pendulum, the restoring force is written as

$$F = -mg\sin\theta \tag{3.4}$$

Here, as usual, the negative sign indicates that the restoring force acts opposite to the displacement.

The radial forces balance each other, such that

$$T = mg\cos\theta + \frac{mv^2}{l}$$

Therefore, there is no motion in the radial direction.

Now, as we mentioned before, the angle θ has to be small. Why do you think this is so? Refer to equation (3.4) which gives the restoring force for the oscillatory motion of the pendulum. If this motion is to be SHM, then the restoring force must be directed toward the mean position and it should also be proportional to the (angular) displacement, θ. The later condition can be met only if we assume that the angle θ is small because, in that condition, $\sin\theta \approx \theta$ where θ is in radians.

Thus, if θ is small, equation (3.4) can be written as

$$F = -mg\theta \tag{3.5}$$

Equation (3.5) is the equation of motion of the simple pendulum executing SHM.

3.5.1 Simple Pendulum as a Linear Simple Harmonic Oscillator

The simple pendulum can be treated as a linear simple harmonic oscillator. It is so because, for small angular displacement, the path traced by the bob is approximately a straight line and is equal to the length of the arc. That is, the path traced by the bob in moving from the mean position to point A can be written as

$$x = l\theta$$

Thus, from equation (3.5), we get

$$F = -\left(\frac{mg}{l}\right)x$$

Comparing the above expression with the equation of motion for a spring-mass system, $F = -kx$, we get

$$k = \frac{mg}{l}$$

And, therefore, the time period of the simple pendulum is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} \qquad (3.6)$$

3.5.2 Simple Pendulum as an Angular Simple Harmonic Oscillator

For considering the simple pendulum as angular harmonic oscillator, we need to write the expression for the restoring torque. The restoring torque is given as

$$\tau = l.F$$

$$\tau = -(mgl)\theta = -K\theta \qquad (3.7)$$

Thus, based on the equation (3.7) we can say, if the amplitude of oscillation is small, the motion of the pendulum is approximately angular simple harmonic.

Further, from equation (3.2), we note that the angular frequency of oscillation depends on the moment of inertia of the osciilating body. The moment of inertia of the bob (assuming it is a point mass) about the axis of rotation passing through point of suspension, P is

$$I = ml^2$$

Therefore, from equation (3.2), we can write the angular frequency of SHM for simple pendulum as

$$\omega = \sqrt{\frac{K}{I}} = \sqrt{\frac{mgl}{ml^2}}$$

$$\therefore \omega = \sqrt{\frac{g}{l}} \qquad (3.8)$$

And the time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \tag{3.9}$$

So, from equations (3.6) and (3.9), it is clear that treatment of simple pendulum as linear or angular harmonic oscillator is equivalent as both the treatments lead to same results.

Example 1: *A simple pendulum provides an easy method to measure the value of 'g' in a laboratory.* An experiment with simple pendulum was carried out on the surface of moon, where it was found that it took 38 s to complete 10 oscillations when the effective length of the pendulum was kept at 60 cm. Calculate the acceleration due to gravity on the moon's surface. How does it compare with the earth's gravity? [For earth's surface $g = 9.8\, ms^{-2}$]

Solution: We are given that the time period is

$T = \frac{36}{20} = 3.8\, s$ (Comment: T=(38/10))

From the expression for the time period of simple pendulum, equation (3.9), we have

$$g = \frac{4\pi^2 l}{T^2}$$

$$= \frac{4\pi^2(0.6)}{3.8^2} = 1.64\, ms^{-2}$$

Comparing it with earth's gravity, we see that the moon's gravity is (9.8/1.64) or about 6 times less than that of earth.

Self Assessment Question (SAQ) 1: Calculate the time period of a simple pendulum of length one meter. The acceleration due to gravity at that place is $\pi^2\, ms^{-2}$.

Self Assessment Question (SAQ) 2: What will happen to the motion of a simple pendulum if the amplitude is large? Is it still SHM? Explain.

Self Assessment Question (SAQ) 3: An astronaut on the surface of the moon finds that the period of a simple pendulum there is much larger than that on the earth and that the pendulum continues to oscillate for much longer time than on the earth. What information regarding the moon could be obtained from these observations?

Self Assessment Question (SAQ) 4: Choose the correct answer -

When a mass undergoes angular simple harmonic motion, there is always a constant ratio between its angular displacement and

(a) time period, (b) angular acceleration, (c) angular velocity, (d) mass

Self Assessment Question (SAQ) 5: Choose the correct answer -

For a particle executing SHM, the kinetic energy is given by $K = K_0 \cos^2 \omega t$. The maximum value of potential energy is

(a) K_0, (b) zero, (c) $K_0/2$, (d) not obtainable

3.6 COMPOUND PENDULUM

Any rigid body suspended from a fixed support constitutes a compound pendulum, which is also known as physical pendulum. As a matter of fact, all real pendulums are compound in nature. The figure below shows a compound pendulum. A rigid body is suspended through a hole at point O. When the center of mass C is vertically below O, the body may remain at rest. This position, when the angle $\theta = 0$, is the equilibrium or the mean position for such oscillating system.

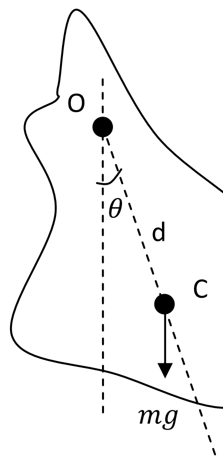


Figure 2: A compound or physical pendulum.

Let the distance between the center of mass C and the point of suspension O be d. If we displace the body to the right (anticlockwise) through a small angle θ, the weight of the body creates a clockwise restoring torque about the axis passing through O.

Similar to what we did before in case of simple pendulum, when treating it as an angular simple harmonic oscillator, we can determine the magnitude of the restoring torque in the case of compound pendulum as well. For the compound pendulum, for small θ, we can write the torque

$$\tau = -(mgd)\theta = -K\theta \tag{3.10}$$

Therefore,

$$K = mgd$$

If I is the moment of inertia of the rigid body about the axis passing through O, then the time period for small oscillations when the motion is nearly angular SHM, is given by

$$T = 2\pi\sqrt{\frac{I}{mgd}} \tag{3.11}$$

Example 2: A uniform rod of length 1 m is suspended through an end and is set into oscillations with small amplitude under gravity. Calculate the time period of the oscillations. [$g = 9.8\, ms^{-2}$]

Solution: The moment of inertia for a uniform rod of length L about the center is

$$I = \frac{mL^2}{12}$$

By linear transformation, the moment of inertia about one end is given by

$$I = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3}$$

Now, for small amplitude, the angular motion is nearly SHM and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{mL^2}{3}}{mg\left(\frac{L}{2}\right)}}$$

where $d = L/2$, because the rod is uniform and therefore, the center of mass lies in the center of the rod. Thus,

$$T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2(1)}{3(9.8)}} = 1.67\ s$$

Example 3: A hollow metal sphere is filled with water and a small hole is made at its bottom. It is hanging by a long thread and is made to oscillate. Explain qualitatively how will the period of oscillation change if the water is allowed to flow through the hole till the sphere is empty?

Solution: Initially and finally, the center of mass will be at the center of the sphere. However, as the water drains out off the sphere, the center of mass of the oscillating system will first move down and then will come up. Due to this, the effective length (d) between the point of suspension

O and the center of mass C, first increase, reaches a maximum and then decreases till it becomes equal to its initial value.

We also know that the time period

$$T \propto \sqrt{L}$$

Therefore, the time period also increases first, reaches a maximum and then will decrease until it becomes equal to its initial value!

Self Assessment Question (SAQ) 6: A girl is swinging in a sitting position. How will the period of swing be affected if

(a) The girl stands up while swinging.
(b) Another girl of the same mass comes and sits next to her.

Self Assessment Question (SAQ) 7: Can a pendulum clock be used in space? Explain.

Self Assessment Question (SAQ) 8: Choose the correct option.

A pendulum clock that keeps correct time on the earth is taken to the moon. It will run

(a) At correct rate
(b) 6 times faster
(c) 6 times slower
(d) $\sqrt{6}$ times faster
(e) $\sqrt{6}$ times slower

3.7 TORSIONAL PENDULUM

In a torsional pendulum, an extended body is suspended by a light thread or a wire as shown in the figure below. To initiate angular oscillations of the hanging extended body, it (the body) is rotated through an angle about the wire as the axis of rotation and then released.

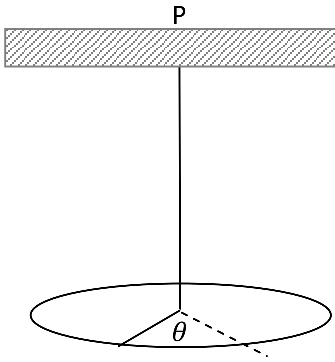


Figure 3: A torsional pendulum.

The wire remains vertical throughout the motion, but a twist is produced in the wire and the body executed angular oscillations. The lower end of the wire is rotated through an angle θ with the body but the upper end of the wire, attached to the fixed point P, remains fixed. The twisted wire exerts a restoring torque on the body to bring it back to its original position, which is the equilibrium or the mean position ($\theta = 0$). This torque has a magnitude proportional to the angle of twist θ, which is equal to the angle rotated by the body. The proportionality constant is called the *torsional constant* (*K*) of the wire. Thus,

$$\tau = -K\theta$$

If I is the moment of inertia of the body about the vertical axis, then the angular acceleration is given by

$$\alpha = \frac{\tau}{I} = -\frac{K}{I}\theta$$

Therefore, comparing it with the familiar expression for the acceleration fir rotational motion, $\alpha = -\omega^2\theta$, we get

$$\omega = \sqrt{\frac{K}{I}}$$

Thus, the time period for torsional pendulum can be written as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{K}} \qquad (3.12)$$

Example 4: A uniform disk of radius 5 cm and mass 200 g is fixed at its center to a metal wire. The other end of the wire is fixed with a clamp. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.2 s, find the torsional constant of the wire.

Solution: The moment of inertia of a uniform disk of radius r about a line (wire) passing through its centre is given by

$$I = \frac{mr^2}{2} = \frac{(0.2)(0.05)^2}{2}$$

$$2.5 \times 10^4 \; kg \; m^2$$

The time period is given by

$$T = 2\pi \sqrt{\frac{I}{K}}$$

Or,

$$K = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2(2.5 \times 10^4)}{(0.2)^2}$$

$$= 0.25 \; kg \; m^2 \; s^{-2}$$

Example 5: A uniform rod of mass M and length L is suspended through a light wire of length L and torsional constant K as shown in Fig. 4. Calculate the time period if the system makes

(a) small oscillations in the vertical plane about the point of suspension P
(b) angular oscillations in the horizontal plane about the center of the rod

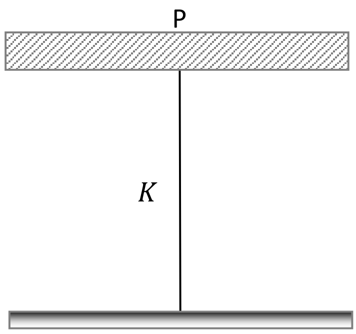


Figure 4: A uniform rod suspended by a wire.

Solution: (a) The oscillations take place about the horizontal line through the point of suspension and perpendicular to the plane of the figure. The moment of inertia of the rod about this line is

$$I = \frac{mL^2}{12} + mL^2 = \frac{13}{12} mL^2$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{13}{12}mL^2}{mgL}}$$

$$= 2\pi\sqrt{\frac{13L}{12g}}$$

(b) The angular oscillations take place about the suspension wire. The moment of inertia about this line is

$$I = \frac{mL^2}{12}$$

The time period is given by

$$T = 2\pi\sqrt{\frac{I}{K}} = 2\pi\sqrt{\frac{mL^2}{12K}}$$

Self Assessment Question (SAQ) 9: Choose the correct option(s). (More than one choice may be correct.)

Which of the following will change the time period as they are taken to moon?

(a) A simple pendulum
(b) A compound pendulum
(c) A torsional pendulum
(d) A spring-mass system

Self Assessment Question (SAQ) 10: Choose the correct option(s). (More than one choice may be correct.)

The motion of a torsional pendulum is

(f) Periodic, (b) oscillatory, (c) linear simple harmonic, (d) angular simple harmonic

3.8 LC CIRCUIT

In this unit, we first studied about the LC circuit having a combination of a pure inductor, which has zero resistance and a pure capacitor, which has infinite resistance, as an example of an electromagnetic system exhibiting SHM. In an ideal system, the energy stored in the magnetic field and the energy stored in the electric field exhibits SHM. Earlier, we knew of only mechanical systems that exhibited simple harmonic motion.

In the earlier units, we have so far restricted ourselves only to mechanical systems. But oscillations are not restricted to mechanical systems only. In electromagnetic systems, we can observe such oscillations as well. The most instructive example of an electromagnetic system is a circuit having a combination of a pure inductor, which has zero resistance and a pure capacitor, which has infinite resistance.

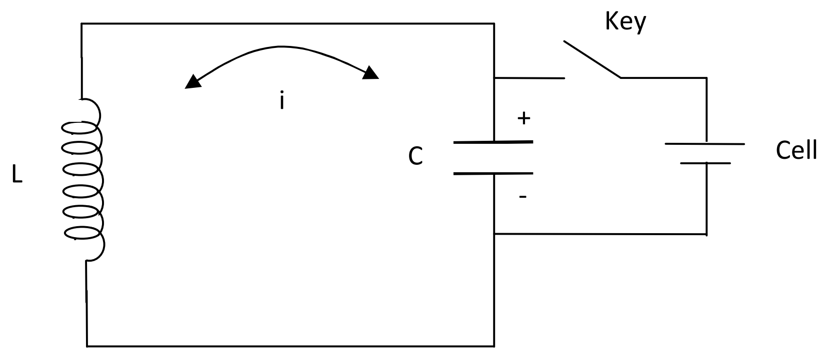


Figure 18: An oscillatory circuit consisting of a capacitor C and an inductor L.

We shall now study harmonic oscillations of an electrical circuit consisting of a pure capacitor C and a pure inductor L (Fig. 1). The equilibrium state of the system is when the capacitor C is uncharged and no current is flowing in the circuit. This state is disturbed when the capacitor is charged by pressing the key.

Let q be the charge on the capacitor at some instant. Then, the voltage across the capacitor plates is given by

$$V = \frac{q}{C}$$

Where, C is the capacitance of the capacitor. When the key is released, the capacitor starts discharging through the inductor and a current begins to flow through the inductor. The current through the inductor is given by

$$i = \frac{dq}{dt}$$

In this circuit, the restoring force is due to the force of repulsion between the electrons. This force tends to distribute electrons equally on the capacitor plates so that there is no net charge. Inductance, on the other hand, tends to oppose this redistribution, i.e. it opposes the increase in current. At any instant, the voltage across the inductor is given by

$$V = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$$

The minus sign is indicative of the fact that the voltage developed across the inductor opposes the increase of current. From Kirchhoff's law, the voltage across the inductor should be equal to the voltage across the capacitor plates, i.e.

$$-L\frac{d^2q}{dt^2} = \frac{q}{C}$$

$$or \quad \frac{d^2q}{dt^2} = -\omega^2 q$$

Where, the angular frequency $\omega = 1/\sqrt{LC}$. Thus, the time period in an electrical circuit consisting of a pure inductor L and a pure capacitor C is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} \quad (4.1)$$

At any instant, the charge q is given by a sine or a cosine function. If the charge q is taken as

$$q = q_0 \cos(\omega t + \delta) \quad (4.2)$$

where q_0 is the maximum value of the charge, the current in the circuit will therefore be given by

$$i = \frac{dq}{dt} = -\omega q_0 \sin(\omega t + \delta)$$

3.8.1 Energy Considerations

The differential equation for the LC oscillator can also be obtained by considering the energy in the system. At any instant, the electrostatic energy stored in the capacitor is given by

$$E_e = \frac{1}{2}\frac{q^2}{C}$$

And, the corresponding magnetic energy of the inductor is given by

$$E_m = \frac{1}{2}Li^2$$

So in an ideal situation, the total energy in the circuit can be expressed as

$$E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\frac{q^2}{C} \quad (4.3)$$

Initially, when the capacitor is fully charged and connected with an inductor, the capacitor starts discharging through the inductor; the charge stored in it will reduce gradually and a current starts flowing in the inductor. Due to the growth of current in the inductor, an EMF will be induced. In due time, the current will reach a maximum value. After attaining the maximum value, the current through the inductor starts decreasing, and in due course of time, its energy becomes zero. At that instant, the charge on the capacitor again reaches a maximum value. This cycle is repeated and the energy of the circuit fluctuates alternatively between electric and the magnetic energies.

Thus, the total energy of the circuit remains constant, therefore

$$\frac{dE}{dt} = 0$$

$\therefore \frac{1}{2}L(2\dot{q}\ddot{q}) + \frac{1}{2}\frac{(2q\dot{q})}{C} = 0$ where $\dot{q} = \frac{dq}{dt}$ $and\ \ddot{q} = \frac{d^2q}{dt^2}$

$$or\ \ \dot{q}\left(L\ddot{q} + \frac{q}{C}\right) = 0$$

Since the current $\dot{q}$ is finite, therefore,

$$L\ddot{q} + \frac{q}{C} = 0$$

$$or\ \ \frac{d^2q}{dt^2} = -\omega^2 q$$

where the angular frequency $\omega = 1/\sqrt{LC}$. So, we can see that from the energy consideration too, we arrive at the same result (4.1) as before.

A comparison of the LC circuit with the mechanical oscillatory systems indicates that mass in mechanical systems and magnetic field inertia in electrical systems play analogous role. The mass controls the velocity change for a given force and the magnetic field controls the rate of change of current for a given voltage.

Example 1: (a) In an oscillating LC circuit, calculate the value of the charge q, in terms of the maximum charge Q, present on the capacitor when the energy is shared equally between the electric and magnetic fields? Assume that L = 12 mH and C = 1.7 µF.

(b) If at time t = 0, calculate the time when this condition will occur.

Solution: (a) According to the given condition, we have

$$\left(\frac{1}{2}\frac{q^2}{C}\right) = \frac{\left(\frac{1}{2}\frac{Q^2}{C}\right)}{2}$$

Or,

$$q = \frac{Q}{\sqrt{2}} = 0.707Q$$

(b) At t = 0, q = Q. Therefore from equation (4.2), we have

$$Q = Q\cos\delta$$

$$\therefore \delta = 0$$

Therefore,

$$q = \frac{Q}{\sqrt{2}} = Q\cos(\omega t)$$

$$\therefore \omega t = \frac{\pi}{4}\ rad$$

Hence,

$$t = \frac{\pi}{4\omega} = \frac{\pi}{4}\sqrt{LC}$$

$$= \frac{\pi}{4}\sqrt{(12 * 10^{-5}\ H)(1.7 * 10^{-6}\ F)} = 1.12 * 10^{-4}\ s$$

Self Assessment Question (SAQ) 11: A charged capacitor and an inductor are connected in series at time t = 0. In terms of the period T of the resulting oscillations, determine how much later the following reach their maximum values:

(a) the charge on the capacitor
(b) the energy stored in the electric field
(c) the current

Self Assessment Question (SAQ) 12: In an LC oscillator, the maximum energy stored in the capacitor is 160 μJ. What is the maximum energy stored in the inductor?

Self Assessment Question (SAQ) 13: In the previous question, if at any time, the energy stored in the capacitor is 100 μJ, what is the energy stored in the magnetic field?

Self Assessment Question (SAQ) 14: In the previous question, if at any time, the energy stored in the capacitor is 100 μJ, what is the energy stored in the electric field?

3.9 SUMMARY

In this unit, we studied about the angular simple harmonic motion and how simple pendulum, compound pendulum and torsional pendulum can be treated as angular simple harmonic oscillators. We also learned how to calculate the time period of these pendulums. For simple pendulum and compound pendulum gravitational pull is responsible for the potential energy and for torsional pendulum it is the elasticity of the wire where potential energy is stored. In the earlier units, we learned that in spring-mass system, it is the spring that stores the potential energy. The kinetic energy is because of the motion of a body with finite mass.

We learned how angular SHM is similar to linear SHM and in angular simple harmonic motion, it is the torque in place of force that is proportional to the magnitude of angular displacement in place of displacement in linear simple harmonic motion. Here too, the torque is always directed towards the mean or the equilibrium position, which is the position where the system rests when it is not oscillating.

3.10 GLOSSARY

Angular acceleration – it is the rate of change of angular velocity. In SI units, it is measured in (rad/s^2), and is usually denoted by the Greek letter alpha (α).

Angular amplitude – it is the maximum angle (disregarding the direction) that a rotating body goes through from the equilibrium position

Angular displacement – it is the angle that a rotating body goes through.

Angular velocity – it is defined as the rate of change of angular displacement and is a vector quantity which specifies the angular speed (rotational speed) of an object and the axis about which the object is rotating. In SI units, it is measured in (rad/s), and is usually denoted by the Greek letter omega (Ω).

Center of mass – it is the point where all of the mass of the object is concentrated. When an object is supported at its center of mass, there is no net torque acting on the body and it will remain in static equilibrium.

Centrifugal force – it is the apparent force that draws a rotating body away from the center of rotation. It is caused by the inertia of the body.

Force – anything that can change the state of motion of an object.

Frequency – the number of complete cycles per second .

Kinetic energy – energy of an object due to motion.

Mechanical energy – it is the sum of the kinetic energy and the potential energy.

Moment of inertia – it is the mass property of a rigid body that determines the torque needed for a desired angular acceleration about an axis of rotation. Moment of inertia depends on the shape of the body and, for the same body, it may have different values for different axes of rotation.

Potential energy – energy due to position.

Tension – the force in an object that has been stretched.

Torque – it is the tendency of a force to rotate an object about an axis, fulcrum, or pivot. Just as a force is a push or a pull, a torque can be thought of as a twist to an object.

Weight – the force of gravity on an object. For a mass, m, its weight = mg.

3.11 TERMINAL QUESTIONS

1. A simple pendulum of length 40 cm oscillates with an angular amplitude of 0.04 rad. Determine [Take $g = 10\ ms^{-2}$]

   (a) the time period,
   (b) the linear amplitude of the bob,
   (c) the speed of the bob when the string makes angle 0.02 rad with the vertical, and
   (d) the angular acceleration when the bob is in momentary rest.

2. What is the period of a pendulum formed by pivoting a meter stick so that it is free to rotate about a horizontal axis passing through the 75 cm mark?

3. The moment of inertia of the disc used in a torsional pendulum about the suspension wire is 0.2 $kg\ m^2$. It oscillates with a period of 2 s. Another disc is placed over the first one and the time period of the system becomes 2.5 s. Calculate the moment of inertia of the second disc about the wire.

4. A solid cylinder is attached to a massless spring so that it can roll without slipping along a horizontal surface. Calculate the period of oscillation made by the cylinder if M is the mass of the cylinder and k is the spring constant.

5. A particle of mass M is located in a one dimensional potential field where the potential energy of the particle depends on the coordinate x as $U(x) = U_0 \sin^2(ax/2)$. If the particle performs small oscillations about the equilibrium position, calculate the period. U_0 and a are constants.

6. The pendulum of a clock is replaced by a spring-mass system with the spring having spring constant 0.1 N/m. What mass should be attached to the spring?

7. A disk, whose radius R is 12.5 cm, is suspended as a compound pendulum, from a hole O at a distance d from its center C as shown in the figure below. When d = R/2, the period T is 0.871 s. Calculate the freefall acceleration 'g' at the location of the pendulum.

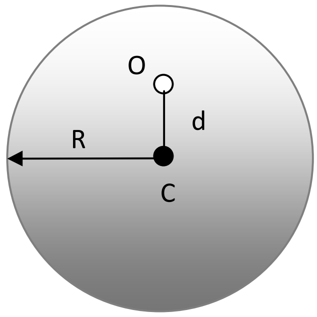


Figure 5: A compound pendulum in the form of a disk.

8. What will be the period of a simple pendulum hanging by the ceiling of an elevator if the elevator is in a free fall?

9. Choose the correct option.

The period of oscillation of a simple pendulum at a place inside the mine is

(a) more than it is on the surface of the earth.
(b) less than it is on the surface of the earth.
(c) the same as it is on the surface of the earth.
(d) the same as it is on the surface of the moon.

10. A simple pendulum is suspended from the ceiling of a truck. When the truck is at rest, the time period T of the pendulum is measured. Thereafter, the truck starts accelerating uniformly on the horizontal road. If the acceleration of the truck is 'a', what will be the new time period with respect to T?

11. Choose the correct option.

Suppose that it takes 1.2 seconds for a simple pendulum to swing from its extream left position to its extream right position. What is the period of the pendulum?

(a) 0.6 s (b) 1.2 s (c) 2.4 s (d) 3.6 s

12. Choose the correct option.

A pendulum with a string of length 2 m has a period of 2.8 s. What would be the period of the pendulum if the length of the string were increased to 8 m?

(a) 1.4 s (b) 2.8 s (c) 5.6 s (d) 11.2 s

13. Write short notes on:

(i) Angular SHM (ii) Simple Pendulum as linear SHM

(iii) Simple Pendulum as angular SHM (iv) Physical Pendulum (v) Torsional Pendulum

14. A 1.5 μF capacitor is charged to 57 V. The charging battery is then disconnected and a 12 mH coil is connected in series with the capacitor so that LC oscillations occur. What is the maximum current in the coil? Assume that the circuit contains no resistance.

3.12 ANSWERS

Selected Self Assessment Questions (SAQs):

1. 2 s

2. No.

3. From the expression for the time period for a simple pendulum, it is clear that the increase in the time period implies that on the moon, the value of 'g' is much smaller than on the earth. Further, longer duration of oscillation on the moon implies that the friction effects are less as compared to earth, i.e. the moon has no atmosphere or if there is any, then it is very thin as compared to the earth's atmosphere.

4. (b)

5. (a)

6. (a) Decreases, because the center of mass goes up and thus reducing the effective length of the swing.

(b) It remains unchanged, because the time period is independent of the mass of the oscillating object. It just depends on the acceleration due to gravity and the effective length.

7. We know that the time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

In space, there is no acceleration due to gravity, i.e. 'g' is zero. Hence, from the above relation for time period, we can see that it will approach to infinity. In other words, there will be no oscillatory motion in space. Hence, a pendulum clock cannot be used in space.

8. (e)

9. (a), (b)

Time period for a simple pendulum and a compound pendulum depends on the acceleration due to gravity 'g.' Hence, when taken to moon, where the value of 'g' is different than that on the earth, their time period will change. On the other hand, for torsional pendulum and spring-mass system, the time period is independent of ‘g.’

10. (a), (b), (d)

12. Using energy conservation, maximum energy is the inductor is equal to the maximum energy in the capacitor = 160 μF.

13. Using energy conservation, the energy is the inductor (magnetic field) is equal to the total energy (160 μF) minus the energy in the capacitor (100 μF), equal to 60 μF.

14. The energy is the capacitor is the same as the energy stored in the electric field = 100 μF.

Selected Terminal Questions:

1. (a) The angular frequency is given by

$$\omega = \sqrt{\frac{g}{l}} = \sqrt{\frac{10}{0.4}} = 5\ s^{-1}$$

The time period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5} = 1.26\ s$$

(b) Linear amplitude = (40)(0.04) = 1.6 cm

(c) Angular speed at displacement 0.02 rad is

$$\Omega = 5\sqrt{0.04^2 - 0.02^2} = 0.17\ rad/s$$

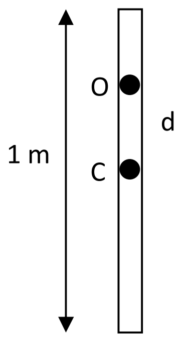
Linear speed of the bob at this instant = (40)(0.17) = 6.8 cm/s

(d) At momentary rest, the bob is in extreme position. Thus, the angular acceleration is

$$\alpha = (0.04)(25) = 1\ rad\ s^{-2}$$

2. We are given that the length of the rod L = 1 m and the distance between O and C is 25 cm or 0.25 m.

Figure 6



By linear transformation, the moment of inertia about point O is given by

$$I = \frac{mL^2}{12} + md^2$$

Now, for small amplitude, the angular motion is nearly SHM and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{mL^2}{12} + md^2}{mgd}}$$

$$T = 2\pi\sqrt{\frac{1 + 12(0.25)^2}{12(9.8)(0.25)}} = 1.53\ s$$

3. Let the torsional constant of the wire be K. The moment of inertia of the first disc about the wire is given. Hence, the time period is

$$2 = 2\pi\sqrt{\frac{0.2}{K}}$$

When the second disc of moment of inertia I is added, the new time period is

$$2.5 = 2\pi\sqrt{\frac{0.2 + I}{K}}$$

From the two equations, we get $I \cong 0.11\ kg\ m^2$.

4. In the mean position of the cylinder, the spring will be in its original length. Let us rotate the cylinder clockwise through an angle θ. If R is the radius of the cylinder, the center of mass of the cylinder will undergo a linear displacement of $R\theta$ towards right as shown in the figure below. Hence, the spring is elongated by $R\theta$.

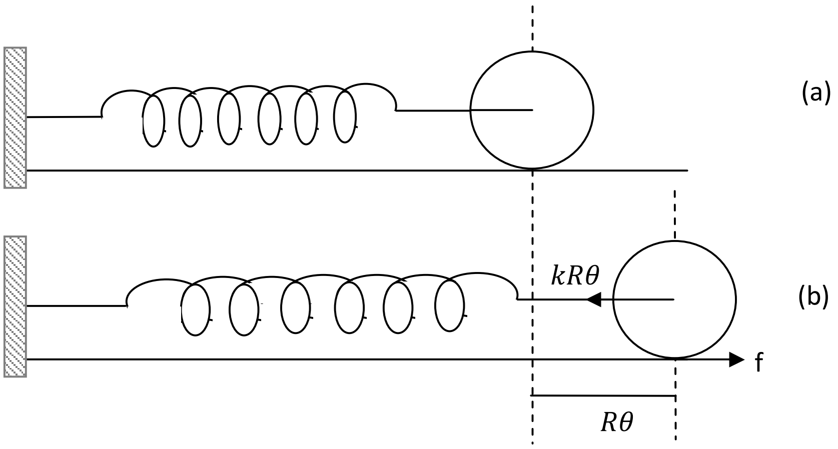


Figure 7: (a) Normal, (b) stretched configurations of a horizontal spring-mass system, where the mass is a solid cylinder of radius R.

The pull of the spring towards left creates a static friction f at the point of contact towards right. The net force on the cylinder is

$$kR\theta - f$$

which is towards the mean or equilibrium position. Hence, by Newton's law we get

$$(1) \quad Ma = kR\theta - f$$

Where "a" is the linear acceleration of the center of mass.

Similarly, the total torque generates an angular acceleration in the cylinder. The equation for torque is given by

$$\tau = fR$$

Therefore, we get

$$(2) \quad fR = I\alpha$$

Here α is the angular acceleration of the cylinder and I is its moment of inertia about the center of mass, which is given by

$$I = \frac{1}{2}MR^2$$

Applying the condition of no slip, we have

$$a = R\alpha$$

From (1) and (2), eliminating f and a we have

$$MR\alpha = kR\theta - \frac{I\alpha}{R}$$

Or,

$$\alpha = \frac{kR^2}{I + MR^2}\theta$$

the magnitude of the angular acceleration is proportional to θ. Hence, the cylinder performs angular SHM, with angular frequency

$$\omega = \sqrt{\frac{kR^2}{I + MR^2}}$$

$$= \sqrt{\frac{kR^2}{\frac{3}{2}MR^2}} = \sqrt{\frac{2k}{3M}}$$

Therefore, the time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3M}{2k}}$$

5. For small oscillations,

$$\sin\left(\frac{ax}{2}\right) \approx \frac{ax}{2}$$

Therefore, the potential energy reduces to

$$U(x) \approx \frac{U_0 a^2 x^2}{2}$$

Comparing it with the familiar form of potential energy for SHM

$$U = \frac{1}{2}kx^2$$

We get

$$k = U_0 a^2$$

Therefore, the time period is given by

$$T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{M}{U_0 a^2}}$$

6. The time period of a pendulum clock is exactly 2 sec. It is also known by the name **seconds pendulum**.

Now when the pendulum is replaced by a spring-mass system the time period should remain the same for clock to function properly. Therefore, if M is the mass, then

$$2 = 2\pi\sqrt{\frac{M}{0.1}}$$

Therefore, M = 0.01 kg or 10 g.

7. The moment of inertia of a disk about its central axis is given by

$$I = \frac{1}{2}MR^2$$

By linear transformation, the moment of inertia is given by

$$I = \frac{1}{2}MR^2 + m\left(\frac{R}{2}\right)^2 = \frac{3}{4}MR^2$$

Now, for small amplitude, the angular motion is nearly SHM and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{3}{4}MR^2}{mg\left(\frac{R}{2}\right)}}$$

$$\therefore T = 2\pi\sqrt{\frac{3R}{2g}}$$

8. Infinite.
The time period for a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

In a freefall, the ‘g’ experienced by the spring pendulum is zero. Therefore, the time period becomes infinite.

9. (a)

The time period for a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The ‘g’ inside the earth’s surface is less than on the surface. Therefore, the time period increases in comparison to the earth’s surface.

10. The initial time period is T, which is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Now, the truck starts accelerating forward, so the bob will experience a pseudo force in the backward direction

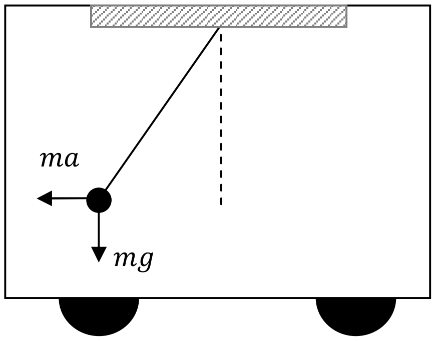


Figure 8: A simple pendulum hanging from the ceiling of an accelerating truck.

The time period depends on the net acceleration that the system experiences. The effective 'g' experienced by the system changes here because of an added acceleration of the truck. Remember we have already solved problem involving elevator going up and down with certain acceleration and we changed the value of effective 'g.' The only difference here is the direction of acceleration.

As the two accelerations are at right angle, the effective 'g' is given by

$$g_{experienced} = \sqrt{g^2 + a^2}$$

The new time period will be

$$T_{new} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}} = T\sqrt{\frac{g}{(g^2 + a^2)^{1/2}}}$$

Note: Here we must understand that the equilibrium position for the SHM will not be vertical anymore and would change depending on the value of a. Thereafter, the oscillations will take place from the new equilibrium position.

11. (c)

The period of a pendulum is the amount of time required for the pendulum to complete one full back-and-forth motion – from its rightmost point back to its rightmost point again, for example.

The time required to swing from its rightmost point to its leftmost point is half of a period. It will require the same amount of time to swing back, so the period is 2.4 s.

12. (c)

The period of a pendulum is proportional to the square root of the length of the string.

$$T \propto \sqrt{l}$$

$$or \quad \frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}}$$

$$T_2 = \frac{2.8}{\sqrt{\frac{2}{8}}} = 2(2.8) = 5.6\ s$$

14. From the principle of conservation of energy, the maximum stored energy in the capacitor must be equal to the maximum stored energy in the inductor. This leads to

$$\frac{Q^2}{2C} = \frac{LI^2}{2}$$

where I is the maximum current and Q is the maximum charge. Therefore,

$$I = \frac{Q}{\sqrt{LC}}$$

Also, we know that

$$V = \frac{Q}{C}$$

Therefore,

$$I = \frac{CV}{\sqrt{LC}} = V\sqrt{\frac{C}{L}}$$

$$= (57)\sqrt{\frac{1.5 * 10^{-6}}{12 * 10^{-3}}} = 0.637\ A$$

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11. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
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3.14 SUGGESTED READINGS

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UNIT 4 SUPERPOSITION Of HARMONIC OSCILLATIONS

Structure

4.1 INTRODUCTION

In the previous unit, you learnt that, under small oscillation approximation, motion of a variety of mechanical oscillators is simple harmonic. However, this is an ideal situation and more often than not, in most of the situations of practical interest, we have to deal with a combination of two or more harmonic oscillations. When two or more harmonic oscillations act on a body simultaneously, the resultant motion of the body can be analysed on the basis of the principle of superposition. We shall study the principle of superposition of harmonic oscillations in this unit.

4.2 OBJECTIVES

After studying this unit, you should be able to

- give example of electromagnetic system exhibiting SHM
- state the principle of superposition
- use the principle of superposition to obtain the resultant of the two collinear harmonic oscillations of same as well as different frequencies
- apply the principle of superposition to obtain the resultant of a number of collinear harmonic oscillations of same frequency

4.3 PRINCIPLE OF SUPERPOSITION

Until now, we studied the oscillatory behavior of isolated physical systems such as spring-mass system and simple pendulum. This is an ideal scenario and quite often, for many practical applications, we have to deal with a combination of two or more harmonic oscillations; e.g. our ear drums receive a complex combination of harmonic oscillations. In order to obtain the resultant effect of two or more harmonic oscillations acting on a body simultaneously, we have to make use of the superposition principle.

The superposition principle states that "*The resultant of two or more harmonic displacements is simply the vector sum of the individual displacements*."

Example 2: Show that if a particle is acted upon by two separate forces each of which can separately produce a simple harmonic motion, the resultant motion of the particle is a combination of two simple harmonic motions.

Solution: Let $\overrightarrow{r_1}$ denote the position of the particle of mass m at time t if the force $\overrightarrow{F_1}$ alone acts on it. Similarly, let $\overrightarrow{r_2}$ denote the position at time t if the force $\overrightarrow{F_2}$ alone acts on it. Newton's second law gives

$$m\frac{d^2\overrightarrow{r_1}}{dt^2} = \overrightarrow{F_1}$$

And,

$$m\frac{d^2\overrightarrow{r_2}}{dt^2} = \overrightarrow{F_2}$$

Adding them, we get

$$m\frac{d^2(\overrightarrow{r_1} + \overrightarrow{r_2})}{dt^2} = \overrightarrow{F_1} + \overrightarrow{F_2}$$

But $\overrightarrow{F_1} + \overrightarrow{F_2}$ is the resultant force acting on the particle and so the resultant position $\vec{r}$ of the particle when the resultant force acts, is given by

$$\vec{r} = \overrightarrow{r_1} + \overrightarrow{r_2} \qquad (4.4)$$

Thus, the actual displacement or position is the vector sum of $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$.

Self Assessment Question (SAQ) 5: Show that the resultant velocity of the particle is the vector sum of the individual velocities of the particle when it is acted upon by the two forces separately, each of which can produce a simple harmonic motion?

4.4 SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS

4.4.1 Oscillations Having Equal Frequencies

Suppose we have two SHMs of equal frequencies but having different amplitudes and phase constants acting on a system in the x-direction. The displacements x_1 and x_2 of the two harmonic motions, of the same angular frequency ω, differing by phase δ are given by

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \delta)$$

There are two methods which can be used to obtain an expression for the resultant displacement due to superposition of the above two harmonic oscillations. Let us discuss them now.

(a) <u>*Analytical Method*</u>

We use the superposition principle which states that the resultant displacement is equal to the vector sum (algebraic sum in this case, because the direction of the two individual oscillation is in the x-direction) of the individual displacements. Therefore, we can write

$$x = x_1 + x_2$$

$$= A_1 \sin \omega t + A_2 \sin(\omega t + \delta)$$

$$= A_1 \sin \omega t + A_2 \sin \omega t \cos \delta + +A_2 \cos \omega t \sin \delta$$

$$= (A_1 + A_2 \cos \delta) \sin \omega t + (A_2 \sin \delta) \cos \omega t$$

$$= C \sin \omega t + D \cos \omega t$$

$$= \sqrt{C^2 + D^2} \left[\frac{C}{\sqrt{C^2 + D^2}} \sin \omega t + \frac{D}{\sqrt{C^2 + D^2}} \cos \omega t\right]$$

where $C = A_1 + A_2 \cos \delta$ and $D = A_2 \sin \delta$.

Now we know that the magnitude of $\frac{C}{\sqrt{C^2+D^2}}$ and $\frac{D}{\sqrt{C^2+D^2}}$ is less than 1. Thus, for an angle ε between 0 to 2π, we can have

$$\sin \varepsilon = \frac{D}{\sqrt{C^2 + D^2}} \quad and \quad \cos \varepsilon = \frac{C}{\sqrt{C^2 + D^2}}$$

Therefore, we get

$$x = \sqrt{C^2 + D^2}[\cos \varepsilon \sin \omega t + \sin \varepsilon \cos \omega t]$$

Or,

$$x = A \sin(\omega t + \varepsilon) \tag{4.5}$$

where

$$A = \sqrt{C^2 + D^2}$$

$$= \sqrt{(A_1 + A_2 \cos \delta)^2 + (A_2 \sin \delta)^2}$$

$$\therefore A = \sqrt{A_1^2 + 2A_1A_2 \cos \delta + A_2^2} \tag{4.6}$$

and

$$\tan \varepsilon = \frac{D}{C} = \frac{A_2 \sin \delta}{A_1 + A_2 \cos \delta} \tag{4.7}$$

Equation (4.5) shows that the resultant of two collinear simple harmonic motions having the same frequency is itself a simple harmonic motion. The amplitude and phase of the resultant SHM depends on the amplitudes of the two individual SHMs as well as the phase difference between them.

Self Assessment Question (SAQ) 6: What is the maximum possible amplitude of the two simple harmonic motions and when does it occur?

Self Assessment Question (SAQ) 7: What is the minimum possible amplitude of two simple harmonic motions and when does it occur?

(b) *Vector Method*

We can arrive at the same results for the superposition of two collinear harmonic oscillations by using vector method too. In Unit 2, you learnt how SHM can be represented as rotating vector. We will make use of this representation to obtain the resultant of the superposition of two harmonic oscillations here. Let us represent the first SHM by a vector of magnitude A_1 and the second SHM by the vector of magnitude A_2. These two vectors are shown in the figure below. We assume here that the phase angle of the first vector is zero and for the second vector it is equal to the phase difference of δ.

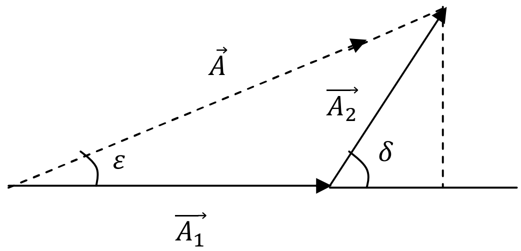

Figure 19: Two vectors A_1 and A_2 representing SHMs along with the resultant vector A.

vector $\vec{A}$. This vector represents the resultant SHM. Further, the magnitude of the resultant vector A is given by

$$A = \sqrt{A_1^2 + 2A_1A_2\cos\delta + A_2^2}$$

which is the same as equation (4.6).

The resultant $\vec{A}$ makes an angle ε with the x-axis and can be expressed as

$$\tan\varepsilon = \frac{Height}{Base} = \frac{A_2\sin\delta}{A_1 + A_2\cos\delta}$$

which is the same as equation (4.7).

The advantage with the vector method is that it can easily be extended to more than two vectors. For example, if we have 3 vectors,

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \delta_1)$$

$$x_3 = A_3 \sin(\omega t + \delta_2)$$

they can be represented as shown in the figure below.

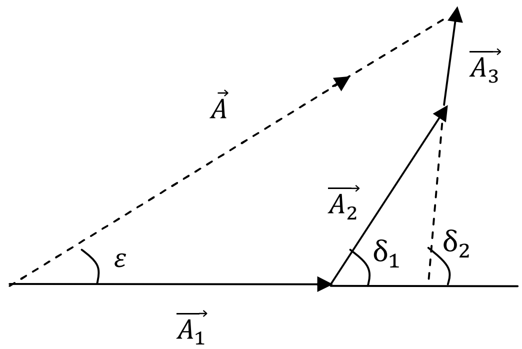

Figure 20: Three vectors representing SHMs along with the resultant vector.

The resultant vector is given by

$$x = A \sin(\omega t + \varepsilon)$$

Using trigonometry, we can see that

$$A = \sqrt{(A_1 + A_2 \cos \delta_1 + A_3 \cos \delta_2)^2 + (A_2 \sin \delta_1 + A_3 \sin \delta_2)^2}$$

And

$$\tan \varepsilon = \frac{Height}{Base} = \frac{A_2 \sin \delta_1 + A_3 \sin \delta_2}{A_1 + A_2 \cos \delta_1 + A_3 \cos \delta_2}$$

Example 3: A particle is subjected to two simple harmonic oscillations

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \frac{\pi}{3})$$

Determine (a) the displacement at t = 0, (b) the maximum speed of the particle and (c) the maximum acceleration of the particle.

Solution: (a) At t = 0,

$$x_1 = A_1 \sin 0 = 0$$

and

$$x_2 = A_2 \sin\left(\frac{\pi}{3}\right) = \frac{A_2\sqrt{3}}{2}$$

Thus, the resultant displacement at t = 0 is

$$x = x_1 + x_2 = \frac{A_2\sqrt{3}}{2}$$

(b) From (4.6), we can write the amplitude of the resultant of the two harmonic oscillations as

$$A = \sqrt{A_1^2 + 2A_1A_2 \cos\left(\frac{\pi}{3}\right) + A_2^2}$$

$$= \sqrt{A_1^2 + A_1A_2 + A_2^2}$$

Therefore, the maximum speed is

$$|v|_{max} = \omega A = \omega\sqrt{A_1^2 + A_1A_2 + A_2^2}$$

(c) The maximum acceleration is

$$a_{max} = \omega^2 A = \omega^2\sqrt{A_1^2 + A_1A_2 + A_2^2}$$

Self Assessment Question (SAQ) 8: Calculate the amplitude and initial phase of the harmonic oscillations obtained by the superposition of two collinear oscillations represented by the following equations:

$$x_1 = (0.02\ m)\sin(5\pi t + \frac{\pi}{2})$$

$$x_2 = (0.03\ m)\sin(5\pi t + \frac{\pi}{4})$$

4.4.2 Oscillations Having Different Frequencies: Beats

Suppose that we have two collinear harmonic oscillations of different frequencies and amplitudes. For simplicity, we assume that the two oscillations have the same phase constant which we take as zero. The two SHMs with displacements x_1 and x_2 and angular frequencies ω_1 and ω_2 ($< \omega_1$) respectively can be written as,

$$x_1 = A_1 \sin \omega_1 t$$

$$x_2 = A_2 \sin \omega_2 t$$

From the superposition principle, the resultant of these two oscillation is given by

$$x = x_1 + x_2$$

$$= A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$$

Now, to simplify the above equation, we define two terms: average frequency, ω_a and modulation frequency, ω_m as

$$\omega_a = \frac{\omega_1 + \omega_2}{2} \quad and \quad \omega_m = \frac{\omega_2 - \omega_1}{2}$$

Thus, we have

$$\omega_1 = \omega_a - \omega_m$$

$$\omega_2 = \omega_a + \omega_m$$

Substituting ω_1 and ω_2, we get

$$x = A_1 \sin(\omega_a - \omega_m)t + A_2 \sin(\omega_a + \omega_m)t$$

$$= A_1\{\sin \omega_a t \cos \omega_m t - \cos \omega_a t \sin \omega_m t\} + A_2\{\sin \omega_a t \cos \omega_m t + \cos \omega_a t \sin \omega_m t\}$$

$$\therefore x = (A_1 + A_2) \sin \omega_a t \cos \omega_m t - (A_1 - A_2) \cos \omega_a t \sin \omega_m t \qquad (4.8)$$

Now, let us define an amplitude $A(t)$ and a phase constant $\varepsilon(t)$, which are both functions of time, such that

$$(A_1 + A_2) \cos \omega_m t = A \cos \varepsilon \qquad (4.9)$$

and

$$-(A_1 - A_2) \sin \omega_m t = A \sin \varepsilon \qquad (4.10)$$

Using the above substitutions in equation (4.8), we get

$$x = A \cos \varepsilon \sin \omega_a t + A \sin \varepsilon \cos \omega_a t$$

$$\therefore x = A \sin(\omega_a t + \varepsilon) \qquad (4.11)$$

Thus, we find that the amplitude A and the phase constant ε, that we defined earlier, is actually the resultant amplitude and the resultant phase constant. The above equation can be misleading.

Although the above equation resembles the equation for SHM, it will be wrong to conclude that the resultant motion is SHM. It is so because the amplitude, A and the phase constant ε are not constant as such; rather, they are dependent on time. This oscillation can, at best, be described as periodic with an angular frequency of ω_a, the average of the two component frequencies.

The resultant amplitude can be calculated squaring equations (4.9) and (4.10) and adding them together

$$A^2(\sin^2\varepsilon + \cos^2\varepsilon) = (A_1^2 + A_2^2)(\sin^2\omega_m t + \cos^2\omega_m t) + 2A_1A_2(\cos\omega_m t - \sin\omega_m t)$$

$$A^2 = (A_1^2 + A_2^2) + 2A_1A_2(\cos^2\omega_m t - \sin^2\omega_m t)$$

$$A^2 = (A_1^2 + A_2^2) + 2A_1A_2\cos(2\omega_m t)$$

$$\therefore A(t) = \sqrt{A_1^2 + 2A_1A_2\cos(2\omega_m t) + A_2^2} \tag{4.12}$$

And the resultant phase constant can be calculated dividing equation (4.10) by (4.9)

$$\tan\varepsilon = \frac{-(A_1 - A_2)\sin\omega_m t}{(A_1 + A_2)\cos\omega_m t} \tag{4.13}$$

Beats

When the component frequencies are nearly equal, i.e. $\omega_1 \approx \omega_2$, the modulated frequency will be very small in comparison with the average frequency,

$$\omega_m \ll \omega_a$$

In such a scenario, the slowly varying modulated amplitude (resultant amplitude) A and the modulated phase constant (resultant phase constant) vary only slightly with time and may be treated as almost constant during the time scale of interest, which, in our case, is the time period given by

$$T = \frac{2\pi}{\omega_a}$$

In such a condition, equation (4.11) will represent an approximate harmonic oscillations having angular frequency ω_a. The resulting oscillations, in the case when the two frequencies of the SHMs are nearly equal, exhibit what are called beats.

Example 4: When is the modulated amplitude maximum?

Solution: From equation (4.12), the maximum amplitude $= (A_1 + A_2)$, when

$$\cos(2\omega_m t) = 1$$

$$\text{or} \quad 2\omega_m t = 0, 2\pi, \ldots$$

$$(\omega_2 - \omega_1)t = 0, 2\pi, \ldots$$

$$t = 0, \frac{2\pi}{(\omega_2 - \omega_1)}, \ldots$$

$$\therefore t = 0, \frac{1}{(\nu_2 - \nu_1)}, \ldots$$

Hence the time interval between two consecutive maxima is $\frac{1}{(\nu_2 - \nu_1)}$. The frequency of the maxima is $(\nu_2 - \nu_1)$.

Example 5: When is the modulated amplitude minimum?

Solution: From equation (4.12), the minimum amplitude $= |A_1 - A_2|$, when

$$\cos(2\omega_m t) = -1$$

$$\text{or} \quad 2\omega_m t = \pi, 3\pi, \ldots$$

$$t = \frac{\pi}{(\omega_2 - \omega_1)}, \frac{3\pi}{(\omega_2 - \omega_1)}, \ldots$$

$$\therefore t = \frac{1}{2(\nu_2 - \nu_1)}, \frac{3}{2(\nu_2 - \nu_1)}, \ldots$$

Hence, the time interval between two consecutive minima is $\frac{1}{(\nu_2 - \nu_1)}$. The frequency of the minima is $(\nu_2 - \nu_1)$.

The periodic variation of the amplitude of the motion, resulting from the superposition of SHMs of slightly different frequencies, is known as the phenomenon of beats. A maxima followed by a minima is technically called a beat. The time period t_b between the successive beats is called the beat period given by

$$t_b = \frac{1}{|\nu_2 - \nu_1|} \tag{4.14}$$

And the beat frequency ν_b is given by

$$\nu_b = |\nu_2 - \nu_1| \tag{4.15}$$

Hence, the beat frequency is equal to the difference between the frequencies of the component oscillations.

The figure below displays graphically the result of superimposing two harmonic oscillations of different frequencies. You may note that superposition of two SHMs of different frequencies results in oscillations that are periodic but not harmonic.

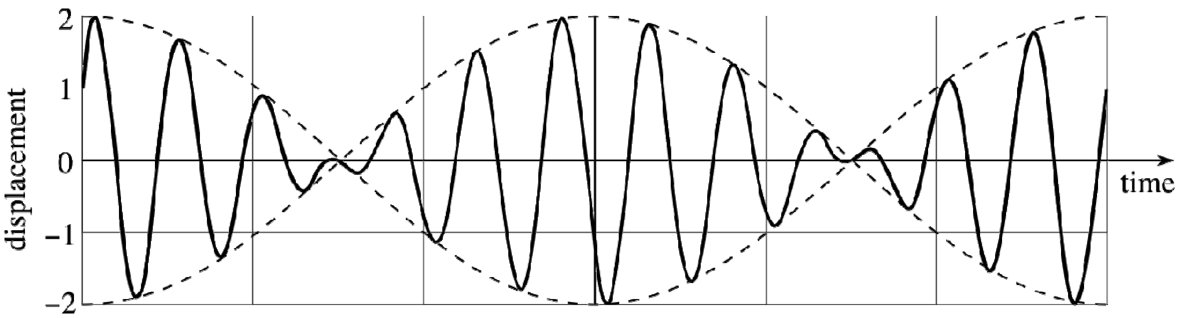

Figure 21: Superposition of two collinear SHMs having same amplitude but differing in angular frequency.

Application of Beats

The phenomenon of beats is of great importance. Beats can be used to determine the small difference between frequencies of two sources of sound. Musicians often make use of beats in tuning their instruments. If the instrument is out of tune, one will hear beats. Sometimes beats are deliberately produced in a particular section of an orchestra to give a pleasing tone to the resulting sound. There are many physical phenomena which involve beats.

Example 6: Two radio stations broadcast their programmes at the same amplitude A, and at slightly different frequencies, where their difference is equal to 10^3 Hz. A detector receives the signals from the two stations simultaneously. Find the time interval between successive maxima of the intensity of the signal received by the detector.

Solution: The time interval between successive maxima of the intensity is equal to the beat period, given by

$$t_b = \frac{1}{|\nu_2 - \nu_1|} = \frac{1}{10^3} = 10^{-3}\ \text{s}$$

When two tuning forks of same frequency are sounded, a continuous sound is heard. When one of the tuning forks is waxed a little, so as to reduce its frequency, beats are heard because now the frequencies of the two tuning forks are slightly different. By counting the number of beats heard in a given interval of time, one can calculate the beat frequency.

Example 7: A tuning fork A produces 4 beats with tuning fork B of frequency 256 Hz. When A is waxed, the beats are found to occur at shorter intervals. What was its original frequency?

Solution: As the tuning fork A produces 4 beats with B of frequency 256 Hz, from equation (4.15), the frequency of A should be

$$\nu_1 = (\nu_2 \pm \nu_b) = 256 \pm 4\ Hz$$

Now when tuning fork A is waxed, we are given that the beat period

$$t_b = \frac{1}{|\nu_2 - \nu_1|}\ decreases$$

Or, in other words, $\nu_b = |\nu_2 - \nu_1|$ increases.

Let the frequency of the tuning fork A is $\nu_1 = 256 + 4 = 260\ Hz$. When tuning fork A is waxed, its frequency ν_1 will decrease, and hence the beat frequency should also decrease. But it is not what we have found above. So, frequency of A can not be 260 Hz.

Let $\nu_1 = 256 - 4 = 252\ Hz$, then its frequency ν_1 decreases due to waxing, the beat frequency increases. Hence, $\nu_1 = 252\ Hz$ is the correct answer.

Self Assessment Question (SAQ) 9: Choose the correct option.

If a tuning fork of frequency 512 Hz is sounded with a string of frequency 500 Hz, the beats produced per second will be

(a) 10 (b) 12 (c) 6 (d) 0 (e) 1012

4.5 SUMMARY

In this unit, we studied how to obtain the resultant oscillation when two or more collinear oscillations are superposed. We discovered that, the resultant oscillation is not SHM, even if the component oscillations are SHM. However, under certain conditions, the resultant oscillation can be considered to be simple harmonic. Further, when the superposing oscillations hace slightly different frequencies, then a phenomenon known as beats is observed. The beats have important practical applications.

4.6 GLOSSARY

Beats – the subjective difference in tone when two sound waves of nearly equal frequencies are simultaneously applied to one ear. It appears as a periodic increase and decrease of the intensity.

Frequency – the number of complete cycles per second made by a vibrating object.

Modulation – the change of amplitude or frequency of a carrier signal of given frequency.

Tuning fork – A fork with two prongs and heavy cross-section, generally made of steel. Specially designed to retain a constant frequency of oscillation when struck. Widely used for tuning musical instruments because its frequency is independent of the changes in temperature, atmospheric pressure and humidity.

Waxing – application of a thin layer of wax.

4.7 TERMINAL QUESTIONS

1. Find the amplitude and the phase angle of the simple harmonic motion obtained by combining the motions $x_1 = (2\ cm)\sin\omega t$ and $x_2 = (2\ cm)\sin(\omega t + \pi/3)$.

2. A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions, find the phase difference between the individual motions.

3. Two particles are oscillating along the same line with the same frequency and the same amplitude. They meet each other at a point midway between the mean position and the extreme position while going in opposite direction. Find the phase difference between their motions.

4. Two simple harmonic motions are represented by the equations

$$y_1 = 10\sin(3t + \pi/4)$$

$$y_2 = 5(\sin 3t + \cos 3t)$$

What is the ratio of their amplitudes?

5. Two vibrations along the same line are described by the equations

$$x_1 = 0.03\cos(10\pi t)$$

$$x_2 = 0.03\cos(12\pi t)$$

where x_1, x_2 are measured in meters and t is in seconds. Obtain the equation describing the resultant motion and hence find the beat frequency.

6. Choose the correct option.

A tuning fork produces 6 beats per sec. with another fork of frequency 384 Hz. If the prongs of the first fork are slightly filed (it is the opposite of waxing and leads to an increase in the individual frequency), 4 beats per sec. are produced. The frequency of the first fork after the filing is

(a) $390\ Hz$ (b) $378\ Hz$ (c) $380\ Hz$ (d) $388\ Hz$

7. Choose the correct option.

Two adjacent piano keys are struck simultaneously. The notes emitted by them have frequencies ν_1 and ν_2. The number of beats heard per second is

(a) $\frac{1}{2}(\nu_1 - \nu_1)$ (b) $\frac{1}{2}(\nu_1 + \nu_1)$ (c) $(\nu_1 - \nu_2)$ (d) $\frac{1}{2}(\nu_1 - \nu_1)$

8. Choose the correct option.

Beats are the result of

(a) Diffraction
(b) Destructive interference
(c) Constructive and destructive interference
(d) Superposition of two waves of nearly equal frequencies

9. Two vibrations along the same line are described by the equations

$$x_1 = 0.03\cos(320\pi t)$$

$$x_2 = 0.03\cos(326\pi t)$$

Calculate the number of beats produced per sec?

10. Write short notes on:

(i) Superposition principle (ii) Beats

4.8 ANSWERS

Selected Self Assessment Questions (SAQs):

5. Differentiate equation (4.4), to get $\vec{v} = \overrightarrow{v_1} + \overrightarrow{v_2}$.

6. If the phase difference, $\delta = 0$, the two SHMs are in phase and from equation (4.6), we have

$$A = \sqrt{A_1^2 + 2A_1A_2 + A_2^2} = A_1 + A_2$$

7. If the phase difference, $\delta = \pi$, the two SHMs are out of phase. From equation (4.6), we have

$$A = \sqrt{A_1^2 - 2A_1A_2 + A_2^2}$$

Since the amplitude cannot be negative, we get

$$A = |A_1 - A_2|$$

If $A_1 = A_2$, the resultant amplitude is zero and the particle does not oscillate at all.

8. The phase difference δ between the two vectors is given by

$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\ rad$$

The resultant amplitude is given by equation (4.6),

$$A = \sqrt{(0.02)^2 + 2(0.02)(0.03)\cos\frac{\pi}{4} + (0.03)^2} = 0.046\ m$$

The resultant angle is given by equation (4.7),

$$\varepsilon = \tan^{-1}\frac{(0.03)\sin\frac{\pi}{4}}{(0.02) + (0.03)\cos\frac{\pi}{4}}$$

$$= \tan^{-1}\frac{3}{2\sqrt{2} + 3}$$

9. (b)

Selected Terminal Questions:

1. The two given equations represent simple harmonic oscillations along the x-axis, both having amplitudes of 2 cm. The phase difference between the two SHMs is, $\delta = \pi/3$. Thus, their resultant will have an amplitude A given as

$$A = \sqrt{A_1^2 + 2A_1A_2\cos\delta + A_2^2}$$

$$= \sqrt{(2)^2 + 2(2)(2)\cos\frac{\pi}{3} + (2)^2} = 3.5\ cm$$

From equation (4.7), the phase constant ε is given by

$$\varepsilon = \tan^{-1}\frac{A_2\sin\delta}{A_1 + A_2\cos\delta} = \tan^{-1}\frac{2\sin\frac{\pi}{3}}{2 + 2\cos\frac{\pi}{3}}$$

$$= \tan^{-1}\frac{2\left(\frac{\sqrt{3}}{2}\right)}{2+2\left(\frac{1}{2}\right)} = \tan^{-1}\frac{1}{\sqrt{3}}$$

$$\therefore \varepsilon = \frac{\pi}{6}$$

2. Let the amplitudes of the individual motions be A each. The resultant amplitude is also A. From equation (4.6), we have

$$A = \sqrt{A^2 + 2A^2\cos\delta + A^2}$$

Or,

$$\cos\delta = -\frac{1}{2}$$
$$\therefore \delta = \frac{2\pi}{3}$$

3. Let t = 0 be the instant when the particles cross each other at x = A/2 with equal and opposite velocities. Let the equation of motion be given by

$$x(t) = A\sin(\omega t + \delta)$$

At t = 0,

$$\frac{A}{2} = A\sin\delta$$

$$\therefore \delta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$\delta = \frac{\pi}{6}$ correspond to a positive initial velocity and $\delta = \frac{5\pi}{6}$ corresponds to a negative initial velocity. The equation for the two particles is thus,

$$x_1(t) = A\sin\left(\omega t + \frac{\pi}{6}\right) \quad and \quad x_2(t) = A\sin\left(\omega t + \frac{5\pi}{6}\right)$$

The phase difference is therefore,

$$= \left(\omega t + \frac{\pi}{6}\right) - \left(\omega t + \frac{5\pi}{6}\right) = \frac{2\pi}{3}$$

4. The amplitude of the first oscillation $y_1 = 10\sin(3t + \pi/4)$ is 10.

The second oscillation $y_2 = 5(\sin 3t + \cos 3t)$ is a combination of two oscillations,

$$y = 5\sin 3t \quad and \quad y = 5\cos 3t = 5\sin\left(3t + \frac{\pi}{2}\right)$$

Hence, the phase difference $\delta = \frac{\pi}{2}$. From equation (4.6), we get the resultant amplitude as

$$A = \sqrt{5^2 + 2(5)(5)\cos\left(\frac{\pi}{2}\right) + 5^2} = 5\sqrt{2}$$

Therefore, the required ratio of the amplitudes is

$$= \frac{10}{5\sqrt{2}} = \sqrt{2}$$

5. Using the superposition principle, the resultant motion is given by

$$x = x_1 + x_2$$

$$= 0.03\{\cos(10\pi t) + \cos(12\pi t)\}$$

Using the trigonometric identity,

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

We get,

$$x = 0.06\cos(\pi t)\cos(11\pi t)$$

which is of the form $x = A\sin(\omega_a t + \varepsilon)$ [equation (4.16)], where $A = 0.06\cos(\pi t)$, $\omega_m = \pi$ and $\omega_a = 11\pi$.

We know

$$\omega_m = \frac{\omega_2 - \omega_1}{2}$$

Or,

$$\omega_2 - \omega_1 = 2\pi$$

Hence the beat frequency is

$$\nu_b = |\nu_2 - \nu_1|$$

$$= \frac{2\pi}{|\omega_2 - \omega_1|} = \frac{2\pi}{2\pi} = 1\ Hz$$

6. (c)

7. (c)

8. (d)

9. $\omega_1 = 320\pi$ and $\omega_2 = 326\pi$. The beat frequency is given by

$$\nu_b = |\nu_2 - \nu_1| = \frac{2\pi}{|\omega_2 - \omega_1|}$$

$$= \frac{2\pi}{6\pi} = 3\ Hz$$

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4.10 SUGGESTED READINGS

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11. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
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UNIT 5 SUPERPOSITION OF TWO MUTUALLY PERPENDICULAR HARMONIC OSCILLATION

Structure

5.1 INTRODUCTION

Our discussion in the earlier units has been confined to collinear harmonic oscillations. However, oscillations can also take place in two, or three dimensions. For example, the simple pendulum that we studied in the earlier units can be actually visualized as a three-dimensional oscillator, because it can swing from north to south or east to west – and it can also moves up and down like a spring pendulum. It can execute all these motions simultaneously, so that the general motion of the bob is a complex path in space. In addition to these swinging motions there could also be the twisting motion around the string; the torsional motion. Taken together, the pendulum may be subjected to four independent oscillations simultaneously, which makes our job of analyzing the motion of the bob quite challenging!

In this unit, we will study the superposition of two mutually perpendicular harmonic oscillations and learn how Lissajous figures can be used to describe the path of the resultant motion when two mutually perpendicular oscillations are superposed.

5.2 OBJECTIVES

After studying this unit, you should be able to

- differentiate between independent and coupled oscillations
- explain what is two-dimensional SHM, with an example
- apply the principle of superposition to obtain the resultant of two mutually perpendicular harmonic oscillations having different amplitudes but same frequencies
- show the equivalence of two mutually perpendicular harmonic oscillations with uniform circular motion
- apply the principle of superposition to obtain the resultant of two mutually perpendicular harmonic oscillations having different amplitudes and different frequencies
- describe how Lissajous figures are drawn

5.3 TWO MUTUALLY PERPENDICULAR HARMONIC OSCILLATIONS

As mentioned before, a pendulum (Fig. 1) may be subjected to four independent oscillations simultaneously.

- It can swing from north to south (i.e. about the y-axis)
- It can swing from east to west (i.e. about the x-axis)
- It can also swing up and down like a spring pendulum (to some extent depending on the elasticity of the string)
- And lastly, it can also exhibit a twisting motion around the string (torsional motion)

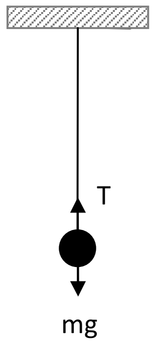


Figure 22: A pendulum, consisting of a bob hanging with a string.

These independent motions of an oscillator are called the modes of vibration of the oscillator and their number is called the number of degrees of freedom of the oscillator. In this case, there are four modes of vibration, including three swinging modes and one torsional mode, and there are, therefore, four degrees of freedom. You will appreciate that when all these modes are in action together, the task of predicting the motion of the bob is quite challenging! Fortunately, the task is made much simpler if we know that the various motions are truly independent.

The condition for independent oscillators is that the oscillators are unaware of each other – either because they are far away from each other or because the displacement of one does not affect the motions of the others. In the case of simple pendulum described above, each of its degrees of freedom represents an independent oscillator. We can be confident of this with our pendulum since the restoring forces in the swinging modes depend on the weight of the bob and this is independent of where the bob is placed. The torsional mode is controlled by the physical properties of the string, (i.e. its elasticity) independent of gravity, and so this mode is independent of the others. On the other hand, if the two oscillators or two modes of a single oscillator are not independent, they are said to be coupled.

In general, it is important to develop a means of combining the effect of two independent oscillations acting on a body simultaneously. In the following, you will learn to do this for the case of two independent, perpendicular SHMs.

5.3.1 Oscillations Having Same Frequencies

Let's now study a simpler case, where we assume that two independent forces are acting on a particle in such a manner that the first alone produces a simple harmonic motion in the x-direction given by

$$x = A_1 \sin \omega t \tag{5.1}$$

and the second would produce a simple harmonic motion in the y-direction given by

$$y = A_2 \sin(\omega t + \delta) \tag{5.2}$$

Thus, we are actually considering the superposition of two mutually perpendicular SHMs which have equal frequencies. The amplitudes may be different and their phases differ by δ. The resultant motion of the particle is a combination of the two SHMs.

The position of the particle at any time t is given by (x, y) where x and y are given by the above equations. The *resultant motion is, thus, two-dimensional* and the path of the particle is, in general, an ellipse. The equation of the path traced by the particle is obtained by eliminating t from equations (5.1) and (5.2).

From equation (5.1), we get

$$\sin \omega t = \frac{x}{A_1} \text{ ; which gives } \cos \omega t = \sqrt{1 - \left(\frac{x}{A_1}\right)^2}$$

Putting these values in equation (5.2), we get

$$y = A_2 \sin(\omega t + \delta) = A_2[\sin \omega t \cos \delta + \cos \omega t \sin \delta]$$

$$= A_2 \left[\left(\frac{x}{A_1}\right) \cos \delta + \left(\sqrt{1 - \left(\frac{x}{A_1}\right)^2}\right) \sin \delta\right]$$

Or,

$$\left(\frac{y}{A_2} - \frac{x}{A_1} \cos \delta\right)^2 = \left(1 - \left(\frac{x}{A_1}\right)^2\right) \sin^2 \delta$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy \cos \delta}{A_1 A_2} = \sin^2 \delta \tag{5.3}$$

As we can see, equation (5.3) is an equation of ellipse. Thus, we may conclude that the resultant motion of the particle is along an elliptical path.

Equation (5.3) shows that x remains between $-A_1$ and A_1 and that of y remains between $-A_2$ and A_2. Thus, the particle always remains inside the rectangle defined by

$$x = \pm A_1 \quad and \quad y = \pm A_2$$

The ellipse given by equation (5.3) is shown in the figure below:

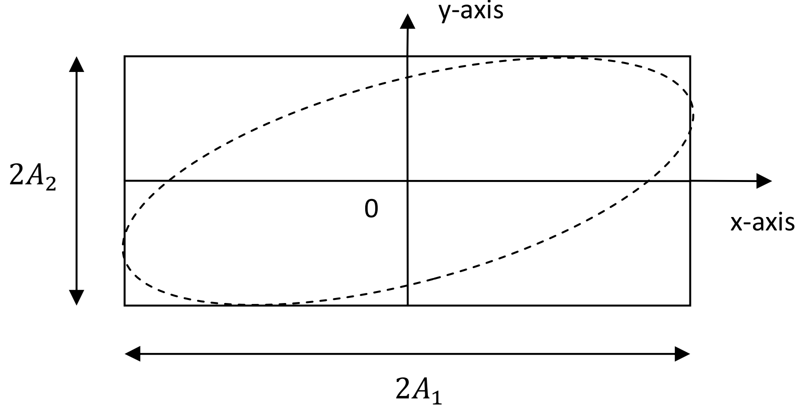


Figure 23: Elliptical path followed by a particle on which two independent SHMs, which are perpendicular to each other, act simultaneously.

Special Cases

- The two component SHMs are in phase, $\delta = 0$
- The two component SHMs are out of phase, $\delta = \pi$
- The phase difference between the two component SHMs, $\delta = \pi/2$

Let us now obtain the resultant motion of the particle under thes special cases.

(a) When the two superposing SHMs are in phase, $\delta = 0$ and equation (5.3) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} - \frac{x}{A_1}\right)^2 = 0$$

$$\therefore y = \frac{A_2}{A_1} x \tag{5.4}$$

Equation (5.4) is an equation of a straight line passing through the origin and having a slope of $\tan^{-1}\left(\frac{A_2}{A_1}\right)$. The figure below shows the path followed by the particle in this case. The particle moves on the diagonal (shown by the dotted line) of the rectangle.

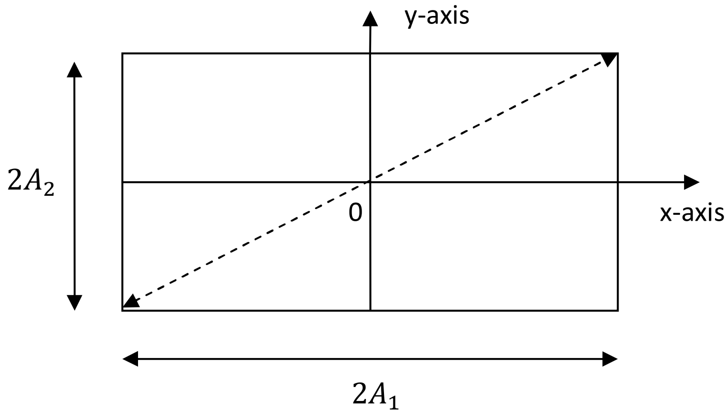


Figure 24: The straight line path traced by the resultant motion of the particle when the phase difference, $\boldsymbol{\delta = 0}$.

Equation (5.4) can also be obtained directly from equations (5.1) and (5.2) putting $\delta = 0$. The displacement of the particle on this straight line at any time t is

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(A_1 \sin \omega t)^2 + (A_2 \sin \omega t)^2} = \sqrt{A_1^2 + A_2^2} \sin \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency and phase as the component motions. However, the amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

(b) When the two superposing SHMs are out of phase, the phase difference between them is $\delta = \pi$. Thus, from equation (5.3), we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} + \frac{x}{A_1}\right)^2 = 0$$

$$\therefore y = -\frac{A_2}{A_1} x \qquad (5.5)$$

Equation (5.5) is an equation of a straight line passing through the origin and having a slope $\tan^{-1}\left(-\frac{A_2}{A_1}\right)$. The figure below shows the path followed by the particle. The particle moves on one of the diagonals (shown by dotted line) of the rectangle.

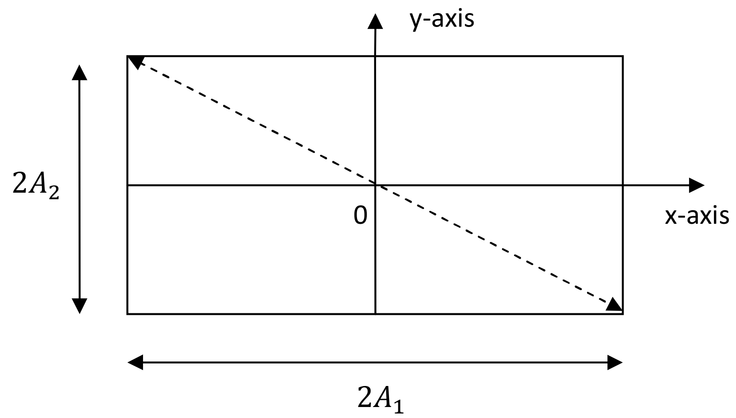


Figure 4: The straight line path traced by the resultant motion of the particle when the phase difference, $\boldsymbol{\delta = \pi}$.

Equation (5.5) can also be obtained directly on the basis of equations (5.1) and (5.2) and putting $\delta = \pi$. Further, the displacement of the particle on this straight line path at a given time t is

$$r = \sqrt{x^2 + y^2} = \sqrt{(A_1 \sin \omega t)^2 + (A_2 \sin(\omega t + \pi))^2}$$

$$= \sqrt{(A_1 \sin \omega t)^2 + (-A_2 \sin \omega t)^2} = \sqrt{A_1^2 + A_2^2} \sin \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency as the component motions. The amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

(c) When the phase difference between the two component SHMs is $\delta = \pi/2$.

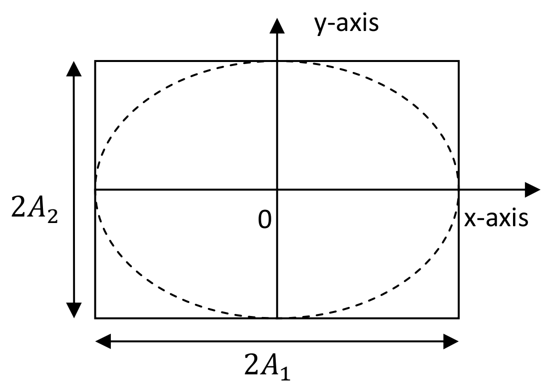


Figure 5: The elliptical path traced by the resultant motion of the particle when the phase difference, $\boldsymbol{\delta = \pi/2}$.

From equation (5.3), we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \qquad (5.6)$$

The above equation is a standard equation of an ellipse with its axes along the x-axis and the y-axis and with its center at the origin. The lengths of the major and the minor axes are $2A_1$ and $2A_2$, respectively. The path traced by the particle (shown by the dotted line) is depicted in Fig. 5.

In case the amplitudes of the two individual SHMs are equal, $A_1 = A_2 = A$, i.e. the major and the minor axes are equal, then the ellipse reduces to a circle.

$$x^2 + y^2 = A^2 \qquad (5.7)$$

Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of $\pi/2$ is a *circular motion*. The circular motion may be clockwise or anticlockwise depending on which component leads the other.

Example 1: Show that the superposition of oscillations represented by

$$x = A \sin \omega t$$

$$y = -A \cos \omega t$$

results in to circular motion traced in the anticlockwise sense.

Solution: Using equations (5.1) and (5.2), taking $A_1 = A_2 = A$ and putting the phase difference between the two perpendicular components, $\delta = -\pi/2$, we get,

$$x = A \sin \omega t$$

$$y = A \sin\left(\omega t - \frac{\pi}{2}\right) = -A \cos \omega t$$

Hence, we can see these are the same coordinates that are given to us. Therefore, $\delta = -\pi/2$.

Putting $\delta = -\pi/2$ in equation (5.3), we get

$$x^2 + y^2 = A^2$$

which is same as equation (5.7) and represents a circle.

For the direction, let us see what are the coordinates of the particle at t = 0 and t = T/4.

$$At\ t = 0, \qquad (x, y) = (0, -A)$$

$$At\ t = \frac{T}{4} = \frac{\pi}{2\omega}, \qquad (x, y) = (A, 0)$$

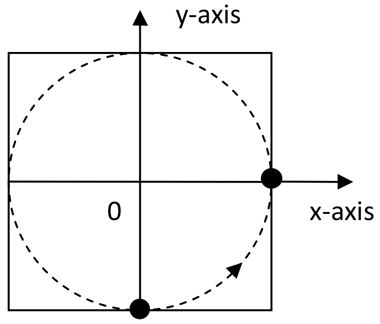


Figure 6

Clearly, one can see that the motion of the particle is anticlockwise.

Example 2: A particle is subjected to two simple harmonic oscillations, one along the x-axis and the other on a line making an angle of $\pi/4$ with the x-axis. The two motions are given by

$$x = A\sin\omega t$$

$$S = B\sin\omega t$$

Calculate the amplitude of the resultant motion.

Solution: The two individual SHMs, with equal frequencies are in phase because the phase difference $\delta = 0$.

Now, let's break the second SHM (shown in the figure below), into its x-component and the y-component.

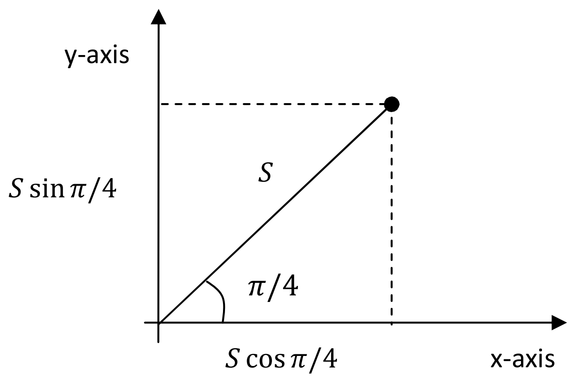


Figure 7

So, the x-component of the second SHM is given by

$$S\cos\pi/4 \;= B\sin\omega t\cos\pi/4$$

$$= \frac{B}{\sqrt{2}}\sin\omega t$$

And, the y-component of the second SHM is given by

$$S\sin\pi/4 \;= B\sin\omega t\sin\pi/4$$

$$= \frac{B}{\sqrt{2}}\sin\omega t$$

Therefore, if we want to see motion in the x-direction and the y-direction only, then we have three separate SHMs, two along the x-axis and one along the y-axis. They are

$$x_1 = A\sin\omega t$$

$$x_2 = \frac{B}{\sqrt{2}}\sin\omega t$$

$$y = \frac{B}{\sqrt{2}}\sin\omega t$$

The amplitude of the resultant oscillation for mutually perpendicular oscillations with zero phase difference is given by

$$A_{resultant} = \sqrt{A_1^2 + A_2^2}$$

$$= \sqrt{\left(A + \frac{B}{\sqrt{2}}\right)^2 + \left(\frac{B}{\sqrt{2}}\right)^2}$$

$$= \sqrt{A^2 + B^2 + \sqrt{2}AB}$$

Self Assessment Question (SAQ) 1: Show that the superposition of oscillations represented by

$$x = A\sin\omega t$$

$$y = A\cos\omega t$$

results in to circular motion traced in the clockwise sense.

Self Assessment Question (SAQ) 2: A particle is subjected to two simple harmonic motions in the perpendicular directions having equal amplitudes (0.01 m) and equal frequencies. If the two SHMs are out of phase, what is the nature of the path followed by the particle?

Self Assessment Question (SAQ) 3: A particle is subjected to two simple harmonic motions in the perpendicular direction having equal amplitudes (0.01 m) and equal frequencies. If the two component SHMs are in phase, what is the nature of path followed by the particle?

Self Assessment Question (SAQ) 4: A particle is subjected to two SHMs

$$y = \sin \omega t$$

$$z = \sin \omega t$$

What is the path that the particle follows?

Self Assessment Question (SAQ) 5: A particle is subjected to two simple harmonic oscillations, one along the x-axis and the other on a line making an angle of $3\pi/4$ with the x-axis. The two motions are given by

$$x = A \sin \omega t$$

$$S = B \sin \omega t$$

Calculate the amplitude of the resultant motion.

5.3.2 Oscillations Having Different Frequencies (Lissajous Figures)

When the frequencies of the two perpendicular SHMs are not equal, the resulting motion becomes more complicated. Let us suppose that the displacements of the two mutually perpendicular (orthogonal) oscillations are given by

$$x = A_1 \sin \omega_1 t$$

$$y = A_2 \sin(\omega_2 t + \delta)$$

The phase difference between them at any instant t, is given by

$$\Delta\varphi = (\omega_2 t + \delta) - \omega_1 t$$

$$= (\omega_2 - \omega_1)t + \delta$$

Since the superimposed orthogonal oscillations are of different frequencies, one of them changes faster than the other and will gain in phase over the other. As a result, the pattern of the resultant motion passes through different phases. It changes with time due to the change in the phase difference, which is also a function of time. However, the general shape traced out by the resultant oscillation is similar to that obtained for the case of equal frequencies, i.e. the motion is confined within a rectangle of sides $2A_1$ and $2A_2$.

When two independent perpendicular oscillations are superposed, the pattern of the resultant motion is described by a *Lissajous figure*, named after French mathematician, J. A. Lissajous (1822-1880) who made an extensive study of these motions. Lissajous figures can be seen by using a cathode ray oscilloscope (CRO) shown in the figure below.

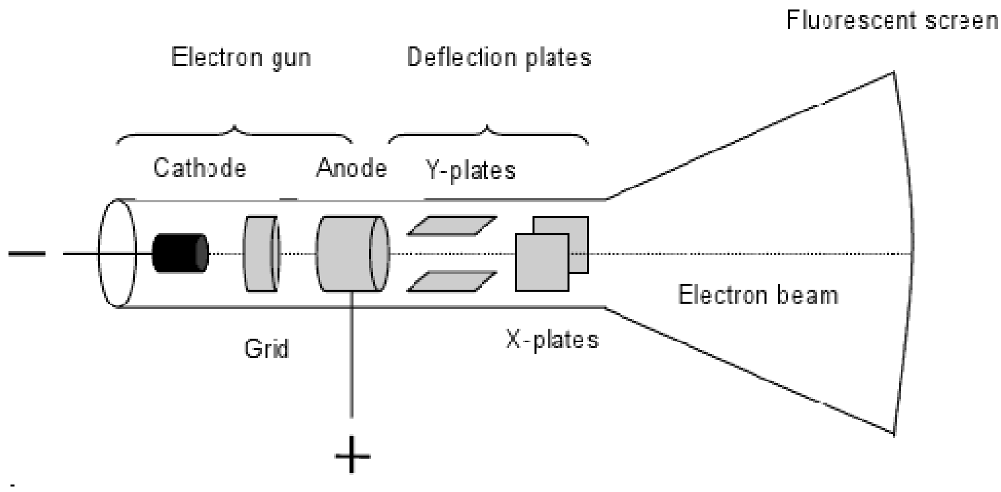


Figure 8: Basic structure of cathode ray oscilloscope.

Here, two rectangular oscillations are simultaneously imposed upon a beam of cathode rays by connecting two sources of electrical oscillations to horizontal plates XX and vertical plates YY of the oscilloscope. We then see the trace of the resultant effect in the form of an electron beam on the fluorescent screen. By adjusting the phases, amplitudes and the ratio of the frequencies of the applied voltage, we obtain various curves as shown in Fig. 9. Lissajous figures may be used to compare two nearly equal frequencies. If the frequencies of the two component oscillations are not exactly equal, the Lissajous figure will change gradually.

Let us now look at few examples to illustrate the shape of the Lissajous figure for some special cases.

Example 3: A particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$x = A_1 \cos \omega t$$

$$y = A_2 \cos(2\omega t + \delta)$$

Trace the trajectory of the motion of the particle using analytical method.

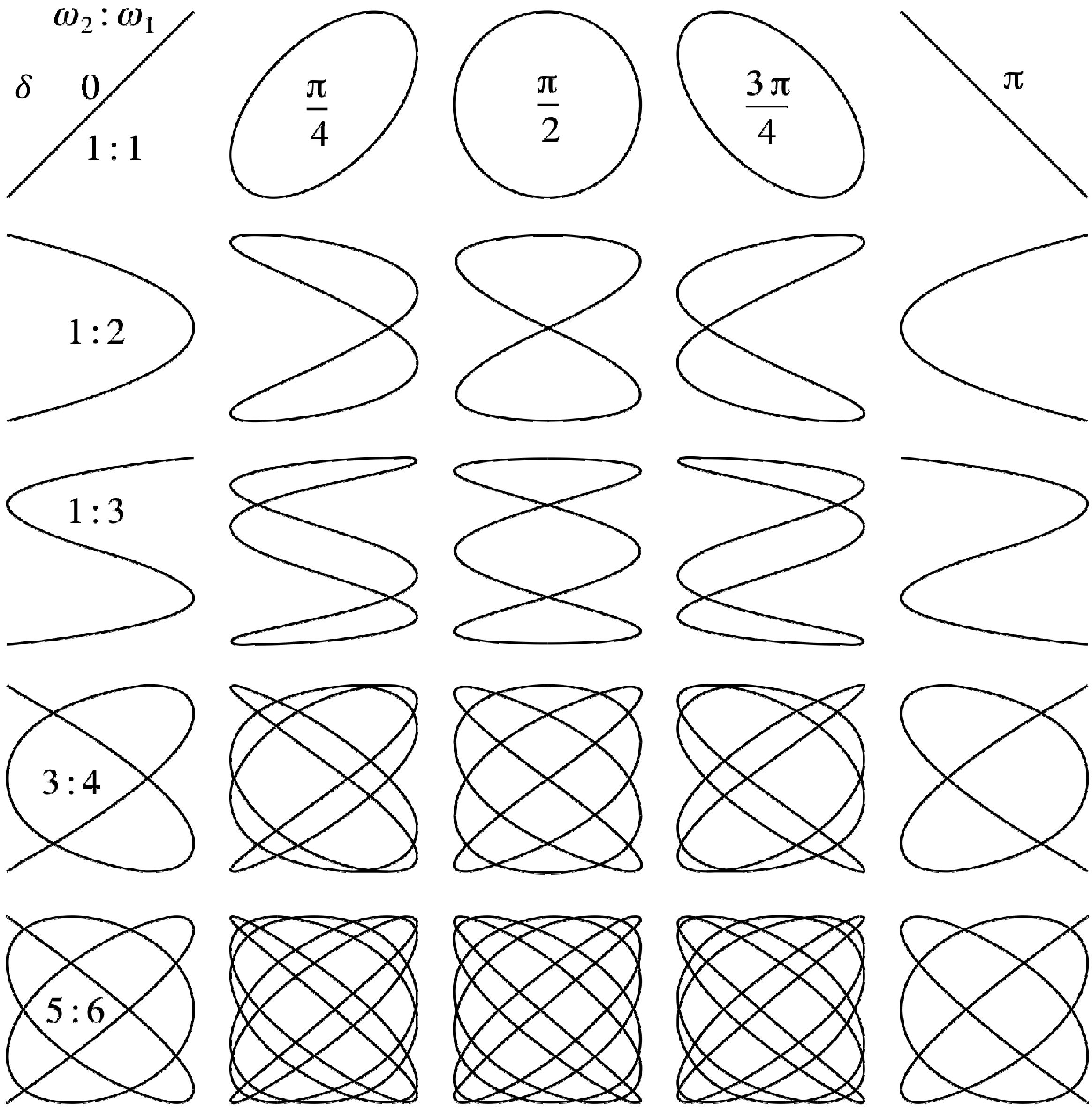


Figure 9: Lissajous figures formed due to superposition of two mutually perpendicular oscillations having different frequency ratios. The amplitudes of the superposing oscillations is equal to each other in this example. The phase difference varies from column to column.

Solution: From the given expressions for the two orthogonal oscillations being superposed, we note that their frequencies are in the ratio 1:2.

<u>*Analytical method*</u>

If we eliminate t from these two equations, we can determine the equation of the trajectory of the particle , i.e. we are looking for an expression for y in terms of x.

From the first SHM equation, we can write

$$\cos\omega t = \frac{x}{A_1}$$

And, expanding the second SHM equation, we get

$$y = A_2\cos(2\omega t + \delta)$$

$$= A_2[(2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t\cos\omega t\sin\delta]$$

Substituting y (x/A_1) for $\cos\omega t$ in the above expression, we get

$$\frac{y}{A_2} = \left\{2\left(\frac{x}{A_1}\right)^2 - 1\right\}\cos\delta - 2\left(\frac{x}{A_1}\right)\sqrt{1-\left(\frac{x}{A_1}\right)^2}\sin\delta$$

Rearranging the terms, we have

$$\left(\frac{y}{A_2} + \cos\delta\right) - 2\left(\frac{x}{A_1}\right)^2\cos\delta = -2\left(\frac{x}{A_1}\right)\sqrt{1-\left(\frac{x}{A_1}\right)^2}\sin\delta$$

On squaring the above expression on both sides and upon simplification, we get

$$\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\cos\delta\right) = 0$$

The above equation is of fourth degree, which, in general, represents a closed curve having two loops. For a given value of δ, the curve corresponding to the above expression can be traced using the knowledge of coordinate geometry.

Let us take the case when $\delta = 0$. Thus, $\cos\delta = 1$. The above expression reduces to

$$\left(\frac{y}{A_2} + 1\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\right) = 0$$

$$\therefore \left(\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2}\right)^2 = 0$$

This represents two coincident parabolas with their vertices at $(0, -A_2)$ as shown in Fig. 10 (using dotted lines). The equation of each parabola is

$$\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2} = 0$$

$$or \quad (y + A_2) = \frac{2A_2}{A_1^2} x^2$$

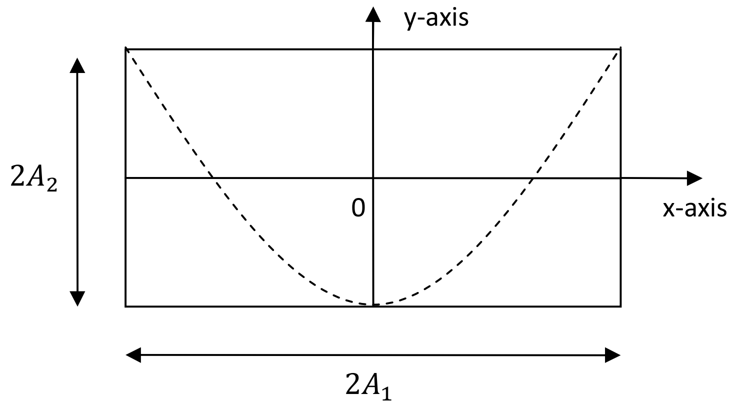


Figure 10: Superposition of two mutually perpendicular SHMs with frequencies in the ratio 1:2 and phase difference equal to zero.

Example 4: A particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$x = 2\cos t$$

$$y = \cos(t + 4)$$

Trace the trajectory of the particle using graphical method.

Solution: The analytical solution becomes very cumbersome if the phase difference δ is non-zero. In such cases, the resultant motion can be constructed quite conveniently by using graphical method.

Graphical method

Here, we set up a table of values to see what is happening. We give each point a "point number" so that it is easier to understand when we graph the curve.

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
x	2	1.4	0	-1.4	-2	-1.4	0	1.4	2
y	-0.6	0.1	0.7	1	0.7	-0.1	-0.8	-1	-0.7
Pt.no.	1	2	3	4	5	6	7	8	9

From the above table, the resulting curve, with the numbered points included, is shown in the following figure. Point 1 is actually equivalent to Point 9.

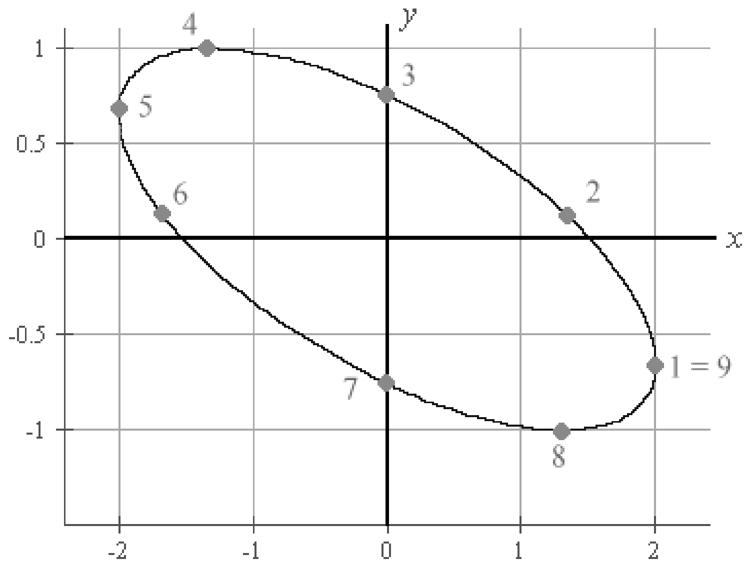


Figure 11

Self Assessment Question (SAQ) 6: Construct the Lissajous figures for the following component oscillations. If you are using graphical method, you may have to take more than 9 points to get the complete graph in some cases.

(a) $x = 2\sin t\,, y = \cos 2t$
(b) $x = \sin t\,, y = \cos(t + \pi/4)$
(c) $x = \sin \pi t\,, y = 2\sin(\pi t + \pi/2)$

Self Assessment Question (SAQ) 7: Any periodic motion, regardless of its complexity, can be reduced to the sum of a number of simple harmonic motions by the application of the superposition principle. Is the above statement true or false?

Self Assessment Question (SAQ) 8: A body is executing simple harmonic motion, and its displacement at time t is given by

$$x = 5\sin 3\pi t$$

Plot the displacement, velocity, and acceleration for two complete periods.

Self Assessment Question (SAQ) 9: A particle is simultaneously subjected to two simple harmonic motions in the same direction in accordance with the following equations:

$$y_1 = 8\sin 2\pi t \quad and \quad y_2 = 4\sin 6\pi t$$

Show graphically the resultant path of the particle.

5.4 SUMMARY

In this unit, we continued the study of superposition of waves. We learnt the superposition of two mutually perpendicular simple harmonic oscillations. By the application of superposition principle, one can combine any number of individual harmonic oscillations if we know that the different motions are not coupled but independent of each other. The condition for independent oscillators is that the oscillators are unaware of each other – either because they are far away from each other or because the motion of one does not affect the motions of the others.

We also learnt that, when the frequencies of the two mutually perpendicular SHMs are not equal, the resulting motion becomes more complicated. The pattern of the resultant motion is described by a Lissajous figure. We can use both analytical and graphical method to trace the resultant motion. If the phase difference δ between the superposing oscillations is non-zero, the analytical solution becomes very cumbersome.. In such cases, it is more convenient to use the graphical method.

5.5 GLOSSARY

Anticlockwise – if something is moving anticlockwise, it is moving in the opposite direction to the direction in which the hands of a clock move.

Clockwise – when something is moving clockwise, it is moving in a circle in the same direction as the hands on a clock.

Coupled – joined together or connected by a link.

Degree of freedom –number of ways in which a body may move or in which a dynamic system may change.

Frequency – the number of complete cycles per second made by a vibrating object.

Independent – if a thing is independent of another, they are separate and not connected, so the first one is not affected or influenced by the second.

Orthogonal – having perpendicular slopes or tangents at the point of intersection.

Oscilloscope – it is an instrument in which the variations in a fluctuating electrical quantity appear temporarily as a visible wave form on the fluorescent screen of a cathode-ray tube.

5.6 TERMINAL QUESTIONS

1. A particle is subjected to three simple harmonic oscillations, one along the x-axis, second along the y-axis and the third along the z-axis. The three motions are given by

$$x = A \sin \omega t$$

$$y = B \sin \omega t$$

$$z = C \sin \omega t$$

Calculate the amplitude of the resultant motion.

2. Choose the correct option.

A particle moves on the x-axis according to the equation

$$x = A + B \sin \omega t$$

The motion is a SHM with amplitude

(a) A (b) B (c) $A + B$ (d) $\sqrt{A^2 + B^2}$

3. Choose the correct option(s). More than one choice can be correct.

Which of the following will change the time period as they are taken to Mars?

(a) A simple pendulum (b) A compound pendulum (c)LC circuit
(d) A torsional pendulum (e) A spring-mass system

4. A particle is under the influence of two simultaneous SHMs in mutually perpendicular directions given by

$$x = \cos \pi t$$

$$y = \cos \frac{\pi t}{2}$$

determine the trajectory of the resulting motion of the particle.

5. Write short note on Lissajous' figures. How are they demonstrated experimentally?

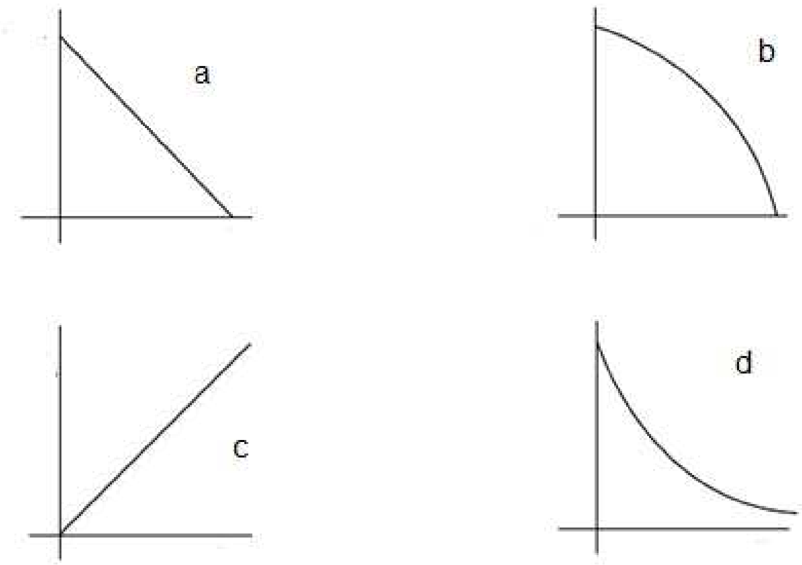
6. Describe and obtain formulae for the superposition of two mutually perpendicular SHMs with equal frequencies.

7. Determine the the shape of the Lissajous figure for the resultant motion, if a particle is subjected to the following SHMs:

$$x = 2 \sin 2\pi t$$

$$y = 3 \sin \pi t$$

8. A body executes simple harmonic motion. Which one of the following graphs best shows the relationship between the kinetic energy (on y-axis) of the body and its distance (on x-axis) from the centre of oscillation?



9. Two pendulums, P and Q, are set up alongside each other. The period of P is 1.90 s and the period of Q is 1.95 s. How many oscillations are made by pendulum Q between two consecutive instants when P and Q move in phase with each other?

10. A student performed the simple pendulum experiment in the laboratory by measuring the time periods for different lengths of the pendulum. If for two readings, he found that the period changes by 50%, what should have been the percentage change in the length of the pendulum?

5.7 ANSWERS

Selected Self Assessment Questions (SAQs):

Solution (SAQ) 1: Using (5.1) and (5.2), taking $A_1 = A_2 = A$ and putting the phase difference between the two perpendicular components, $\delta = \pi/2$, we get,

$$x = A \sin \omega t$$

$$y = A \sin\left(\omega t + \frac{\pi}{2}\right) = A \cos \omega t$$

Hence, we can see these are the same coordinates that are given to us. Therefore, $\delta = \pi/2$.

Putting $\delta = \pi/2$ in equation (5.3), we get

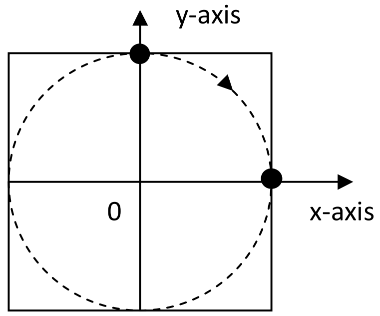
$$x^2 + y^2 = A^2$$

which is the equation of a circle.

For the direction, let us see what are the coordinates of the particle at t = 0 and t = T/4.

$$At\ t = 0, \qquad (x, y) = (0, A)$$

$$At\ t = \frac{T}{4} = \frac{\pi}{2\omega}, \qquad (x, y) = (A, 0)$$



Clearly, one can see that the motion of the particle is clockwise.

2. Solution (SAQ) $\delta = \pi$. The particle follows the path $y = -x$. The amplitude of the resultant oscillation is

$$\sqrt{A^2 + A^2} = \sqrt{2}(0.01) = 0.014\, m$$

3. Solution (SAQ) $\delta = 0$. The particle follows the path $y = x$. The amplitude of the resultant oscillation is

$$\sqrt{A^2 + A^2} = \sqrt{2}(0.01) = 0.014\, m$$

4. Solution (SAQ) As per the problem, the two individual SHMs are in the y and the z directions, which we know are perpendicular to each other. Hence, the resultant oscillation will be superposition of these two mutually perpendicular SHMs. Since, $\delta = 0$, they are in phase. Therefore, from equation (5.4), we can say that the path followed by the resultant oscillation is

$$y = x$$

The amplitude of the resultant oscillation is

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

Therefore, the particle oscillates on $y = x$ line between $\pm\sqrt{2}$.

5. Solution (SAQ): This problem is similar to Example 2 in this unit. We resolve the second SHM, S into x and y components.

The x-component of S is given by

$$S\cos 3\pi/4 \ = B\sin\omega t\cos 3\pi/4$$

$$= -B\sin\omega t\sin\pi/4 \ = -\frac{B}{\sqrt{2}}\sin\omega t$$

And, the y-component of S is given by

$$S\sin 3\pi/4 \ = B\sin\omega t\sin 3\pi/4$$

$$= B\sin\omega t\cos\pi/4 \ = \frac{B}{\sqrt{2}}\sin\omega t$$

Therefore, if we want to see motion in the x-direction and the y-direction only, then we have three separate SHMs, two along the x-axis and one along the y-axis. They are

$$x_1 = A\sin\omega t$$

$$x_2 = -\frac{B}{\sqrt{2}}\sin\omega t$$

$$y = \frac{B}{\sqrt{2}}\sin\omega t$$

Solution(SAQ) 6:

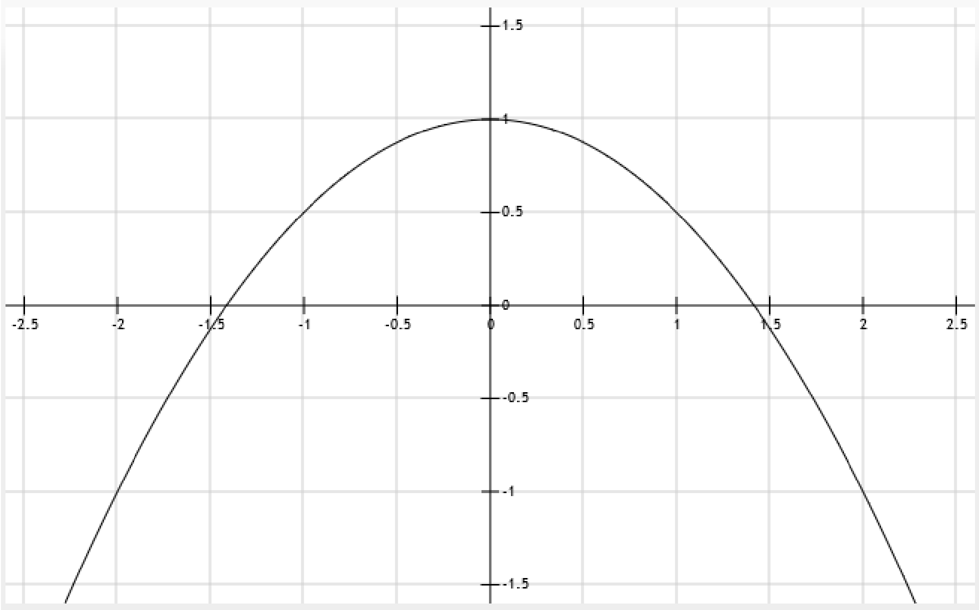
(a) If we eliminate t between these two equations we can find the equation of the trajectory – i.e. an expression for y in terms of x. this will be termed as analytical method.

We can write the y-component oscillation as

$$y = \cos 2t = 1 - 2\sin^2 t$$

$$or \qquad y = 1 - \frac{x^2}{2}$$

This represents a parabola, as shown in the figure below:



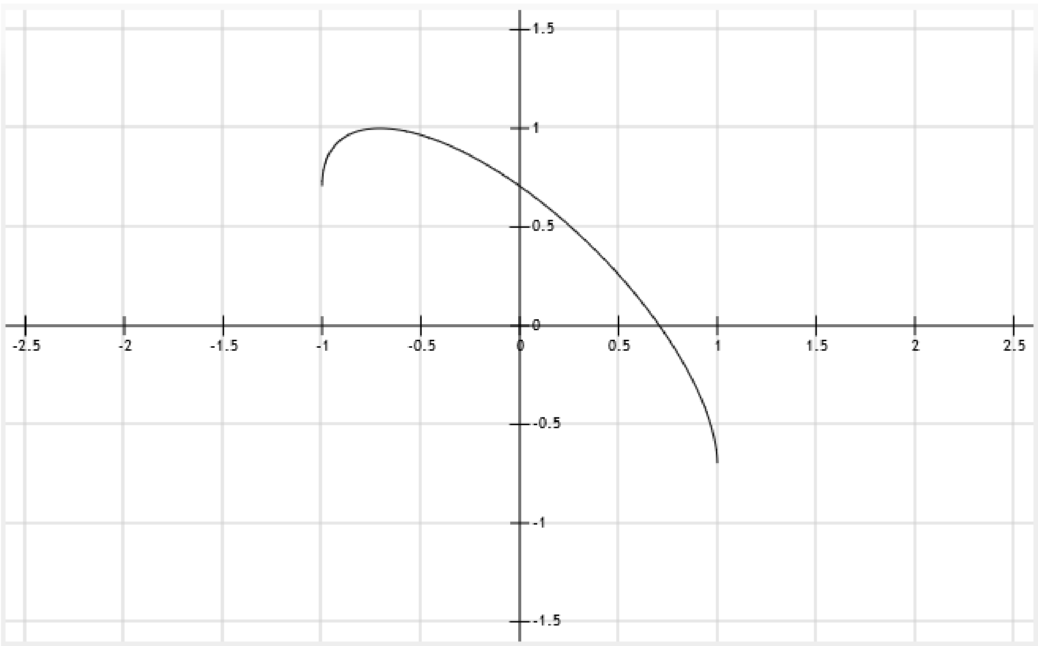
(b) We can write the y-component oscillation as

$$y = \cos(t + \pi/4) = \cos t \cos \pi/4 - \sin t \sin \pi/4$$

$$or \qquad y = \frac{1}{\sqrt{2}}(\cos t - \sin t)$$

$$or \qquad y = \frac{1}{\sqrt{2}}\left(\sqrt{1 - \sin^2 t} - \sin t\right) = \frac{1}{\sqrt{2}}\left(\sqrt{1 - x^2} - x\right)$$

This represents the curve as shown in the following figure:



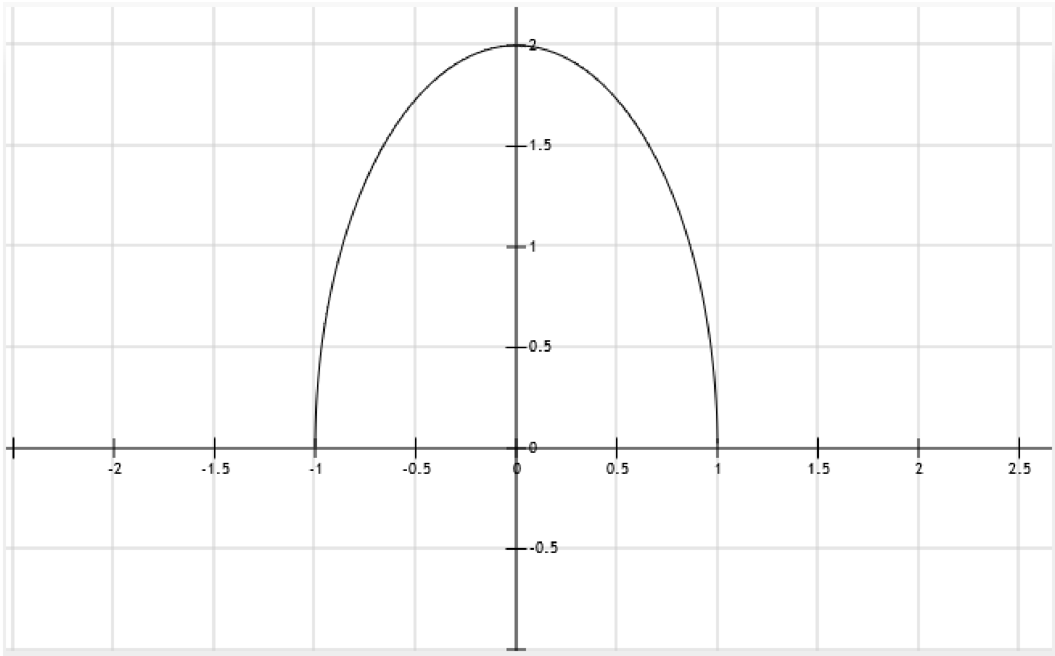
(c) We can write the y-component oscillation as

$$y = 2\sin(\pi t + \pi/2) = 2\left(\sin \pi t \cos\frac{\pi}{2} + \cos \pi t \sin\frac{\pi}{2}\right)$$

$$or \qquad y = 2\cos \pi t$$

$$or \qquad y = 2\left(\sqrt{1-\sin^2 \pi t}\right) = 2\left(\sqrt{1-x^2}\right)$$

This represents the curve as shown in the following figure:



Solution(SAQ 7. True.

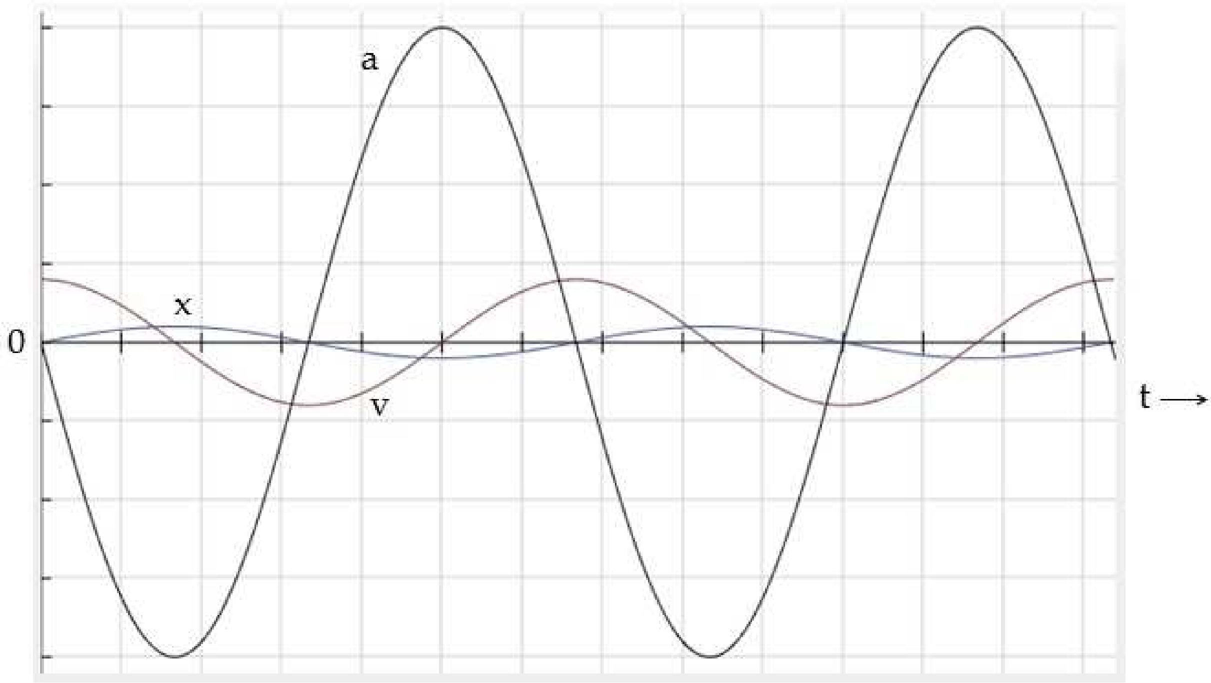
Solution(SAQ) 8:
The displacement is given by $x = 5\sin 3\pi t$. The velocity can be found out by its first derivative, i.e.

$$\frac{dx}{dt} = v = 15\pi \cos 3\pi t$$

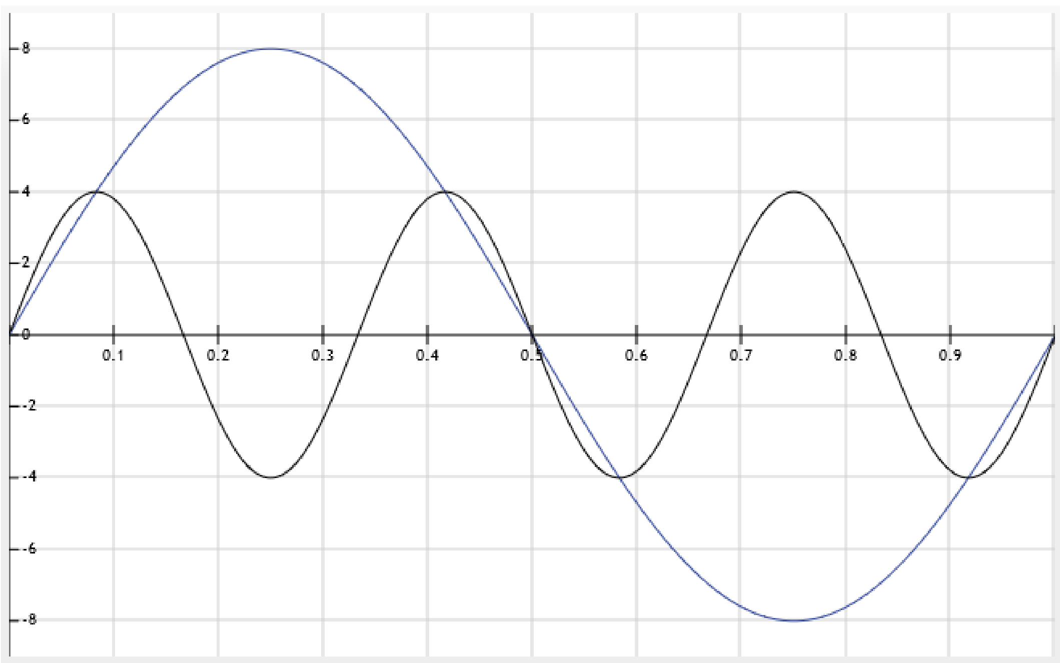
The acceleration is the second derivative of the displacement function or the first derivative of the velocity function, i.e.

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = a = -45\pi^2 \sin 3\pi t$$

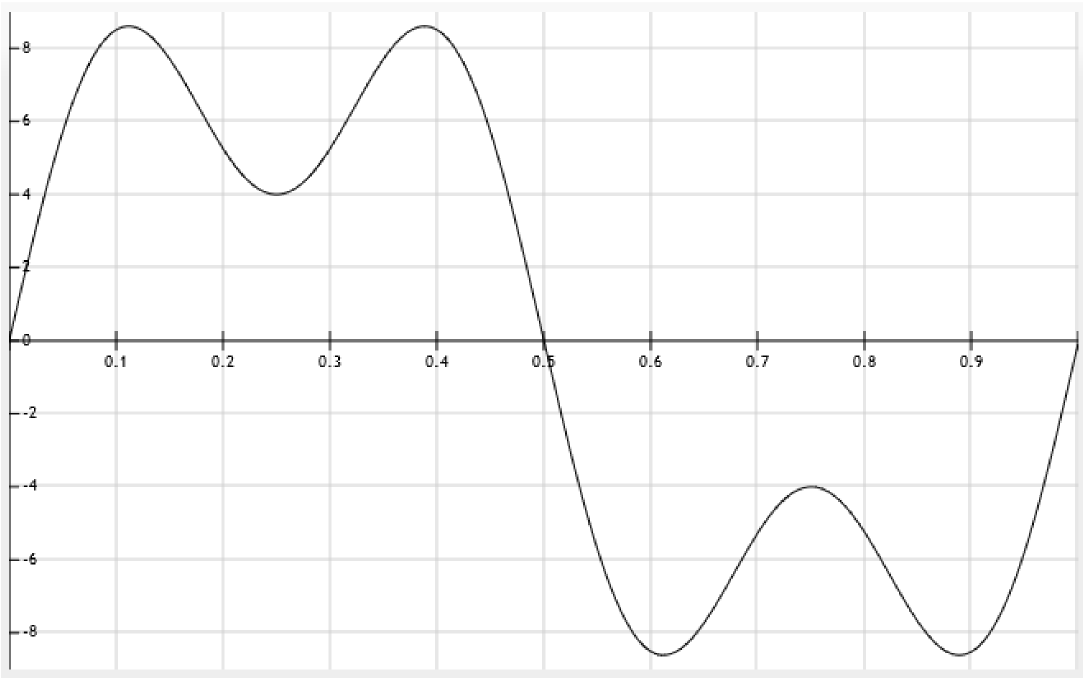
The following plot shows all three of them (displacement, velocity and acceleration) together.



Solution(SAQ) 9:
We first plot the two functions separately and note down the values of y_1 and y_2 for different values of t.



Now, we add the two functions together, and plot the resultant $y_1 + y_2$ to get the following path.



The amplitude of the resultant oscillation for mutually perpendicular oscillations with zero phase difference is given by

$$A_{resultant} = \sqrt{A_1^2 + A_2^2}$$

$$= \sqrt{\left(A - \frac{B}{\sqrt{2}}\right)^2 + \left(\frac{B}{\sqrt{2}}\right)^2}$$

$$= \sqrt{A^2 + B^2 - \sqrt{2}AB}$$

Selected Terminal Questions:

1. Let us first consider the first two oscillations. They are mutually perpendicular with equal frequencies and phase difference of zero. Hence the resultant oscillation from the combination of these two is going to be a SHM and will oscillate on the path [from equation (5.4)],

$$y = x$$

And the amplitude of the resultant of these two SHMs is given by

$$A_{12} = \sqrt{A^2 + B^2}$$

The 3rd SHM is in the z-direction and hence is always going to be perpendicular to resultant SHM of first two SHMs, which is in the x-y plane along the $y = x$ direction (let's denote it as S-axis). Since the SHMs along the z- and S - axis are in phase, we can write their equations as

$$S = A_{12} \sin \omega t$$

$$z = C \sin \omega t$$

Thus, the resultant amplitude will be

$$A_{resultant} = \sqrt{(A_{12})^2 + C^2}$$

$$= \sqrt{A^2 + B^2 + C^2}$$

2. (b)

The given equation of motion, $x = A + B \sin \omega t$, can be written as

$$(x - A) = B \sin \omega t$$

If $(x - A) = X$, using a simple time independent linear translation, we get

$$X = B \sin \omega t$$

which is the familiar equation of motion for SHM.

3. (a) and (b)

The time periods of simple pendulum and compound pendulum depend on the acceleration due to gravity and hence, their periods will change as the acceleration due to gravity on Mars is different from that on the Earth.

4. Let us substitute $\frac{\pi t}{2} = \theta$. We get,

$$x = \cos 2\theta$$

$$y = \cos \theta$$

From trigonometry, we know that

$$\cos 2\theta = 2\cos^2 \theta - 1$$

$$\therefore x = 2y^2 - 1$$

which is the equation of a parabola. Hence, the point traces a parabolic path.

7. The Lissajous figure resembles the number 8 (eight).

8. The kinetic energy is related to velocity by

$$K = \frac{1}{2}mv^2$$

And for SHM, the velocity as a function of x is given as

$$v = \pm\omega\sqrt{(A^2 - x^2)}$$

Therefore,

$$K = \frac{1}{2}m\omega(A^2 - x^2)$$

Hence, the correct choice is (b).

9. The difference in time period is 0.05 s - therefore to make up a complete extra swing P will move one more period than Q does in the same time.

Let number of swings of Q = n. Then,

$$1.95n = 1.90(n + 1)$$

$$\therefore n = \frac{1.90}{0.05} = 38$$

10. We know that the period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Therefore,

$$T^2 = \frac{4\pi^2}{g}l$$

Or,

$$\left(\frac{T_2}{T_1}\right)^2 = \frac{l_2}{l_1}$$

Now, we are given that

$$T_2 = 150\%(T_1) = 1.5T_1$$

Therefore,

$$l_2 = l_1(1.5)^2 = 2.25l_1$$

Percentage increase in length is

$$= \frac{(l_2 - l_1)}{l_1} \times 100 = (2.25 - 1) \times 100 = 125\%$$

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UNIT 6 DAMPED HARMONIC OSCILLATOR

Structure

6.1 INTRODUCTION

When we consider an oscillator executing simple harmonic motion, it is assumed that the oscillations will continue for infinite time. In other words, we treat the whole system as an *idealized frictionless system*. But, in reality, this does not happen, and the oscillating object gradually loses its energy due to several factors. One such major factor is the frictional forces which diminish the amplitude of oscillation and the system ultimately comes to rest.

The decrease in amplitude caused by the dissipative forces is called **damping** and these oscillations with decreasing amplitude are called **damped oscillations**. In the present unit, we shall discuss the effects of damping on the oscillatory systems.

6.2 OBJECTIVES

After studying this unit, you should be able to

- understand the various types of damping effects;
- write the differential equation of a damped harmonic oscillator;
- analyze the weakly damped, critically damped and over damped motions;
- explain various parameters characterizing weak damping; and
- understand the damping in LCR circuit.

6.3 FRICTIONAL EFFECTS (DAMPING)

The oscillatory motion we considered so far, have been for ideal systems. It means that such oscillatory systems will oscillate indefinitely under the action of only one force – a linear restoring force. In the basic analysis of harmonic oscillators, we completely ignore the effect of frictional forces in it. But, in real situations, the oscillator is in a resistive medium like air, oil etc. In such conditions, part of the energy of the oscillator is spent in opposing frictional or viscous forces. At ordinary velocities, the opposing, resistive or damping force is to a first approximation, proportional to velocity and may be represented by

$$F = -\gamma v = -\gamma \frac{dx}{dt}$$

Where γ is a positive constant, called **damping coefficient** of the medium and may be termed as resistive force per unit velocity.

So, if there is no other force other than this resistive or damping force acting on the oscillating body of mass m, then Newton's second law of motion gives

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = -\gamma v$$

Or $$\frac{dv}{dt} + \frac{\gamma}{m}v = 0 \qquad \text{-----------------------------} (1)$$

Here, $\frac{m}{\gamma}$ is usually denoted by a constant, having dimensions of time and is called as relaxation time (τ).

Therefore,

$$\frac{dv}{dt} + \frac{1}{\tau}v = 0 \qquad \text{------------------------------------} (2)$$

The constant $\frac{1}{\tau} = \frac{\gamma}{m}$, or the resistive force per unit mass per unit velocity, is often denoted by 2b, where b is called damping constant of the medium.

Now rewriting and integrating equation (2), we get

$$\int \frac{dv}{v} = -\frac{1}{\tau}\int dt$$

which gives

$$\ln v = -\frac{t}{\tau} + C$$

Where C is a constant of integration to be determined from the initial conditions.

Puting $t=0,\ v=v_0$ we get

$$\ln v_0 = C$$

therefore-

$$v = v_0 e^{-\frac{t}{\tau}} \qquad \text{------------------------------------} (3)$$

Above equation clearly shows that the velocity decreases exponentially with time, as shown by the curve in Fig 1 below.

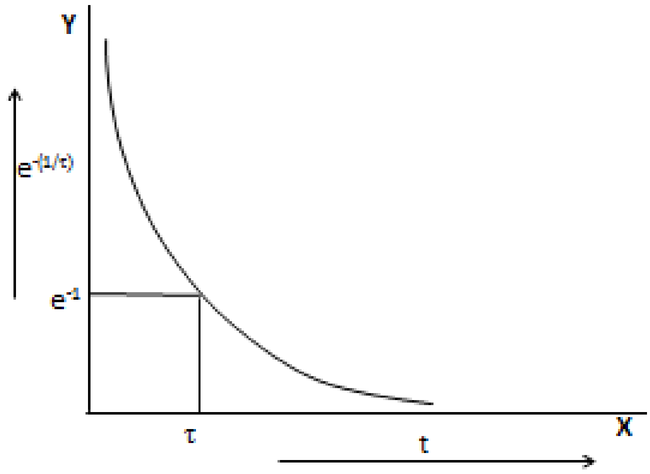

Figure 1

We can express this by saying that velocity is damped by a time constant τ. From the above expression for v, you may note that, at $t = \tau$, $v = v_0 e^{-1} = v_0 / e = v_0 / 2.718 = 0.368 v_0$

Therefore, the time constant, τ (also called relaxation time) may be defined as *the time in which the velocity of the oscillating particle falls to 1/e times (i.e., 0.368 times) of its initial value.*

6.4 TYPES OF DAMPING FORCES

Every physical system experiences damping, which depends upon the system under consideration. A familiar example is a spring – mass system executing longitudinal oscillations in a horizontal surface. The mass which has to move on the horizontal surface experiences frictional force from the surface and this frictional force opposes its motion. So, the friction due to the surface acts like damping force for the oscillating spring-mass system.

Another example is Millikan's oil drop experiment. In this experiment, a charged oil drop falling freely in an electric field experiences a viscous drag. According to Stoke's Law, the viscous (or the damping) force is directly proportional to the velocity of the body. The direction of viscous (resistive or damping) force is opposite to velocity. This resistive force become quite relevant when a spacecraft comes back in to the earth atmosphere during its return journey or an aircraft begins it's descend. In both cases, the magnitude of the upwards thrust, which acts as a resistive force, can be very large. Usually the spacecraft or the aircraft experiences large stress, which ultimately raises its outside body temperature. The unfortunate accident of the US spacecraft *Columbia,* in which all astronauts were killed including Indian astronaut Kalpana Chawla, was mainly due to the viscous drag only.

Note that the damping is not restricted to mechanical systems only. It is encountered in an LCR – Circuit, ballistic galvanometer, fluxmeter etc. In these cases, the damping is electromagnetic in nature. You will learn about it later in the unit.

In general, inclusion of damping force makes mathematical analysis somewhat difficult. But for simplicity, it is customary to model it by an equivalent viscous damping. In our discussion, we make the simplifying assumption that velocity of the moving part of the system is small so that the damping force can be taken to be linear in velocity.

6.5 DIFFERENTIAL EQUATION OF A DAMPED OSCILLATOR

For studying the effect of damping on a one dimensional oscillator, we can consider the representative case of a spring-mass system, as shown in figure below.

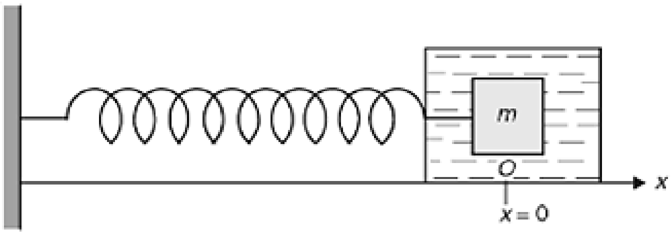

Fig. 2 A damped spring-mass system; the oscillating mass is immersed in a viscous medium

The spring-mass system in which the oscillating mass is executing oscillations in a viscous medium which causes its amplitude progressively decreasing to zero is called a *damped harmonic oscillator.* Obviously, in case of such an oscillator, in addition to the restoring force –*kx*, a resistive or damping force also acts upon it. This damping force is proportional to the velocity, v $(= dx/dt)$. We, therefore, can write the equation of the damped spring-mass system as

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - kx$$

or $$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad \text{-------------------------- (5)}$$

This can further be written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0{}^2 x = 0 \qquad \text{------------------------------ (6)}$$

where $\frac{k}{m} = \omega_0{}^2$ is the natural frequency of oscillating particle (i.e. its frequency in the absence of damping), $\frac{\gamma}{m} = 2b$ (k is the damping constant of the resistive medium)

Above equation is called the differential equation of a damped harmonic oscillator.

6.6 SOLUTION OF THE DIFFERENTIAL EQUATION OF DAMPED HARMONIC OSCILLATOR

The above differential equation is a second order linear homogeneous differential equation. Therefore, it will have at least one solution of type $x = Ae^{\alpha t}$

Here α and t both are arbitrary constants.

Therefore,

$$\frac{dx}{dt} = \alpha Ae^{\alpha t} \quad \text{and} \quad \frac{d^2x}{dt^2} = \alpha^2 Ae^{\alpha t}$$

Substituting these values in the differential equation (6) above we get

$$\alpha^2 Ae^{\alpha t} + 2b\alpha Ae^{\alpha t} + \omega_0{}^2 Ae^{\alpha t} = 0$$

Or

$$\alpha^2 + 2b\alpha + \omega_0{}^2 = 0 \qquad \text{-------------------------------- (7)}$$

This is a quadratic equation in α having its solution of the form

$$\alpha = -b \pm \sqrt{b^2 - \omega_0^2}$$

Thus the original differential equation is satisfied by following two values of x

$$x = Ae^{(-b+\sqrt{b^2-\omega_0^2})t}$$

and $x = Ae^{(-b-\sqrt{b^2-\omega_0^2})t}$

Since the equation being a linear one, the linear sum of two linearly independent solutions will also be a general solution.

Therefore,

$$x = A_1 e^{(-b+\sqrt{b^2-\omega_0^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega_0^2})t} \qquad \text{-------------------- (8)}$$

Here A_1 and A_2 are arbitrary constants.

$$\text{Or} \quad x = A_1 e^{-\frac{t}{2\tau}+\beta t} + A_2 e^{-\frac{t}{2\tau}-\beta t} \qquad \text{------------------------ (9)}$$

where $b = \dfrac{1}{2\tau}$ and $\beta = \sqrt{b^2 - \omega_0^2}$

The values of the constants A_1 and A_2 can be determined as given below:

Differentiating Eq. (9) with respect to t, we get

$$\frac{dx}{dt} = (-\frac{1}{2\tau} + \beta) A_1 e^{-\frac{t}{2\tau}+\beta t} + (-\frac{1}{2\tau} - \beta) A_2 e^{-\frac{t}{2\tau}-\beta t} \qquad \text{---------------- (10)}$$

Now at t=0, displacement must be maximum, i. e. $x_{max}=a_0=A_1+A_2$ and $\dfrac{dx}{dt} = 0$

Putting t=0 in Eq. (10)

$$(-\frac{1}{2\tau} + \beta) A_1 + (-\frac{1}{2\tau} - \beta) A_2 = 0$$

$$-\frac{1}{2\tau}(A_1 + A_2) + \beta(A_1 - A_2) = 0$$

$$-\frac{1}{2\tau}(a_0) + \beta(A_1 - A_2) = 0$$

$$\beta(A_1 - A_2) = \frac{a_0}{2\tau}$$

$$\text{Or} \quad (A_1 - A_2) = \frac{a_0}{2\tau\beta} \qquad \text{--- --------------- (11)}$$

As we know $A_1 + A_2 = a_0$

Adding it with (11), we get

$$A_1 = \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

And $A_1 = (A_1 + A_2) - A_1$

$$= a_0 - \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

$$= \frac{a_0}{2}\left[1 - \frac{1}{2\tau\beta}\right]$$

Putting these values in equation (9), we get-

$$x = \frac{a_0 e^{-\frac{t}{2\tau}}}{2}\left[\left(1 + \frac{1}{2\tau\beta}\right)e^{\beta t} + \left(1 - \frac{1}{2\tau\beta}\right)e^{-\beta t}\right] \quad \text{- - -- - - - - - - - - - - - - (12)}$$

For analysis purpose, above equation may be written as

$$x = \frac{a_0 e^{-\frac{t}{2\tau}}}{2}\left[\left(1 + \frac{1}{2\tau\beta}\right)e^{(\sqrt{b^2-\omega_0^2})t} + \left(1 - \frac{1}{2\tau\beta}\right)e^{-(\sqrt{b^2-\omega_0^2})t}\right] \quad \text{- - - - - - - - - (13)}$$

Now Eq. (13) can be discussed according to following three cases.

6.6.1 CASE I: WHEN b (OR $\frac{1}{2\tau}$)>ω_0, CASE OF OVERDAMPING

In such case $\sqrt{(b^2 - \omega_0^2)}$ is a real quantity, with a positive value. This means that each term in the R. H. S. of Eq. (13), has an exponential term with a negative power.Therefore, the displacement of the oscillator, after attaining a maximum, dies off exponentially with time. Thus, after some time, there will be no oscillations. Such kind of oscillatory motion is called **overdamped** or **aperiodic** motion. Such kind of motion we see in case of dead beat galvanometer.

6.6.2 CASE II: WHEN b (OR $\frac{1}{2\tau}$)= ω_0, CASE OF CRITICAL DAMPING

In such case $\sqrt{(b^2 - \omega_0^2)} = 0$ .Therefore, each term on R. H. S. of Eq. (13) becomes infinite.

Still we can assume that, $\sqrt{(b^2 - \omega_0^2)} = h$ (where h is a very small quantity but not zero obviously).

Therefore Equation (8) gives-

$$x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t}$$

$$= e^{-bt} (A_1 e^{ht} + A_2 e^{-ht})$$

$$= e^{-bt}[A_1(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots\ldots\ldots\ldots) + A_2(1 - ht + \frac{h^2t^2}{2!} - \frac{h^3t^3}{3!} + \ldots\ldots\ldots)]$$

Neglecting the terms containing higher powers of h, we obtain-

$$x = e^{-bt}[A_1(1 + ht) + A_2(1 - ht)]$$

$$= e^{-bt}[(A_1 + A_2) + (A_1 - A_2)ht]$$

$$= e^{-bt}[M + Nt] \quad \text{-------------------------------------- 14)}$$

Here $(A_1 + A_2) = M$ and $(A_1 - A_2)h = N$

Further at $t = 0, x = x_{max} = a_0$

And $\frac{dx}{dt} = 0$

Therefore, the above equation becomes

$$a_0 = M$$

Differentiating Eq. (14), we get

$$\frac{dx}{dt} = \frac{d}{dt}(\mathrm{M\ e^{-bt}}) + \frac{d}{dt}(\mathrm{Nte^{-bt}})$$

$$0 = - b M e^{-kt} + Ne^{-bt} - Nte^{-bt}$$

$$= -bM + N$$

Or $N = ba_0$

Putting these values of M and N in equation (14) above

$$x = e^{-bt}(a_0 + ba_0t)$$

$$= a_0e^{-bt}(1 + bt)$$

$$= a_0 e^{-\frac{t}{2\tau}}(1 + \frac{t}{2\tau})$$

$$= a_0 e^{-\frac{t}{2\tau}} + a_0 e^{-\frac{t}{2\tau}}(\frac{t}{2\tau})$$

An important feature of the above expression is that its second term decays less rapidly as compared to its first term. In such cases, the displacement of the oscilator first increases, then quickly return back to its equlibrium position. This kind of oscillatory motion is known as *just aperiodic* (it just ceases to oscillate), or non oscillatory. This case is known as the **critical damping**.

Critical damping finds many applications in many pointer type instruments like, galvenometers, where the pointer moves to and stays at, the correct position, without any further oscillations.

6.6.3 CASE III: WHEN b (OR $\frac{1}{2\tau}$) < ω_0 ,CASE OF WEAK (UNDER) DAMPING

In such cases, the quantity $\sqrt{(b^2 - \omega_0^2)}$ will be imaginary one.

Let $\sqrt{(b^2 - \omega^2)} = i\omega$, where $i = \sqrt{(-1)}$ and $\omega = \sqrt{(\omega_0^2 - b^2)}$ is a real quantity

Putting the values –

$$x = A_1 e^{(-b+iw)t} + A_2 e^{(-b-iw)t}$$

$$= e^{-bt}[A_1(\cos \omega t + i \sin\omega t) + A_2(\cos \omega t - i\sin\omega t)]$$

$$= e^{-bt}[\cos \omega t\,(A_1 + A_2) + \sin \omega t\{i\,(A_1 - A_2)\}]$$

$$= e^{-bt}[A \cos \omega t + B \sin \omega t]$$

where $(A_1 + A_2) = A$ and $i(A_1 - A_2) = B$

$$= e^{-kt}[\, a_0 \cos \omega t \,.\, \frac{A}{a_0} + a_0 \sin \omega\, t \,.\, \frac{B}{a_0}]$$

Considering a right angle triangle as below in Fig. 3.

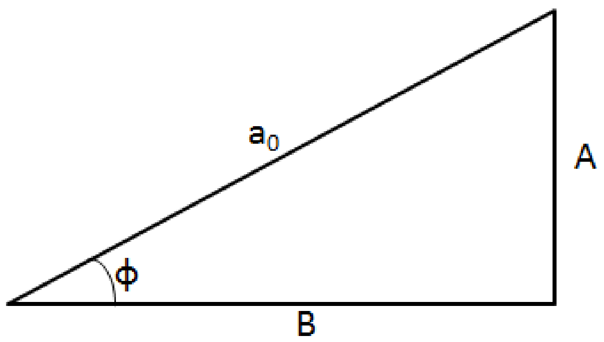

Figure 3

Therefore, we can write

$sin\ \varphi = \dfrac{A}{a_0}$, $cos\ \varphi = \dfrac{B}{a_0}$

so the above expression can be rewritten as-

$$x = e^{-bt}[\, a_0 \{cos\ \omega t\ .\ sin\ \varphi + sin\ \omega\ t\ .cos\ \varphi\}]$$

$$= a_0 e^{-bt}\ sin\ (\omega t + \varphi)$$

$$or\ \ x = a_0 e^{-b\frac{t}{2x}} \sin(\omega t + \varphi)$$

This is the equation of a damped harmonic oscillator with amplitude $a_0 e^{-bt}$ or $x = a_0 e^{-b\frac{t}{2x}}$.

The sine term in the equation suggests that the motion is oscillatory whereas, the exponential term implies that the amplitude is decreasing gradually.

Therefore, we may conclude that the damping produces two effects:

(i) The frequency of damped harmonic oscillator, $\dfrac{\omega}{2\pi}$ is smaller than its natural frequency $\dfrac{\omega_0}{2\pi}$, or damping somewhat decreases the frequency or increases the time period of oscillator.

(ii) The amplitude of the oscillator does not remain constant at a_0, which represents amplitude in the absence of damping, but decays exponentially with time, according to the value of term e^{-bt}.

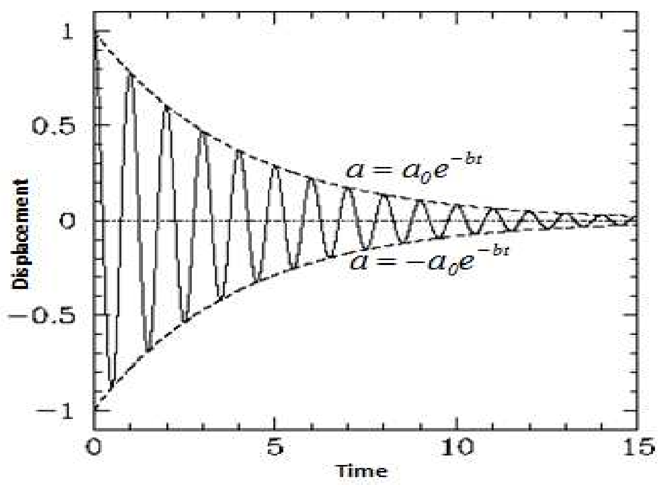

Figure 4 Decay of amplitude of a damped oscillatory motion with time

6.7 CHARACTERIZING WEAK DAMPING

The following three parameters characterize the weak damping:

1- Relaxation time, τ
2- Logarithmic Decrement, λ
3- Quality factor, Q

Each of these parameter is defined in terms of ω_0 and k. Depending upon the nature of the system under consideration, one or more of these parameters may suitably be used to quantify damping.

6.7.1 RELAXATION TIME, τ

It refers to the time in which the amplitude of a weakly damped system reduces to 1/e times of the original value. In other words, it is the time in which the mechanical energy of an oscillator decays to 1/e times its initial value.

The energy of a damped harmonic oscillator is given by

$$E = E_0 e^{-\frac{t}{\tau}}$$

Here E_0= the initial value of energy

E= Energy at time t

At $t = \tau, E = \frac{E_0}{e}$

6.7.2 LOGRATHMIC DECREMENT

Due to damping, the amplitude of a damped harmonic oscillator decreases exponentially with time. Suppose that a_n and a_{n+1} be the two successive amplitudes of the oscillations of the particles on two sides of the equilibrium position respectively. The time interval between these two successive amplitudes clearly would be T/2 - half the time period (T) of oscillations. We can further write-

$a_n = a_{0e}\text{-}bkt$

and $a_{n+1}=a_0e^{-b(t+\frac{T}{2})}$

Therefore, $\frac{a_n}{a_{n+1}} = e^{\frac{bt}{2}} = d$ (16)

Here d is a constant, and it refers to decrease in successive amplitudes. It is known as the decrement for that motion.

Further, on taking the natural log of Eq.(16), we obtain

$$\ln d = \frac{kT}{2} = \lambda$$

or, $d = e^{\lambda}$

The constant λ, which is the natural logarithm of decrement or the ratio between two successive amplitudes of the oscillations is referred to as logarithmic decrement for that oscillatory motion.

6.7.3 QUALITY FACTOR

As the name suggests, quality factor is a measures the quality of a harmonic oscillator, as far as damping is concerned."Lesser the damping, better will be the quality of harmonic oscillator as an oscillator". Therefore, an harmonic oscillator with low damping will have high value of its quality factor, Q. It is also referred to as the figure of merit of a harmonic oscillator and is defined as the 2π times the ratio between the energy stored and the energy lost per period. Being a ratio, it is a dimensionless quantity.

Thus $Q = 2\pi \frac{Energy\ stored}{Energy\ lost\ per\ period}$

$= \frac{2\pi E}{PT}$ (Here P= Average loss of energy per cycle= $\frac{E}{\tau}$)

And since, $\frac{2\pi}{T} = \omega$, we have

$$Q = \frac{E\omega}{P} = \frac{E\omega}{E/\tau} = \omega\tau$$

In case of low damping, $\omega=\omega_0$ and we can rewrite the above equation as

$$Q = \omega_0 \tau$$

But, as we know

$\omega_0 = \sqrt{\frac{k}{m}}$ and $\tau = \frac{m}{\gamma}$, so that,

$$Q = \sqrt{\frac{k}{m}}.\frac{m}{\gamma} = \sqrt{\frac{km}{\gamma}}$$

Clearly, if γ is small (i.e. if the damping is low), the value of Q will be large.

Further, the energy of a damped harmonic oscillator is

$$E = E_0 e^{-\frac{t}{\tau}}$$

Hence, at $t = \tau$, we have

$$E = E_0 e^{-1} = \frac{E_0}{e}$$

6.8 EXAMPLE OF WEAKLY DAMPED SYSTEM: LCR CIRCUIT

A circuit consisting of an inductor (L), a capacitor (C) and a resistor (R) connected in series (Fig. 1) is called an LCR circuit. The resistance of the resistor R represents all of the resistance in the circuit. With the resistance R present, the total electromagnetic energy U of the circuit (the sum of the electrical energy and magnetic energy) decreases with time because some portion of this energy is transformed in to thermal energy due to the resistor. Because of this loss of energy, the amplitude of oscillations of charge, current and potential difference continuously decreases, and the oscillations are said to be damped.

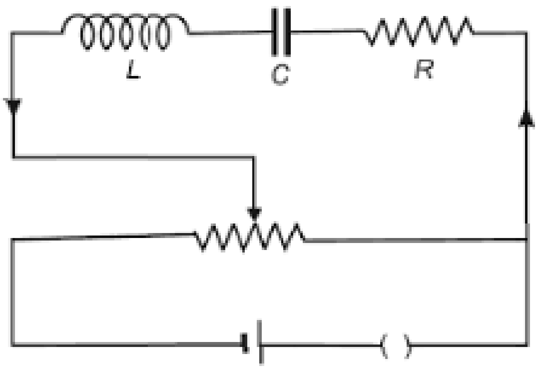

Figure 5: A series LCR Circuit

The capacitor C in circuit is charged using an external battery. Thereafter the battery connection is removed and the charged capacitor is connected in series with an inductor L and resistor R as shown in Fig 5. The capacitor begins to discharge through the inductor and resistor. Let us assume that, at any instant of time t, current flows in the circuit and charge q(t) resides on the capacitor. The induced EMF across the inductance would be

$$L\frac{dI}{dt} = L\frac{d^2q(t)}{dt^2}$$

And the potential difference across the resistance would be

$$RI = R\frac{dq(t)}{dt}$$

Therefore, using Kirchoff's rule, the equation of motion of charge can be written as

$$\frac{q(t)}{c} = -L\frac{d^2q(t)}{dt^2} - R\frac{dq(t)}{dt}$$

or

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{q(t)}{C} = 0 \qquad \text{-------------------------- (17)}$$

On comparing Eq. (17) with the differential equation of damped harmonic oscillator, we discover that L, R and 1/C are respectively analogous to m, γ and k. This means that the effect of a resistor in an electric circuit is exactly analogous to that of the viscous force in a mechanical oscillatory system.

$$\frac{d^2q(t)}{dt^2} + \frac{R}{L}\frac{dq(t)}{dt} + \frac{1}{LC}q(t) = 0 \qquad \text{-------------------------- (18)}$$

This equation is analogous to (17), if we identify-

$$\omega_0^{\ 2} = \frac{1}{LC} \qquad \text{----- ------------------------- (19)}$$

And $b = \dfrac{R}{2L}$ ------------------------------- (20)

Equation (4) suggests that damping in a series LCR circuit is determined by the resistance and inductance. We know that the b has dimension of time inverse. It means that R/L also has the unit of s^{-1}, which is same as that of ω_0. It implies that $\omega_0 L$ will be measured in ohms.

With these analogies, all the results of weak damping apply to equation (18). Therefore for a weakly damped LCR circuit, the instantaneous value of charge on the capacitor plates can be expressed as

$$q(t) = q_0 \exp(-\frac{R}{2L}t)\cos(-\omega_d t + \phi) \qquad \text{------------------- (21)}$$

Where ω_d of the damped circuit is given by

$$\omega_d = \sqrt{\omega_0{}^2 - b^2}$$

$$= \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad \text{- (22)}$$

Equation (21) shows that the rate of decay of charge amplitude $q_0 \exp(-\frac{R}{2L}t)$ depends on the resistance and inductance. However, only resistance acts as the dissipative element in a series LCR Circuit; an increase in R increases the rate of decay of charge and decreases the frequency of oscillation.

When $\frac{1}{LC} >> \frac{R^2}{4L^2}$, Eqn (22) gives

$$\omega_d^2 = \omega_0^2 = \frac{1}{LC}$$

Or $\omega_0 L = \frac{1}{\omega_0 C}$

Sinceω_0L is measured in ohms, 1/ ω_0C will also be measured in ohms. They are respectively known as inductive reactance and capacitive reactance

Self Assessment Question (SAQ) 1: A Harmonic oscillator is represented by the equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = C$$

With m = 0.25 kg, γ=0.070 kg/s and k =85 N/m, calculate the period of oscillation.

Self Assessment Question (SAQ) 2: For the harmonic oscillator given in problem 1, calculate (i) the number of oscillations in which its mechanical energy will drop to one-half of its initial value. Also calculate its quality factor.

Self Assessment Question (SAQ) 3: The amplitude of a damped harmonic oscillator reduces from 25 cm to 2.5 after 100 complete oscillations, each of period 2.3 seconds. Calculate logarithmic decrement of the system.

6.9 SUMMARY

This chapter presents the various effects of damping and gives, and presents a procedure to incorporate in the differential equation of a damped harmonic oscillator. This differential equation has been solved to get the cases of underdamped, critically damped and overdamped motion. The characterization process of weakly damped motion, in terms of relaxation time, logarithmic decrement and quality factor also have been discussed.

6.10 GLOSSARY

Damped – Slowed down, being stopped

Dissipative- continuously loosing energy / amplitude

Overdamped – Having high value of damping effects

Underdamped- Having small value of damping

Friction- resistance

Conservation- protection, preservation or restoration

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6.13 TERMINAL QUESTIONS

6.13.1 OBJECTIVE TYPE

Q1. Which one of the following statements is not true for a body executing simple harmonic motion when damping is present?

A. The damping force is always in the opposite direction to the velocity.
B. The damping force is always in the opposite direction to the acceleration.
C. The presence of damping gradually reduces the maximum potential energy of the system.
D. The presence of damping gradually reduces the maximum kinetic energy of the system.

Q2. Oscillations become damped due to

A. normal force
B. friction
C. tangential force
D. parallel force

Q3. The amplitude of damped oscillator becomes half in one minute. The amplitude after 3 minutes will be 1/x times the original value, where x is

A. 2 x 3
B. 2^2
C. 2^3
D. 3 x2^2

Q4. In a damped harmonic oscillator, the damping force is proportional to

A. displacement
B. Acceleration
C. velocity
D. none of these

Q5. In cars springs are damped by

A. shock absorbers
B. engine
C. tyres
D. brake pedals

Q6. In which of the following oscillations, the amplitude varies with time-

A. Damped oscillator
B. Forced oscillator
C. Undamped oscillators
D. None of these

Q.7 The periodic motion which is not oscillatory is

A. Simple pendulum
B. Compound pendulum
C. Acoustic Harmonic Oscillator
D. Motion of the earth around sun

6.13.2 LONG ANSWER TYPE

1.Why does the amplitude of oscillations go on decreasing in case of damped harmonic oscillator? Assuming damping to be proportional to the velocity, find an expression for the frequency of oscillations.
2. A system executing damped harmonic motion is subjected to an external periodic force. Investigate the forced vibration and obtain the condition of resonance.

3. Show that the ratio of two successive maxima in the displacement of a damped harmonic oscillator is constant.

6.13.3 NUMERICAL QUESTIONS

1. If the amplitude of a damped harmonic oscillator decreases to 1/e of its initial value after n($\gg$ 1) periods, show that the ratio of the period of oscillation to the period of the oscillation with no damping is

$$\left(1+\frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}} \approx 1+\frac{1}{8\pi^2 n^2}$$

2. A spring-mass system is subjected to restoring and frictional forces of magnitude kx and $\gamma \frac{dx}{dt}$ respectively. It oscillates with a frequency of 0.5 Hz. Its amplitude reduces to half in 2 seconds. Calculate the damping coefficient γ and spring constant k, in terms of mass, m. Also write the differential equation of motion.

3. The period of a simple pendulum is 2 seconds and its amplitude is 5^0. After 20 complete oscillations, its amplitude is reduced to 3^0. Calculate the damping constant and relaxation time.

4. The quality factor of a tuning fork of frequency 512 Hz is 6 x 10^4. Calculate the time in which its energy drops to E_0e^{-1}. How many oscillations will the tuning fork make in this time.

5. A vertically hanging spring is extended by 9.8 cm when an object is suspended from it. The object is then pulled down and released to make it oscillate. For what value of damping coefficient will (i) the amplitude of oscillation become 1% of its initial value in 10 seconds? (ii) the weight return to equilibrium aperiodically ?

6. Express amplitude, energy, logarithmic decrement and relaxation time in terms of the quality factor, Q

6.14 ANSWERS

6.14.1 Self Assessment Question (SAQ)

Solution (SAQ) 1: The period of oscillation of a damped oscillator is given as

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}}$$

$$= \frac{2\pi}{\sqrt{\frac{85}{0.25} - \left(\frac{0.07}{2\times 0.25}\right)^2}}$$ =**0.34 seconds**

Solution (SAQ) 2- The average energy associated with a damped harmonic oscillator is given by

$$< E >= E_0 e^{-2bt} = E_0 e^{-\frac{\gamma t}{m}}$$

$$\frac{< E >}{E_0} = e^{-\frac{\gamma t}{m}}$$

$$for \frac{< E >}{E_0} = \frac{1}{2} \quad , we\, have$$

$$e^{-\frac{\gamma t}{m}} = \frac{1}{2}$$

Taking natural algorithm on both sides and rearranging the terms, we can rewrite it as

$$t = \frac{m \ln 2}{\gamma} = \frac{0.25 \times 0.693}{0.070} = 2.48s$$

No of oscillation in this time interval= $\frac{2.48}{0.34} \cong 7 oscillations$

(ii) The quality factor Q is given as

$$Q = \frac{\omega_d \tau}{2} \cong \frac{\omega_0 m}{\gamma}$$

Since $\omega_d = \omega_0 \ and \ \tau = \frac{1}{b} = \frac{2m}{\gamma}$

$$Q = \frac{18.43 \times 0.25}{0.07} = 66$$

Solution (SAQ) 3: Here, the amplitude ration of oscillation separated by 100 oscillations is

$$= \frac{25\,cm}{2.5\,cm} = 10$$

Therefore, logarithmic decrement $= \frac{1}{100} \ln 10 = \frac{2.3}{100} = 0.023$

6.14.2 OBJECTIVE TYPE

1 (B), 2 (B), 3 (C), 4 (C), 5 (A), 6 (A), 7 (D)

6.14.3 NUMERICAL QUESTIONS

2. γ= 0.693, k= 9.98, $\frac{d^2x}{dt^2} + 0.693\frac{dx}{dt} + 9.98x = 0$,
3. b = 5.57 x 10^{-3} s^{-1},τ = 179.5 seconds,
4. t= 18.7 s, n = 9570,
5. (i) 0.46 s^{-1} (ii) 10 s^{-1}, 6. $A = A_0 e^{-\frac{\omega}{2Q}t}$, $E = E_0 e^{-\frac{\omega t}{Q}}$, $\lambda = \frac{\pi}{Q}$, $t_r = \frac{2Q}{\omega}$]

UNIT 7 FORCED HARMONIC OSCILLATIONS AND RESONANCE

Structure

7.1 INTRODUCTION

As you have studied earlier in this course, a harmonic oscillator is a system which keeps oscillating about its mean or equilibrium position. It is an ideal system for studying periodic motions. But, in real oscillating systems, other factors such as damping forces need to be considered which results in what is called a damped harmonic oscillator. You have also studied the differential equation for weakly damped harmonic oscillator and its solution. In addition, in real system there are other factors also which add up to modify the differential equation of the oscillator and hence its solution. These may be the systems in which, beside the dissipating force, there an externally applied periodic force acts on the oscillator. Such an oscillator is called damped forced harmonic oscillator ad you will study about such oscillators in this unit.

7.2 OBJECTIVES

After studying this unit, you should be able to-

- Define forced harmonic oscillations;
- Write the differential equation for weakly damped forced harmonic oscillator;
- solve the differential equation for weakly damped forced harmonic oscillator;
- describe the phenomenon of resonance;
- apply the solution of weakly damped forced harmonic oscillator to explain resonance;
- obtain an expression for the power absorbed by a forced oscillator;
- define the quality factor of a forced oscillator; and
- apply the concept of quality factor in real applications.

7.3 FORCED DAMPED HARMONIC OSCILLATOR

A damped harmonic oscillator on which an external periodic force is applied is called a forced damped harmonic oscillator. Such an oscillator is also called a driven harmonic oscillator. In such an oscillator, the frequency of the externally applied periodic force is not necessarily the same as the natural frequency of the oscillator. In such a case, there is a sort of tussle between the damping forces tending to retard the motion of the oscillator and the externally applied periodic force which tend to continue the oscillatory motion. As a result, after some initial erratic movements, the oscillator ultimately succumbs to the applied or the driving force and settles down to oscillating with the driving frequency and a constant amplitude and phase so long as the applied force remains operative.

7.3.1 DIFFERENTIAL EQUATION FOR FORCED DAMPED HARMONIC OSCILLATOR

When an external periodic force *F(t)* is applied to a damped harmonic oscillator, the differential equation for the oscillator will have one additional term for the applied time dependent periodic force and we can write

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x + F(t) = 0 \qquad (1)$$

If the applied external force is represented as $F(t) = f cos(nt)$, where *f* and *n* are constants, then Eq. (1) becomes

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f\cos(nt) \qquad (2)$$

We can further simplify the equation as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = \mathrm{a}\cos(nt) \qquad (3)$$

where a = f/m, $2b = \gamma$ and $\omega_0^2 = k/m$ is the natural frequency of the oscillator

Eq. (3) represents the differential equation for damped forced harmonic oscillator.

7.3.2 SOLUTION OF DIFFERENTIAL EQUATION

You may note that the differential equation (Eq. (3)) for the damped forced harmonic oscillator is a linear inhomogeneous second order ordinary differential equation; the inhomogeneous term is represented by the externally applied time dependent periodic force.

The solution to this equation comprises two parts: the general solution and the particular solution. Let us learn about them now.

7.3.3 GENERAL SOLUTION

The general solution of the differential equation for damped forced harmonic oscillator comprises of two terms - one representing the homogeneous ordinary differential equation part and the other representing the particular integral part.

If X_i is a particular solution of an inhomogeneous differential equation, and X_n is a solution of a complementary homogeneous equation then $X(t) = X_i(t) + X_n(t)$ is a general solution.

Thus, the general solution of this linear inhomogeneous ODE can be expressed as

$$x(t) = x_H(t) + x_p(t) \qquad (4)$$

$x_H(t)$ is the solution of the corresponding homogeneous part of the equation. The homogeneous part is same as the differential equation for the solution of damped harmonic oscillator and its solution is given as

$$x_H(t) = \frac{1}{2}a_0 e^{-\lambda t}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-\omega_0^2}}\right)e^{\sqrt{(\lambda^2-\omega_0^2\,)}t} + \left(1-\frac{\lambda}{\sqrt{\lambda^2-\omega_0^2}}\right)e^{-\sqrt{(\lambda^2-\omega_0^2\,)}t}\right] \quad (5)$$

To obtain the particular solution, $x_p(t)$, let us assume the solution of the form

$$x_p(t) = Acos(nt - \emptyset) \qquad (6)$$

Ø is the possible phase difference between the applied force and the displacement of the oscillator and *n* is the frequency of the applied force.

Now we have to obtain *dx/dt* and d^2x/dt^2 and substitute in Eq. (3). We have

$$\frac{dx}{dt} = -Ansin(nt - \emptyset)$$

$$\frac{d^2x}{dt^2} = -An^2cos(nt - \emptyset)$$

Substitution in Eq. (3) gives

$$-An^2\cos(nt - \emptyset) - 2\lambda An\sin(nt - \emptyset) + A\omega_0^2\cos(nt-\emptyset)$$
$$= \mathrm{a}\cos[(nt - \emptyset) + \emptyset] \qquad (7)$$

Expanding the R.H.S gives

$$-An^2\cos(nt - \emptyset) - 2bAn\sin(nt - \emptyset) + A\omega_0^2\cos(nt-\emptyset)$$
$$= a[\cos(nt - \emptyset)\cos\emptyset - \sin(nt - \emptyset)\sin\emptyset \qquad (8)$$

Rearranging we get

$$A(\omega_0^2 - n^2)\cos(nt - \emptyset) - 2bAn\sin(nt - \emptyset) = a[\cos(nt - \emptyset)\cos\emptyset - \sin(nt - \emptyset)\sin\emptyset$$

If this equation is to hold true, then the coefficient of cos(nt – Ø) and sin(nt – Ø) on either sides must be equal

i.e. $A(\omega_0^2 - n^2) = acos\emptyset \;\; and \;\; 2bn = asin\emptyset$

Squaring and adding these two we get

$$A^2(\omega_0^2 - n^2)^2 + 4b^2n^2 = a^2 \qquad (9)$$

Hence

$$A^2 = \frac{a^2}{(\omega_0^2 - n^2)^2 + 4b^2n^2} \qquad (10)$$

The amplitude of driven or forced oscillator is given as

$$A = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} \qquad (11)$$

We have taken only the positive value of the square root. The negative value will mean opposite phase but then Ø will also change by π and there would, therefore be no effect on the value of *A*. Further, the phase is given by

$$\tan\emptyset = \frac{2bn}{(\omega_0^2 - n^2)} \qquad (12)$$

The particular solution of Eq. (3) is thus given by

$$x_p(t) = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}\cos(nt - \emptyset) \qquad (14)$$

Thus, we can write the general solution as

$$x(t) = \frac{1}{2}a_0e^{-bt}\left[\left(1 + \frac{b}{\sqrt{b^2 - \omega_0^2}}\right)e^{\sqrt{(b^2 - \omega_0^2)}t} + \left(1 - \frac{b}{\sqrt{b^2 - \omega_0^2}}\right)e^{-\sqrt{(b^2 - \omega_0^2)}t}\right] + \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}\cos(nt - \emptyset) \qquad (15)$$

Where $\frac{a_0}{2}$ and Ø need to be determined by initial conditions.

7.3.4 STEADY STATE SOLUTION

When the tussle between the damping and the externally applied forces ends and the oscillator has settled down to oscillate with the frequency of the applied periodic force, it is said to be in the steady state. In the steady state, the homogeneous term vanishes as $t \to \infty$ whereas the particular solution does not. Thus we have a distinction between the transient state , which is a function of the initial conditions, and a steady state , which depends on the external force. Thus, we can write the steady state solution as

$$x(t) = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}\cos(nt - \emptyset) \qquad (16)$$

$$x(t) = A\cos(nt - \emptyset) \tag{17}$$

where $A = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} = \frac{f}{m\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}$ (18)

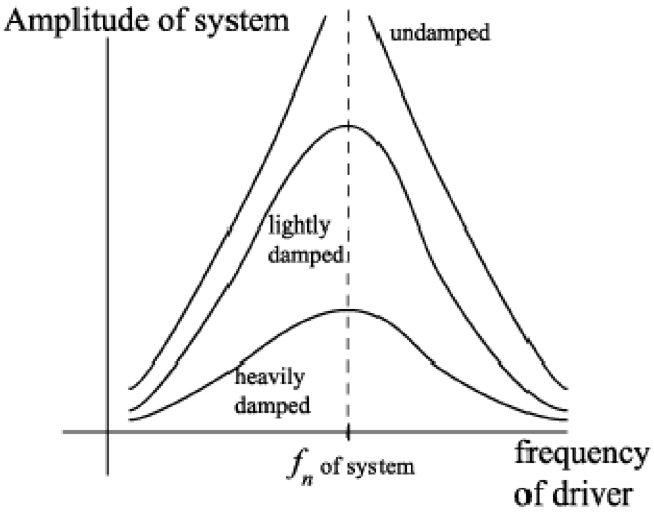

Fig.1: The variation of amplitude of oscillation of a forced oscillator with the frequency of the externally applied periodic force

7.4. EXAMPLE OF FORCED OSCILLATIONS - A DRIVEN LCR CIRCUIT

Consider an LCR circuit consisting of an inductor, *L,* a capacitor, *C*, and a resistor *R*, connected in series with a sinusoidal voltage source, *V(t)*, as shown in Fig 2 below.

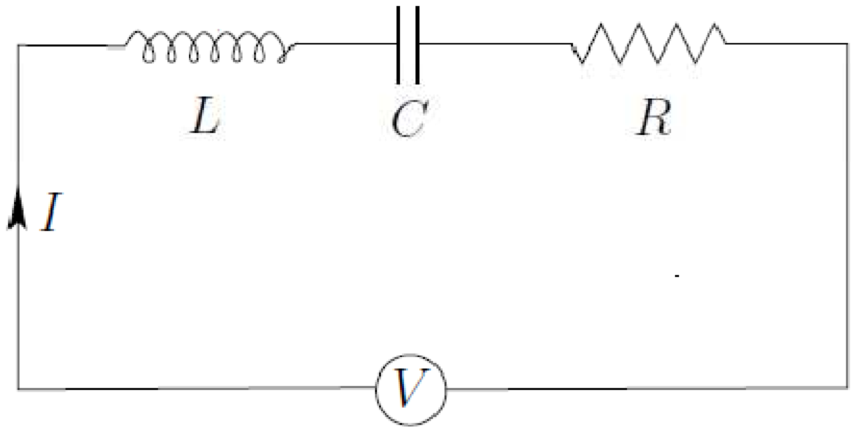

Fig. 2: A driven *LCR* circuit

Let *I(t)* be the instantaneous current flowing through this circuit, Now, as per Kirchoff's second circuital law, the sum of the potential drops across the various components of a closed circuit loop is equal to zero. Thus, since the potential drop across an emf is *minus* the associated voltage, we obtain

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V \qquad (19)$$

Where $\frac{dQ}{dt} = I$ and $\frac{d^2Q}{dt^2} = \dot{I}$. Suppose that the emf is such that its voltage oscillates sinusoidally at the angular frequency ω (> 0) with a peak value V_0 (> 0) so that

$$V(t) = V_0 \sin(\omega t) \qquad (20)$$

Substituting (20) in (19), we get

$$L\dot{I} + RI + \frac{Q}{C} = V_0 \sin(\omega t) \qquad (21)$$

Dividing equation (21) by L and differentiating with respect to time, we get

$$\ddot{I} + \gamma\dot{I} + \omega_0^2 I = \frac{\omega V_0}{L}\cos(\omega t) \qquad (22)$$

Where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\gamma = \frac{R}{L}$

Equation (22) is similar to the differential equation representing a driven damped harmonic oscillator. The current driven in the circuit by the oscillating emf is given as

$$I(t) = I_0 \cos(\omega t - \emptyset) \qquad (23)$$

where $$I_0 = \frac{\omega V_0}{L\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} \qquad (24)$$

$$\emptyset = tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) \qquad (25)$$

In the expression for I_0, the denominator $\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ functions as the effective resistance in the circuit. It is called the impedance of the circuit.

7.5 RESONANCE

In general, resonance may be defined as a tendency of a vibrating / oscillating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration / oscillation.

For a weakly damped forced (driven) oscillator, after a transitory period, the object will oscillate with the same frequency as that of the driving force. The plot of amplitude *x*(ω) versus angular frequency is shown in Fig. 3 below. If the angular frequency is increased from zero, the amplitude, *x*(ω) will increase until it reaches a maximum when the angular frequency of the driving force is the same as the natural frequency of the undamped oscillator. This phenomenon is called resonance.

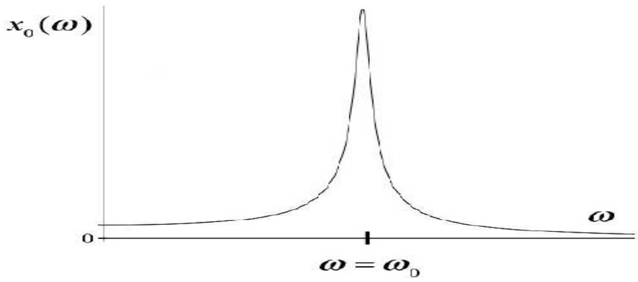

Fig. 3: Plot of amplitude *x*(ω) with driving angular frequency ω of a weekly damped harmonic oscillator

From Eq.(18), it is clear that, for a damped forced harmonic oscillator, the amplitude of the oscillator in the steady state depends not only on the amplitude of the driving force, but also on the relation between the frequency, *n* of the driving force and the natural frequency, ω of the oscillator, as well as on the damping parameter *b*.

For $n \to 0$ we have $A \to a/\omega$. For $n \to \infty$ we obtain $A \to 0$. In between these two extremes, the amplitude may reach a maximum which we refer to as the resonance frequency.

To obtain an expression for resonance frequency, we differentiate the denominator of Eq. (18) with respect to *n* and then equate it to zero.

$$\frac{d}{dn}[\omega_0^2 - n^2)^2 + 4b^2n^2] = -4n[\omega^2 - n^2] + 8b^2n = 0$$

The non - trivial solution is:

$$n = n_r = \sqrt{\omega_0^2 - 2b^2} \qquad (26)$$

This is the resonance frequency.

As we have already studied that resonance is defined mathematically using the differential Eq.(2 6) for a forced driven harmonic oscillator where the resonance is defined as the existence of a solution that is unbounded as $t \to \infty$. This corresponds to what we call as pure resonance. It occurs exactly when the natural internal frequency matches the natural external frequency, in which case all solutions of the differential equation are unbounded.

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching characterizes the event. This ideal situation never happens in the physical world, because damping is always present. In the presence of damping only bounded solutions exist for Eq. (26).

7.5.1 EXAMPLES OF RESONANCE

The Helmholtz Resonator

Helmholtz resonator is an instrument which works on the principle of resonance and it is used to determine the frequency of a vibrating bodye. The resonator consists of either a spherical or a cylindrical air cavity with a small neck as shown in Fig. 4 below. The dimension of the cavity is small in comparison with the wavelength of sound to be detected. In case of spherical cavity, the volume of the cavity is fixed whereas the volume is variable in case of cylindrical cavity.

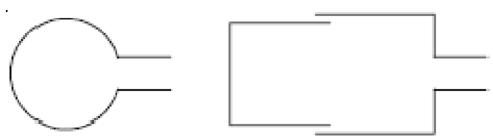

Fig. 4: Helmholtz resonator

The air contained at the neck of the resonator acts like a piston alternately compressing and rarefying the air within the cavity of the resonator. The natural frequency of vibration of Helmholtz resonator is given as:

$$\vartheta = \frac{v}{2\pi}\sqrt{\frac{S}{lV}}$$

v is the velocity of propagation of sound in air, 'l' is the length of the neck of the resonator S is the area of cross-section of the neck and V is the volume of the resonator. The natural frequency of the resonator can be changed by changing the volume V of the resonator.

When the sound wave of frequency resonant with the natural frequency of the resonator is incident on it, the resonator will produce sharp response. The frequency of the vibrating body is then equal to the natural frequency of the resonator.

Desirable Resonance

LCR circuits: The differential equation describing a driven LCR circuit, as discussed earlier in the unit, is exactly analogous to the damped driven oscillator. It is due to resonance in this circuit which allows us to pick out a certain frequency and ignore all the others. This is how radios, cell phones, etc. work.

If you have a radio on your desk, it is being bombarded by radio waves with all sorts of frequencies. If you want to pick out a certain frequency, then you can tune your radio to that frequency by changing the radio's natural frequency (normally done by changing the capacitance C in the internal circuit). Assuming that the damping in the circuit is small (this is determined by R), then there will be a large oscillation in the circuit at the radio station's frequency, but a negligible oscillation at all the other frequencies that are bombarding the radio.

Musical instruments: The pipe of a flute has various natural frequencies (depending on which keys are pressed), and these are the ones that survive when you blow air across the opening.

The ear: The hair-like nerves in the cochlea have a range of resonant frequencies which depend on the position in the cochlea. Depending on which ones vibrate, a signal is (somehow) sent to the brain telling it what the pitch is. It is quite remarkable how this works.

Undesirable Resonance

Vehicle vibrations: This is particularly relevant in aircraft. Even the slightest driving force (in particular from the engine) can create havoc if its frequency matches up with *any* of the resonant frequencies of the plane. There is no way to theoretically predict every single one of the resonant frequencies, so the car/plane/whatever has to be tested at all frequencies by sweeping through them and looking for large amplitudes.

Millennium Bridge in London: This pedestrian bridge happened to have a lateral resonant frequency on the order of 1 Hz. So when it started to sway (for whatever reason), people began to walk in phase with it (which is the natural thing to do). This had the effect of driving it more and further increasing the amplitude. Dampers where added to protect the bridge from damage.

Tall buildings: A tall building has a resonant frequency of swaying (or actually a couple, depending on the direction; and there can be twisting, too). If the effects from the wind or earthquakes happen to drive the building at this frequency, then the sway can become noticeable.

Space station: In early 2009, a booster engine on the space station changed its direction at a frequency that happened to match one of the station's resonant frequencies (about 0.5 Hz). The station began to sway back and forth, made noticeable by the fact that free objects in the air were moving back and forth. Left unchecked, larger and larger amplitude would of course be very bad for the structure. It was fortunately stopped in time.

7.6 POWER ABSORBED BY A FORCED OSCILLATOR

Whenever an oscillator is driven by an external force, energy is absorbed by the oscillator. The energy absorbed by the oscillator is equal to the energy dissipated due to damping. The rate of energy absorption or power absorbed is a function of driving frequency. It is maximum at resonance *i.e.*, when the frequency of the periodic force is equal to that of the natural frequency of the oscillator. The power absorbed by the oscillator is given by:

$$\text{Power absorbed} = (\text{Damping Force}) \times (\text{Velocity})$$

Since the damping force is proportional to the velocity, we can write

$$\text{Power absorbed} = \gamma \left(\frac{dx}{dt}\right)^2$$

We have

$$x = (f_0/m) \cos (nt)$$

Let $f_0/m = A$, then

$dx/dt = - An \sin (nt)$

Substituting this in above equation, we get

Power absorbed = $\gamma A^2 n^2 \sin^2 nt$

Average Power absorbed by oscillator in one complete cycle is given by

$$P_{av} = \int_0^{\frac{2\pi}{n}} \gamma\, (A\, n \sin nt)^2\, dt$$

$$P_{av} = n^2\gamma A^2 \int_0^{\frac{2\pi}{n}} \sin^2 nt \; dt$$

$$P_{av} = n^2\gamma A^2 \int_0^{\frac{2\pi}{n}} \left(\frac{1-\cos 2nt}{2}\right) dt$$

$$P_{av} = n^2\gamma A^2 \left[\int_0^{\frac{2\pi}{n}} \frac{1}{2} dt - \int_0^{\frac{2\pi}{n}} \frac{\cos 2nt}{2} dt\right]$$

$$P_{av} = n^2\gamma A^2 \left[\frac{\pi}{n} + 0\right]$$

$$P_{av} = n\pi\gamma A^2$$

Total power absorbed by the oscillator = (No. of cycles per second) × (Average power per cycle)

$$P_t = \frac{n}{2\pi} \times n\pi\gamma A^2$$

$$P_t = \frac{n^2\gamma A^2}{2}$$

The energy absorbed by the oscillator is equal to the energy dissipated due to damping.

7.7 QUALITY FACTOR

The energy loss rate of a weakly damped harmonic oscillator is characterized in terms of a parameter which is known as *quality factor*. This quantity is defined as 2π times the energy stored in the oscillator, divided by the energy lost in a single oscillation period. If the oscillator is weakly damped, then the energy lost per period is relatively small, and quality factor is therefore much larger than unity. Roughly speaking, it is the number of oscillations that the oscillator typically completes, after being set in motion, before its amplitude decays to a negligible value.

7.7.1 EXPRESSION FOR QUALITY FACTOR

For a weakly damped oscillator, the quality factor, Q is defined as

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy lost per radian of oscillation}}$$

While the Q of an oscillator relates to the energy loss due to damping, this links directly to the bandwidth of the resonator with respect to its central frequency. The lesser the damping, the better is the quality of the harmonic oscillator as an oscillator.

For a driven harmonic oscillator, the energy, *E(t)* of the oscillator is time dependent (oscillating with decaying amplitude ~ $e^{-T/\tau}$), So, its quality factor, Q would be

$$Q = 2\pi \frac{E(t)}{E(t) - E(t+T)} = \omega_d \frac{E(t)}{\langle P \rangle (t)}$$

Here, $T = 2\pi/\omega_d$ is the period, $\omega_d = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}$ is the frequency of damped oscillations and $\langle P \rangle (t)$is the average power loss due to damping,

Now *since* $E(t + T) = E(t)e^{-2\pi/\omega_d \tau}$, we have

$$Q = \frac{2\pi}{1 - e^{-\frac{2\pi}{\omega_d \tau}}} \approx \omega_d \tau$$

This expression for *Q* is under the assumption of weak damping. The quality factor, *Q* is a measure of sharpness of resonance in the case of a driven harmonic oscillator as shown in Fig. 5 below.

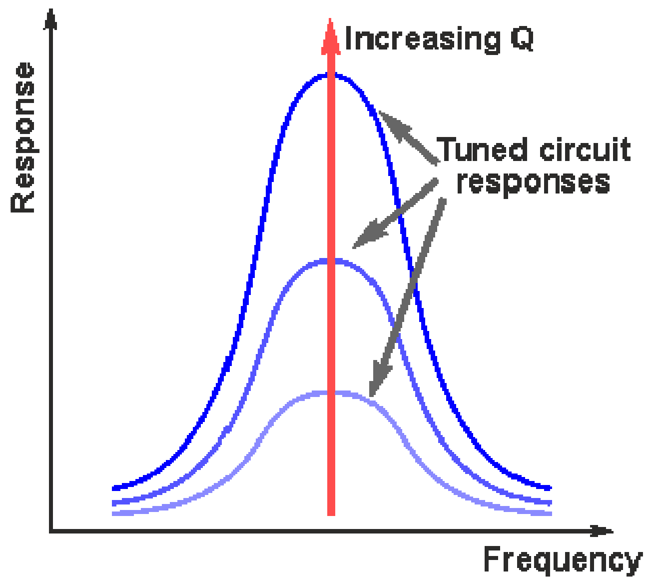

Fig. 5: Variation of *Q* with frequency

7.7.2 SOME APPLICATIONS OF QUALITY FACTOR

As already mentioned, the quality factor of an oscillator indicates how damped a resonator or an oscillator is, and characterizes a a resonator's bandwidth relative to its center frequency.

Higher Q value indicates a lower rate of energy loss relative to the stored energy of the resonator; the oscillations die out more slowly.

For an electrically resonant system, the Q factor represents the effect of electrical resistance and, for electromechanical resonators such as quartz crystals it represents the mechanical friction.

7.8. SUMMARY

The unit describes damped forced harmonic oscillator. We studied that the differential equation for damped forced harmonic oscillator is a second order non homogeneous linear ordinary differential equation. We obtained the differential equation and discussed the solution which has two components: one is the general solution and the other being the steady state solution. The general solution has two parts. Further, the steady state solution is obtained in the time domain $t \rightarrow \infty$. For the forced oscillator in steady state, we studied the concept of resonance. Further, in this unit, we studied the examples of forced vibrations and resonance in our day to day life. Aspects of forced damping related to electrical circuits were also discussed in detail. We also studies examples of resonance which are advantageous as well as which are undesirable. We further studied the power absorbed by a forced oscillator and obtained the expression for it. The quality factor of forced damped harmonic oscillator was also discussed and the expression for it was obtained. We concluded the unit with some real life applications of the quality factor.

7.9 GLOSSARY

Damping: reduction in the amplitude of oscillation as a result of energy being drained from the system to overcome frictional or other resistive foeces

Driven Oscillator: An oscillator to which an external periodic force is applied

Transient state: The state of the driven harmonic oscillator prior to achieving the steady state.

Steady state: The state of the harmonic oscillator which is independent of the initial state and depends only upon the driving frequency and the damping ratio.

Resonance: The condition when the oscillator under the influence of external driving force oscillates with greater amplitude at a specific preferential frequency.

7.10 SUGGESTED READINGS

1. SCHAUM'S OUTLINE SERIES **"THEORY AND PROBLEMS OF THEORETICAL MECHNICS"** Murray R Spiegel Mc Graw Hill Education Publications.

2. **MECHANICS** by D S Mathur S Chand and Company Ltd.

3. **"WAVES AND OSCILLATIONS"** R N Chaudhuri, New Age International Limited Publishers.

7.11 TERMINAL QUESTIONS

Q.1. A particle of mass m moves under the influence of external periodic force F sin pt along x axis in addition to the restoring force $-kx$ (also along x - axis) and damping force $-\beta xi$ along x axis . Set up the differential equation of motion and find the steady-state solution.

Q.2. Show that in case of a system undergoing a forced oscillation, the response is independent of its mass if $n << \omega_0$ and is independent of spring constant if $n >> \omega_0$

Q.3. A damped harmonic oscillator consists of a block (m = 2 kg), a spring (k = 30 N/m), and a damping force ($F = -bv$). Initially, it oscillates with amplitude of 25 cm; because of the damping, the amplitude falls to three-fourths of this initial value at the completion of four oscillations.
(a) What is the value of b?
(b) How much energy has been "lost" during these four oscillations?

MCQ

1. Which among the following is an example of forced harmonic oscillator:

a) sound produced by a flute.

b) sound produced by an organ pipe.

c) vibrations produced in violin string.

d) vibrations produced in telephone transmitter during conversation.

2. As the amplitude of resonant vibrations decreases, degree of damping

a) increases b) decreases c) remains same d) varies

3. In the case of forced simple harmonic vibrations, the body generally vibrates with
a) its natural frequency of vibration and its amplitude is small

b) its natural frequency of vibration but its amplitude is large

c) the frequency of external force with a small amplitude

d) the frequency of external force with a large amplitude

4. Consider the following statement:

A body vibrating due to forced oscillations is acted upon by

1) A restoring force which is directly proportional to its displacement

2) A retarding force which is directly proportional to its velocity

3) An external periodic force of constant amplitude and frequency

Choose correct statement

a) 1 and 2 are correct

b) 2 and 3 are correct

c) 1 and 3 are correct

d) 1,2 and 3 are correct

5. The quality factor for an LCR circuit is

a) ωR/L b) ωL/R c) ω/LR d) R/ωL

6. For a resonating system, it should oscillate

a) bound b) only for some time c) freely d) for infinite time

7. For a weekly damped harmonic oscillator with damping frequency ω and time period τ, the quality factor equals

a) ω/τ b) ωτ c) 1/ωτ d) τ/ω

8. The power absorbed by a forced oscillator is proportional to

a) square of the amplitude b) cube of amplitude c) amplitude d) inverse of amplitude

9. For a series LCR circuit driven by a sinusoidal voltage, the damping constant is

a) R/L b) RL c) 1/RL d) L/R

10. The amount of power supplied to a system is equal to the rate of dissipation of energy in

a) forced vibration

b) damped vibration

c) simple harmonic motion

d) oscillatory motion

7.12 ANSWERS

Solution 3: (a) We assume that b is small compared to √km and we take T = 2π/√(m/k) ≈ 1.62 s. It is given that at t = 4T, the amplitude falls to 3A/4, i.e.

$e^{-bt/2m} = 3/4$

-2bT/m = ln(3/4)

or b = 0.18 kg/s.

(b) Energy lost during these four oscillations = ½ k(A^2 – $(3A/4)^2$) = 7kA^2/32 = 0.410 J

MCQ Answers

1. d 2. a 3. d 4. d 5. b 6. c 7. b 8. a 9. a 10. A

UNIT 8 WAVE MOTION

Structure

8.1 INTRODUCTION

Generally speaking, there are two ways in which the energy can be transported from one place to other. The first of these methods involves the actual transportation of matter. For example, a bullet fired from a firearm carries its kinetic energy as it travels to the other location. The second method by which one can transport energy is much more important and useful. It involves what are known as waves. The waves transfer energy but there may not be any transportation of matter in the process. For example, when a violinist plays violin, its sound is heard at distant locations. The *sound waves* carry with them energy, with which they are able to move the diaphragm of the ear. When a stone is dropped in the still water in a lake, ripples are formed on the surface of the water body and the *water waves* move steadily in the outward direction. *Electromagnetic waves* are vibrating electric and magnetic fields that travel through space without the need for a medium. The electromagnetic waves include the visible light that, for example, comes from a bulb in our houses and the radio waves that come from a radio station. The other types of electromagnetic waves are microwaves, infrared light, ultraviolet light, X-rays and gamma rays. *Seismic waves* are vibrations of the earth, which become quite significant in the events such as earthquakes.

Although, these various processes of transport of energy are different yet they have a common feature, which we shall from now on refer to as the wave motion. In simple terms, we can say that the wave motion involves the transfer of disturbance (energy) from one point to the other with particles of the medium oscillating about their mean positions. The particles themselves oscillate only over a short distance about their initial positions, and as a result a wave moves through the medium. The medium as a whole does not go in the direction of the motion of the wave.

In the present unit, you will learn about wave motion including the formation and propagation of waves, characteristic features of a wave and the distinction between longitudinal and transverse waves.

8.2 OBJECTIVES

After studying this unit, you should be able to

- explain the formation and propagation of waves,
- describe different types of waves and their uses,
- represent a wave graphically at a fixed position and at a fixed time,
- explain what is meant by the amplitude, wavelength, frequency and speed of a wave,
- relate the speed to the frequency and the wavelength of a wave,
- describe what is meant by a transverse wave and give examples,
- describe what is meant by a longitudinal wave and give examples, and
- mathematically describe the wave motion.

8.3 WAVE FORMATION AND PROPAGATION

Let us first consider the example of water wave, which is the most familiar kind of wave that we can generate and observe easily. When we drop a stone in a lake or a water tub, we observe circular ripples that spread out from the point where the stone strikes the water surface, as shown in Fig. 1.

Figure 1: Waves generated on the water surface.

Looking at these ripples, you may wrongly get an impression that water moves with them. But, if you observe carefully, you will notice that water actually does not move along with the ripples that are generated. You can easily verify this fact by placing a paper boat or a dry leaf on the water surface and observing how it moves. You will notice that the paper boat or the dry leaf just bounces up and down at the same place on the surface of water and does not move with the ripples. This means that water particles do not have any translational motion. However, water particles do undergo oscillatory motion caused due to dropping of the stone in the still water. The disturbance caused at the point of contact of the stone with water surface is progressively transferred to adjacent water particles due to the oscillatory motion. The term "disturbance" refers to the deformation in the shape of the water surface (or any other medium such as air, string etc.) with respect to its undisturbed surface.

You can produce a mechanical wave using a thin and long elastic string with its one end fixed to a wall. By holding the other end of the string with your hand so that the string is stretched and taut and quickly moving your hand up and down once, you may observe a disturbance travelling along the length of the string (Figure 2). If you keep your hand moving up and down, you will observe a series of disturbances moving along the string giving rise to a wave.

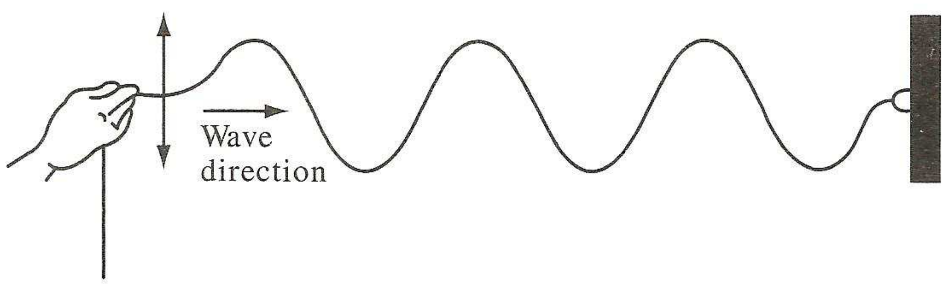

Figure 2: A mechanical wave.

From the above descriptions of waves, one may conclude that:

1- A wave is generated due to two simultaneous, at the same time, distinct motions. The first one is the oscillatory motion of the particles of the medium and the second is the linear motion of the disturbance.
2- In wave motion, the propagation of a disturbance does not take place due to the physical movement of the particles in the medium. The disturbance actually propagates because of the transfer of energy from one particle to the other progressively. Thus, we may conclude that the waves transport energy and not the matter.

The oscillations of the particle of a medium and the propagation of wave in the medium are intimately connected. To appreciate the nature of this relationship, refer to Figure 3, which shows a thin elastic string tied to a spring-mass system executing vertical oscillations. The other end of the string is tied to a rigid support. We assume that the motion of the spring-mass system is without any friction and that the vertical oscillations by the mass are without any lateral movement. Figure 4, further breaks down the waveform shown in Figure 3b and shows the snapshots of the waveform on the string taken at intervals of T/8, i.e. at time t = 0, T/8, T/4, 3T/8, T/2, 5T/8, 3T/4, 7T/8 and T. The arrows attached to each of the nine particles indicate the directions along which these particles are about to move at a given instant. At t = 0, all the particles are at their mean position as shown in Figure 4a.

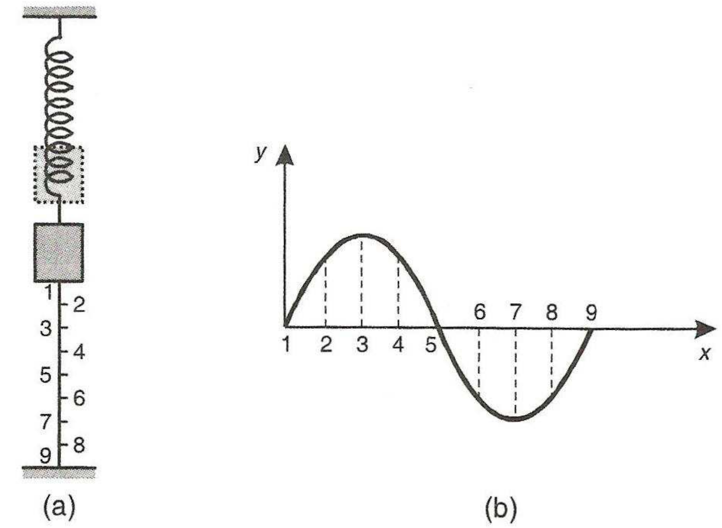

Figure 3: (a) A vertically oscillating spring-mass system fastened to a string, and (b) waveform of the motion of the string.

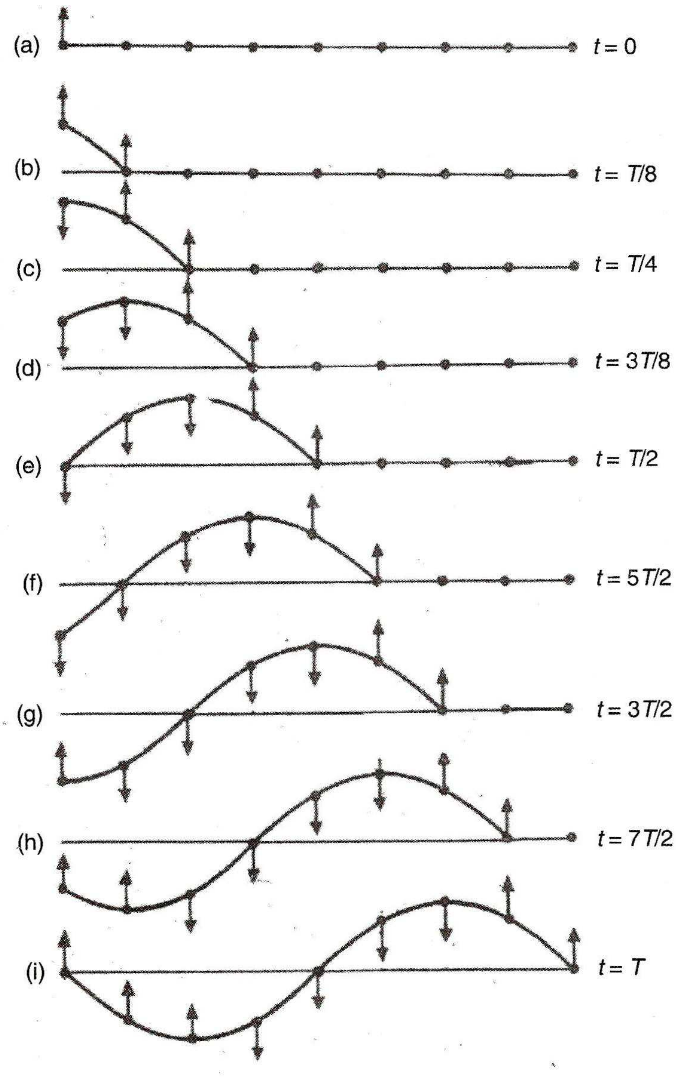

Figure 4: Snapshots of the motion of the particles 1 to 9 in the string beginning at the instant t = 0 and up to the instant t = T at intervals of T/8.

The particles in the string begins to oscillate due to the transfer of mechanical energy and momentum from the spring-mass system and their motion is sustained due to the elasticity of the medium, in this case string. One particle transfers its energy and momentum to another particle and then it transfers its energy and momentum to the third particle and so on. This process continues as long as the spring-mass system keeps oscillating. When the energy that initially activated particle 1 reaches particle 9 at time T, we say that a wave has been generated in the string. We notice that all the particles in the string oscillate up and down about their respective mean positions with time period T and the wave moves along the string with the same time period.

In our discussion until now, we have considered the propagation of mechanical waves on strings and springs for introducing the wave motion. Mechanical waves require material medium such as water, air, etc. to transfer mechanical energy and momentum from one point to another. Therefore, seismic waves, water waves, sound are all examples of mechanical waves. One should note here that sound waves travelling in air columns and on a string, both are examples of mechanical waves, but there is an important difference between the two. While the former is an example of longitudinal waves, the latter are transverse waves. We will briefly study about these waves in the next section.

8.3.1 Transverse Waves

In transverse waves, the particles of the medium oscillate perpendicular to the direction in which the wave travels. Travelling waves on a taut string, which we discussed in the previous section, are transverse waves. When the one end of the string is rigidly fixed and the other end is given periodic up and down jerks, the disturbance propagates along the length of the rope but the particles oscillate up and down. The disturbance travels along the rope in the form of crests (upward peak) and troughs (valley) as shown in Figure 3.

Secondary seismic waves are an example of transverse waves. They travel more slowly than the primary seismic waves. Secondary seismic waves shake the material they travel through from side to side. Transverse waves require that there should be a shearing force in the medium. Hence, they can be propagated only in the medium which will support a shearing stress, i.e. mainly solids. For this reason, mechanical transverse waves cannot pass though a liquid because liquid molecules slide past each other.

Electromagnetic waves, which do not require any medium to propagate, are also an example of transverse waves. The electric and the magnetic field of an electromagnetic wave vibrate at right angles to the direction of propagation and also at right angles to each other.

8.3.2 Longitudinal Waves

In longitudinal waves, the oscillation of the particles is parallel to the direction in which the wave travels. Disturbance travelling in a spring parallel to its length, a pressure variation propagating in a liquid are examples of longitudinal waves. Longitudinal waves do not require shearing stress and hence can travel in any elastic medium – solid, liquid and gas.

Consider a stretched spring. If one end of the spring is suddenly given an in and out oscillation parallel to the length of the spring, the coils of the spring start exerting forces on each other and the compression and the expansion points travel along the length of the spring. The coils oscillate right and left parallel to the spring as shown in Figure 5. Compressions, which is the crowding together of the molecules, and rarefactions, which is the spreading out of the molecules away from each other, travel along the spring. The pressure at the compression point is higher and the pressure at a rarefaction point is lower.

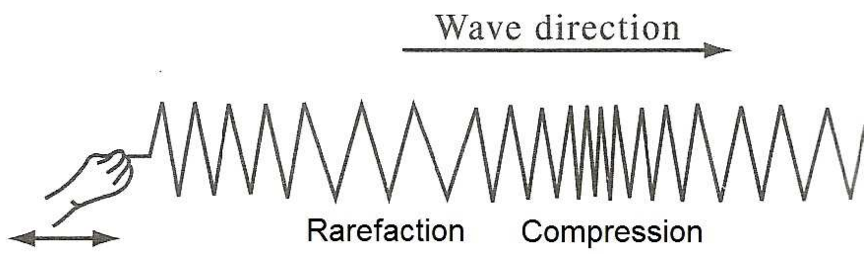

Figure 5: Longitudinal wave generated in a stretched spring.

The spring in the above example can be replaced by a long tube of air with a piston at the left end. The piston is set into oscillation along the length of the tube. The molecules of air oscillate right and left, i.e. parallel to the wave propagation as shown in Figure 6a.

Sound waves are also longitudinal waves as shown in Figure 6b. A loudspeaker supplied with alternating current creates sound waves because the diaphragm of the loudspeaker is forced to move to and fro. The diaphragm compresses the surrounding air in front of it as it moves forward and then it moves back before creating another compression. Effectively, the air which is the medium of propagation in this case, moves to and fro as the sound waves pass through it. Primary seismic waves are another example of longitudinal waves. They travel faster than the secondary waves, and can travel through solids and liquids as they push and pull on the medium they travel though.

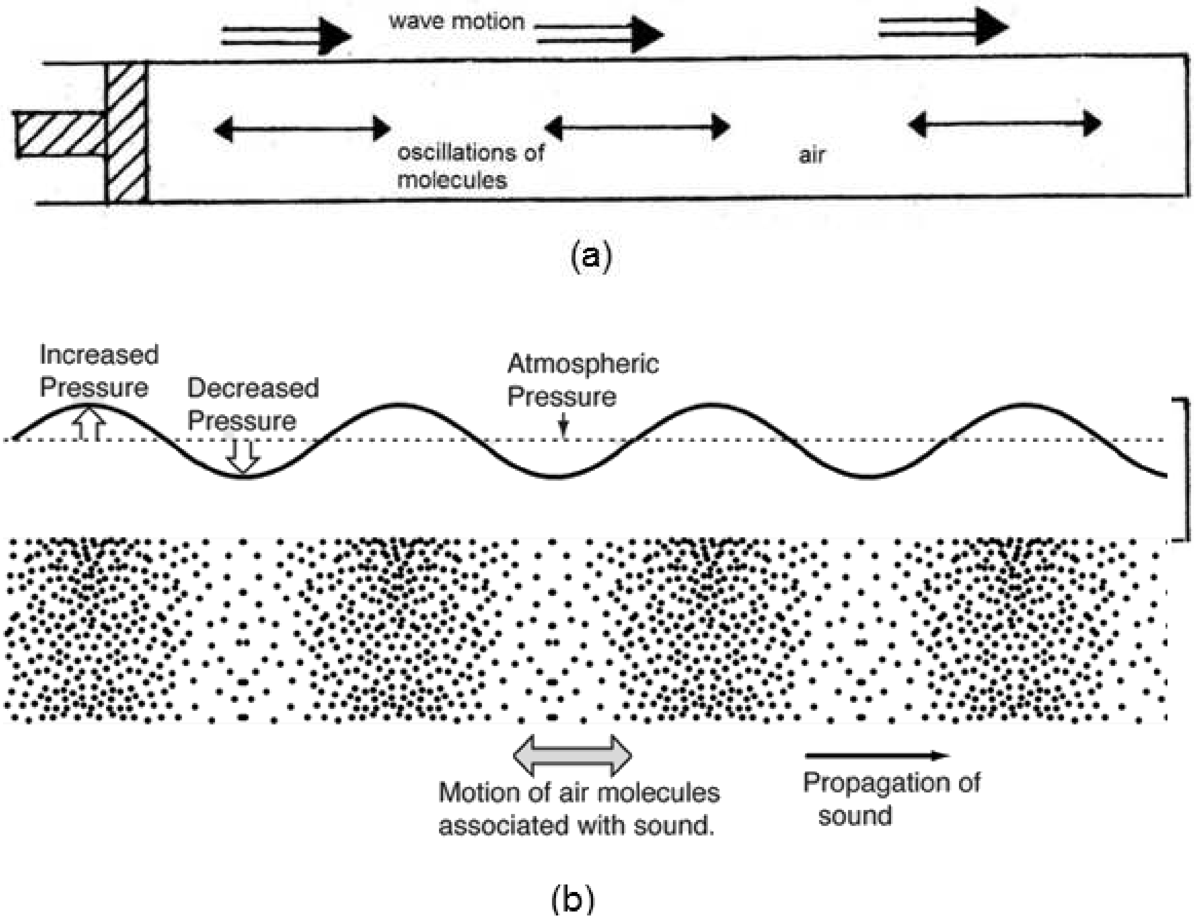

Figure 6: (a) Longitudinal waves generated in a tube of air with a piston at one end. (b) Sound waves in air.

Water waves are a combination of longitudinal and transverse waves. Each particle near the surface moves in a circular orbit, so that a succession of crests and troughs occur. At a crest, the water at the surface moves in the direction of the wave and at trough, it moves in the opposite direction.

Self Assessment Question (SAQ) 1: What type of mechanical waves do you expect to exist in (a) vacuum, (b) air, (c) water, (d) rock?

Self Assessment Question (SAQ) 2: Choose the correct option:

Elastic waves in solid are

(a) Transverse (b) Longitudinal

(c) Either transverse or longitudinal (d) Neither transverse and longitudinal.

Self Assessment Question (SAQ) 3: Give evidence in support of the fact that sound is a mechanical wave.

Self Assessment Question (SAQ) 4: Choose the correct option:

Mechanical waves on the surface of a liquid are

(a) Transverse (b) Longitudinal (c) Torsional (d) Both transverse and longitudinal.

8.4 WAVE PROPERTIES

In the preceding sections, we saw that when a wave moves, the displacements of the particles change with time as well as with the position. In one complete cycle of oscillation, the particles in the medium are displaced in one direction from their mean position to a position of maximum displacement, come back to the mean position and move in the opposite direction to the other extreme, and again come back to their mean position. In the following sections, we will be discussing some of the terms that are useful in characterizing the waves.

8.4.1 Wave Speed

The speed of a wave is the distance it covers in one second. It should be carefully noted that the wave speed is completely different from the particle speed. Particle speed is the speed of the vibrating particles in the medium. On the other hand, wave speed is the speed with which the disturbance (or wave) propagates in the medium.

8.4.2 Wave Frequency

The frequency with which the particles of the medium (through which the wave is passing) oscillate is known as wave frequency. In transverse waves, frequency is the number of crests (or troughs) that pass through a point in one second. In longitudinal waves, frequency is the number of compressions (or rarefactions) that pass through a point in one second. It is denoted by the symbol f. The SI unit of frequency is hertz (Hz), which is equal to 1 cycle per second.

We already know that the wave motion requires a source which moves or vibrates with a particular frequency. So an important point to keep in mind is that the frequency of a wave is a property of the source, not of the medium through which it propagates.

8.4.3 Time Period

The time period of the oscillation of the particles in the medium is the time period of the wave and is depicted in Figure 7. It is denoted by the symbol *T*. The frequency of a wave is the reciprocal of the time period, i.e.

$$f = \frac{1}{T} \tag{10.1}$$

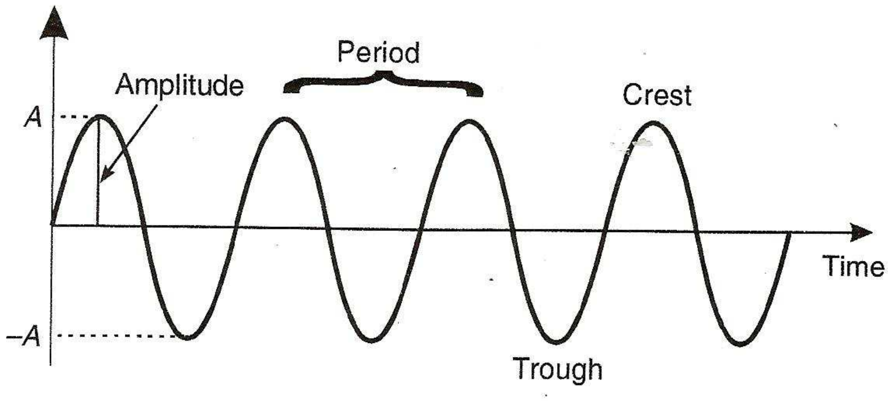

Figure 7: The vibration graph of a wave.

8.4.4 Amplitude

The amplitude of the wave is equal to the maximum positive displacement of the particles from their mean position. Thus, the amplitude of the wave is the same as the amplitude of the oscillating particles. It is depicted in Figures 7 and 8 and is denoted by the symbol *A*.

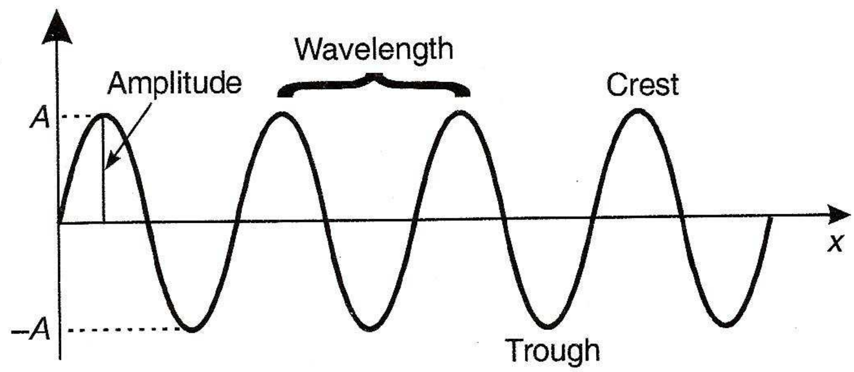

Figure 8: The waveform graph of a wave.

8.4.5 Wavelength

The distance between any two points in the same state of motion defines the wavelength of a wave. Physically, this means that the wavelength is equal to the distance between two consecutive crests (or troughs) and is depicted in Figure 8. Wavelength is denoted by the symbol λ. The wave speed is given by

$$v = \frac{\lambda}{T} \tag{10.2}$$

Since, the frequency f of a wave is the reciprocal of its period T, the above equation can also be written as

$$v = f\lambda \tag{10.3}$$

The above equation predicts that in a given medium, the wave speed of a wave of given frequency is constant. Note that equation (10.3) holds for a transverse as well as a longitudinal wave.

Thus, we can see that the wavelength and the time period represent the spatial and the temporal properties of a wave, respectively. When a wave propagates in a medium, it travels with the same amplitude, time period (or frequency) as those of the particles oscillating in the medium. Hence, we can infer that in a wave, the variation with the position and the time follows the same pattern as that of the oscillating particles. This means that we can represent wave motion both graphically as wells as mathematically. In the graphical representation, the information can be displayed in the following two ways:

1- Keeping the position x fixed and varying the time t.
2- Keeping the time t fixed and varying the position x.

The first type of graph is referred to as the vibration graph of a wave. The vibration graph shows the wave behavior at one position in the path of a wave with time. One can obtain it by fixing a slit at one spot and observing the motion of the wave at different times. Figure 7 shows the vibration graph of a wave. The vibration graph of a wave can be represented as

$$y(t) = A\sin\left(\frac{2\pi t}{T}\right) \tag{10.4}$$

On the other hand, when the time is kept fixed and the position can vary, the graph obtained is called a waveform graph. It is analogous to a snapshot at any instant of time, such as t = T. A waveform graph displays the wave behavior simultaneously at different locations as shown in Figure 8. We can represent the waveform graph of a wave as

$$y(x) = A\sin\left(\frac{2\pi x}{\lambda}\right) \tag{10.5}$$

Although, there are similarities in the shapes of vibration and waveform graphs, they should not be confused. While the vibration graph tells us about the shape of the wave, its amplitude and time period, the waveform graph gives us information about the shape of the wave, its amplitude and wavelength.

Example 1: An observer standing at sea coast observes 54 waves reaching the coast per minute. If the wavelength of the waves is 10 m, find the velocity. What type of waves did he observe?

Solution:

Since, 54 waves reach the shore per minute,

$$f = \frac{54}{60} = 0.9\ Hz$$

And as the wavelength of waves is 10 m, therefore,

$$v = f\lambda = 0.9 \times 10 = 9\ m/s$$

The waves on the surface of water are combined transverse and longitudinal waves called ripples. In case of surface waves, the particles of the medium move in elliptical paths in a vertical so that the vibrations are simultaneously back and forth and up and down as shown in Figure 9.

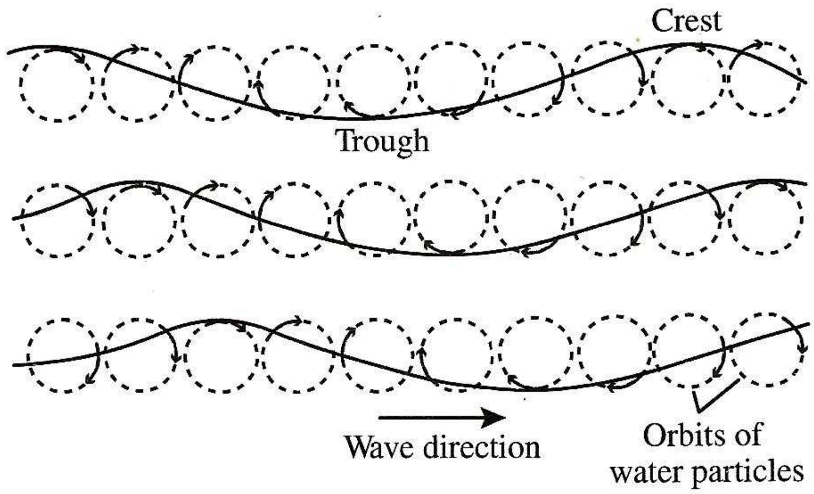

Figure 9: Ripples at different times. At a crest, the surface water moves in the direction of the wave and at trough, it moves in the opposite direction.

Example 2: A light pointer fixed to one prong of a tuning fork touches a vertical plate. The fork is set vibrating and the plate is allowed to fall freely. 8 complete oscillations are counted when the plate falls through 0.1 m. What is the frequency of the tuning fork? Take g = 9.8 m/s^2.

Solution:

Time taken by the plate to fall 0.1 m freely under gravity is given by

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(0.1)}{9.8}} = \frac{1}{7} s$$

And in this time, 8 oscillations are recorded on the plate. Therefore, the number of oscillations per second, or in other words, the frequency of the tuning fork will be

$$f = 7 \times 8 = 56\ Hz$$

Example 3: Certain radar emits 9400-MHz radio waves in groups 0.08 μs in duration. The time needed for these groups to reach a target, be reflected and return back to the radar is indicative of the distance of the target. The velocity of these waves, like other electromagnetic waves is $c = 3 \times 10^8\ m/s$. Find

(c) the wavelength of these waves,
(d) the length of each wave group, which governs how precisely the radar can measure distances of the target, and
(e) the number of waves in each group.

Solution:

(c) Since, 1 MHz = 10^6 Hz,

$$9400\ MHz = 9.4 \times 10^9\ Hz$$

Therefore, the wavelength

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8\ m/s}{9.4 \times 10^9\ Hz} = 3.19 \times 10^{-2}\ m$$

(d) The length s of each wave group is

$$s = ct = (3 \times 10^8\ m/s)(8 \times 10^{-8}\ s) = 24\ m$$

(e) There are two ways to find the number of waves n in each group:

$$n = ft = (9.4 \times 10^9\ Hz)(8 \times 10^{-8}\ s) = 752\ waves$$

Or

$$n = \frac{s}{\lambda} = \frac{24\ m}{3.19 \times 10^{-2}\ m} = 752\ waves$$

Self Assessment Question (SAQ) 5: Choose the correct option:

Which of the following cannot travel through vacuum?

(a) Light waves, (b) heat waves, (c) X-rays, or (d) sound waves.

Self Assessment Question (SAQ) 6: A body vibrating with a certain frequency sends waves of wavelength 15 cm in medium A and 20 cm in medium B. If the velocity of wave in A is 120 m/s, that in B will be ____________ m/s.

Self Assessment Question (SAQ) 7: Challenge Question:

An anchored boat is observed to rise and fall through a total range of 2 m once every 4 s as waves whose crests are 30 m apart pass it. Find

(d) the frequency of the waves,
(e) their velocity,
(f) their amplitude, and
(g) the velocity of an individual water particle at the surface.

Self Assessment Question (SAQ) 8: An object oscillates in a simple harmonic motion with a frequency of 100 Hz. Calculate its time period.

Self Assessment Question (SAQ) 9: Sound travels in air with a speed of 332 m/s. The upper limit of audible range is 20,000 Hz. Calculate the corresponding wavelength in cm.

8.5 MATHEMATICAL DESCRIPTION OF WAVE MOTION

If a mathematical equation describes a wave, it must be able to give the position of any particle of the medium at any given instant of time. Consider a transverse wave travelling toward right in a tight string lying on the x-axis. Figure 10 shows the snapshots of a wave travelling along the positive x-axis at the instant t = 0 and at time t. If the wave velocity is v, then as the wave travels the y-coordinate of point C (at x’) at t = 0 is the same as the y-coordinate of point D (at x = x’ + vt) at time t, i.e.

$$y(x,t) = y(x',0) \tag{10.6}$$

From equation (10.5), we have

$$y(x',0) = A\sin\left(\frac{2\pi x'}{\lambda}\right) = A\sin\left(\frac{2\pi(x - vt)}{\lambda}\right) \tag{10.7}$$

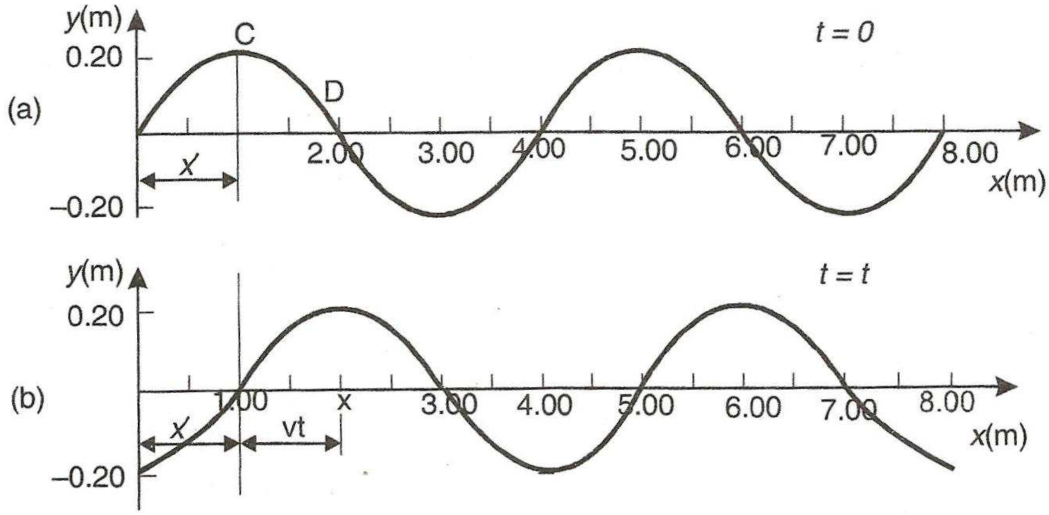

Figure 250

Therefore, from equations (10.6) and (10.7), we have

$$y(x,t) = A\sin\left(\frac{2\pi(x - vt)}{\lambda}\right)$$

Replacing v by λ/T in the above equation, the displacement y(x, t) of any particle located at some x-coordinate at any instant of time t is given by

$$y(x,t) = A\sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right) \tag{10.8}$$

Equation (10.8) is known as the wave equation. It can also be written in the following equivalent form:

$$y(x,t) = A\sin(kx - \omega t) \tag{10.9}$$

where $k = 2\pi/\lambda$ is known as the wave number, which signifies how quickly the wave oscillates is space, and $\omega = 2\pi/T$ is known as the angular frequency, which tells us how quickly the wave oscillates in time.

Also, since the wave velocity is given as λ/T, from equations (10.8) and (10.9), we can write

$$v = \frac{\omega}{k} \tag{10.10}$$

Equation (10.9) or its other equivalent forms describes a monochromatic wave, since it has a single constant frequency. Note that these equations describe 1-dimensional transverse as well as longitudinal sinusoidal waves travelling in the positive x-direction. This leads to one important difference between the displacement of the particles of the medium and the displacement y(x, t) of any point on the waveform: while the former changes periodically, the latter remains constant. As the wave travels, the entire waveform shifts. Hence, the displacement of a point on the waveform remains the same and this holds for all points on the waveform.

The following equation can easily be derived by replacing v with $-v$, if we want to describe a wave travelling in the negative x-direction.

$$y(x,t) = A\sin(kx + \omega t) \qquad (10.11)$$

Example 4: A wave is represented by

$$y(x,t) = [8\ cm]\sin[(10\ rad/cm)x - (10\ rad/s)t]$$

Determine the amplitude, wavelength, angular frequency, wave number and the velocity of the wave.

Solution:
Comparing the given wave equation with equation (10.9), we find that the wave is travelling in the positive x-direction, with amplitude A = 8 cm, angular frequency $\omega = 10\ rad/s$ and the wave number k = 10 rad/cm.

From the definition of the wave number, we have

$$k = \frac{2\pi}{\lambda}$$

$$\Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{10} = 0.63\ cm$$

Further, using equation (10.10), we have

$$v = \frac{\omega}{k} = \frac{10\ rad/s}{10\ rad/cm} = 1\ cm/s$$

Example 5: A transverse wave is travelling along a string from left to right. The figure below represents the shape of the string at a given instant. At this instant

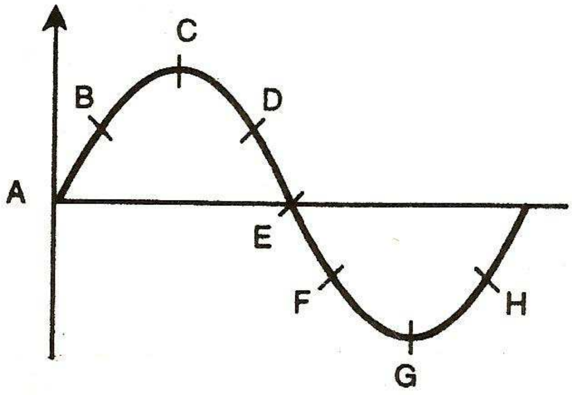

Figure 261

(a) Which points have an upward velocity?
(b) Which points have a downward velocity?
(c) Which points have zero velocity?
(d) Which points have maximum magnitude of velocity?

Solution:

For a wave travelling in positive x-direction, the particle velocity v_p at any instant is given by

$$v_p = \left(\frac{dy}{dt}\right)_x$$

$$\Rightarrow v_p = -\omega A \cos(kx - \omega t) \tag{10.12}$$

Further, the slope of the wave is given as

$$\frac{dy}{dx} = Ak \cos(kx - \omega t) \tag{10.13}$$

From equations (10.12) and (10.13), we get that the particle velocity v_p is equal to the negative of the product of the wave velocity with the slope of wave at that point,

$$v_p = -\frac{\omega}{k} \times (slope) = -v \times (slope) \tag{10.14}$$

(a) For upward velocity, $v_p = positive$, so the slope must be negative which is at points D, E and F.

(b) For downward velocity, $v_p = negative$, so the slope must be positive which is at points A, B and H.
(c) For zero velocity, the slope must be zero which is at C and G.
(d) For maximum magnitude of velocity, $|slope| = maximum$ which is at A and E.

Self Assessment Question (SAQ) 10: A simple harmonic wave having an amplitude A and time period T is represented by the equation $y = 5\sin\pi(t + 4)\ m$. What are the values of A (in m) and T (in s)?

Self Assessment Question (SAQ) 11: Choose the correct option:

Waves whose crests are 30 m apart reach an anchored boat once every 3 s. The wave velocity (in m/s) is

(g) 0.1 (b) 5 (c) 10 (d) 900

8.6 SUMMARY

In this unit, we have studied about the different waves that are familiar to us and are part of our everyday life. Then we studied what is meant by the wave motion, the formation and propagation of waves in a medium. We learned about the difference between transverse and longitudinal waves, and how to represent a wave at a fixed position and at a fixed time graphically. We wrote the mathematical expression of a progressive wave corresponding to a given set of wave parameters and travelling along +x /–x directions.

We defined the terms that are needed to describe a wave such as amplitude, time period, wavelength, frequency, wave number, angular frequency, wave velocity etc and understood how they are related to each other. Finally, we derived the relationship between the velocity of a particle in the medium and the velocity of the wave at any instant.

8.7 GLOSSARY

Amplitude – the maximum displacement of a wave from equilibrium (e.g. height of a transverse wave from the middle).

Displacement – net change in location of a moving body. It is measured from the equilibrium position.

Elasticity – ability of a material to regain its shape after being distorted.

Force – any interaction that, when unopposed, can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a wave. The SI unit of frequency is the hertz (Hz), which is equal to 1 cycle per second.

Longitudinal waves – waves in which the vibrations are parallel to the direction of travel of the wave.

Microwaves – electromagnetic waves of wavelength between about 0.1 mm and 10 mm.

Molecule – the smallest amount of a compound or element that can exist independently.

Momentum – mass multiplied by velocity.

Pressure – force per unit area applied at right angles to a surface. The SI unit of pressure is the pascal (Pa), which is equal to 1 N/m^2.

Radio waves – electromagnetic waves of wavelength longer than about a millimeter.

Sound – vibrations in a substance that travel through the substance.

Speed – the ratio of distance traveled and time. The SI unit of speed is m/s.

Transverse waves – waves in which the vibrations are at right angles to the direction of propagation of wave.

Ultraviolet radiation – electromagnetic waves between the violet end of the visible spectrum (wavelength ~400 nm) and X-rays (wavelength less than ~1 nm).

Velocity – speed in a given direction.

Wavelength – the distance between two adjacent wave-crests.

X-rays – electromagnetic waves of wavelength less than about 1 nm.

8.8 TERMINAL QUESTIONS

1. A tuning fork vibrating at 300 Hz is placed in a tank of water. (a) Find the frequency and wavelength of the sound waves in the water. (b) Find the frequency and wavelength of the sound waves produced in the air above the tank by the vibrations of the water surface. The velocity of the sound is 4913 ft/s in water and 1125 ft/s in air.

2. The visible region of the electromagnetic spectrum begins from 400 nm. Calculate the corresponding frequency.

3. The equation for the displacement of a stretched string is given by

$$y = 4\sin 2\pi\left[\frac{t}{0.02} - \frac{x}{100}\right]$$

where y and x are in cm and t is in seconds. Determine the

(a) direction in which the wave is propagating
(b) amplitude
(c) time period
(d) frequency
(e) angular frequency
(f) wavelength
(g) velocity of wave
(h) wave number

4. What quantity is carried off by all types of waves from their source to the place where they are eventually absorbed?

5. A wave of frequency f_1 and wavelength λ_1 goes from a medium in which its velocity is v to another medium in which its velocity is 2v. Find the frequency and wavelength of the wave in the second medium.

6. A violin string is vibrating at a frequency of 440 Hz. How many vibrations does the string make while its sound travels 200 m in air?

7. Lower the frequency of a wave

(a) higher is its velocity.

(b) longer is its wavelength

(c) smaller is its amplitude

(d) shorter is its period

8. Which of the following is an entirely longitudinal wave?

(a) Water wave

(b) Sound wave

(c) Electromagnetic wave

(d) A wave in a stretched string

9. Sound cannot travel through

(a) vacuum

(b) liquid

(c) gas

(d) solid

10. Of the following properties of a wave, the one that is independent of the others is

(a) velocity

(b) frequency

(c) wavelength

(d) amplitude

11. Write notes on:

(i) Wave Formation and Propagation (ii) Transverse and Longitudinal Waves

(iii) Wave Properties

12. What is meant by wave equation? Derive the wave equation when a wave is travelling in the negative x-direction.

8.9 ANSWERS

Selected Self Assessment Questions (SAQs):

1. (a) no wave, (b) longitudinal, (c) longitudinal, (d) either transverse or longitudinal

2. (c)

3. Sound requires medium for propagation.

4. (d)

5. (d)

6. 160 m/s

7. (a) $f = \frac{1}{T} = \frac{1}{4} = 0.25\ Hz$

(b) $v = f\lambda = (0.25\ Hz)(30\ m) = 7.5\ m/s$

(c) The amplitude is half the total range, so A = 1 m.

(d) As each wave passes, the water particles at the surface move in circular orbits of radius r = A = 1 m (see Figure 9). The circumference of such an orbit is $s = 2\pi r = 2\pi(1\ m) = 6.28\ m.$ The waves have time period of 4 s, which means that each surface water particle must move through its 6.28 m orbit in 4 s. The velocity of such a water particle is, therefore,

$$v_p = \frac{s}{T} = \frac{6.28\ m}{4\ s} = 1.57\ m/s$$

Note that the wave velocity here is nearly five times greater than the water particle velocity. This signifies that the motion of a wave can be much faster than the motions of the individual particles of the medium in which the wave travels.

8. 0.01 s

9. 1.66 cm

10. The wave is travelling in the negative x-direction and the wave equation is given as

$$y(x,t) = A\sin\left(\frac{2\pi x}{\lambda} + \frac{2\pi t}{T}\right)$$

Comparing with the above wave equation, the amplitude A = 5. Comparing the second term inside the sine term in the above equation, we get

$$\frac{2\pi}{T} = \pi \Rightarrow T = 2$$

11. (c)

Selected Terminal Questions:

1. (a) In the water, the frequency of the sound waves is the same as the frequency of their source, and their wavelength is

$$\lambda_1 = \frac{v_1}{f} = \frac{4931\ ft/s}{300\ Hz} = 16.4\ ft$$

(b) In the air, the frequency of the sound waves is the same as the frequency of their source, but the wavelength differs from that in the water

$$\lambda_2 = \frac{v_2}{f} = \frac{1125\ ft/s}{300\ Hz} = 3.75\ ft$$

2. $f = c/\lambda = (3 \times 10^8\ m/s)/(400 \times 10^{-9}\ m)$

$$= 7.5 \times 10^{14}\ Hz$$

3. (a) As there is a negative sign between t and x terms, the wave is propagating along the positive x-axis.

(b) A = 4 cm

(c) $T = 0.02$ s

(d) $f = 1/T = 50$ Hz

(e) $\omega = 2\pi f = 100\pi$ rad/s

(f) $\lambda = 100$ cm

(g) $v = f\lambda = 50$ m/s

(h) $k = 2\pi/\lambda = \pi/50$ cm^{-1}

4. Energy

5. The frequency of the wave remains constant, therefore, $f_2 = f_1$. The wavelength meanwhile will change according to the relation

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}$$

$$\Rightarrow \lambda_2 = \frac{v_2}{v_1}\lambda_1 = 2\lambda_1$$

6. The speed of sound wave (v) in air about 330 m/s. Therefore, the wavelength of the sound wave produced by the violin string will be given as

$$\lambda = \frac{v}{f} = \frac{330\ m/s}{440\ Hz} = 0.75\ m$$

Hence, to travel 200 m, the number of vibrations will be

$$= \frac{200}{0.75} = 266$$

7. (b)

8. (b)

9. (a)

10. (d)

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UNIT 9 PHASE, ENERGY AND INTENSITY OF WAVE

Structure

9.1 INTRODUCTION

In the previous unit (Unit 8), you learnt about mechanical waves which transport energy from one place where it is produced to the other place where it is desired to be utilized. We studied how waves are formed and how they propagate. The different types of waves can be categorized into two types - longitudinal wave and transverse wave.

We then learned about the different characteristics of waves, such as amplitude, wave speed, frequency, wavelength, time period, which are essential to describe waves. Later we delved into the mathematical description of a wave propagating in the + x and – x directions. We also came to understand that in a wave, the particles of the medium do not actually physically travel in the direction of the propagation of wave, but just oscillate and only the energy is transferred in the form of a wave. We then derived a relationship between the particle velocity and the velocity of the wave.

In this unit, we will be looking into more properties of the wave such as phase and phase difference, and phase velocity. We will also study how the energy is transported by the progressive waves. Lastly, we will see how the intensity of wave is defined and how it varies with the distance from the source.

9.2 OBJECTIVES

After studying this unit, you should be able to

- explain the concept of phase and phase difference in relation to waves,
- explain what is meant by the phase velocity,
- derive the expression for the energy carried by a wave,
- describe the concept of intensity of a wave,
- derive the expression for the wave intensity at a point in space,
- understand the idea of intensity levels and decibels, and
- explain what is meant by power gain of a system.

9.3 PHASE OF A WAVE

In the preceding unit, we studied about the wave parameters such as amplitude, time period, frequency, wavelength and wave speed that are used to characterize a wave. To complete the mathematical description of a wave, we also need to know about the phase of a wave and the concept of phase difference. This is what we will be discussing in this section.

The word phase is synonymous to what we call "state;" e.g. in thermodynamics, it refers to the solid, liquid, vapor and plasma states of a substance. You may recall how phase was defined for a system exhibiting simple harmonic motion. Similarly, for a wave, which arises due to periodic motion of particles around their mean position, we can extend the concept of phase of SHM, to define the phase of a wave.

The argument of the periodic function representing a periodic wave is called the phase of the wave. We denote it by the symbol $\varphi(x,t)$. It describes the state of motion of a particle on the wave. Thus, the phase of a sinusoidal wave, at point x and at instant t, represented by $y(x,t) = A\sin(kx - \omega t)$, is the argument of the sine function and is given by

$$\varphi(x,t) = kx - \omega t \qquad (11.1)$$

Since, the phase of a wave is an angle; it is measured in degrees or radians with 360^0 or 2π radian being equivalent to a phase difference of one wavelength. Also note, that phase is a function of position as well as time and varies with both x and t.

From the definition of phase, it follows that all the points on the wave separated by one wavelength or its integral multiples are in the same phase. To appreciate this statement, we recall that for a sinusoidal function at a given instant of time t,

$$\sin(kx - \omega t + 2\pi) = \sin(kx - \omega t)$$

And for x' = x + λ, using the relation k = 2π/λ, we can write

$$\sin(kx' - \omega t) = \sin[k(x + \lambda) - \omega t]$$

$$= \sin[kx - \omega t + 2\pi]$$

$$= \sin(kx - \omega t)$$

This result shows that at any given instant of time t, the phase of particles on the wave at a point x = λ is the same as the phase at a point x = 0. In fact, we can demonstrate that all other particles on the wave separated by integral multiples of the wavelength ought to have the same phase. To prove this, all one has to do is substitute $x' = x \pm n\lambda$ and use the result $\sin(kx - \omega t \pm 2n\pi) = \sin(kx - \omega t)$ for n = 0, 1, 2 … We can generalize this result as follows:

Particles on a wave separated by one wavelength λ or its integral multiples are in-phase. However particles at a point x has a finite phase difference with all other particles at points $x' \neq x$, for which $x' \neq x \pm n\lambda$ where n = 0, 1, 2 … In terms of the angle, the in-phase points are separated by nπ radians, where n = 0, 1, 2 …, while out-of-phase points can be any number of degrees other than nπ radians. Physically, the phase of a wave is indicative of the instantaneous position of the wave relative to a reference position.

We can also extend this concept to a situation when more than one wave is travelling in space and time. Two waves are said to be in-phase when the corresponding points on each wave reach their respective maximum and minimum displacements simultaneously. Thus, if the crests and troughs of the two waves coincide, they are said to be in-phase. If the crest of one wave coincides with the trough of the other wave, their phases differ by π radians and the waves are said to be of opposite phase. The phase difference between two waves can vary from 0 to 2π radians.

For waves travelling in the positive or negative x-directions, the arguments of the most general equation contain an additional factor to factor in the initial phase. If we take it as φ_0, then we can write

$$y(x,t) = A\sin(kx - \omega t + \varphi_0) \quad (11.2a)$$

$$y(x,t) = A\sin(kx + \omega t + \varphi_0) \quad (11.2b)$$

for waves travelling in the positive or negative x-directions, respectively, where the waves represented by (11.2a) and (11.2b) are shifted by an angle φ_0. Figure 1 depicts the phase shifts of π/2 and π for a wave from the case, when there is no phase shift.

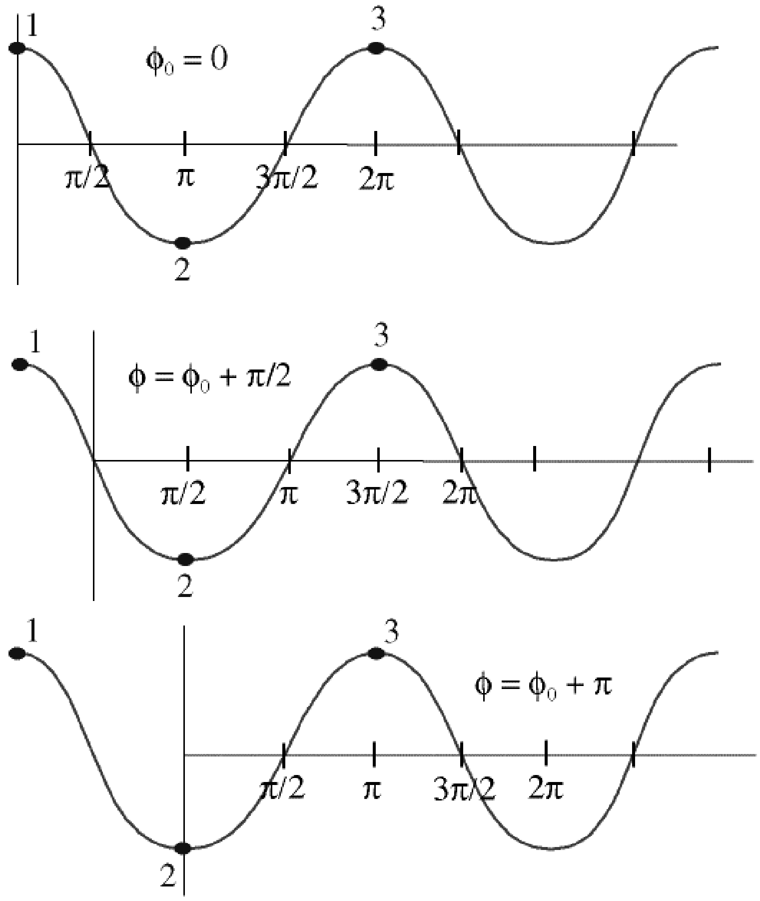

Figure 27: Phase shifts of π/2 and π of a wave from the case, when there is no phase shift.

9.3.1 Phase Velocity

We now know that for a travelling wave, the entire waveform shifts with time. If the medium is isotropic[2] and its characteristics remain constant with time, the wave propagation takes place at constant velocity. For harmonic travelling waves, this velocity is called the phase velocity, for example, the ripples on the surface of water will travel with a constant velocity if the depth of water remains the same.

Let us now derive the expression for the phase velocity of a wave. Consider a sinusoidal wave travelling in the positive x-direction. In the simple case of a pure sinusoidal wave we can imagine a rigid profile being physically moved in the positive x-direction with speed v as illustrated below.

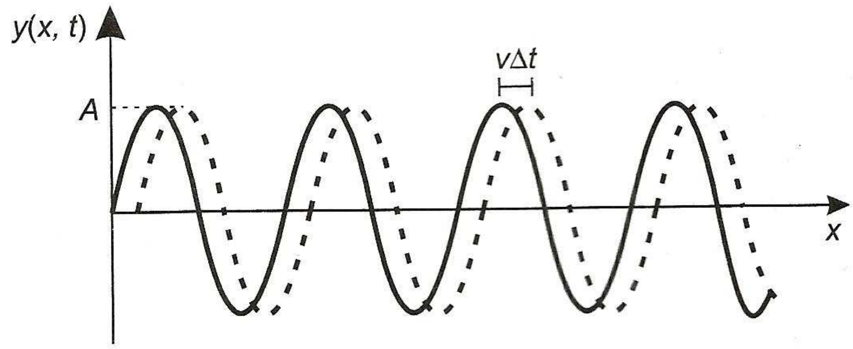

Figure 28: Wave profile of a sinusoidal wave travelling along positive x-direction.

The phase velocity of the wave defines the speed with which the wave pattern travels in space. In other words, the phase velocity of a wave is defined as the velocity with which a point of constant phase on the wave travels. For a point of constant phase, we can write

$$\varphi(x,t) = kx - \omega t = constant \tag{11.3}$$

To obtain the expression for the phase velocity, we first express the infinitesimal change in $\varphi(x,t)$ in terms of changes in x and t as

$$d\varphi = \left(\frac{\partial\varphi}{\partial t}\right)_x dt + \left(\frac{\partial\varphi}{\partial x}\right)_t dx$$

Substituting $\left(\frac{\partial\varphi}{\partial t}\right)_x = -\omega$ and $\left(\frac{\partial\varphi}{\partial x}\right)_t = k$, from equation (11.3), we get

$$d\varphi = -\omega dt + k dx$$

[2] An object or a substance having a physical property which has the same value when measured in different directions.

Since, the phase velocity is the velocity of a point of constant phase,

$$d\varphi = 0$$

This gives,

$$\omega dt = kdx$$

$$\Rightarrow v_p = \left(\frac{dx}{dt}\right)_\varphi = \frac{\omega}{k} \tag{11.4}$$

Equation (11.4) gives the phase velocity v_p. Recall the wave velocity that we defined in Unit 8, which is also given by the same expression as the phase velocity of a wave.

In terms of other wave parameters, the phase velocity of a wave is given as

$$v_p = \frac{\omega}{k}$$

$$= \lambda f$$

$$= \frac{\lambda}{T} \tag{11.5}$$

Example 1: A 1-D plane progressive wave of amplitude 1 cm is generated at one end (x = 0) of a long string by a tuning fork. At some instant of time, the displacements of the particles at x = 10 cm and at x = 20 cm are -0.5 cm and 0.5 cm, respectively. The speed of the wave is 100 m/s.

(a) Calculate the frequency of the tuning fork.
(b) If the wave is travelling along the positive x-direction and the end x = 0 is at equilibrium position at t = 0, write the displacement of wave in terms of amplitude, frequency and the wavelength.

Solution:

(a) The equation of a plane progressive wave in 1-D is given as

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(vt - x)\right]$$

We are given $A = 1$ cm. From the first condition, we know that at $x = 10$ cm, $y = -0.5$ cm. Therefore, we have

$$-0.5 = \sin\left[\frac{2\pi}{\lambda}(vt - 10)\right]$$

$$\Rightarrow \frac{2\pi}{\lambda}(vt - 10) = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow \frac{2\pi}{\lambda}(vt - 10) = \frac{7\pi}{6}$$

$$\Rightarrow vt - 10 = \frac{7\lambda}{12} \quad -----------(I)$$

From the second condition, we know that at x = 20 cm, y = +0.5 cm. Therefore, we have

$$0.5 = \sin\left[\frac{2\pi}{\lambda}(vt - 20)\right]$$

$$\Rightarrow \frac{2\pi}{\lambda}(vt - 20) = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{2\pi}{\lambda}(vt - 20) = \frac{\pi}{6}$$

$$\Rightarrow vt - 20 = \frac{\lambda}{12} \quad -----------(II)$$

Subtracting (II) from (I), we get

$$10 = \frac{7\lambda}{12} - \frac{\lambda}{12}$$

$$\Rightarrow \frac{\lambda}{2} = 10 \qquad or \qquad \lambda = 20\ cm$$

Since, the wave velocity $v = f\lambda$, the frequency of the tuning fork is given as

$$f = \frac{v}{\lambda} = \frac{100\ m/s}{0.2\ m} = 500\ Hz$$

(b) The wave equation in the desired form is given as

$$y(x,t) = (0.01\ m)\sin\left[\frac{2\pi}{0.2}(100t - x)\right]$$

$$= (0.01\ m)\sin[10\pi(100t - x)]$$

Self Assessment Question (SAQ) 1: A progressive wave of frequency 500 Hz is travelling with a velocity of 360 m/s. How far apart are two points 60^0 out of phase?

Self Assessment Question (SAQ) 2: Choose the correct option:

Three progressive waves A, B and C are shown in the figure below. With respect to wave A,

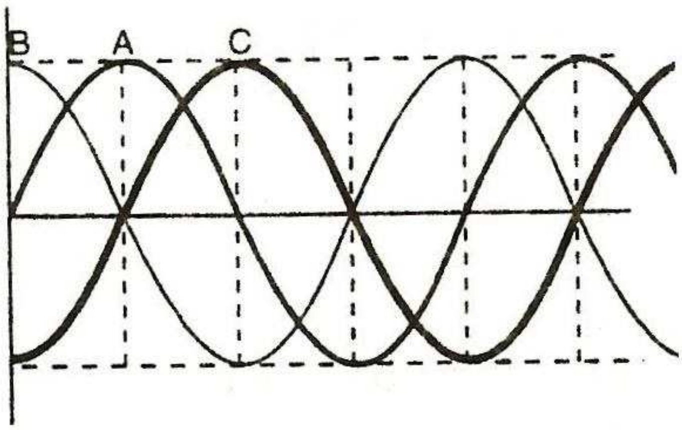

Figure 29

(a) The wave C lags behind in phase by π/2 and B leads by π/2.
(b) The wave C leads in phase by π and B lags behind by π.
(c) The wave C leads in phase by π/2 and b lags behind by π/2.
(d) The wave C lags behind in phase by π and B leads by π.

Self Assessment Question (SAQ) 3: Choose the correct option:

Two waves are respectively $y(x,t) = A\sin[\omega t - kx)]$ and $y(x,t) = B\cos[\omega t - kx)]$. The phase difference between the two waves is

(a) π/2 (b) π/4 (c) π (d) 3π/4

Self Assessment Question (SAQ) 4: Consider the two transverse waves travelling along a tight string in opposite directions

$$y_1 = A\sin(kx - \omega t) \quad travelling\ towards + x\ direction$$

$$y_2 = A\sin(kx + \omega t) \quad travelling\ towards - x\ direction$$

What is the phase difference between the two waves?

9.4 ENERGY TRANSPORTED BY PROGRESSIVE WAVES

Consider a 1-D mechanical wave travelling along the +x direction in air or on a string. Such a wave is described by the equation

$$y(x,t) = A\sin(kx - \omega t)$$

We already know that when a sinusoidal wave moves in a medium, it transports energy which is characterized by the wave velocity. For the sake of simplicity, consider a thin segment of the medium and calculate the rate at which total energy is transferred. For mechanical waves, the mechanical energy transported by these can be expressed as the sum of the kinetic energy and the potential energy of the segment under consideration.

Suppose that the thin segment under consideration has thickness dx, cross-section area A and is situated at a distance x from the source generating the progressive waves. If the density of the medium is ρ, the mass, dm of the layer will be equal to ρA(dx). Thus, the kinetic energy (KE) imparted by the wave propagating with velocity v to the thin layer of mass dm of the medium is given by

$$d(KE) = \frac{1}{2}(dm)v^2 \tag{11.6}$$

The velocity v is calculated by differentiating $y(x,t)$ with respect to t. But $y(x,t)$ is a function of two variables: position x and time t. Therefore, we determine v by partially differentiating $y(x,t)$ with respect to t, while treating x as constant, i.e.

$$v = \left(\frac{\partial y}{\partial t}\right)_x$$

Thus, we can write

$$v(x,t) = A\omega\cos(kx - \omega t) \tag{11.7}$$

Substituting the expression for $v(x,t)$ in equation (11.6), we get the expression for KE transferred to the segment of mass dm of the medium,

$$d(KE) = \frac{1}{2}(dm)A^2\omega^2\cos^2(kx - \omega t) \tag{11.8}$$

To calculate the average KE transported by the wave in one time period or one wavelength, we integrate equation (11.8) over an interval and divide the resultant expression by the length of that interval. By definition, the average KE transported over one wavelength is given by

$$\langle KE\rangle = \frac{1}{2}dmA^2\omega^2\left[\frac{\int_0^\lambda \cos^2(kx)dx}{\int_0^\lambda dx}\right] \tag{11.9}$$

For convenience, we set t = 0 in the above expression. To solve the above integral, we introduce a change of variable by substituting $kx = \varphi$, so that $kdx = d\varphi$. The limits of integration change accordingly to 0 and kλ = 2π. From the knowledge of elementary calculus, we know that the average value of the square of a cosine function over one full cycle is equal to 1/2. Therefore, the expression (11.9) becomes

$$\langle KE\rangle = \frac{1}{4}dmA^2\omega^2 \tag{11.10}$$

The potential energy (PE) stored in mass dm displaced from the equilibrium position by a distance of y is given by

$$PE = -\int_0^y Fdy$$

where F is the force acting on the layer of mass dm and is given as

$$F = mass \times acceleration = dm\frac{\partial v(x,t)}{\partial t}$$

Differentiating equation (11.7) with respect to t while keeping x fixed and inserting the expression in the above equation, we get

$$F = -dm(\omega^2 A)\sin(kx - \omega t)$$

$$= -dm\omega^2 y(x,t)$$

Substituting the above expression for force, the expression for PE becomes

$$PE = dm\omega^2\int_0^y ydy$$

$$\Rightarrow PE = \frac{1}{2} dm\omega^2 y^2 = \frac{1}{2} dm\omega^2 A^2 \sin^2(kx - \omega t) \qquad (11.11)$$

Just like we calculated the average KE, the average PE over one wavelength can be calculated to get

$$\langle PE \rangle = \frac{1}{2} dmA^2\omega^2 \left[\frac{\int_0^\lambda \sin^2(kx)dx}{\int_0^\lambda dx} \right]$$

$$\Rightarrow \langle PE \rangle = \frac{1}{4} dmA^2\omega^2 \qquad (11.12)$$

Comparing equations (11.12) and (11.10), we notice that the average PE of mass dm of a layer in the medium is equal to its average KE. The total average energy of the segment of the medium under consideration at any instant of time is the sum of its average KE and PE, i.e.

$$E = \langle KE \rangle + \langle PE \rangle = \frac{1}{2} dmA^2\omega^2 \qquad (11.13)$$

The above equation demonstrates the fact that half of the average energy transported by a progressive wave per cycle through a thin layer of mass dm is kinetic and the other half is potential. The energy is transferred to successive layers of the medium and in this process, energy is transported in the medium by a progressive wave.

We may also calculate the average rate of energy flow in the medium per cycle, or in other words, the power transmitted by the wave. Since, $dm = \rho A \Delta x$, we can write the expression for power P of the wave as

$$P = \frac{E}{\Delta t} = \frac{2\pi^2 A^2 f^2 \rho A \Delta x}{\left(\frac{\Delta x}{v}\right)}$$

$$\Rightarrow P = 2\pi^2 A^2 f^2 \rho A v \qquad (11.14)$$

where we have used the expression $\omega = 2\pi f$ for the frequency and $\Delta t = \Delta x / v$ for the time taken by the wave to cross the layer of thickness Δx by the wave travelling with velocity v. This result shows that the rate at which the energy is transported by a wave varies linearly with wave velocity and as the square of its amplitude and frequency.

Example 2: A plane progressive wave of amplitude 0.02 cm is generated when a musical instrument is played. If a note of frequency 300 Hz is produced, calculate the rate at which the energy is generated per unit volume, if the density of air is 1.29 kg/m^3.

Solution: From equation (11.13), we know that the energy per unit volume, or in other words, the energy density is given by

$$\frac{E}{A\Delta x} = 2\pi^2 A^2 f^2 \rho$$

$$= 2\pi^2 (2 \times 10^{-4}\ m)^2 (300\ Hz)^2 (1.29\ kg/m^3)$$

$$= 9.2 \times 10^{-2}\ J/m^3$$

Self Assessment Question (SAQ) 5: Half of the energy transported by a sinusoidal wave is kinetic and the other half is potential. Is this statement true or false? Give reason.

9.5 INTENSITY OF A WAVE

While energy and power are useful parameters, these do not account for observations related to the variations in the strength of a progressive wave with distance from the source. Consider progressive waves generated by a stationary source and spreading out in the surrounding medium. If there were no loss of energy of the wave, its strength should remain the same everywhere away from the source. This is, however, not true in practice. From our common experience, we know that the chirping of the birds, the noise of the traffic, the sound of firecrackers or the light from a bulb fade out beyond a certain distance.

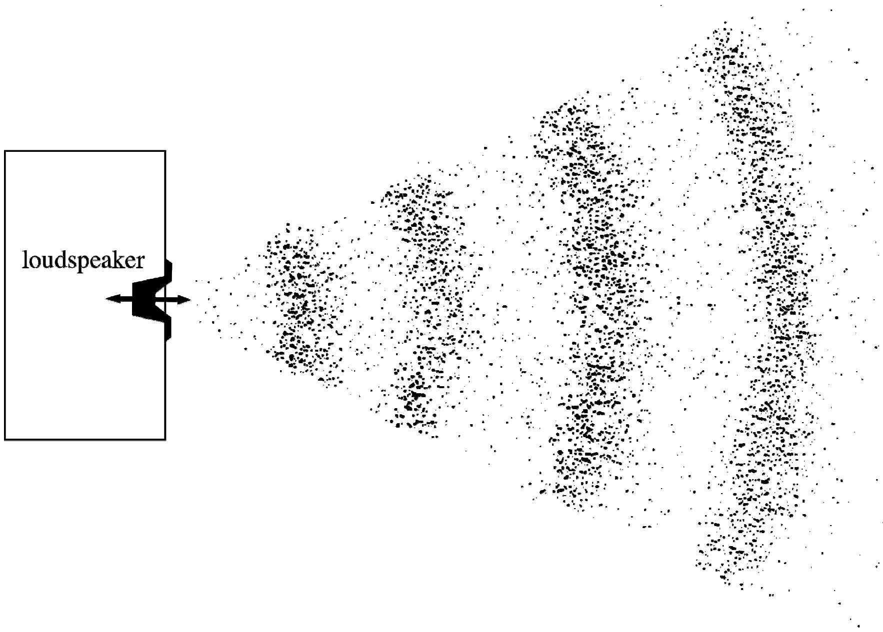

Figure 30: Sound waves spreading out from a loudspeaker.

Therefore, it makes more sense to describe the strength of a wave at a given point in space by specifying its intensity, which is defined as the rate of energy transfer by the wave per unit area A normal to the direction of propagation. So by definition, we can write the intensity I of the wave as

$$I = \frac{P}{A} \tag{11.15}$$

where P is the power (energy transfer rate per unit time) and A is the area of the surface intercepting the wave.

In the context of the sound waves, the intensity refers to the loudness (strength of sound). The loudness of sound decreases as we move away from its source, as shown in Figure 4. Although, in the figure we have depicted the spreading of sound waves in 2-D, the sound waves will actually spread out in 3-D space over a spherical surface. Note that as we move away from the source of the sound, the sound energy or acoustic energy transported by the sound waves will be distributed over a greater area. This spreading of the waves means that the energy supplied by the speaker to a given compression or rarefaction is spread over an increasingly large area as that compression or rarefaction moves away from the source of the sound.

Let us now derive an expression for the intensity of wave as a function of the distance r from the source.

9.5.1 Inverse Square Law

At a distance r_1, the energy is distributed over a spherical surface of area $4\pi r_1^2$, whereas at a distance $r_2(> r_1)$, the same amount of energy will be distributed over a larger spherical surface having surface area $4\pi r_2^2$. This is depicted in Figure 5 with the help of three different distances.

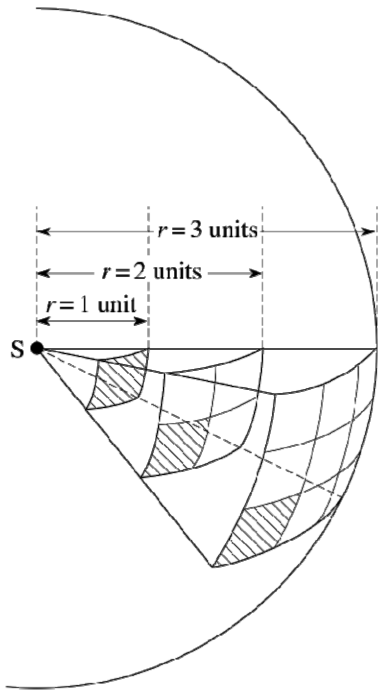

Figure 31: Sound waves spreading uniformly from a point source.

So if we assume that there is no energy loss due to any dissipative mechanism, the energy available per unit cross-sectional area will be more at r_1 as compared to r_2. That is, the energy per unit cross-sectional area will decrease as we move away from the source. In other words, the intensity of sound decreases as the distance from the source increases. Thus, from equation (11.15), we can write the intensity of sound at a distance r from the source as

$$I = \frac{P}{4\pi r^2} \tag{11.16}$$

In other words, we can say that for the case of sound emitted uniformly in all directions from a point source, without any loss of acoustic energy due to heating, absorption or any other effect, the intensity, I, is inversely proportional to the square of distance, r, from the source,

$$I \propto \frac{1}{r^2} \tag{11.17}$$

This result is known as the *inverse square law*.

From equation (11.14), we note that

$$P \propto A^2$$

where A is the amplitude of the wave. So we can say that

$$I \propto A^2 \tag{11.18}$$

From equations (11.18) and (11.17), we get

$$A \propto \frac{1}{r} \tag{11.19}$$

Thus, the amplitude of the wave is inversely proportional to the distance from the source. One should bear in mind that this is only true if the wave is not obstructed, or absorbed, and if the source is small enough compared with the distances from the source to enable it to be considered as a point source emitting sound uniformly in all directions. Equation (11.19) explains why we can be heard only up to a certain distance.

9.5.2 Intensity Levels and Decibels

The perceived loudness of sound is partly determined by its intensity, but also depends on its frequency, and on other factors such as the age of the listener. However, for healthy young adults, the faintest sounds that can be heard at a frequency of 1000 Hz have an intensity of about 10^{-12} W m^{-2} known as the threshold of hearing (ToH) for human beings. The normal conversations involve intensities of 10^{-6} W m^{-2} or so, and sounds become painful at an intensity of around 1 W m^{-2}.

In view of this wide range of intensities, it is often convenient to describe a sound of intensity I in terms of a quantity called the intensity level, β, which is measured in unit called decibel (dB), and defined by the following relation

$$\beta = (10\ dB)\log_{10}\left(\frac{I}{I_0}\right) \tag{11.20}$$

where I_0 is the intensity corresponding to ToH (1×10^{-12} W m^{-2}). The advantage of this scale is that the faintest audible sound has an intensity level of

$$\beta = (10\, dB)\log_{10}\left(\frac{10^{-12}}{1\times 10^{-12}}\right) = 0\, dB$$

normal speech has an intensity level of about

$$\beta = (10\, dB)\log_{10}\left(\frac{10^{-6}}{1\times 10^{-12}}\right) = 60\, dB$$

and the threshold of pain is at

$$\beta = (10\, dB)\log_{10}\left(\frac{1}{1\times 10^{-12}}\right) = 120\, dB$$

There are clear advantages to using a scale that ranges from 0 to 120 units rather than one that goes from 10^{-12} to 1. However, there is also the disadvantage of having to use a logarithmic scale, since an increase in intensity by a factor of ten only leads to an addition of ten dB to the intensity level.

Example 3: A loud shout has an intensity of 8×10^{-5} W m^{-2} at a distance of 1 m from the source.

(a) Given that the threshold of human hearing is about 10^{-12} W m^{-2} at voice frequencies, and that the sound spreads out evenly in all directions, how far away could such a shout be heard in open space?

(b) What is the ratio of the amplitude of the sound wave at this distance to the amplitude 1 m from the source?

Solution:

(a) The intensity, $I \propto 1/r^2$, where r is the distance from the source.

We are given that $I_1 = 8 \times 10^{-5}$ W m^{-2}, $I_2 = 10^{-12}$ W m^{-2} and $r_1 = 1$ m. so we can find out r_2 from the following relation,

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

Therefore,

$$\frac{8 \times 10^{-5} W m^{-2}}{10^{-12} W m^{-2}} = \frac{r_2^2}{(1\, m)^2}$$

$$\Rightarrow r_2 = \sqrt{\frac{8 \times 10^{-5}}{10^{-12}}}\, m = 9 \times 10^3\, m$$

Hence, the sound could be heard about 9 km away.

(b) The amplitude of the sound wave is inversely proportional to the distance from the source of the sound. Hence

$$\frac{A(at\ 1)}{A(at\ 2)} = \frac{r_2}{r_1}$$

So the ratio of the amplitude of the sound wave at a distance of 9 km to the amplitude at a distance of 1 m is

$$\frac{1}{9 \times 10^3} = 1.1 \times 10^{-4}$$

Self Assessment Question (SAQ) 6: The intensity level of heavy traffic heard from street level is very roughly 70 dB. What is the intensity of the traffic noise?

Self Assessment Question (SAQ) 7: Choose the correct option:

An 80-dB sound relative to a 30-dB sound is more intense by a factor of

(g) 5 (b) 50 (c) 500 (d) 10^5

Self Assessment Question (SAQ) 8: The sound intensity 0.25 m away from the speakers at an open-air disco is 10^{-3} W m^{-2}. How far away from the speakers should you stand in order that the music you hear has the same intensity as ordinary conversation with an intensity of approximately 3×10^{-6} W m^{-2}, assuming that no energy is absorbed?

9.5.3 Power Gain

If the power input to an amplifier or other signal processing device is P_{in} and the power output of the device is P_{out}, the power gain G of the system in decibels is defined as

$$G = (10\ dB)\log_{10}\left(\frac{P_{out}}{P_{in}}\right) \tag{11.21}$$

A change in audio power output of 1 dB is about the minimum that can be detected by a person with good hearing; usually the change must be 2 or 3 dB to be apparent.

Self Assessment Question (SAQ) 9: What is the intensity in watts per square meter of the 70-dB noise of a truck passing by?

Self Assessment Question (SAQ) 10: An audio system is made up components with the following power gains: preamplifier, +35 dB; attenuator, -10 dB; amplifier, +70 dB. What is the overall gain of the system?

9.6 SUMMARY

In this unit, we studied about what is meant by the phase of a wave and what is meant by the phase difference and phase velocity. We derived the relationship for the energy transported by progressive waves and also calculated the average rate of energy flow in the medium per cycle, or in other words, the power transmitted by the wave.

Thereafter, we learned that although energy and power are useful parameters, these do not account for observations related to the variations in the strength of a progressive wave with distance from the source. Therefore, there was a need to describe the strength of a wave at a given point in space by specifying its intensity, which is defined as the rate of energy transfer by the wave per unit area. In view of the wide range of intensities need was felt to describe a sound of intensity I in terms of a quantity called the intensity level, β, which is measured in unit called decibel (dB). Finally, we learnt the concept of power gain of a system.

9.7 GLOSSARY

Amplitude – the maximum displacement of a wave from equilibrium (e.g. height of a transverse wave from the middle).

Displacement – net change in location of a moving body. It is measured from the equilibrium position.

Force – any interaction that, when unopposed, can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a wave. The SI unit of frequency is the hertz (Hz), which is equal to 1 cycle per second.

Intensity – power transferred per unit area, where the area is an imagined surface that is perpendicular to the direction of propagation of the energy. In the SI system, it has units of watts per square meter (W/m^2).

Kinetic energy – energy of a moving object.

Mechanical energy – it is the sum of the kinetic energy and the potential energy.

Phase difference – the fraction of a cycle between the motion of two waves propagating at the same frequency.

Potential energy – energy possessed by a body by virtue of its position relative to others, stresses within itself, and other factors.

Power – the rate of doing work. It is the amount of energy consumed per unit time. Having no direction, it is a scalar quantity. In the SI system, the unit of power is the joule per second (J/s), known as the watt (W).

Velocity – speed in a given direction.

Wavelength – the distance between two adjacent wave-crests.

9.8 TERMINAL QUESTIONS

1. A sound wave in air is represented as

$$y = 0.05\sin(100t - 50x)\ m$$

where t is expressed in seconds and x in m, and y represents the displacement. Determine the phase velocity of the wave.

2. Choose the correct option:

The relation between phase difference and the path difference is

(a) $\Delta\varphi = \frac{\pi}{\lambda}\Delta x$
(b) $\Delta\varphi = 2\pi\lambda\Delta x$
(c) $\Delta\varphi = \frac{2\pi\lambda}{\Delta x}$
(d) $\Delta\varphi = \frac{2\pi}{\lambda}\Delta x$

3. When a plane wave traverses a medium, the displacement of the particles is given by

$$y(x,t) = 0.01\sin(4\pi t - 0.02\pi x)$$

where y and x are expressed in meters and t is in seconds. Calculate the

(a) amplitude, wavelength, velocity and the frequency of the wave.
(b) The phase difference between two positions of the same particle at time interval of 0.25 s.
(c) The phase difference, at a given instant of time, between two particles 50 m apart.

4. A transverse harmonic wave travelling in positive x-direction is represented as

$$y(x,t) = 5\sin(4t - 0.02x)$$

Calculate the velocity of the wave, maximum particle velocity, acceleration and intensity. Note that y is measured in cm, t in seconds and density of the media is 1.25 g/cm^3.

5. In normal conversation, the intensity of sound is $5 \times 10^{-6}\ Wm^{-2}$. The amplitude and velocity of the sine wave are respectively $2.4 \times 10^{-8}\ m$ and $332\ m/s$. If the density of air at STP is 1.29 kg/m^3, calculate the frequency of normal human voice.

6. Sunlight strikes the earth with an intensity of 2 cal/cm^2 per minute. How many watts of power must an electrical lamp radiate in order to produce, at 1 m, the brightness of sunlight?

7. Show that the average kinetic energy transported by a progressive wave per cycle through a thin layer of mass dm is equal to the average potential energy of the mass dm of a layer in the medium.

8. Choose the correct option:
The loudness of sound depends upon

(a) Amplitude
(b) Pitch
(c) Velocity
(d) Wavelength

9. Choose the correct option:
An amplifier has power input of 0.2 W and power output of 80 W. It's power gain is

(a) 400 dB
(b) 26 dB
(c) about 100 dB
(d) None of the above

10. Choose the correct option:
A sound intensity level of 55 dB is produced by 10 flutes. The number of flutes needed to produce a level of 65 dB under the same circumstances is

(a) 20 (b) 60 (c) 100 (d) 200

11. Write short notes on:

    (i) Intensity of a wave (ii) Intensity Levels and Decibels
    (iii) Energy transported by progressive waves

12. Define the terms phase, phase difference and phase velocity. How are they related?

13. Deduce the expression for the power transmitted by a sinusoidal wave travelling with velocity v.

9.9 ANSWERS

Selected Self Assessment Questions (SAQs):

1. We know that for a wave

$$v = f\lambda$$

$$\Rightarrow \lambda = \frac{v}{f} = \frac{360}{500} = 0.72\, m$$

Now in a wave, the path difference is related to the phase difference by the relation,

$$phase\ difference\ \Delta\varphi = \frac{2\pi}{\lambda}(path\ difference\ \Delta x)$$

We are given that the phase difference is 60^0, which is equal to π/3 rad. Therefore,

$$\Delta x = \frac{\lambda}{2\pi}(\Delta\varphi) = \frac{\lambda}{2\pi}\left(\frac{\pi}{3}\right) = 0.12\, m$$

2. (a)

3. (a)

$y = B\cos[\omega t - kx)]$ can be written as $y = B\sin[\omega t - kx + \pi/2]$. Since, the amplitude does not come into play for the determination of the phase difference between two waves, therefore, the phase difference between the given waves is equal to

$$\left(\omega t - kx + \frac{\pi}{2}\right) - (\omega t - kx) = \frac{\pi}{2}$$

4. The two waves are out-of-phase and therefore, the phase difference between them is π rad.

5. The statement is false, since, the ratio of KE and PE varies as the wave moves. It is only the average KE and average PE over a cycle that is equal.

6. Since, traffic noise has an intensity level about 10 dB greater than conversation it follows from the above discussion that its intensity must be about ten times that of normal speech, i.e. about 10^{-5} W m^{-2}. A more general technique for finding this answer is to use the antilog$_{10}$ function (the inverse of the log$_{10}$ function), since it follows from the definition of β (Equation 11.20), that

$$I = I_0 \times 10^{\beta/(10\ \mathrm{dB})} = I_0 \times \mathrm{antilog}_{10}\left(\frac{\beta}{10\ \mathrm{dB}}\right)$$

7. (d)

8.

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

Therefore,

$$\frac{10^{-3}Wm^{-2}}{3 \times 10^{-6}Wm^{-2}} = \frac{r_2^2}{(0.25\ m)^2}$$

$$\Rightarrow r_2 = 21\ m$$

9. An intensity of 0 dB is equivalent to $1 \times 10^{-12}Wm^{-2}$. Since, a sound of 70 dB is 10^7 times more intense, it is equivalent to a rate of energy flow of

$$I = (10^7)(1 \times 10^{-12}Wm^{-2}) = 1 \times 10^{-5}Wm^{-2}$$

10. Since power gains in dB are logarithmic quantities, the overall gain in dB of a system of several devices is equal to the sum of the separate gains in dB of the devices:

$$G\ (overall) = G_1 + G_2 + G_3 + \cdots$$

$$\Rightarrow G\ (overall) = +35 - 10 + +70\ dB = +95\ dB$$

Selected Terminal Questions:

1. The phase velocity of a wave is given by

$$v_p = \frac{\omega}{k}$$

From the given wave equation, we can deduct that $\omega = 100$ and $k = 50$. Therefore,

$$v_p = \frac{100}{50} = 2\ m/s$$

2. (d)

3. (a) The given equation can be rewritten in the form

$$y(x,t) = 0.01 \sin\left[\frac{2\pi}{100}(200t - x)\right]$$

Comparing this with the wave equation,

$$y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(vt - x)\right]$$

We have amplitude A = 0.01 m, wavelength λ = 100 m, and the wave velocity v = 200 m/s. The frequency can be calculated as

$$f = \frac{v}{\lambda} = \frac{200}{100} = 2\,Hz$$

(b) Phase change in a time interval of Δt is

$$\frac{2\pi}{T}.\Delta t = 2\pi f.\Delta t$$

Therefore, the phase difference is

$$2\pi \times 2 \times 0.25 = \pi$$

In other words, the particle phase is reversed in a time 0.25 s.

(c) Phase difference for the path difference of Δx is

$$-\frac{2\pi}{\lambda}.\Delta x = -\frac{2\pi}{100} \times 50 = -\pi$$

In other words, the particle located 50 m (which is half the wavelength) ahead of another particle lags in phase by π rad.

4. The equation of transverse wave travelling in +x direction is given as

$$y(x,t) = 5\sin(4t - 0.02x)$$

Comparing it with the standard wave equation,

$$y(x,t) = A\sin(\omega t - kx)$$

We get, $\omega = 4\,rad/s$, k = 0.02 rad/cm and A = 5 cm. Hence, the velocity of the wave is given by

$$v = \frac{\omega}{k} = \frac{4}{0.02} = 200\ cm/s$$

Instantaneous velocity of the particle is given as

$$\frac{dy(x,t)}{dt} = 5 \times 4\cos(4t - 0.02x) = 20\cos(4t - 0.02x)$$

Therefore, the maximum velocity of the particle = 20 cm/s. The particle acceleration is given as

$$\frac{d^2y(x,t)}{dt^2} = -20 \times 4\sin(4t - 0.02x) = -80\sin(4t - 0.02x)$$

Therefore, the maximum acceleration of the particle = 80 cm/s^2.

The intensity of the wave is given by

$$I = 2\pi^2 A^2 f_0^2 \rho v$$

$$= \frac{1}{2} A^2 \omega^2 \rho v$$

$$= \frac{1}{2}(0.05\ m)^2(4\ rad/s)^2(1250\ kg/cm^3)(2\ m/s)$$

$$= 50\ W/m^2$$

5. The expression for intensity is

$$I = 2\pi^2 A^2 f_0^2 \rho v$$

Therefore,

$$f_0 = \frac{1}{\pi A}\sqrt{\frac{I}{2\rho v}}$$

$$= \frac{1}{\pi(2.4 \times 10^{-8}\ m\ )}\sqrt{\frac{5 \times 10^{-6}\ Wm^{-2}}{2(1.29\ kg\ m^{-3})(332\ m/s)}}$$

$$= 1000\ Hz$$

. The amplitude and velocity of the sine wave are respectively and 332 m/s. If the density of air at STP is 1.29 kg/m^3

6. The intensity I of sunlight on earth is

$$2\ cal. cm^{-2} = \frac{2 \times 4.2}{10^{-4} \times 60} = 1.4 \times 10^3\ Wm^{-2}$$

The power P of a lamp producing an intensity of $1.4 \times 10^3\ Wm^{-2}$ at r = 1 m is given by

$$P = I \times 4\pi r^2$$

$$= 1.4 \times 10^3 \times 4\pi \times (1)^2 = 17.6\ kW$$

8. (a)

9. (b)

10. (c)

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9.11 SUGGESTED READINGS

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UNIT 10 ONE DIMENSIONALWAVE EQUATION

Structure

10.1 INTRODUCTION

Till now, you studied about harmonic waves and its basic characteristics like amplitude, time period and frequency. You also learnt about the concept of phase, phase difference and phase velocity. We have also discussed how the energy is transported by sinusoidal waves and how the rate at which energy is transported by a wave varies linearly with wave velocity and as square of its amplitude and frequency. Since, energy and power do not account for observations related to the variations in the strength of a progressive wave with distance from the source, intensity of wave was specified. We also discussed the idea of intensity levels and power gain of a system.

In this unit, we will study the different kinds of waves that can propagate in various media. Do you know what determines whether or not waves can propagate in a medium and when the waves do travel, how fast it can go in that medium? Experimental investigations have shown that the speed of a wave does not depend on its wavelength or period. This means that answers to such questions must lie in the physical properties of the medium in which a wave propagates. To discover this, we consider propagation of waves in some typical medium such as a stretched string, solid rod, and a gaseous medium.

10.2 OBJECTIVES

After studying this unit, you should be able to

- derive the expression for one dimensional wave equation for waves travelling on a stretched string,
- derive the expression for one dimensional wave equation for longitudinal waves travelling in a uniform rod,
- derive the expression for one dimensional wave equation for longitudinal waves in a gas,
- write down the one dimensional wave equation for waves travelling in an absorbing medium, and
- write down the equations for two and three dimensional waves.

10.3 ONE DIMENSIONAL WAVE EQUATION

10.3.1 Waves on a Stretched String

Consider a uniform stretched string, having mass per unit length m. Under equilibrium conditions, it can be considered to be straight. The x-axis is chosen along the length of the stretched string in its equilibrium state. Let the string be displaced perpendicular to its length by a small amount so that a small section of length Δx is displaced through a distance y from its mean position, as shown in Figure 1. When the string is released, it results in wave motion. Let's see how.

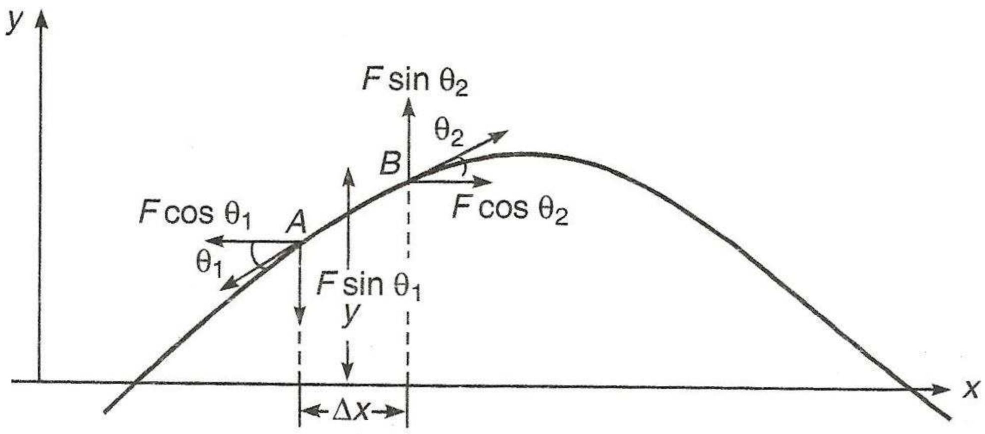

Figure 32: Forces acting on a small element of a string displaced perpendicular to its length.

We have studied that the wave disturbance travels from one particle to another due to their masses (or inertia) and the factor responsible for the periodic motion of the particle is the elasticity of the medium. For a stretched string, the elasticity is measured by the tension F in it and the inertia is measured by mass per unit length or linear mass density, m.

Suppose that the tangential force on each end of a small element AB, as shown in Figure 1, is F; the force on the end B is produced by the pull of the string to the right and the one at A is due to the pull of the string to the left. Due to the curvature of the element AB, the forces are not directly opposite to each other. Instead, they make angles θ_1 and θ_2 with the x-axis. This means that the forces pulling the element AB at opposite ends, though of equal magnitude, do not exactly cancel each other. In order to calculate the net force along the x- and y-axes, the forces are resolved into rectangular components. The net force in the x and the y directions are respectively given by

$$F_x = F\cos\theta_2 - F\cos\theta_1$$

$$and \quad F_y = F\sin\theta_2 - F\sin\theta_1$$

For small angle approximation, $\cos\theta \approx 1$ and $\sin\theta \approx \theta \approx \tan\theta$. This implies that if the displacement of the string perpendicular to its length is relatively small, the angles θ_1 and θ_2 will be small and there is no net force in the x-direction, and the element AB is only subjected to a net upward force F_y. Under the action of this force, the string element will move up and down. Therefore, the y-component of the force on element AB can be written as

$$F_y = F\tan\theta_2 - F\tan\theta_1$$

We know that the tangent of an angle actually defines the slope at that point. In other words, the tangent define the derivative dy/dx. Using this result, the y-component of force on the element can be approximated as

$$F_y = F\left(\frac{dy}{dx}\bigg|_{x+\Delta x} - \frac{dy}{dx}\bigg|_x\right) \tag{10.1}$$

Note that the perpendicular displacement y(x, t) of the string is both a function of the position x and time t. However, equation (10.1) is valid at a particular instant of time. Therefore, the derivative in this expression should be taken by keeping the time fixed. Therefore, equation (10.1) can be rewritten as

$$F_y = F\left(\frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - \frac{\partial y}{\partial x}\bigg|_x\right) \tag{10.2}$$

For the sake of convenience, let us put

$$f(x) = \frac{\partial y}{\partial x}\bigg|_x \qquad and \qquad f(x+\Delta x) = \frac{\partial y}{\partial x}\bigg|_{x+\Delta x}$$

in equation (10.2). Thus, equation (10.2) becomes

$$F_y = F[f(x+\Delta x) - f(x)] \tag{10.3}$$

To simplify the above expression, we make use of Taylor series expansion of the function $f(x+\Delta x)$ about the point x:

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\bigg|_x \Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_x \Delta x^2 + \cdots$$

Since, Δx is small , we can ignore the second and the higher order terms in Δx to obtain,

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\bigg|_x \Delta x$$

$$= f(x) + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Delta x$$

$$\Rightarrow f(x+\Delta x) - f(x) = \frac{\partial^2 y}{\partial x^2}\Delta x$$

Inserting the above result in equation (10.3), we get

$$F_y = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

This equation gives the net force on the element AB. We use Newton's second law of motion to obtain the equation of motion of this element, by equating this force to the product of mass and acceleration of the element AB. The mass of the element AB is $m\,\Delta x$. Therefore, we can write

$$m\,\Delta x\frac{\partial^2 y}{\partial t^2} = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{m}{F}\frac{\partial^2 y}{\partial t^2} \qquad (10.4)$$

Note that even though equation (10.4) has been obtained for a small element AB, it can be applied to the entire string, since there is nothing special about this particular element of the string. In other words, equation (10.4) can be applied to all the elements of the string.

Now, let us go back to the sinusoidal wave propagating on the string described by the equation

$$y(x,t) = A\sin(\omega t - kx)$$

If this mathematical form is consistent with equation (10.4), then we can be sure that such a wave can indeed move on the string. To check this, we calculate the spatial and the temporal partial derivatives of particle displacement y(x, t):

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(\omega t - kx)$$

$$\text{and} \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$$

Substituting these partial derivatives in equation (10.4), we get

$$-k^2 A\sin(\omega t - kx) = \frac{m}{F}[-\omega^2 A\sin(\omega t - kx)]$$

$$\Rightarrow \frac{F}{m} = \left(\frac{\omega}{k}\right)^2 \qquad (10.5)$$

But, we know that ω/k is the wave speed v, therefore, from the above relation, we get

$$v = \frac{\omega}{k} = \sqrt{\frac{F}{m}} \qquad (10.6)$$

The above relation tells us that velocity of a transverse wave on a stretched string depends on tension and mass per unit length of the string. Using equation (10.6), we can write equation (10.4) as

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{10.7}$$

This result expresses one-dimensional wave equation. It holds as long as we deal with small amplitude waves. Elasticity provides the restoring force and the inertia determines the response of the medium.

10.3.2 Longitudinal Waves in a Uniform Rod

Consider a cylindrical metal rod of uniform cross-sectional area. When the rod is struck with a hammer at one end, the disturbance will propagate along it with a speed determined by its physical properties. For simplicity, we assume that the rod is fixed at the left end as shown in Figure 2.

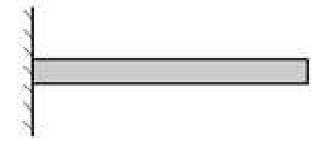

Figure 33: Uniform cylindrical rod fixed at left end.

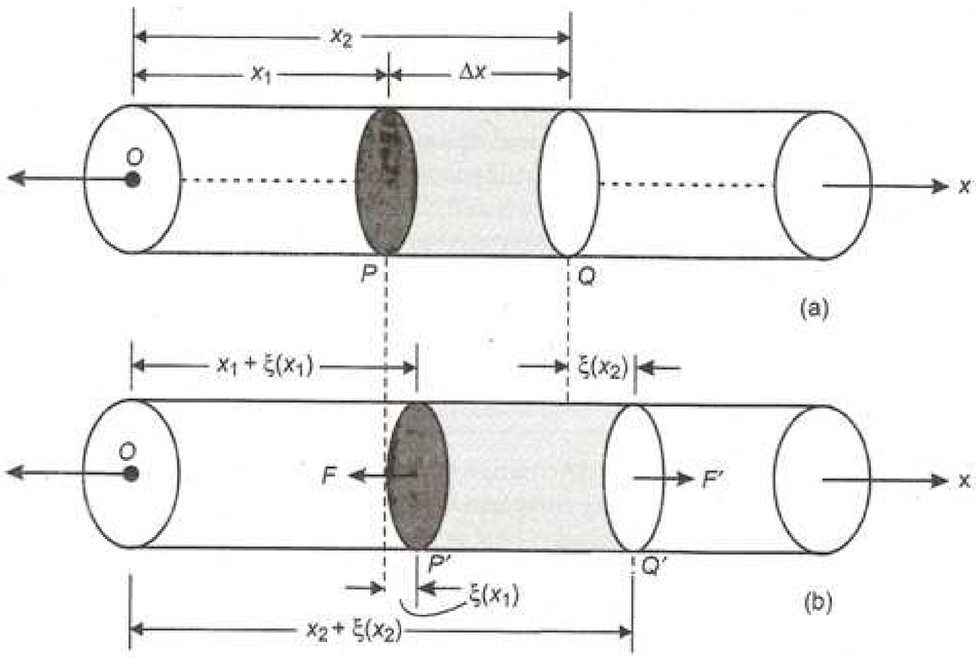

Figure 34: Longitudinal wave propagating in a uniform cylindrical rod. Element PQ in (a) equilibrium state, and (b) deformed state.

We choose x-axis along the length of the rod with origin O at the left end. We divide the rod in a large number of small elements, each of length Δx. Let us consider one such element PQ, as shown in Figure 3a. Since, the rod has been struck at end O lengthwise, the section at P, which is at a distance x_1 from O, will be displaced along x-axis. Since, the force experienced by different sections of the rod is a function of distance, the displacements of particles in different sections will also be function of position. Let us denote is by ξ(x).

Figure 3b shows the deformed state of the rod and displaced position of the element under consideration. Let us denote the x-coordinate of the element in the displaced position by $x_1 + \xi(x_1)$ so that $\xi(x_1)$ represents the displacement of the particles in the section P. Similarly, the new x-coordinate of the particles initially located in the section at Q (x = x_2) be denoted by $x_2 + \xi(x_2)$, so that $\xi(x_2)$ signifies the displacement of the particles in section at Q. Hence, the change in length of the element is $\xi(x_2) - \xi(x_1)$. Using Taylor series expansion of $\xi(x_2)$ around x_1 and retaining the first order terms, just like we did in the case of the string, we can write

$$\xi(x_2) - \xi(x_1) = \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \Delta x$$

The linear strain produced in the element PQ can be expressed as

$$\varepsilon(x_2) = \frac{Change\ in\ length}{Original\ length} = \frac{\left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \Delta x}{\Delta x}$$

$$\Rightarrow \varepsilon(x_2) = \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \tag{10.8}$$

The net force $F' - F$ on the element P'Q' at points P' and Q', as shown in Figure 3b, is toward right. Due to this force, the element under consideration will experience stress, which is the restoring force per unit area. You may recall that the ratio of stress to longitudinal strain defines the Young's modulus Y,

$$Y = \frac{Stress}{Strain}$$

$$\Rightarrow Stress = Y \times Strain$$

In view of the spatial variation of force, we can say that the sections P and Q of the element under consideration will develop different stresses. Therefore, we can write

$$\sigma(x_1) = Y \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1}$$

$$and \qquad \sigma(x_2) = Y\left(\frac{\partial\xi}{\partial x}\right)_{x=x_2}$$

The net stress on the element PQ is

$$\sigma(x_2) - \sigma(x_1) = Y\left[\left(\frac{\partial\xi}{\partial x}\right)_{x=x_2} - \left(\frac{\partial\xi}{\partial x}\right)_{x=x_1}\right]$$

$$= Y[f(x_2) - f(x_1)]$$

where we have put $f(x) = \partial\xi/\partial x$.. As before, using Taylor series expansion for $f(x_2)$ about x_1, we can easily see

$$\sigma(x_2) - \sigma(x_1) = Y\left(\frac{\partial f}{\partial x}\right)\Delta x$$

$$= Y\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial x}\right)\Delta x$$

$$\Rightarrow \sigma(x_2) - \sigma(x_1) = Y\left(\frac{\partial^2\xi}{\partial x^2}\right)\Delta x \qquad (10.9)$$

If the cross-sectional area of the rod is A, the net force on the elements in the x-direction is given by

$$F(x_2) - F(x_1) = A[\sigma(x_2) - \sigma(x_1)]$$

$$\Rightarrow F(x_2) - F(x_1) = Y\left(\frac{\partial^2\xi}{\partial x^2}\right)\Delta x \qquad (10.10)$$

Under dynamic equilibrium condition, the equation of motion of the element PQ, using Newton's second law of motion, can be written as

$$Y\left(\frac{\partial^2\xi}{\partial x^2}\right)\Delta x = \rho A\Delta x\left(\frac{\partial^2\xi}{\partial t^2}\right) \qquad (10.11)$$

where ρ is the density of the material of the rod and $\rho A\Delta x$ signifies the mass of the element PQ. On simplification, we find that the displacement $\xi(x, t)$ satisfies the equation

$$\frac{\partial^2\xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2\xi}{\partial x^2} \qquad (10.12)$$

which is of the form of wave equation (10.7) with

$$v = \sqrt{\frac{Y}{\rho}} \tag{10.13}$$

Equations (10.12) and (10.13) show that the deformation propagates along the rod as a wave and the velocity of the longitudinal waves is independent of the cross-sectional area of the rod.

10.3.3 Longitudinal Waves in a Gas

Since a gaseous medium lacks rigidity, transverse waves cannot propagate in it; only solids can sustain transverse waves. However, longitudinal waves can propagate in all media such as gas, solid and liquid in the form of compressions and rarefactions. We now discuss longitudinal waves in a gaseous medium.

Sound waves in air columns perhaps are the most familiar one-dimensional waves in a gas. These can be easily excited by placing a vibrating tuning fork at the open end of an air column. What is the basic difference between longitudinal waves in a solid rod that we studied in the last section and a gas column? We know that gases being compressible, the pressure variations in a gas are accompanied by fluctuations in the density, while the density of a solid rod remains essentially constant.

In order to understand the propagation of one-dimensional longitudinal waves in a gas, consider a gas column in a long pipe or cylindrical tube of uniform cross-sectional area A. As before, we conveniently choose x-axis along the length of the tube and divide the column of the gas into small elements or slices, each of small length Δx. Figure 4 shows one such volume element PQRS. Thus, the volume of this element is $V = A\Delta x$.

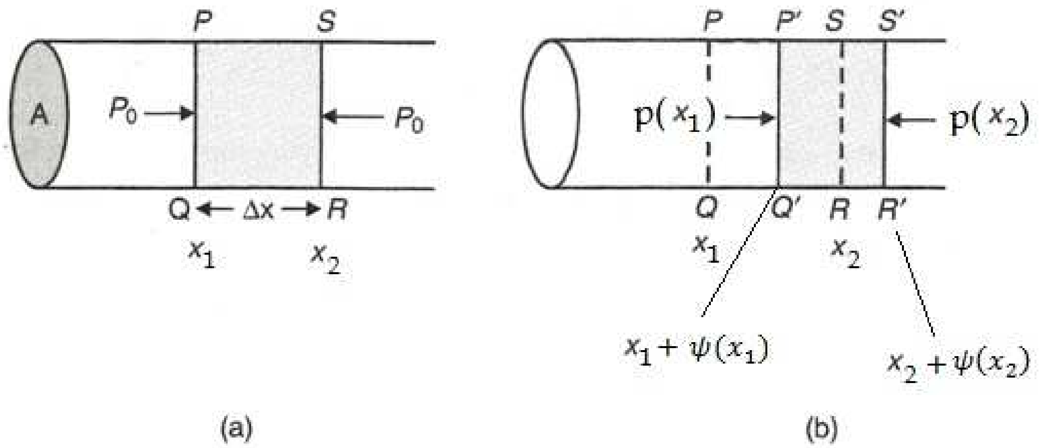

Figure 35: (a) Equilibrium state of the column PQRS of a gas contained in a long tube of cross-sectional area A, and (b) displaced position of column under pressure difference.

Under equilibrium condition, pressure and density of the gas remains the same throughout the volume of the gas, independent of the x-coordinate. Let the equilibrium pressure be denoted by p_0. If the pressure of the gas in the tube is changed, the volume element PQRS will be set in motion giving rise to a net force. Let us choose the origin of the coordinate system so that the particles in plane PQ are at a distance x_1 and those in plane SR are at a distance x_2 from it. Figure 4b shows the displaced position of the volume element when PQ is shifted to P'Q' and SR is shifted to S'R' Let the new coordinates be denoted by $x_1 + \psi(x_1)$ and $x_2 + \psi(x_2)$, respectively. It means that $\psi(x_1)$ and $\psi(x_2)$ respectively, denote the displacements of the particles originally at x_1 and x_2. Therefore, the change in thickness Δl is given by

$$\Delta l = \psi(x_2) - \psi(x_1)$$

If Δl is positive, there is increase in length, and hence the volume of the element also increases and vice versa. Using Taylor series expansion for $\psi(x_2)$ about $\psi(x_1)$, we can write

$$\Delta l = \psi(x_2) - \psi(x_1) = \left(\frac{\partial \psi}{\partial x}\right) \Delta x$$

This means that the change in volume ΔV is

$$\Delta V = A\Delta l = A\Delta x \left(\frac{\partial \psi}{\partial x}\right)$$

The volume strain, which is defined as the change in volume per unit volume, is given by

$$\frac{\Delta V}{V} = \frac{A\Delta x}{A\Delta x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial \psi}{\partial x} \qquad (10.14)$$

This increase in volume of the element is due to the decrease in pressure and vice versa.

It should be noted that until now all the steps that have been followed are identical to the case of the solid rod. However, as mentioned earlier, due to comparatively large compressibility of the gas, change in volume is accompanied by changes in density. This implies that the pressure in the compressed/rarefied gas varies with distance. To proceed further, let us suppose that the pressure at P'Q' is $p_0 + p(x_1')$. Hence, the pressure difference across the ends of the element P'Q'R'S' can be expressed in terms of the pressure gradient,

$$p(x_2') - p(x_1') = \left(\frac{\partial p(x)}{\partial x}\right)_{x=x_1'} \Delta x$$

$$= \frac{\partial (p_0 - \Delta p)}{\partial x} \Delta x$$

Since, p_0 is a constant

$$\Rightarrow p(x_2') - p(x_1') = -\frac{\partial (\Delta p)}{\partial x} \Delta x \qquad (10.15)$$

To express the above result in a familiar form, we note that Δp is connected to the bulk modulus of elasticity by the relation

$$E = \frac{Stress}{Volume\ Strain} = -\frac{\Delta p}{\Delta V / V}$$

The negative sign is included to account for the fact that when the pressure increases, the volume decreases. This ensures that E is positive. We can write the above relation as

$$\Delta p = -E\left(\frac{\Delta V}{V}\right)$$

On substituting for $\Delta V / V$ from equation (10.14), we get

$$\Delta p = -E\left(\frac{\partial \psi}{\partial x}\right)$$

Using equation (10.15), we find that the pressure difference at the ends of the displaced column is given by

$$p(x_2') - p(x_1') = -\frac{\partial}{\partial x}\left(-E\frac{\partial \psi}{\partial x}\right)\Delta x = E\left(\frac{\partial^2 \psi}{\partial x^2}\right)\Delta x$$

The net force acting on the volume element is obtained by multiplying this expression for pressure difference by the cross-sectional are of the column,

$$F = [p(x_2') - p(x_1')]A$$

$$= EA\Delta x\left(\frac{\partial^2\psi}{\partial x^2}\right)$$

Under the action of this force, the volume element under consideration shall be set in motion. Using Newton's second law of motion, we find that the equation of motion of the element under consideration can be expressed as

$$\rho\Delta x A\frac{\partial^2\psi}{\partial t^2} = EA\Delta x\left(\frac{\partial^2\psi}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial^2\psi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2\psi}{\partial x^2} \tag{10.16}$$

If we identify the speed of the longitudinal wave as

$$v = \sqrt{\frac{E}{\rho}} \tag{10.17}$$

equation (10.16) becomes identical to equation (10.7). One must note that the wave speed is determined only by the bulk modulus of elasticity and density – two properties of the medium through which the wave is propagating.

When a longitudinal wave propagates through a gaseous medium such as air, the volume elasticity is influenced by the thermodynamic changes that take place in it. These changes can be isothermal or adiabatic. Newton gave the first theoretical expression of the velocity of sound wave in a gas. He assumed that when sound wave travels through a gaseous medium, the temperature variations in the regions of compression and rarefaction are negligible. For sound waves propagating in air, Newton assumed that isothermal changes take place in the medium. For an isothermal change, the volume elasticity equals atmospheric pressure,

$$E = E_T = p$$

Then we can write,

$$v = \sqrt{\frac{p}{\rho}} \tag{10.18}$$

This is known as the Newton's formula for velocity of sound. For air at STP, $\rho = 1.29\ \text{kgm}^{-3}$ and $p = 1.01 \times 10^5\ \text{Nm}^{-2}$. Hence, velocity of sound in air at STP, using the Newton's formula comes out to be

$$v = \sqrt{\frac{1.01 \times 10^5\ Nm^{-2}}{1.29\ kgm^{-3}}} = 280\ m/s$$

But experimental results paint a different picture and show that the speed of sound in air at STP is actually around 332 m/s, which is about 15% higher than the value predicted by Newton's formula. This implies that something was wrong with the assumption of isothermal change.

The discrepancy was resolved when Laplace pointed out that sound waves produced adiabatic changes; the regions of compression are hotter while the regions of rarefaction are cooler, i.e. local changes in temperature occur when sound propagates in air. Since, the thermal conductivity of a gas is small and these thermal change occur so rapidly that the heat developed in compression and cooling produced in rarefaction is not transferred out during the short time-scale. The time-scale is the time required by sound to travel from compression to rarefaction. However, the total energy of the system is conserved. This means that the adiabatic changes occur in air when sound propagates.

For an adiabatic change, E_s is γ times the pressure, where γ is the ratio of specific heat capacities of a gas at constant pressure and at constant volume, i.e.

$$E_s = \gamma p$$

Then, equation (10.18) becomes

$$v = \sqrt{\frac{\gamma p}{\rho}} \qquad (10.19)$$

This is known as the Laplace's formula. For air, γ = 1.4 and the velocity of sound in air at STP based on equation (10.19) comes out to be 331 m/s, which is in close agreement with the experimentally measured value, thereby establishing the correctness of Laplace's explanation.

At a given temperature, p/ρ is constant for a gas. So, equation (10.19) shows that the velocity of a longitudinal wave is independent of pressure.

The question arises that why is the thermal energy unable to flow from a compression to a rarefaction and equalize the temperature creating isothermal conditions? To answer this question, we notice that to attain this condition, thermal energy must flow through a distance of one-half wavelength in a time much shorter than one-half of the period of oscillation of the particles. Thermodynamically, this means that we would need,

$$v_{sound} \ll v_{thermal}$$

One may recall from kinetic theory of gases that the root mean square speed of air molecules is given by

$$v_{rms} = \sqrt{\frac{2k_BT}{m}} \tag{10.20}$$

where m is the mass of air molecules and T is the absolute temperature in K. We can similarly write the expression for the speed of sound

$$v_{sound} = \sqrt{\frac{\gamma k_BT}{m}} \tag{10.21}$$

As liquids, in general, are incompressible, the speed of sound in liquids must be significantly higher than in gases. For example, in water whose $E = 2.22 \times 10^9\ Nm^{-2}$, using equation (10.17) the wave speed comes out to be about 1500 m/s. Even though water is about 1000 times denser than air, sound propagates faster in water than air.

Example 1: Transverse waves are generated in two uniform steel wires A and B of diameters 0.001 m and 0.0005 m, respectively, by attaching their free end to a vibrating source of frequency 500 Hz. Find the ratio of the wavelengths if they are stretched with the same tension.

Solution: The density ρ of a wire of mass M, length L and diameter d is given by

$$\rho = \frac{M}{L\left(\frac{\pi d^2}{4}\right)} = \frac{m}{\left(\frac{\pi d^2}{4}\right)}$$

where m is the linear mass density (mass per unit length). Now, we know that the velocity of a transverse wave in a stretched wire is given by

$$v = \sqrt{\frac{F}{m}}$$

Since, the tension is the same for both the steel wires A and B, therefore, we have

$$\frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}}$$

$$\Rightarrow \frac{v_A}{v_B} = \frac{d_B}{d_A}$$

Since, both the wires are made of steel, and have the same densities. Also, we know that the wave velocity v = fλ , where f is the frequency of the source, therefore the above relation can be written as

$$\frac{\lambda_A}{\lambda_B} = \frac{d_B}{d_A} = \frac{0.0005}{0.001} = \frac{1}{2}$$

Self Assessment Question (SAQ) 1: Using dimensional analysis, show that the wave speed v is given by

$$v = K\sqrt{\frac{F}{m}}$$

where K is a dimensionless constant, F is the tension in the string and m is the linear mass density of the string.

Self Assessment Question (SAQ) 2: A one meter long string weighing one gram is stretched with a force of 10 N. Calculate the speed of transverse wave.

Self Assessment Question (SAQ) 3: For a steel rod, $Y = 2 \times 10^{11}\ Nm^{-2}$ and ρ = 7800 kg/m^3. Calculate the speed of the longitudinal waves.

Self Assessment Question (SAQ) 4: In a laboratory experiment (room temperature being 15^{0}C) the wavelength of a note of sound of frequency 500 Hz is found to be 0.68 m. If the density of air at STP is 1.29 kg/m^3, calculate the ratio of the specific heats of air.

Self Assessment Question (SAQ) 5: Write the expressions for speed of mechanical waves in

(a) a string, (b) rod, (c) liquid, (d) gas

10.4 WAVES IN TWO AND THREE DIMENSIONS

So far we have restricted our discussion to the waves propagating along one-dimension, for example, waves in a stretched string. But, waves on a stretched membrane will make the disturbance spread over the surface and such waves are two-dimensional waves. When the membrane is suddenly disturbed in a direction normal to the plane of the membrane, the particles of the membrane vibrate along the direction of the applied force. But tension in the membrane makes the disturbance to spread over the surface. That is to say, the waves on the membrane are two-dimensional. Similarly, surface waves caused by dropping a stone into a quiet lake are two-dimensional.

In such cases, the displacement is a function of x, y and t and we write the displacement as $\psi = \psi(x, y, t)$. We obtain the two-dimensional wave equation for a stretched membrane by using arguments similar to those used for a stretched string. Suppose σ is the mass per unit area of the membrane and F is the uniform tension per unit length. This means that, if a line of unit length is drawn in the surface of the membrane, then the material on one side of this line exerts a force F on the material on the other side in a direction normal to that of the line. This F is the surface tension of the membrane. Figure 7 shows a small rectangular element ABCD of a stretched membrane of sides Δx and Δy in the xy plane, vibrating in the z-direction. The forces Fδx and Fδy are acting on the sides of the element during vibrations. The components of these forces in direction normal to xy plane constitute the restoring force tending to bring the element back to its equilibrium position.

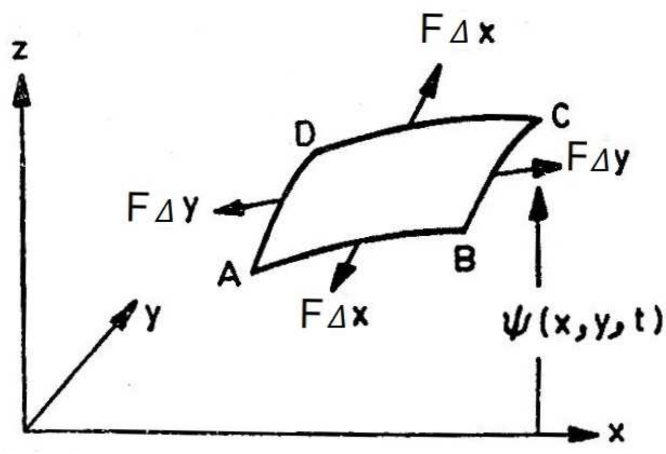

Figure 36: Forces acting on a small element ABCD of a stretched membrane vibrating in the z-direction.

Without going into the mathematical details, which are very similar to the case of 1-D waves on a stretched string, we will directly write the wave equation for the case of a stretched membrane. Since the forces along x- and y-axes can be taken to act independently; each one will contribute analogous term to the wave equation. So, we can write

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\sigma}{F}\frac{\partial^2 \psi}{\partial t^2} \tag{10.22}$$

where the wave speed is given by

$$v = \sqrt{\frac{F}{\sigma}}$$

The solution of this equation has the form,

$$\psi(x,y,t) = A\sin(\omega_0 t - \vec{k}.\vec{r}) \quad (10.23)$$

where the position vector $\vec{r} = x\hat{\imath} + y\hat{\jmath}$.

Extending the preceding arguments for three-dimensional waves such as seismic waves, we can write

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(\vec{r},t) = \frac{1}{v^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t^2} \quad (10.24)$$

where the position vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. For an isotropic medium (a medium in which the wave velocity is the same in every direction) the quantity ψ is a scalar. For non-isotropic media, ψ becomes a vector.

Example 2: A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope? [$g = 9.8\ ms^{-2}$]

Solution: As the rope has mass and a mass is also suspended from the lower end, the tension in the rope will be different at different points. We know that the speed of a transverse wave is given by

$$\mathrm{v} = \sqrt{\frac{\mathrm{F}}{\mathrm{m}}}$$

Therefore, the ratio of speeds of transverse wave at the top and at the bottom of the rope is

$$\frac{\mathrm{v_T}}{\mathrm{v_B}} = \sqrt{\frac{\mathrm{F_T}}{\mathrm{F_B}}} = \sqrt{\frac{(6+2)\times 9.8\ \mathrm{N}}{2\times 9.8\ \mathrm{N}}}$$

$$\Rightarrow \frac{\mathrm{v_T}}{\mathrm{v_B}} = 2$$

Since, the frequency is the characteristic of the source producing the waves, we have frequency of the wave at the top is the same as the frequency at the bottom of the rope. Since, wave speed $v = f\lambda$, we have

$$\lambda_\mathrm{T} = 2\lambda_\mathrm{B} = 2\times 0.06 = 0.12\ \mathrm{m}$$

Self Assessment Question (SAQ) 6: Explain why the velocity of sound is generally greater in liquids than in gases.

Self Assessment Question (SAQ) 7: Choose the correct option.

The speed of sound in air is

(f) $\propto \sqrt{pressure\ of\ air}$
(g) $\propto pressure\ of\ air$
(h) $\propto square\ of\ pressure\ of\ air$
(i) $independent\ of\ pressure\ of\ air$

Self Assessment Question (SAQ) 8: Choose the correct option:

Velocity of sound is measured in hydrogen and oxygen gases at a given temperature. The ratio of the two velocities will be:

(h) 1:4, (b) 4:1, (c) 1:1, (d) 32:1

10.5 SUMMARY

In this unit, we studied one-dimensional transverse and longitudinal waves propagating in different media such as stretched string, in a uniform solid rod, in a gas and in an absorbing media. We gained an understanding of how the speed of a wave is determined by the interplay of elasticity and inertia. Elasticity gives rise to restoring force and inertia determines the response of the medium. We also learned that since, a gas lacks rigidity, transverse waves cannot propagate in a gaseous medium; only solids can sustain transverse waves. However, longitudinal waves can propagate in all media – gaseous, liquid and solid. Then we also briefly studied two and three dimensional waves.

While studying speed of sound in air, we came across Newton's formula for determining the velocity of sound and its limitation, which was later resolved by Laplace who pointed out that sound waves produced adiabatic changes and that local changes in temperature occur when sound propagates in air. In the last part of the text, we also discussed briefly waves in two and three dimensions.

10.6 GLOSSARY

Amplitude – the maximum displacement or distance moved by a point on a vibrating body or wave measured from its equilibrium position.

Displacement – net change in location of a moving body. It is measured from the equilibrium position.

Elasticity – ability of a material to regain its shape after being distorted.

Force – any interaction that, when unopposed, can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a vibrating object.

Inertia – a property of matter that causes it to resist changes in its state of rest or motion.

Longitudinal waves – waves in which the vibrations are parallel to the direction of travel of the wave.

Pressure – force per unit area applied at right angles to a surface. The SI unit of pressure is the pascal (Pa), which is equal to 1 N/m^2.

Sound – a vibration that propagates as a typically audible mechanical wave of pressure and displacement, through a medium such as air or water.

Speed – the ratio of distance traveled and time. The SI unit of speed is m/s.

Tension – the force in an object that has been stretched.

Transverse waves – waves in which the vibrations are at right angles to the direction of propagation of wave.

Velocity – speed in a given direction.

Wave – an oscillation accompanied by a transfer of energy that travels through medium (space or mass).

10.7 TERMINAL QUESTIONS

1. Deduce the expression for the velocity of transverse waves on a stretched string.

2. (a) Deduce the expression for the velocity of longitudinal waves in a column of a gas and hence obtain Newton's formula.

(b) What is the Laplace's correction to Newton's formula?

3. How will you describe displacements in a two-dimensional wave? Write down the wave equation in the case of two-dimensional waves on a stretched membrane.

4. Calculate the speed of sound in (a) water, and (b) steel. Given: Density of steel = 7800 kg/m^3, Young's modulus of steel = $20 \times 10^{10}\ Nm^{-2}$ and bulk modulus of water = $0.2 \times 10^{10}\ Nm^{-2}$.

5. Compare the velocities of sound in hydrogen and carbon dioxide. The ratio of specific heats of hydrogen and carbon dioxide are respectively, 1.4 and 1.3.

6. Use the formula $v = \sqrt{\gamma p/\rho}$ to explain why the speed of sound in air is independent of pressure.

7. Choose the correct option:

The velocity of sound in air is not affected by changes in:

(a) moisture of the air
(b) temperature of the air
(c) atmospheric pressure
(d) composition of air

8. Choose the correct option.

The velocity of sound in a gas is:

(b) Independent of the temperature on absolute scale
(c) Proportional to the square root of temperature
(d) Inversely proportional to the square root of temperature
(e) Proportional to the square of temperature

9. Choose the correct option.

The Laplace's correction in the expression for the velocity of sound given by Newton is needed because sound waves:

(a) Are longitudinal
(b) Propagate isothermally
(c) Propagate adiabatically
(d) Are of long wavelengths

10.8 ANSWERS

Selected Self Assessment Questions (SAQs):

1. We have to check if the following dimensional formula for wave speed is correct or not, i.e. it has units of velocity or not.

$$v = K\sqrt{\frac{F}{m}}$$

- K is the dimensionless constant.
- F has units of N or kg m/s^2. Therefore,

$$[F] = \frac{[M][L]}{[T]^2}$$

- m has units of kg/m. Therefore,

$$[m] = \frac{[M]}{[L]}$$

Hence, it can be shown that the formula for the wave speed is dimensionally correct.

$$[v] = \sqrt{\frac{\frac{[M][L]}{[T]^2}}{\frac{[M]}{[L]}}} = \frac{[L]}{[T]}$$

2. We know that the velocity of a transverse wave on a stretched string is related to tension and mass per unit length of the string by the following relation

$$v = \sqrt{\frac{F}{m}} = \sqrt{\frac{10\ N}{0.001\ kg/m}}$$

$$\Rightarrow v = 100\ m/s$$

3. We know that the velocity of a longitudinal wave in a uniform solid rod is related to the Young's modulus of the material of the rod and its density by the following relation

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2 \times 10^{11}\ Nm^{-2}}{7800\ kg/m^3}}$$

$$\Rightarrow v = 5060\ m/s$$

4. Velocity of sound at 15^0C = fλ = 500 X 0.68 = 340 m/s.

Velocity of sound at 0^0C is given as

$$= (340\ m/s)\sqrt{\frac{273\ K}{(273+15)\ K}} = 331\ m/s$$

Since, the velocity of sound in air in a gas is given by

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

$$\Rightarrow \gamma = v^2 \left(\frac{\rho}{p}\right)$$

$$= \left(331 \frac{m}{s}\right)^2 \left(\frac{1.29\ kg/m^3}{1.01 \times 10^5\ Pa}\right) = 1.39$$

5. (a) $\sqrt{\frac{F}{m}}$ (b) $\sqrt{\frac{Y}{\rho}}$ (c) $\sqrt{\frac{E}{\rho}}$, (d) $\sqrt{\frac{\gamma p}{\rho}}$

6. Refer to the text

7. (d)

8. (b)

Selected Terminal Questions:

1. Refer to the text

2. Refer to the text

3. Refer to the text

4. (a) The speed of sound is water is given by

$$v_w = \sqrt{\frac{E_w}{\rho_w}} = \sqrt{\frac{0.2 \times 10^{10}\ N/m^2}{1000\ kg/m^3}}$$

$$\Rightarrow v_w = 1414\ m/s$$

(e) The speed of sound in the steel bar is given by

$$v_s = \sqrt{\frac{Y_s}{\rho_s}} = \sqrt{\frac{20 \times 10^{10}\ N/m^2}{7800\ kg/m^3}}$$

$$\Rightarrow v_s = 5060\ m/s$$

5. Since, the density of a gas is proportional to its molecular weight, therefore,

$$\frac{\rho_{CO_2}}{\rho_{H_2}} \cong \frac{44}{2} = 22$$

Also, we know that velocity of sound in air in a gas is given by

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

Therefore,

$$\frac{v_{H_2}}{v_{CO_2}} = \sqrt{\frac{\gamma_{H_2}}{\gamma_{CO_2}}\frac{\rho_{CO_2}}{\rho_{H_2}}} = \sqrt{\frac{1.4}{1.3} \times 22}$$

$$\Rightarrow \frac{v_{H_2}}{v_{CO_2}} \cong 4.85$$

6. Refer to the text

7. (c)

8. (b)

9. (c)

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UNIT 11 THE DOPPLER EFFECT

Structure

11.1 INTRODUCTION

In the earlier units, we studied the formation and propagation of waves in a medium. We studied the different wave parameters such as amplitude, frequency, wavelength, phase, period etc. and derived the mathematical expression of a progressive wave. We also derived expressions for the energy carried by a wave and its intensity at a point in space. After deriving one-dimensional wave equations in different media, we also wrote wave equations for two and three dimensional systems.

However, until now, we have only discussed wave motion in a homogenous medium travelling in different directions. In all such cases, the wave speed and frequency were taken to be constant. Now the question arises, are there any situations where the frequency of a wave either changes or appears to change? This is the topic we will address in this unit.

11.2 OBJECTIVES

After studying this unit, you should be able to

- explain Doppler effect
- give examples of usage of Doppler effect
- obtain expressions for apparent frequency of sound when the source or the listener or both are in motion
- describe shock waves

11.3 THE DOPPLER EFFECT

You must have heard the whistle of a train steaming in and steaming out as the train passes by. The pitch (frequency) of the whistle seems to rise when the train comes closer and falls when it goes away. While standing near a highway, you may have listened to the sound of vehicles. While approaching you, they make a relatively high-pitched sound but as they recede, the pitch drops abruptly and stays low. *This apparent change of frequency due to the relative motion between the source and the observer is known as Doppler effect.*

Usually, when the source approaches the observer or the listener approaches the source or both approach each other, the apparent frequency is higher than the actual frequency of sound produced by the source. On the other hand, when the source moves away from the listener or when the listener moves away from the source, or when both move away from each other, the apparent frequency is lower than the actual frequency of the sound produced by the source.

There are many applications of the phenomenon of Doppler effect. Do you know that Doppler shift in ultrasound waves reflected from moving body tissues is commonly used by physician to detect heartbeat of the baby inside the womb. As the heart muscle pulsates, the frequency of reflected ultrasound waves is different from the frequency of incident waves. Similarly, sonar, which is an equipment on a ship, calculates the depth of the sea or the position of an underwater object such as submarine using sound waves making use of Doppler effect.

Electromagnetic waves also exhibit Doppler effect. In aircraft navigation, the radar works by measuring the Doppler shift of high frequency radio waves reflected from moving airplanes. The Doppler shift of star-light allows us to study the motion of the stars. When the stellar light is examined in a spectrograph, we observe several spectral lines. These lines are slightly shifted as compared to the corresponding lines from the same elements on the earth. This shift is generally towards the red end of the electromagnetic spectrum, which implies the apparent frequency of light radiated by the star increases. This observation is interpreted as if the star is receding, and the universe is expanding.

In order to study the Doppler effect, we usually consider the following situations:

1- The source or the observer or both are in motion.
2- The motion is along the line joining the source and the observer.
3- The direction of motion of the medium is along or opposite to the direction of propagation of sound.
4- The speed of source is greater or smaller than the speed of sound produced by it.

Let us consider some of these possibilities in the subsequent sections to study Doppler effect in detail.

11.3.1 Source Stationary and Observer in Motion

The waves emitted by a point source spread out as spherical wave fronts of sound as shown in Figure 1. Wave front is defined as the locus of all the particles of a medium vibrating in the same phase at a given instant. The phase difference between any two points situated on the same wave front is zero the shape of a wave front depends upon the shape of the source of disturbance. If the source disturbance is a point source, then the wave front is spherical.

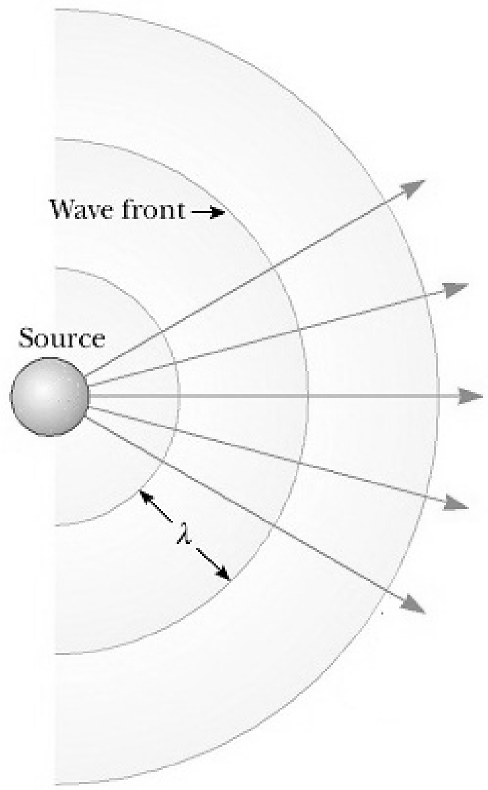

Figure 37: Spherical waves emitted by a point source. The circular arcs represent the spherical wave fronts that are concentric with the source. The sound waves move outward, perpendicular to the wave fronts.

Let us suppose that a stationary source S is producing sound of frequency f and wavelength $\lambda = v/f$. This means that the length of the block of waves passing a stationary observer per second is v and contains f waves as shown in Figure 2. But when the observer moves away from the source at a velocity v_0, he will be at O' after one second and find that the length of the block of waves passing him in one second is not v but $v - v_0$. In other words, the sound waves will appear to the moving observer to have a speed $v' = v - v_0$. However, the distance between two successive wave maxima in the observer's moving reference frame remains the same as for the stationary reference frame of the source, equal to λ.

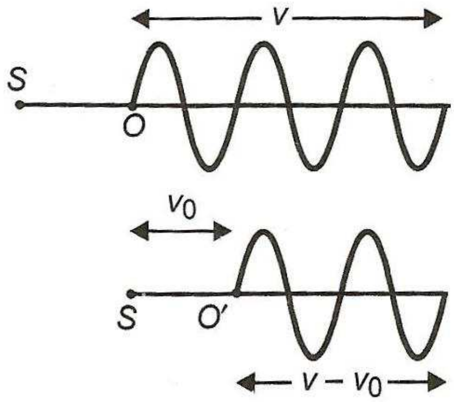

Figure 38: Representation of waves received by the observer in motion at interval of one second.

The frequency heard by the observer is given by

$$f' = \frac{v'}{\lambda}$$

where $v' = v - v_0$ and $\lambda = v/f$. Therefore, we find that for an observer moving away from a source, the apparent frequency is given by

$$f' = f\left(\frac{v - v_0}{v}\right) \qquad (11.1)$$

On the other hand, if the observer approaches the source, v_0 is to be regarded as negative. Then the apparent frequency is given by

$$f' = f\left(\frac{v + v_0}{v}\right) \qquad (11.2)$$

From equations (11.1) and (11.2), it can be seen that the frequency heard by the observer is lower than the source frequency when the observer is moving away from the stationary source. But the perceived frequency is greater than the emitted frequency when the observer approaches a stationary source. Note the difference in perceived and emitted frequencies lasts only as long as there is relative motion between the source and the observer.

11.3.2 Source in Motion and Observer Stationary

Let us now consider the case when a source S producing sound of frequency f and wavelength λ moves with a speed of v_s (which is less than the wave speed v) towards an observer, who is standing at rest in a stationary medium. The wave fronts emitted by the moving source when the observer is at rest on either side are shown in Figure 3. It must be noted that the concentric circles when the source is stationary (see Figure 1), are not concentric anymore. The wave fronts are more closely spaced in the direction of motion of the source and widely separated on the opposite side. To the observer standing on either side, this corresponds, respectively to a shorter and longer effective wavelength.

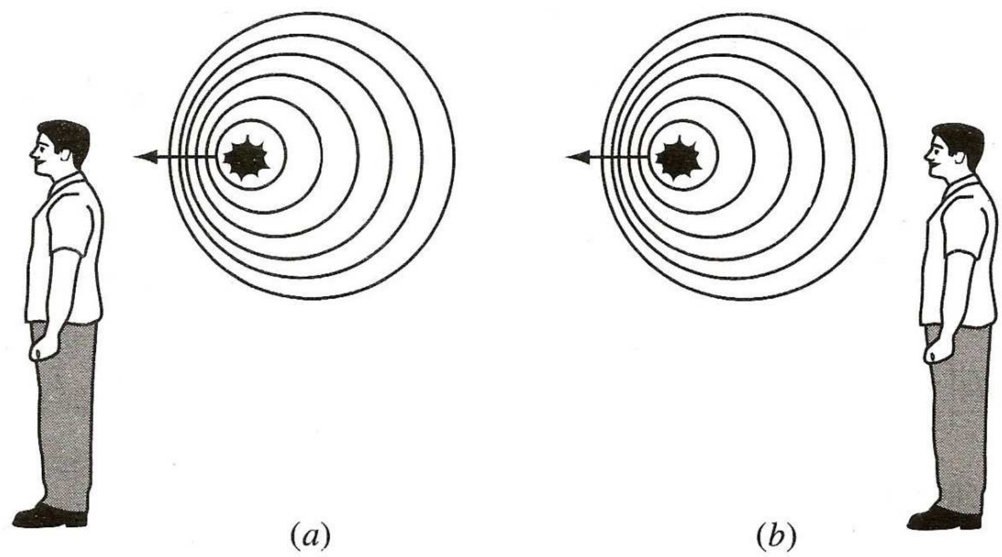

Figure 39: (a) Wave fronts emitted by the moving source as it approaches the stationary observer. The apparent wavelength shortens or in other words, the apparent frequency heard by the listener increases. (b) Wave fronts emitted by the moving source as it leaves the stationary observer. The apparent wavelength lengthens or in other words, the apparent frequency heard by the listener decreases.

We recall that for a stationary source, f waves occupied a length v in one second. When the source moves a distance v_s towards the stationary observer in one second, these f waves are crowded together in the direction of motion in a length $v - v_s$ as shown in Figure 4. This means that the distance between successive crests decreases from $\lambda = v/f$ to $\lambda' = (v - v_s)/f$. In other words, the wavelength of waves perceived by the observer shortens and manifests as apparent change in the frequency of sound.

The apparent frequency f' is given by

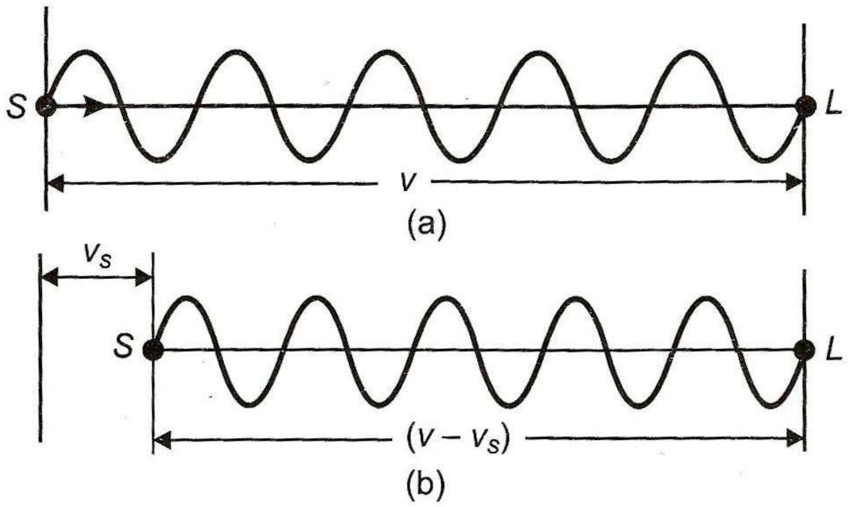

Figure 40: Representation of crowding of waves when a source moves towards a stationary observer.

$$f' = \left(\frac{v}{\lambda'}\right) = f\left(\frac{v}{v - v_s}\right) \tag{11.3}$$

On the other hand, if the source moves away from the observer, v_s is to be regarded as negative. Then the equation (11.3) takes the form

$$f' = f\left(\frac{v}{v + v_s}\right) \tag{11.4}$$

From equations (11.3) and (11.4), it can be seen that the frequency heard by the observer is lower than the source frequency when the source is moving away from the stationary observer. But the perceived frequency is greater than the emitted frequency when the source approaches a stationary observer. Again, the difference in perceived and emitted frequencies lasts only as long as there is relative motion between the source and the observer.

11.3.3 Source and Observer both in Motion

When both the source and the observer move in the same direction, we have to combine the results of the earlier sections. When the source is in motion, it causes a change in the wavelength. And, when the observer is in motion, it causes change in the number of waves received. In such a case, the apparent frequency f' is given by

$$f' = \frac{Length\ of\ block\ of\ waves\ received}{Reduced\ wavelength}$$

$$f' = f\left(\frac{v - v_0}{v - v_s}\right) \tag{11.5}$$

From equation (11.5), we note that the magnitude of apparent frequency with respect to the emitted frequency will be determined by both the velocities of the observer v_0 and the source v_s. If the observer recedes faster, f' will be less than f, but if the source approaches faster, f' will be greater than f. Equation (11.5) also tells us that the apparent frequencies will be different in the cases when the source approaches stationary observer or the observer approaches the stationary source with the same velocity.

The Doppler effect holds not only for sound (mechanical waves) but also for the electromagnetic waves including microwaves, radio waves and visible light. However, as electromagnetic waves do not require medium for their propagation and motion of source relative to detector or of detector relative to source represents same physical situation (as speed of light is independent of relative motion between the source and the observer), the formulae are different from that of sound. Here when either source or the detector or both are in motion, only two cases are possible, i.e. of approach and recession and for these apparent frequencies, respectively, are given by

$$f_A = f\sqrt{\frac{c+v}{c-v}} \qquad and \qquad f_R = f\sqrt{\frac{c-v}{c+v}} \tag{11.6}$$

where c is the speed of light and v is the relative speed of approach or recession between the source and the observer. In case of approach, the frequency increases while the wavelength decreases, i.e. there is shift towards blue end of the electromagnetic spectrum while in case of recession the frequency decreases and the wavelength increases, i.e. the shift is towards the red end of the electromagnetic spectrum.

This effect has allowed astronomers to determine the speeds of stars and galaxies relative to the earth by studying the wavelength (frequency) of the light coming from them. Police use the effect in 'radar guns' that send out pulses of short-wavelength radio waves that are reflected from a moving vehicle. When the reflected pulses return to the radar gun, the Doppler shift in their frequencies is measured and reveals the speed of the vehicle.

Example 1: The siren of fire engine has a frequency of 500 Hz.

(a) The fire engine approaches a stationary car at 20 m/s. What frequency does a person in the car hear?

(b) The fire engine stops and the car drives away from it at 20 m/s. What frequency does the person in the car hear now?

Solution: (a) Here $f = 500\ Hz$, Speed of sound in air, $v = 343\ m/s$, $v_s = 20\ m/s$ and $v_0 = 0$. Hence, the perceived frequency is given as

$$f' = f\left(\frac{v}{v - v_s}\right)$$

$$= (500\ Hz)\left(\frac{343}{343 - 20}\right) = 531\ Hz$$

(b) In this case, $v_s = 0$ and $v_0 = 20\ m/s$. Therefore, the perceived frequency of the siren is

$$f' = f\left(\frac{v - v_0}{v}\right)$$

$$= (500\ Hz)\left(\frac{343 - 20}{343}\right) = 471\ Hz$$

Example 2: A locomotive approaching a crossing at a speed of 80 miles/hr, sounds a whistle of frequency 400 Hz when 1 mile from the crossing. There is no wind, and the speed of sound in air is 0.2 miles/s. What frequency is heard by an observer 0.6 miles from the crossing on the straight road which crosses the rail road at right angles?

Solution: The situation is shown in the figure below:

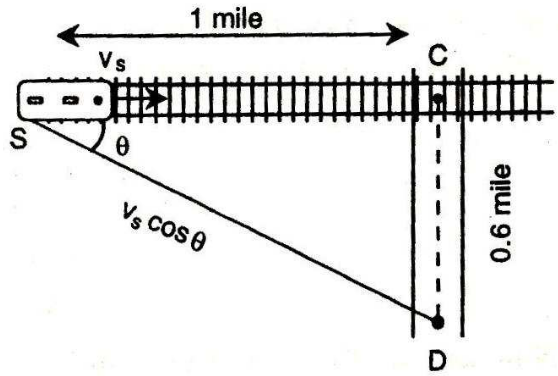

Figure 41

Until now, all the Doppler effect equations that have been derived, have assumed that the motion is along the line joining the source and the detector (observer). If the motion is along some other direction, the component of velocities along the line joining the source and the observer are considered for v_0 and v_s, i.e. if at any instant the line joining the moving source and the stationary observer makes an angle θ with the direction of motion of source, then v_s is replaced by $v_s \cos\theta$.

And so, the apparent frequency heard by the stationary observer is given by

$$f' = f\left(\frac{v}{v - v_s \cos\theta}\right) \tag{11.7}$$

Note that in such situations, the apparent frequency f' is not constant and depends on the angle θ and may be greater, equal or less than f depending on the value of θ.

From Figure 5, we can see that

$$SD = \sqrt{1^2 + 0.6^2} = 1.166\, miles$$

And

$$\cos\theta = \frac{SC}{SD} = \frac{1}{1.166} = 0.857$$

So, the speed of source along the line of sight (line joining the source and the observer) is

$$v_s \rightarrow v_s \cos\theta$$

$$= 80 \times 0.857\, miles/hr$$

$$= \frac{80 \times 0.857}{60 \times 60}\, miles/s = 0.019\, miles/s$$

Therefore, from equation (13.7),

$$f' = f\left(\frac{v}{v - v_s \cos\theta}\right) = (400\, Hz)\left(\frac{0.2}{0.2 - 0.019}\right)$$

$$= 442\, Hz$$

Self Assessment Question (SAQ) 1: Choose the correct option:

A spacecraft is approaching the earth. Relative to the radio signal it sends out, the signals received on the earth have

(a) A lower frequency
(b) A shorter wavelength
(c) A higher velocity
(d) All of the above

Self Assessment Question (SAQ) 2: A railway engine moving with a speed of 60 m/s approaches a platform on which there is a stationary listener. The real frequency of the whistle of the engine is 400 Hz. Calculate the apparent frequency of the whistle heard by the listener, when

(a) The engine is approaching the listener.
(b) The engine is moving away from the listener.

[Given: Speed of sound in air = 340 m/s]

Self Assessment Question (SAQ) 3: Choose the correct option(s). More than one option may be correct:

If the shift in a star light is towards the red end of the electromagnetic spectrum,

(a) The star is approaching the earth
(b) The star is receding from the earth
(c) The apparent frequency is smaller than the actual frequency
(d) The apparent wavelength is shorter than the actual

Self Assessment Question (SAQ) 4: Choose the correct option(s). More than one option may be correct:

In case of the Doppler effect for sound,

(a) Change in frequency is independent of the distance between the source and the observer.
(b) Change in frequency depends on the fact that the source is moved towards the observer or the observer is moved towards the source.
(c) The frequency heard will change if wind starts blowing between stationary source and stationary observer.
(d) Passenger sitting in a moving train will hear different pitch than produced by the engines of the same train.

Self Assessment Question (SAQ) 5: Choose the correct option:

The change in frequency due to Doppler effect,

(a) Depends on the distance between the source and the observer.
(b) Depends on the fact that source is moving towards the listener or listener is moving towards the source.

11.4 SUMMARY

In this unit we studied the Doppler effect, which is a phenomenon observed whenever the source of waves is moving with respect to an observer. The Doppler effect can be described as the effect produced by a moving source of waves in which there is an apparent upward shift in frequency for the observer and the source are approaching and an apparent downward shift in frequency when the observer and the source is receding. The Doppler effect can be observed to occur with all types of waves - most notably sound waves and light waves. A common experience is the shift in apparent frequency of the sound of a train horn. As the train approaches, the sound of its horn is heard at a high pitch and as the train moved away, the sound of its horn is heard at a low pitch.

11.5 GLOSSARY

Doppler effect for sound – the change in pitch or frequency that you hear when the source of sound is moving with respect to you. When the source of sound is moving toward you, the frequency is higher and the sound has a higher pitch. When the source is moving away from you, the sound has a lower frequency.

Electromagnetic waves – waves that are propagated by simultaneous periodic variations of electric and magnetic field intensity and that include radio waves, infrared, visible light, ultraviolet, X rays, and gamma rays.

Frequency – the number of complete cycles per second made by a wave. The SI unit of frequency is the hertz (Hz), which is equal to 1 cycle per second.

Sound – vibrations in a substance that travel through the substance.

Speed – the ratio of distance traveled and time. The SI unit of speed is m/s.

Tuning fork – A fork with two prongs and heavy cross-section, generally made of steel. Specially designed to retain a constant frequency of oscillation when struck. Widely used for tuning musical instruments because its frequency is very insensitive to changes in temperature, atmospheric pressure and humidity.

Velocity – speed in a given direction.

Wave front – the locus of points characterized by propagation of position of the same phase: a propagation of a line in 1-D, a curve in 2-D or a surface for a wave in 3-D.

Wavelength – the distance between two adjacent wave-crests.

11.6 TERMINAL QUESTIONS

1. A person arriving late at a concert hurries toward her seat so fast that the middle note (262 Hz) appears 1 Hz higher in frequency to her. How fast is she moving?

2. A distant galaxy of stars in the constellation Hydra is moving away from the earth at $6.1 \times 10^7\ m/s$. One of the characteristic wavelengths in the light the galaxy emits is $5.5 \times 10^{-7}\ m$. What is the corresponding wavelength measured by astronomers on the earth?

3. A train approaching a hill at a speed of 40 km/hr sounds a whistle of frequency 580 Hz when it is at a distance of 1 km from the hill. A wind with speed 40 km/hr is blowing in the direction of motion of the train. Find

(a) The frequency of the whistle as heard by an observer on the hill.

(b) The distance from the hill at which the echo from the hill is heard by the driver and its frequency. [Given: Velocity of sound in air = 1200 km/hr]

4. Sound waves of frequency f send at speed v from a stationary transmitter are received back at the site of the transmitter from a distant object moving towards the transmitter with a speed u. Show that the frequency of the reflected waves received by the transmitter will be given by

$$f' = f\left(\frac{v+u}{v-u}\right)$$

5. If the source and the observer both move in the same direction with the same speed, i.e. $v_s = v_0$, will there be any Doppler effect, i.e. will there be any apparent change in frequency?

6. If both the source and the observer are at rest and a wind blows at speed w, will there be any Doppler effect, i.e. will there be any apparent change in frequency?

7. Choose the correct option.

The shrillness (pitch) of a sound note depends upon:

(a) Amplitude (b) Frequency (c) Wavelength (d) Velocity

8. Choose the correct option.

A vibrating tuning fork of frequency 150 Hz is moving in the direction of a person with a velocity of 110 m/s. The frequency heard by the person will be: [Given: velocity of sound = 330 m/s]

(a) 100 (b) 90 (c) 80 (d) 900

9. Write short notes on:

(i) Doppler effect for sound (ii) Doppler effect for light

11.7 ANSWERS

Selected Self Assessment Questions (SAQs):

1. (b)

2. (a) When the engine is approaching the listener, the apparent frequency heard by the listener is

$$f' = f\left(\frac{v}{v - v_s}\right)$$

$$= (400\ Hz)\left(\frac{340}{340 - 60}\right) = 485.7\ Hz$$

(b) When the engine is moving away from the stationary listener, the apparent frequency heard by the listener is given as

$$f' = f\left(\frac{v}{v + v_s}\right)$$

$$= (400\ Hz)\left(\frac{340}{340 + 60}\right) = 340\ Hz$$

3. (b) and (c)

4. (a) and (b)

5. (a) No.

(b) In case of sound, yes. While, in case of light, no.

6. Shock waves are produced. No, the Doppler formula will not hold as it is valid only if $v_s < v$.

7. True.

8. (a)

Selected Terminal Questions:

1. We need to solve for v_0. Since $v_s = 0$, and the middle note appears of higher frequency to the observer, we have

$$f' = f\left(\frac{v + v_0}{v}\right)$$

$$\Rightarrow 263\ Hz = (262\ Hz)\left(\frac{343\ m/s + v_0}{343\ m/s}\right)$$

$$\Rightarrow v_0 = (343\ m/s)\left(\frac{263}{262} - 1\right) = 1.3\ m/s$$

2. The apparent frequency in case of recession is given as

$$f_R = f\sqrt{\frac{c - v}{c + v}}$$

Since, $f = c/\lambda$, the above equation may be written in terms of the wavelength as follows:

$$\frac{c}{\lambda_R} = \frac{c}{\lambda}\sqrt{\frac{c - v}{c + v}}$$

$$\Rightarrow \lambda_R = \lambda\sqrt{\frac{c + v}{c - v}}$$

$$= (5.5 \times 10^{-7}\ m)\sqrt{\frac{3 \times 10^8 \frac{m}{s} + 6.1 \times 10^7 \frac{m}{s}}{3 \times 10^8 \frac{m}{s} - 6.1 \times 10^7 \frac{m}{s}}}$$

$$= 8.3 \times 10^{-7}\ m$$

3. (a) For an observer at rest (on the hill) and source (engine) moving towards the observer, the apparent frequency is given by

$$f' = f\left(\frac{v}{v - v_s}\right)$$

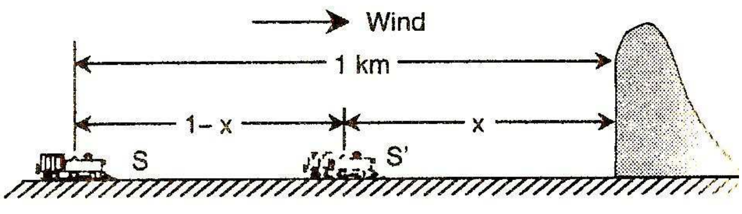

Figure 42

Now as the wind is blowing from towards the observer, the apparent speed of sound is not v but $v + w$, where w is the wind speed. Therefore, the apparent frequency heard by the observer on the hill is given as (taking into the wind effect)

$$f' = f\left[\frac{(v+w)}{(v+w)-v_s}\right]$$

Substituting the given values in the above formula, we get

$$f' = (580\ Hz)\left[\frac{(1200+40)\ km/hr}{(1200+40)-40\ km/hr}\right] = 599.33\ Hz$$

(b) If ‘x’ is the required distance from the hill, the distance moved by the train will be (1 – x) and hence the time taken by the train to travel this distance will be equal to

$$t = \frac{1-x}{40}$$

In this time, sound travels a distance of 1 km at a speed of (1200 + 40) km/hr and comes back a distance x at a speed of (1200 – 40) km/hr. Therefore,

$$\frac{1-x}{40} = \frac{1}{1240} + \frac{x}{1160}$$

$$\Rightarrow x = \frac{29}{31}\ km = 933.3\ m$$

Now the engine will act as the observer and the hill as source, so the frequency heard by the moving observer towards the stationary source will be

$$f' = f\left(\frac{v+v_0}{v}\right)$$

But in this situation, as the wind is blowing opposite to the direction of motion of sound, the wind speed will have to subtracted from the speed of sound, i.e. $v \to v - w$. Thus, the apparent frequency heard by the driver will be

$$f' = f\left[\frac{(v-w)+v_0}{(v-w)}\right]$$

Substituting the given values in the above formula, we get

$$f' = (599.33\ Hz)\left[\frac{(1200-40)+40}{(1200-40)}\right]$$

$$= 599.33\ \times \frac{1200}{1160} = 620\ Hz$$

4. The apparent frequency heard by the distant object from the stationary transmitter will be given by equation (11.2),

$$f_R = f\left(\frac{v+u}{v}\right) \quad ----------(1)$$

Now, this frequency will be reflected back by the distant moving object towards the transmitter, therefore, the apparent frequency of the reflected waves heard at the site of the transmitter will be given by equation (13.3),

$$f' = f_R\left(\frac{v}{v-u}\right) \quad ---------- (2)$$

From (1) and (2), we get

$$f' = f\left(\frac{v+u}{v}\right)\left(\frac{v}{v-u}\right)$$

$$= f\left(\frac{v+u}{v-u}\right)$$

5. If both the source and the observer move with the same speed $v_s = v_0 = u$ and in the same direction, from equation (13.5), we get

$$f' = f\left(\frac{v-u}{v-u}\right)$$

$$\Rightarrow f' = f$$

So, there will be no Doppler effect.

6. Depending on the direction of the wind, it will be added or subtracted from the speed of sound v. The apparent frequency in this situation will be given by

$$f' = f\left[\frac{(v \pm w) \pm 0}{(v \pm w) \pm 0}\right]$$

$$\Rightarrow f' = f$$

So, there will be no Doppler effect.

7. (b)

8. (a)

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UNIT 12 THE PRINCIPLE OF SUPERPOSITION AND STATIONARY WAVES

Structure

12.1 INTRODUCTION

The principle of superposition was first observed experimentally by Thomas Young in 1801. When two or more wave passes through the same medium, the resultant disturbance at any point at any instant is the vector (algebraic) sum of the instantaneous values of the disturbances produced by the individual waves at that point. It is an experimental fact that each component moves independently as if the others were not present at all i.e. their individual shapes and other characteristics are not changed due to the presence of one another. An applied example is the simultaneous transmission radio waves of different frequencies by different radio stations pass through radio antennas. These waves are superimposed and currents set up in the antenna which is very complex in terms of electromagnetic waves. But, when we adjust (tune) the radio to a particular station, we receive its frequency as if all other stations stopped broadcasting. Another example is when we listen to an orchestra; we receive a complex sound due to superposition of many sound waves, even though we can recognize the separate sounds produced by individual instruments and voices.

When two wave trains of nearly the same frequency are travelling along the same line in the same direction, superposition takes place and the resultant amplitude does not remain constant but varies with time. Since intensity depends on amplitude, so intensity also fluctuations with time. This phenomenon of periodic rise and fall of amplitude is called beats.

In the present unit, you will study the principle of superposition of waves and how this principle can be used to analyse the formation of stationary waves and related phenomena.

12.2 OBJECTIVES

After studying this unit, you should be able to

- understand Principle of superposition of waves
- explain Stationary Waves
- Properties of Stationary Waves
- Calculate the velocity of a Particle at any Point in a Stationary Wave
- understand Harmonics in Stationary Waves
- understand Beats

12.3 THE PRINCIPLE OF SUPERPOSITION OF WAVES

According to the superposition principle, the net response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually. In other words, if input A produces response X and input B produces response Y then input (A + B) produces response (X + Y).

For waves, the above general statement of the superposition principle can be stated as follows: When two or more waves of the same type cross at some point, the resultant displacement at that point is equal to the sum of the displacements due to each individual wave.Let two waves of same frequency and a constant phase difference, δ are travelling along the same direction. These waves can be expressed as

$$y_1 = A\sin(kx - \omega t) \qquad (12.1)$$

$$y_2 = A\sin(kx - \omega t + \delta) \qquad (12.2)$$

Then, according to the principle of superposition, the resultant displacement de to superposition of these two waves is given as

$$y = y_1 + y_2$$

$$y = A\sin(kx - \omega t) + A\sin(kx - \omega t + \delta)$$

$$y = 2A\cos\left(\frac{\delta}{2}\right).\sin\left(kx - \omega t + \frac{\delta}{2}\right) \qquad (12.3)$$

where, we have used the trigonometric identity

$$\sin\theta_1 + \sin\theta_2 = 2\cos(\frac{1}{2}(\theta_1 - \theta_2)).\sin(\frac{1}{2}(\theta_1 + \theta_2))$$

Eq. (12.3) indicates important consequences of superposition of waves. It shows that the resultant wave has amplitude $2A\cos\left(\frac{\delta}{2}\right)$. It also indicates that the resultant wave is sinusoidal. Further, if the two superposing waves are in phase, that is, if $\delta = 0$, then the resultant wave has amplitude $2A$; twice the value of the amplitude of the individual superposing wave. On the other hand, if the two waves are out of phase, that is, when $\delta = \pi$, the amplitude of the resultant wave becomes zero as $\cos(\pi/2) = 0$. These features of the superposition of waves are shown in Fig. 1 below.

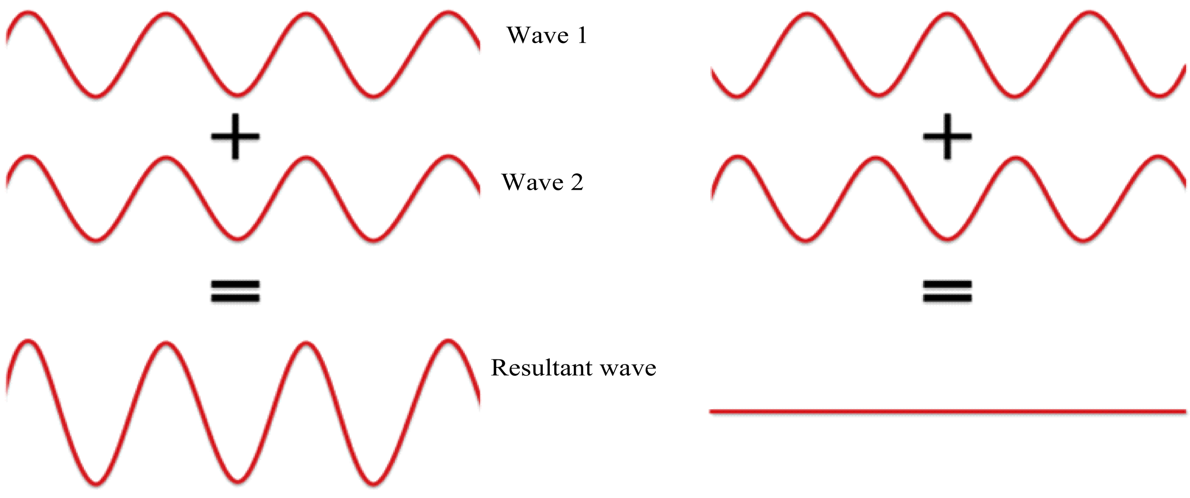


Fig. 1: Superposition of two waves of same amplitude and frequency.

The phenomena of superposition of waves travelling in opposite directions gives rise to what is called stationary or standing waves. You will study about it now.

12.4 STATIONARY WAVES

When two waves with the same frequency, wavelength and amplitude traveling in opposite directions superpose, a stationary wave or stationary wave is generated. To appreciate the phenomenon, refer to Fig 2 in which depicts a wave traveling to the right along a taut string. When this wave meets the end of the string, it will be reflected back in the opposite direction along the string. So, we have two waves on the string travelling in opposite directions. And the two waves will superpose to produce a standing wave. The reflected wave has the same amplitude and frequency as the incident wave.

If the string is held at both ends, forcing zero movement at the ends, the ends become positions of zero displacement and such points on the stationary waves are called nodes of the wave. On the other hand, between two nodes, there will be positions where the displacement is maximum and such points on the stationary waves are called antinodes. The positions of nodes and antinodes are shown in Fig. 2 below.

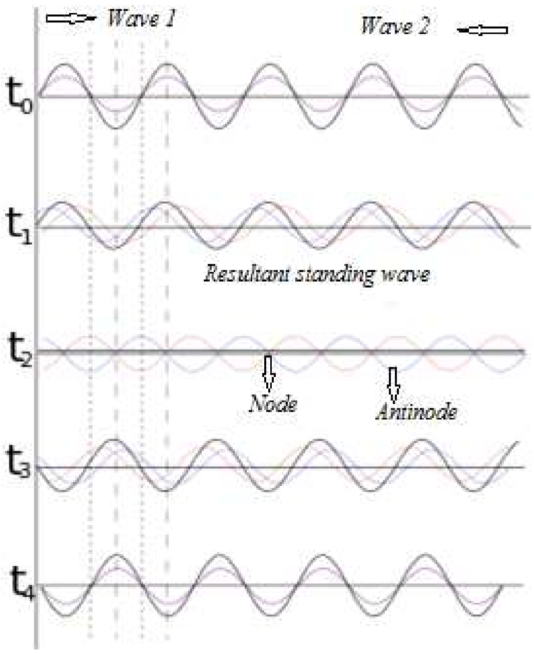


Fig. 2: Formation of stationary waves on a string due to the superposition of two waves travelling in opposite directions.

Two harmonic waves travelling in opposite directions can be represented by :

$$y_1 = A\sin(kx - \omega t) \qquad (12.4)$$

and

$$y_2 = A\sin(kx + \omega t) \qquad (12.5)$$

Where A is the amplitude of the wave, ω is the angular frequency, k is the wave number.

According to the superposition principle, the resultant wave will be given as

$$y = y_1 + y_2$$

$$y = A\sin(kx - \omega t) + A\sin(kx + \omega t) \qquad (12.6)$$

Using the trigonometric identity, we can write:

$$y = 2A\cos(\omega t)\sin(kx) \qquad (12.7)$$

Eq. (12.7) describes a wave that oscillates in time, but has a spatial dependence that is stationary: sin(kx). At locations x = 0, λ/2,λ, 3λ/2, ... called the nodes the amplitude is always zero, whereas at locations x = λ/4, 3λ/4, 5λ/4, ... called the anti-nodes, the amplitude is maximum. The distance between two conjugative nodes or anti-nodes is λ/2. These features of the stationary waves are shown in Fig. 3 below.

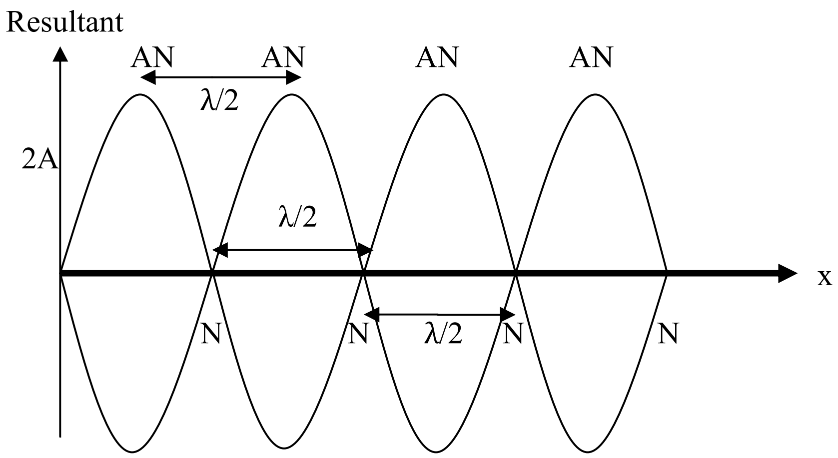


Fig. 3: Stationary waves and positions of nodes (N) and antinodes (AN) on it.

From Fig. 3, you may note that

- separation of adjacent nodes is half a wavelength *(λ*/2)
- separation of adjacent antinodes is also *λ*/2
- hence separation of adjacent nodes and antinodes is *λ*/4
- the maximum amplitude is 2a (twice that of a single wave)
- a standing wave does not transfer energy (its two components, however, do transfer energy in their respective directions.)

12.4.1 Properties of Stationary Waves

The following are the characteristic properties of stationary waves:

- In stationary waves, there are certain points called nodes where the particles are permanently at rest and certain other points called antinodes where the particles vibrate with maximum amplitude. The nodes and antinodes are formed alternately.
- All the particles of the medium, except those at the nodes, vibrate simple harmonically with a time period equal to that of the component waves.

- The amplitude of vibration increases gradually from zero to maximum from a node to an antinode.
- The medium is split up into segments. The particles in a segment vibrate in phase. The particles in one segment are out of phase with the particles in the neighbouring segment by 180^o.
- In a given segment, the particles attain their maximum or minimum velocity and acceleration at the same instant.
- There is no net transport of energy in the medium.

During each vibration, all the particles pass simultaneously through their mean positions twice, with maximum velocity which is different for different particles.

12.4.2 Velocity of a Particle at any Point in a Stationary Wave

When two identical waves, either transverse or longitudinal, travel through a medium along the same line in opposite direction, they superpose to produce a new type of waves which appear stationary in space. From Eq. (12.7), we know that the resultant disturbance of particles in stationary wave is given by

$$y = 2A\cos(\omega t)\sin(kx) \qquad (12.7)$$

Therefore the resultant particle velocity is

$$\frac{dy}{dt} = -2\omega A\sin(\omega t)\sin(kx) \qquad (12.8)$$

Self Assessment question (SAQ) 1:

Write the equations of the waves which would form stationary wave after being superimposed with the following waves

(i) $y = 5\, sin(5t - 0.01x)$

(ii) $y = 10\sin\pi\,(4t - 0.01x)$

(ii) $y = 15\sin\pi\,(0.20x - 0.8t)$

Self Assessment question (SAQ) 2:

The constituent waves of a stationary wave have amplitude, frequency and velocity as 8cm, 30Hz and 180cm/s respectively. Find out the equation of stationary wave.

12.4.3 Harmonics in Stationary Waves

A harmonic or an overtone of a stationary wave is any frequency higher than the fundamental frequency. One of the characteristics of stationary waves is that it can have only certain fixed frequencies called resonance frequencies. The lowest resonance frequency is called fundamental frequency or the first harmonic. The higher harmonics have frequencies f_n which are equal to harmonic number, n times the fundamental frequency f_1, that is, $f_n = n f_1$.

The relation between the length, L of the string on which stationary waves are set up and the frequency and wavelength that can be excited depends on the fact whether the string is fixed at both ends or fixed at one end and free at the other.

When the string is fixed at both the ends, the frequency and wavelength of the harmonics are given as

$$L = n\left(\frac{\lambda_n}{2}\right)$$

which gives

$$\lambda_n = \left(\frac{2L}{n}\right) \qquad (12.9)$$

And, frequency of the harmonics is given by

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} = nf_1 \qquad (12.10)$$

The stationary waves corresponding to different harmonics for a string fixed at both ends are depicted in Fig. 4 below.

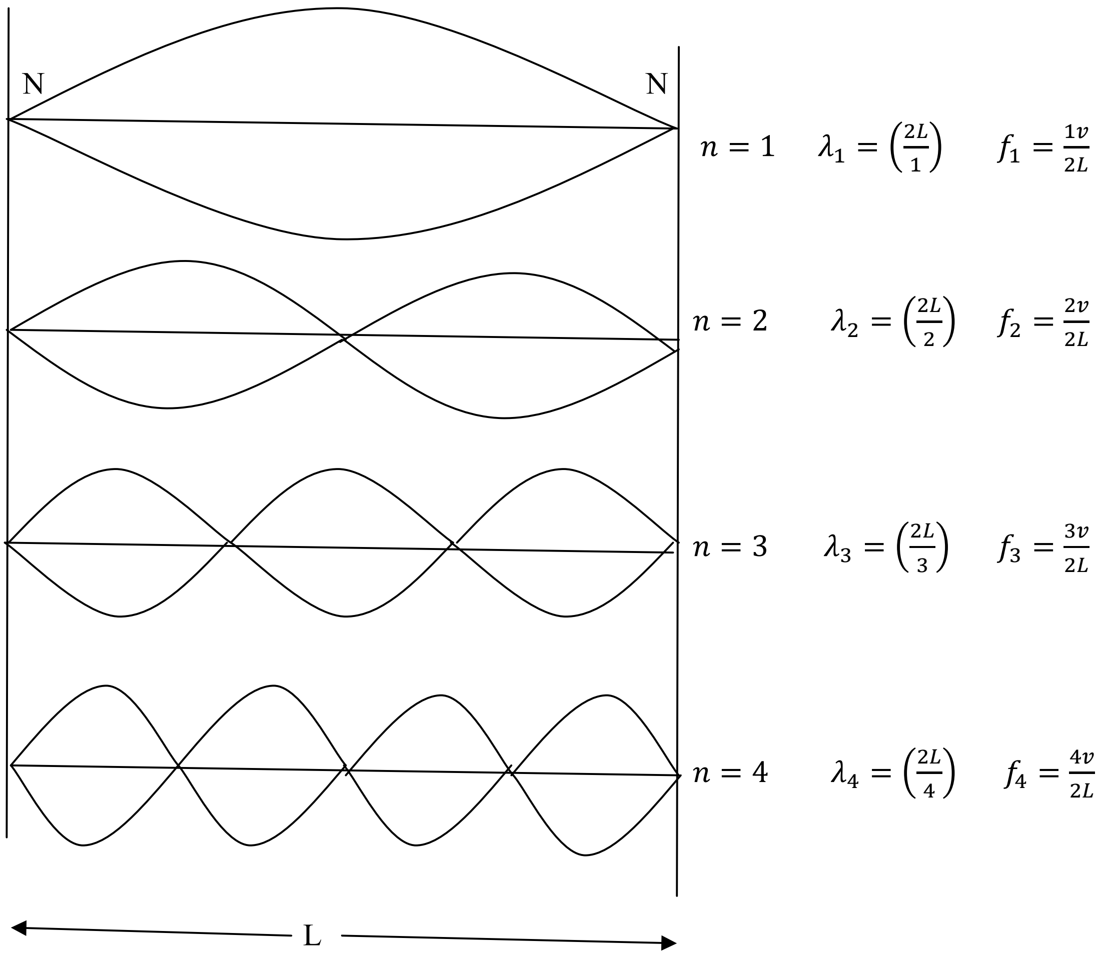


Fig. 4: Harmonics for stationary waves on a string fixed at both ends.

When the string is fixed at one end and free at the other, the the frequency and wavelength of the harmonics are given as

$$L = n\left(\frac{\lambda_n}{4}\right); \quad n = 1, 3, 5, 7, ...$$

which gives

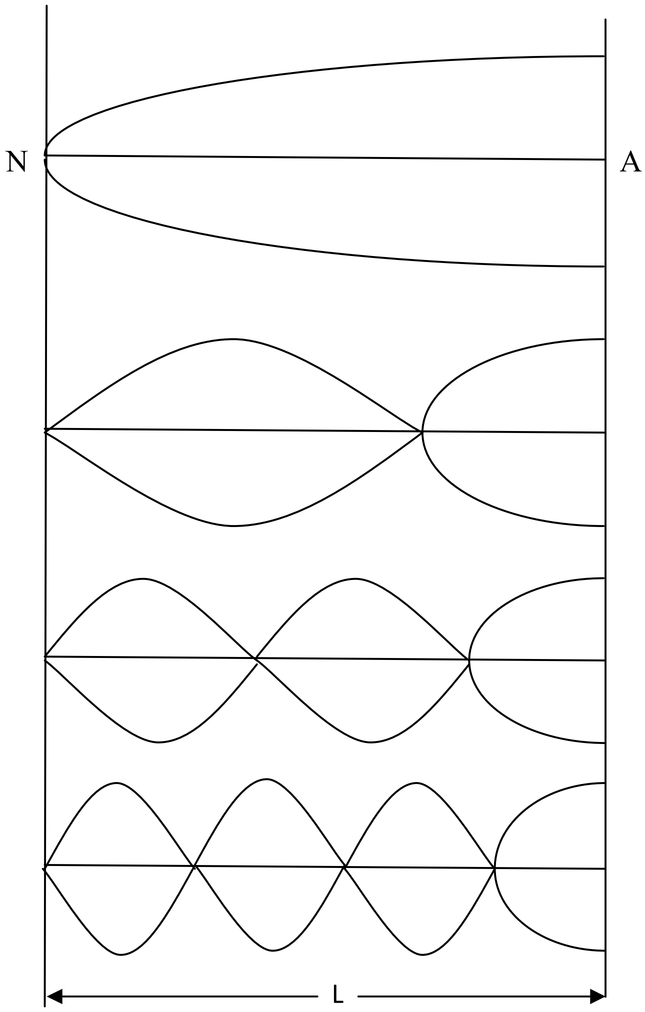
$$\lambda_n = \left(\frac{4L}{n}\right) \qquad (12.11)$$

And, frequency of the harmonics is given by

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{4L} = nf_1; \quad (12.12)$$

$$\text{where,} \qquad f_1 = \frac{v}{4L}$$

The harmonics and the corresponding stationary waves on a string fixed at one end and free at the other are depicted in Fig. 5 below.



$n = 1 \qquad \lambda_1 = \left(\frac{4L}{1}\right) \qquad f_1 = \frac{v}{4L}$

Fundamental or First Harmonic

$n = 3 \qquad \lambda_3 = \left(\frac{4L}{3}\right) \qquad f_3 = \frac{3v}{4L}$

Third Harmonic

$n = 5 \qquad \lambda_5 = \left(\frac{4L}{5}\right) \qquad f_5 = \frac{5v}{4L}$

Fifth Harmony

$n = 7 \qquad \lambda_7 = \left(\frac{4L}{7}\right) \qquad f_3 = \frac{7v}{4L}$

Seventh Harmony

Fig. 5: Harmonics of stationary waves formed on a string fixed at one end and free at the other.

12.5 BEATS

When two sound waves of slightly different frequencies are superposed (that is, they are sounded simultaneously), the intensity of the resultant sound rises and falls alternatively with time. This phenomena is called ‘beats’. It arises due to the superposition of the sound waves emitted from the two sources.

To understand the phenomenon qualitatively, suppose that, at a certain instant, the two waves having same phase but slightly different frequencies are meeting each other at a particular point in space. Due to superposition, their superposition will produce maximum sound intensity at that point. However, since the two waves are of slightly different frequencies, they will get further and further out of step as time passes until one wave, having higher frequency, gains half an oscillation on the other. Then, at the same point in space, the two waves will superpose in opposite phase and produce minimum sound intensity. Again, after some time, one wave will gain one full oscillation on the other. Then, they will once meet in phase and produce maximum sound, and so on. This periodic increase and decrese in the sound intensity is what is called beats.

If the difference in frequencies of the two waves is *n*, then in 1 second, one wave will gain *n* oscillations over the other. Hence the intensity of sound will rise and fall '*n*' times per second, giving *n* beats per second. Thus the number of beats per second is equal to the differences in frequencies of the two superposing waves.

Let two waves having same amplitude *A*, but slightly different frequencies f_1 and f_2 are superposed. Let us represent the displacement at a point produced by one waves as

$$y_1 = A\sin 2\pi f_1 t$$

and the displacement produced by the other wave as

$$y_2 = A\sin 2\pi f_2 t$$

According to the superposition principle, the resultant displacement at the point is given by

$$y = y_1 + y_2 = A\,(sin2\pi f_1 t + sin2\pi f_2 t) \qquad (12.13)$$

Applying the trigonometric identity

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

On the above expression, we get,

$$y = 2A\sin\{2\pi(\tfrac{f_1+f_2}{2})t\}\cos\{(\tfrac{f_1-f_2}{2})t\}$$

$$= a\cos\{(\tfrac{f_1-f_2}{2})t\}\sin\{2\pi(\tfrac{f_1+f_2}{2})t\} \qquad (12.14)$$

where

$$a = 2a\cos\{2\pi(\tfrac{f_1+f_2}{2})t\} \qquad (12.15)$$

Thus, the resultant vibration of the point has a frequency $\frac{f_1+f_2}{2}$ and amplitude a, which is given by

$a = 2a\cos\{(\frac{f_1-f_2}{2})t\}$

This shows that a, is not constant but periodically varies with time between a maximum value $2A$ and a minimum value 0. Hence the intensity of the resultant sound, being proportional to the square of the amplitude, rises and falls alternately.

Now, the amplitude a is maximum when

$\cos\{2\pi(\frac{f_1-f_2}{2})t\} = \pm1$

$$2\pi\left(\frac{f_1-f_2}{2}\right)t = k\pi$$

where $k = 0,1,2,3\ldots\ldots$

$$t = \frac{k}{f_1-f_2} = 0, \frac{1}{f_1-f_2}, \frac{2}{f_1-f_2}, \frac{3}{f_1-f_2}, \text{etc}$$

Amplitude maxima occur at intervals of $\frac{1}{f_1-f_2}$ second, or $(f_1 - f_2)$ times per second.

Similarly, the amplitude is minimum when

$$\cos\{2\pi(f_1 - f_2)t\} = 0$$

or $$2\pi(\frac{1}{f_1-f_2})\, t = (2k+1)\frac{\pi}{2}, \text{ where } k = 0,1,2,3\ldots.\text{etc.}$$

$$t = \frac{(2k+1)}{2(f_1-f_2)} = \frac{1}{2(f_1-f_2)}, \frac{3}{2(f_1-f_2)}, \frac{5}{2(f_1-f_2)}, \text{etc}$$

Amplitude minimum occur between the maxima, at intervals of $\frac{1}{f_1-f_2}$ second or $(f_1 - f_2)$ times per second. Thus beat frequency $= f_1 - f_2$.

Self Assessment question (SAQ) 3:

On sounding two tuning forks A and B together 5 beats per seconds are produced. The frequency of B is 512 Hz. If the prong of A is slightly scraped, the beat frequency increases. Find the frequency of A.

Self Assessment question (SAQ) 4:

A sound wave of unknown frequency gives 10 beats per second with a wave frequency 300Hz and 15 beats per second with a wave of frequency 325 Hz. What is the frequency of unknown wave?

12.6 SUMMARY

In this unit, we studied about the principle of superposition of waves, stationary waves, and their mathematical description. We also discussed the velocity of a particle at any point in a stationary wave

12.7 GLOSSARY

Frequency - number of repetition in a second

Amplitude - maximum displacement from the mean position

Superposition - waves adding/subtracting

Wavelength - distance between two consecutive crest or trough - represented by λ

Stationary waves - no net transport of energy in the medium

12.8 TERMINAL QUESTIONS

12.8.1 Multiple choice questions

1. What type of wave is produced when the particles of the medium are vibrating to and fro in the same direction of wave propagation?

a. longitudinal wave. b. sound wave. c. standing wave. d. transverse wave.

Answer: A

This is the definition of a longitudinal wave. A longitudinal wave is a wave in which particles of the medium vibrate to and fro in a direction **parallel** to the direction of energy transport.

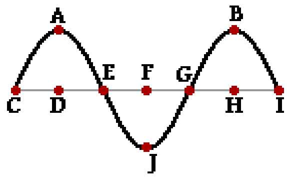
2. When the particles of a medium are vibrating at right angles to the direction of energy transport, the type of wave is described as a _____ wave.

a. longitudinal b. sound c. standing d. transverse

Answer: D

This is the definition of a transverse wave. A transverse wave is a wave in which particles of the medium vibrate to and fro in a direction perpendicular to the direction of energy transport

3. A transverse wave is traveling through a medium. See diagram below. The particles of the medium are moving.



a. parallel to the line joining AD.

b. along the line joining CI.

c. perpendicular to the line joining AD.

d. at various angles to the line CI.

e. along the curve CAEJGBI.

Answer: A

In transverse waves, particles of the medium vibrate to and fro in a direction perpendicular to the direction of energy transport. In this case, that would be parallel to the line AD.

4. If the energy in a longitudinal wave travels from south to north, the particles of the medium ____.

a. move from north to south, only.

b. vibrate both north and south.

c. move from east to west, only.

d. vibrate both east and west.

Answer: B

In longitudinal waves, particles of the medium vibrate to and from in a direction parallel to the direction of energy transport. If the particles only moved north and not back south, then the particles would be permanently displaced from their rest position; this is not wavelike.

5. The main factor which effects the speed of a sound wave is the ____.

a. amplitude of the sound wave

b. intensity of the sound wave

c. loudness of the sound wave d. properties of the medium

e. pitch of the sound wave

Answer: D

The speed of a wave is dependent upon the properties of the medium and not the properties of the wave.

6. As a wave travels into a medium in which its speed increases, its wavelength ____.

a. decreases b. increases c. remains the same

Answer: B

As a wave crosses a boundary into a new medium, its speed and wavelength change while its frequency remains the same. If the speed increases, then the wavelength must increase as well in order to maintain the same frequency.

12.8.2 Short answer type question:

1) Discuss the characteristics of stationary waves.
2) Describe the concept of beats as a consequence of the superposition principle.
3) What do you mean by beats? What are their applications?
4) What is sound? Give a brief scientific description.

12.8.3 Long answer type question

1. What are beats? Explain graphically and mathematically their production and derive expression for the frequency of beats.

2. **Differentiate between progressive waves and stationary waves.**

Answer of Self Assessment question:

Solution (SAQ) 1:

For the formation of stationary waves, the two waves of sane frequency and same amplitude traverse in the opposite directions, therefore for given waves the required another wave should be same but with x should be replaced by –x for opposite direction. Further, we $\pm$ sign is placed before amplitude for representing the reflection from free boundary or from closed boundary.

Therefore the required equation of waves are

(i) $y = \pm 5\ sin(5t + 0.01x)$

(ii) $y = \pm 10 \sin \pi\,(4t + 0.01x)$

(ii) $y = \pm 15 \sin \pi\,(-0.20x - 0.8t)$

Solution (SAQ) 2:

For the formation of standing wave the constituent wave can be considered as

$y_1 = a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)$ and $y_2 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$

The equation of stationary wave formed by above waves

$$y = y_1 + y_2 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) + a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

$$y = 2a \sin \frac{2\pi \left(\frac{t}{T} - \frac{x}{\lambda} + \frac{t}{T} + \frac{x}{\lambda}\right)}{2} \cos \frac{2\pi \left(\frac{t}{T} - \frac{x}{\lambda} + \frac{t}{T} + \frac{x}{\lambda}\right)}{2}$$

$$y = 2a \sin \frac{2\pi t}{T} \cos \left(-\frac{2\pi x}{\lambda}\right)$$

$$y = 2a \cos \frac{2\pi x}{\lambda} \sin \left(\frac{2\pi t}{T}\right)$$

Given a=8 cm, n=30Hz, v=180cm/s

T=1/n= 1/30s , λ=v/n=180/30=6cm therefore eq. becomes

$$y = 16 \cos \frac{\pi x}{3} \sin(60\pi t)$$

Solution (SAQ) 3:

The frequency of B $n_{\boldsymbol{B}} = 512\ Hz$, number of beats x=5

The frequency of A, $n_{\boldsymbol{A}} = 512 \pm 5 = 517\ or\ 507\ Hz$,

When prong of A is scraped its frequency increases and sounding with B the beat frequency also increases. In this case the frequency of A should be 517Hz.

Solution (SAQ) 4:

The number of beat per second is equal to the difference in the frequencies two waves. Therefore the frequencies of the unknown sound wave are

$$n = 300 \pm 10 = 310, 290\ Hz$$

$$n = 325 \pm 15 = 340, 310\ Hz$$

Thus the frequency of unknown sound wave is 310Hz.

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