The next example is even more complicated, but it can be simplified considerably with the aid of the following lemma.

Lemma 5.1.3. *Let $m \geq 2$ and $n \geq 1$ and consider an $m \times 2^n$ matrix game of the following form. Player 1 has m strategies, labeled $1, 2, \ldots, m$. Player 2 has 2^n strategies, labeled by the subsets $T \subset \{1, 2, \ldots, n\}$. Furthermore, for $i = 1, \ldots, m$, there exist $p_i(0) \geq 0$, $p_i(1) > 0$, $\ldots$, $p_i(n) > 0$ with $p_i(0) + p_i(1) + \cdots + p_i(n) = 1$ together with a real number $a_i(0)$, and for $l = 1, 2, \ldots, n$, there exist $m \times 2$ payoff matrices*

$$\begin{pmatrix} a_{11}(l) & a_{12}(l) \\ \vdots & \vdots \\ a_{m1}(l) & a_{m2}(l) \end{pmatrix}, \tag{5.13}$$

such that the $m \times 2^n$ matrix game has payoff matrix with (i, T) entry given by

$$a_{i,T} := p_i(0)a_i(0) + \sum_{l \in T} p_i(l)a_{i1}(l) + \sum_{l \in T^c} p_i(l)a_{i2}(l) \tag{5.14}$$

for $i \in \{1, 2, \ldots, m\}$ and $T \subset \{1, 2, \ldots, n\}$. Here $T^c := \{1, 2, \ldots, n\} - T$.

We define

$$\begin{aligned} T_1 &:= \{1 \leq l \leq n : a_{i1}(l) < a_{i2}(l) \text{ for } i = 1, 2, \ldots, m\}, \\ T_2 &:= \{1 \leq l \leq n : a_{i1}(l) > a_{i2}(l) \text{ for } i = 1, 2, \ldots, m\}, \\ T_3 &:= \{1, 2, \ldots, n\} - T_1 - T_2, \end{aligned} \tag{5.15}$$

and put $n_0 := |T_3|$. Then, given $T \subset \{1, 2, \ldots, n\}$, strategy T is strictly dominated unless $T_1 \subset T \subset T_2^c$. Thus, the $m \times 2^n$ matrix game can be reduced to an $m \times 2^{n_0}$ matrix game.

Remark. The game can be thought of as follows. Player 1 chooses a strategy $i \in \{1, 2, \ldots, m\}$. Let Z_i be a random variable with distribution $\mathrm{P}(Z_i = l) = p_i(l)$ for $l = 0, 1, \ldots, n$. Given that $Z_i = 0$, the game is over and player 1's conditional expected profit is $a_i(0)$. If $Z_i \in \{1, 2, \ldots, n\}$, then player 2 observes Z_i (but not i) and based on this information chooses a "move" $j \in \{1, 2\}$. Given that $Z_i = l$ and player 2 chooses move 1 (resp., move 2), player 1's conditional expected profit is $a_{i1}(l)$ (resp., $a_{i2}(l)$). Thus, player 2's strategies can be identified with subsets $T \subset \{1, 2, \ldots, n\}$, with player 2 choosing move 1 if $Z_i \in T$ and move 2 if $Z_i \notin T$.

Proof. Suppose that the condition $T_1 \subset T \subset T_2^c$ fails. There are two cases. In case 1, there exists $l_0 \in T_1$ with $l_0 \notin T$. Here define $T' := T \cup \{l_0\}$. In case 2, there exists $l_0 \in T$ with $l_0 \notin T_2^c$ (or $l_0 \in T_2$). Here define $T' := T - \{l_0\}$. Then, for $i = 1, 2, \ldots, m$,

$$a_{i,T'} = p_i(0)a_i(0) + \sum_{l \in T'} p_i(l)a_{i1}(l) + \sum_{l \in (T')^c} p_i(l)a_{i2}(l)$$

$$
\begin{aligned}
&= p_i(0)a_i(0) + \sum_{l\in T} p_i(l)a_{i1}(l) + \sum_{l\in T^c} p_i(l)a_{i2}(l) \\
&\qquad \pm p_i(l_0)(a_{i1}(l_0) - a_{i2}(l_0)) \\
&< p_i(0)a_i(0) + \sum_{l\in T} p_i(l)a_{i1}(l) + \sum_{l\in T^c} p_i(l)a_{i2}(l) \\
&= a_{i,T},
\end{aligned}
\tag{5.16}
$$

where the $\pm$ sign is a plus sign in case 1 and a minus sign in case 2. This tells us that strategy T for player 2 is strictly dominated by strategy T', as required. ♠

Example 5.1.4. *Chemin de fer*. The game of chemin de fer ("railway" in French) is a variant of baccarat that is still played in Monte Carlo. However, present-day rules are more restrictive than they once were, and it will serve our purposes to consider the game as it was played early in the 20th century. Chemin de fer is a two-person game played with a six-deck shoe comprising six standard 52-card decks, hence 312 cards. We will refer to player 1 and player 2 as player and banker, respectively. Denominations A, 2–9, 10, J, Q, K have values 1, 2–9, 0, 0, 0, 0, respectively. The value of a hand, consisting of two or three cards, is the sum of the values of the cards, modulo 10. In other words, only the final digit of the sum is used to evaluate a hand. For example, $5 + 7 \equiv 2 \pmod{10}$ and $5 + 7 + 9 \equiv 1 \pmod{10}$.

Two cards are dealt face down to player and two to banker, and each may look only at his own hand. The object of the game is to have the higher-valued hand (closer to 9) at the end of play. A two-card hand of value 8 or 9 is a *natural*. If either hand is a natural, the game is over and the higher-valued hand wins. Hands of equal value result in a push (no money changes hands). If neither hand is a natural, player then has the option of drawing a third card. If he exercises this option, his third card is dealt face up. Next, banker has the option of drawing a third card. This completes the game, and the higher-valued hand wins. A win for player pays even money. Again, hands of equal value result in a push.

Since nonplayers can bet on player's hand, player's strategy is restricted. He must draw to a hand valued 4 or less and stand on a hand valued 6 or 7. When his hand has value 5, he is free to draw or stand as he chooses. Banker, on whose hand no one can bet, has no restrictions on his strategy. Bets on player's hand pay even money. (We again emphasize that current rules are more restrictive as regards banker's strategy.)

To keep calculations to a minimum, we will assume that cards are dealt *with replacement*, recognizing that this is only an approximation. The probability that a two-card hand has a value of 0 is

$$
\left(\frac{4}{13}\right)^2 + 9\left(\frac{1}{13}\right)^2 = \frac{25}{169}, \tag{5.17}
$$

as we see by conditioning on the value of the first card dealt, while the probability that a two-card hand has the value $i \in \{1, 2, \ldots, 9\}$ is

$$\frac{4}{13} \cdot \frac{1}{13} + \frac{1}{13} \cdot \frac{4}{13} + 8\left(\frac{1}{13}\right)^2 = \frac{16}{169}. \tag{5.18}$$

Let X denote the value of player's two-card hand and let Y denote the value of banker's two-card hand. On the event $\{X \le 7,\ Y \le 7\}$, let X_3 denote the value of player's third card if he draws, and let $X_3 := \varnothing$ if he stands. Similarly, let Y_3 denote the value of banker's third card if he draws, and let $Y_3 := \varnothing$ if he stands. As the rules specify, player has only two strategies, which we will denote by $S_5 := \{0, 1, 2, 3, 4, 5\}$ (draw to 5) and $S_4 := \{0, 1, 2, 3, 4\}$ (stand on 5). In general, $S \subset \{0, 1, \ldots, 7\}$ denotes the strategy in which player draws if $X \in S$ and stands otherwise. Banker, on the other hand, has a strategy for each subset $T \subset \{0, 1, \ldots, 7\} \times \{0, 1, \ldots, 9, \varnothing\}$. Specifically, suppose that $X \le 7$ and $Y \le 7$. Then banker draws if $(Y, X_3) \in T$ and stands otherwise. It follows that chemin de fer is a 2×2^{88} matrix game.

Our first step is to show that Lemma 5.1.3 applies, allowing us to reduce the game to a much more manageable 2×2^4 matrix game. Let us denote by G_{ST} player's profit from a one-unit bet when he adopts strategy S and banker adopts strategy T, so that $a_{ST} := \mathrm{E}[G_{ST}]$ is the (S, T) entry in the payoff matrix. Then

$$\begin{aligned} a_{ST} &= \mathrm{E}[G_{ST}] \\ &= \mathrm{P}(X \in \{8,9\},\ X > Y) - \mathrm{P}(Y \in \{8,9\},\ Y > X) \\ &\qquad + \mathrm{E}[G_{ST}\, 1_{\{X \le 7,\ Y \le 7\}}] \\ &= \mathrm{E}[G_{ST}\, 1_{\{X \le 7,\ Y \le 7\}}] \\ &= \sum_{j=0}^{7} \sum_{k=0}^{9} \mathrm{P}(X \in S,\ Y = j,\ X_3 = k)\mathrm{E}[G_{ST} \mid X \in S,\ Y = j,\ X_3 = k] \\ &\quad + \sum_{j=0}^{7} \mathrm{P}(X \in S^c,\ Y = j,\ X_3 = \varnothing)\mathrm{E}[G_{ST} \mid X \in S^c,\ Y = j,\ X_3 = \varnothing] \end{aligned} \tag{5.19}$$

for $S = S_5$ and $S = S_4$, and for $T \subset \{0, 1, \ldots, 7\} \times \{0, 1, \ldots, 9, \varnothing\}$, where $S^c := \{0, 1, \ldots, 7\} - S$.

Let us now define, for $j \in \{0, 1, \ldots, 7\}$ and $k \in \{0, 1, \ldots, 9\}$,

$$\begin{aligned} a_{S,l}(j,k) &:= \mathrm{E}[G_{ST} \mid X \in S,\ Y = j,\ X_3 = k] \\ a_{S,l}(j,\varnothing) &:= \mathrm{E}[G_{ST} \mid X \in S^c,\ Y = j,\ X_3 = \varnothing] \end{aligned} \tag{5.20}$$

for $S = S_5$ and $S = S_4$; $l = 1$ if (j, k) (or $(j, \varnothing)$) belongs to T; and $l = 2$ if (j, k) (or $(j, \varnothing)$) belongs to $T^c := (\{0, 1, \ldots, 7\} \times \{0, 1, \ldots, 9, \varnothing\}) - T$. Defining also

$$\begin{aligned} p_S(j,k) &:= \mathrm{P}(X \in S,\ Y = j,\ X_3 = k), \\ p_S(j,\varnothing) &:= \mathrm{P}(X \in S^c,\ Y = j,\ X_3 = \varnothing), \end{aligned} \tag{5.21}$$

we have

$$\begin{aligned} a_{ST} = &\sum_{(j,k)\in T,\ k\neq\varnothing} p_S(j,k)a_{S,1}(j,k) + \sum_{(j,\varnothing)\in T} p_S(j,\varnothing)a_{S,1}(j,\varnothing) \\ &+ \sum_{(j,k)\in T^c,\ k\neq\varnothing} p_S(j,k)a_{S,2}(j,k) + \sum_{(j,\varnothing)\in T^c} p_S(j,\varnothing)a_{S,2}(j,\varnothing), \end{aligned} \tag{5.22}$$

which has the form (5.14) with $m = 2$, $n = 88$, $p_S(0) = \mathrm{P}(X \in \{8,9\}$ or $Y \in \{8,9\})$, and $a_S(0) = 0$. It remains to evaluate T_1, T_2, and T_3 of the lemma. For this we need to evaluate $a_{S,l}(j,k)$ and $a_{S,l}(j,\varnothing)$ in (5.20).

Observe that

$$\begin{aligned} &a_{S,l}(j,k) \\ &\quad = \mathrm{E}[G_{ST} \mid X \in S,\ Y = j,\ X_3 = k] \\ &\quad = \sum_{i\in S} \frac{\mathrm{P}(X = i,\ Y = j,\ X_3 = k)}{\mathrm{P}(X \in S,\ Y = j,\ X_3 = k)} \mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = k] \\ &\quad = \sum_{i\in S} \mathrm{P}(X = i \mid X \in S)\mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = k] \end{aligned} \tag{5.23}$$

if $k \neq \varnothing$ and that

$$\begin{aligned} &a_{S,l}(j,\varnothing) \\ &\quad = \mathrm{E}[G_{ST} \mid X \in S^c,\ Y = j,\ X_3 = \varnothing] \\ &\quad = \sum_{i\in S^c} \frac{\mathrm{P}(X = i,\ Y = j,\ X_3 = \varnothing)}{\mathrm{P}(X \in S^c,\ Y = j,\ X_3 = \varnothing)} \mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = \varnothing] \\ &\quad = \sum_{i\in S^c} \mathrm{P}(X = i \mid X \in S^c)\mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = \varnothing]. \end{aligned} \tag{5.24}$$

Let us define the function $M : \mathbf{Z}_+ \mapsto \{0,1,\ldots,9\}$ by $M(r) \equiv r \pmod{10}$. Then there are four cases to consider:

Case 1. $i \in S$, $(j,k) \in T$, $k \neq \varnothing$. Here

$$\begin{aligned} &\mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = k] \\ &\quad = \mathrm{P}(M(i+k) > M(j+Y_3) \mid X = i,\ Y = j,\ X_3 = k) \\ &\qquad\quad - \mathrm{P}(M(i+k) < M(j+Y_3) \mid X = i,\ Y = j,\ X_3 = k) \\ &\quad = \frac{M(i+k) + 3\cdot 1_{\{M(i+k)>j\}}}{13} - \frac{9 - M(i+k) + 3\cdot 1_{\{M(i+k)<j\}}}{13} \\ &\quad = \frac{2M(i+k) - 9 + 3\,\mathrm{sgn}(M(i+k)-j)}{13}. \end{aligned} \tag{5.25}$$

Case 2. $i \in S$, $(j,k) \in T^c$, $k \neq \varnothing$. Here

$$\begin{aligned}\mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = k] &= 1_{\{M(i+k)>j\}} - 1_{\{M(i+k)<j\}} \\ &= \operatorname{sgn}(M(i+k) - j). \end{aligned} \tag{5.26}$$

Case 3. $i \in S^c$, $(j,\varnothing) \in T$. Here

$$\begin{aligned}&\mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = \varnothing] \\ &\quad = \mathrm{P}(i > M(j+Y_3) \mid X = i,\ Y = j,\ X_3 = \varnothing) \\ &\qquad - \mathrm{P}(i < M(j+Y_3) \mid X = i,\ Y = j,\ X_3 = \varnothing) \\ &\quad = \frac{i + 3 \cdot 1_{\{i>j\}}}{13} - \frac{9 - i + 3 \cdot 1_{\{i<j\}}}{13} \\ &\quad = \frac{2i - 9 + 3\operatorname{sgn}(i-j)}{13}. \end{aligned} \tag{5.27}$$

Case 4. $i \in S^c$, $(j,\varnothing) \in T^c$. Here

$$\mathrm{E}[G_{ST} \mid X = i,\ Y = j,\ X_3 = \varnothing] = 1_{\{i>j\}} - 1_{\{i<j\}} = \operatorname{sgn}(i-j). \tag{5.28}$$

Finally, by (5.17) and (5.18), we have

$$\mathrm{P}(X = i \mid X \in S) = (16 + 9\delta_{i,0})/(16|S| + 9), \qquad i \in S, \tag{5.29}$$

$$\mathrm{P}(X = i \mid X \in S^c) = 1/|S^c|, \qquad i \in S^c. \tag{5.30}$$

This suffices to complete the evaluation of (5.23) and (5.24). For example, substituting (5.29) and either (5.25) or (5.26) into (5.23), we find that

$$\begin{aligned} a_{S_4,1}(5,4) &= \sum_{i \in S_4} \mathrm{P}(X = i \mid X \in S_4)\mathrm{E}[G_{S_4T} \mid X = i,\ Y = 5,\ X_3 = 4] \\ &= \sum_{i=0}^{4} \frac{16 + 9\delta_{i,0}}{89}\ \frac{2M(i+4) - 9 + 3\operatorname{sgn}(M(i+4) - 5)}{13} \\ &= \frac{300}{1{,}157} \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} a_{S_4,2}(5,4) &= \sum_{i \in S_4} \mathrm{P}(X = i \mid X \in S_4)\mathrm{E}[G_{S_4T} \mid X = i,\ Y = 5,\ X_3 = 4] \\ &= \sum_{i=0}^{4} \frac{16 + 9\delta_{i,0}}{89} \operatorname{sgn}(M(i+4) - 5) = \frac{23}{89} = \frac{299}{1{,}157}. \end{aligned} \tag{5.32}$$

The distinction between the first sums in (5.31) and (5.32), which appear identical, is that $(5,4) \in T$ in (5.31), while $(5,4) \notin T$ in (5.32).

In Table 5.3, we display (1,365)$a_{S_5,1}(j,k)$ and (1,365)$a_{S_5,2}(j,k)$, and in Table 5.4, we display (1,157)$a_{S_4,1}(j,k)$ and (1,157)$a_{S_4,2}(j,k)$, in both cases for $j = 0, 1, \ldots, 7$ and $k = 0, 1, \ldots, 9, \varnothing$. This tells us what T_1, T_2, and T_3 are, and the results are summarized in Table 5.2. T_1 (resp., T_2) is the set of pairs (j,k) for which the first (j,k) entry is greater than (resp., is less than) the second in both Table 5.3 and Table 5.4. In particular, $|T_3| = 4$.

Table 5.2 Banker's optimal move in chemin de fer, indicated by D (draw) or S (stand), except in the four cases indicated by $*$ in which it depends on player's strategy.

banker's two-card total	player's third card ($\varnothing$ if player stands)										
	0	1	2	3	4	5	6	7	8	9	$\varnothing$
0–2	D	D	D	D	D	D	D	D	D	D	D
3	D	D	D	D	D	D	D	D	S	$*$	D
4	S	$*$	D	D	D	D	D	D	S	S	D
5	S	S	S	S	$*$	D	D	D	S	S	D
6	S	S	S	S	S	S	D	D	S	S	$*$
7	S	S	S	S	S	S	S	S	S	S	S

Requiring that banker make the optimal move in the 84 cases that do not depend on player's strategy, we have reduced the game to a 2×2^4 matrix game, and our next step is to find the resulting payoff matrix. Banker's 16 remaining strategies are described by whether he draws or stands in the four uncertain cases, namely $(j,k) = (3,9), (4,1), (5,4), (6,\varnothing)$ (in that order). For example, the strategy DSDS corresponds to banker drawing with $(j,k) \in \{(3,9),(5,4)\}$ and standing with $(j,k) \in \{(4,1),(6,\varnothing)\}$.

It suffices to use equations (5.22), etc., together with

Table 5.3 Assuming player draws to 5, first entry is player's conditional expected profit when banker draws, multiplied by 1,365, and second entry is player's conditional expected profit when banker stands, multiplied by 1,365. Value of banker's two-card hand is j, and player's third card is k ($= \varnothing$ if player stands).

	k										
j	0	1	2	3	4	5	6	7	8	9	$\varnothing$
0	−225	60	270	480	690	532	422	312	202	92	735
	1,040	1,365	1,365	1,365	1,365	1,157	1,157	1,157	1,157	1,157	1,365
1	−348	−15	270	480	690	484	326	216	106	−4	735
	507	1,040	1,365	1,365	1,365	949	741	741	741	741	1,365
2	−444	−138	195	480	690	484	278	120	10	−100	735
	91	507	1,040	1,365	1,365	949	533	325	325	325	1,365
3	−540	−234	72	405	690	484	278	72	−86	−196	735
	−325	91	507	1,040	1,365	949	533	117	−91	−91	1,365
4	−636	−330	−24	282	615	484	278	72	−134	−292	735
	−741	−325	91	507	1,040	949	533	117	−299	−507	1,365
5	−732	−426	−120	186	492	409	278	72	−134	−340	735
	−1,157	−741	−325	91	507	624	533	117	−299	−715	1,365
6	−780	−522	−216	90	396	286	203	72	−134	−340	577
	−1,365	−1,157	−741	−325	91	91	208	117	−299	−715	682
7	−780	−570	−312	−6	300	190	80	−3	−134	−340	262
	−1,365	−1,365	−1,157	−741	−325	−325	−325	−208	−299	−715	−682

Table 5.4 Assuming player stands on 5, first entry is player's conditional expected profit when banker draws, multiplied by 1,157, and second entry is player's conditional expected profit when banker stands, multiplied by 1,157. Value of banker's two-card hand is j, and player's third card is k ($= \varnothing$ if player stands).

	k										
j	0	1	2	3	4	5	6	7	8	9	$\varnothing$
0	−289	−36	142	320	498	676	486	344	202	60	534
	832	1,157	1,157	1,157	1,157	1,157	949	949	949	949	1,157
1	−412	−111	142	320	498	676	438	248	106	−36	534
	299	832	1,157	1,157	1,157	1,157	741	533	533	533	1,157
2	−508	−234	67	320	498	676	438	200	10	−132	534
	−117	299	832	1,157	1,157	1,157	741	325	117	117	1,157
3	−604	−330	−56	245	498	676	438	200	−38	−228	534
	−533	−117	299	832	1,157	1,157	741	325	−91	−299	1,157
4	−700	−426	−152	122	423	676	438	200	−38	−276	534
	−949	−533	−117	299	832	1,157	741	325	−91	−507	1,157
5	−748	−522	−248	26	300	601	438	200	−38	−276	445
	−1,157	−949	−533	−117	299	832	741	325	−91	−507	771
6	−748	−570	−344	−70	204	478	363	200	−38	−276	267
	−1,157	−1,157	−949	−533	−117	299	416	325	−91	−507	0
7	−748	−570	−392	−166	108	382	240	125	−38	−276	89
	−1,157	−1,157	−1,157	−949	−533	−117	−117	0	−91	−507	−771

$$\begin{aligned} p_S(j,k) &:= \frac{16|S|+9}{169}\;\frac{16+9\delta_{j,0}}{169}\;\frac{1+3\delta_{k,0}}{13}, \\ p_S(j,\varnothing) &:= \frac{16|S^c|}{169}\;\frac{16+9\delta_{j,0}}{169}, \end{aligned} \tag{5.33}$$

to obtain the 2×16 payoff matrix, of which, for typographical reasons, we display the transpose multiplied by $(13)^6/16$ in Table 5.5. Keep in mind that the entries are player's expected profits.

We can reduce this payoff matrix using strict dominance. Specifically, strategies DDSS, SDDS, and SDSS are strictly dominated by strategy DSDS. In addition, strategies DDSD, SDDD, and SDSD are strictly dominated by strategy DSDD. This leaves us with the 2×10 payoff matrix of Table 5.6.

In Example 5.2.7 on p. 184 we will find the optimal strategies for player and banker. ♠

It would be instructive to include some examples of game theory applied to poker, but we postpone them to Section 22.2.

Table 5.5 Transpose of 2×16 payoff matrix, multiplied by $(13)^6/16$, for chemin de fer, obtained by application of Lemma 5.1.3.

	player draws to 5	player stands on 5
DDDD	−2,585	−4,126
DDDS	−4,457	−3,710
DDSD	−2,586	−4,111
DDSS	−4,458	−3,695
DSDD	−2,692	−4,121
DSDS	−4,564	−3,705
DSSD	−2,693	−4,106
DSSS	−4,565	−3,690
SDDD	−2,656	−4,021
SDDS	−4,528	−3,605
SDSD	−2,657	−4,006
SDSS	−4,529	−3,590
SSDD	−2,763	−4,016
SSDS	−4,635	−3,600
SSSD	−2,764	−4,001
SSSS	−4,636	−3,585

Table 5.6 Transpose of 2×10 payoff matrix, multiplied by $(13)^6/16$, for chemin de fer, obtained from Table 5.5 by strict dominance.

	player draws to 5	player stands on 5
DDDD	−4,126	−2,585
DDDS	−3,710	−4,457
DSDD	−4,121	−2,692
DSDS	−3,705	−4,564
DSSD	−4,106	−2,693
DSSS	−3,690	−4,565
SSDD	−4,016	−2,763
SSDS	−3,600	−4,635
SSSD	−4,001	−2,764
SSSS	−3,585	−4,636