CHAPTER 8

Sequences and series

- What is a sequence?
- How do we make an arithmetic sequence?
- How can we generate a sequence recursively using a graphics calculator?
- What is the rule used to find the *n*th term of an arithmetic sequence?
- What is the rule for the sum of *n* terms in an arithmetic sequence?
- How is a geometric sequence different from an arithmetic sequence?
- What is the rule for the *n*th term of a geometric sequence?
- What is the rule for finding the sum of *n* terms in a geometric sequence?
- When does the sum of the terms of a geometric sequence increase but never exceed a limiting value?
- How does a difference equation help us to generate the terms of a sequence recursively?

8.1 Sequences

A **sequence** is a list of numbers in a particular order. The numbers or items in a sequence are called the **terms** of the sequence. They may have been generated randomly or by a particular rule.

Recording the numbers obtained while tossing a die would give a *randomly generated* sequence, such as:

$$3, \quad 1, \quad 2, \quad 2, \quad 6, \quad 4, \quad 3, \quad \ldots$$

Writing down odd numbers starting at 1 would result in a sequence *generated by a rule*:

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad \ldots$$

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Each new number in the sequence of odd numbers can be made by adding 2 to the previous odd number. The dots at the end of the sequence are there to show that the sequence continues. In this chapter we will look at sequences that can be generated by a rule.

Example 1 Looking for a pattern or rule

Look for a pattern or rule in each sequence and find the next number.

a 2, 8, 14, 20, . . . **b** 5, 15, 45, 135, . . . **c** 7, 4, 1, −2, . . .
d 1, 4, 9, 16, . . . **e** 1, 1, 2, 3, 5, . . .

Solution

a 2, 8, 14, 20, . . .

1. Each term is made by adding 6 to the previous term.
2. Add 6 to 20 to make the next term, 26.

$$+6 \quad +6 \quad +6 \quad +6$$
$$2, \quad 8, \quad 14, \quad 20, \quad 26, \quad \ldots$$

b 5, 15, 45, 135, . . .

1. Multiply by 3 to make each new term.
2. Multiply 135 by 3 to make the next term, 405.

$$\times 3 \quad \times 3 \quad \times 3 \quad \times 3$$
$$5, \quad 15, \quad 45, \quad 135, \quad 405, \quad \ldots$$

c 7, 4, 1, −2, . . .

1. Subtract 3 each time to make the next term.
2. Subtract 3 from −2 to get the next term, −5.

$$-3 \quad -3 \quad -3 \quad -3$$
$$7, \quad 4, \quad 1, \quad -2, \quad -5, \quad \ldots$$

d 1, 4, 9, 16, . . .

1. Each term is the square of its position number in the sequence.
2. The fifth term is $5^2 = 25$.

1st, 2nd, 3rd, 4th, 5th
$$1^2, \quad 2^2, \quad 3^2, \quad 4^2, \quad 5^2,$$
$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad \ldots$$

e 1, 1, 2, 3, 5, . . .

1. Add the previous two terms to make each new term.
2. The next term is $3 + 5 = 8$.
   Note: This is called a **Fibonacci sequence**.

$$1+1 \quad 1+2 \quad 2+3 \quad 3+5$$
$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad \ldots$$

The meaning of $t_1, t_2, t_3, \ldots, t_n$

The symbols $t_1, t_2, t_3, \ldots$ are used as labels or names for the first, second and third terms in the sequence. In the labels $t_1, t_2, t_3, \ldots$, the numbers 1, 2, 3 are called **subscripts**. They tell us the position of each term in the sequence. So t_{10} is just a short way of talking about the tenth term in the sequence. The term in the nth position has the label $\boldsymbol{t_n}$, where $\boldsymbol{n}$ can be any number we choose.

Don't confuse the subscripts, which give the position of each term with the powers we see in say, x^2, which means x multiplied by x.

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Also notice that t_3 is just the label or name for the term in the third position. It is *not* saying that the value of the term is 3. For example, in the sequence:

$$\begin{array}{cccccccc} 2, & 8, & 14, & 20, & 26, & 32, & 38, & \ldots \\ t_1, & t_2, & t_3, & t_4, & t_5, & t_6, & t_7, & \ldots \end{array}$$

we have $t_1 = 2$, $t_2 = 8$, $t_3 = 14$ and $t_4 = 20$. If someone asks us to find t_5, they want to know the fifth term in the sequence. In our example, $t_5 = 26$.

Exercise 8A

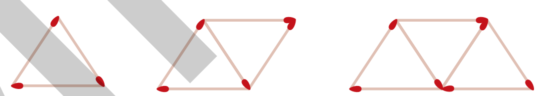
1 Find the next term in each sequence.

a 3, 7, 11, 15, . . .
b 10, 9, 8, 7, . . .
c 4, 8, 16, 32, . . .
d January, February, . . .
e 1, 2, 1, 2, . . .
f 48, 24, 12, 6, . . .
g 11, 4, −3, −10, . . .
h 1, 8, 27, 64, . . .
i a, c, e, g, . . .
j ♣, ♦, ♥, ♠, . . .
k 1, 3, 9, 27, . . .
l ⇒, ⇐, ⇒, ⇐, . . .
m 3, −6, 12, −24, . . .
n ↑, →, ↓, ←, . . .
o A, C, B, D, . . .

2 Find the required terms from the sequence: 6, 11, 16, 21, . . .

a t_1 **b** t_3 **c** t_2 **d** t_4 **e** t_5 **f** t_6

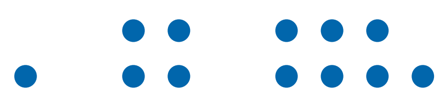
3 a Draw the next shape in this sequence of matchstick shapes:



b Write the sequence for the number of matchsticks used in each shape shown.

c Find the next two terms in the sequence of matchstick shapes.

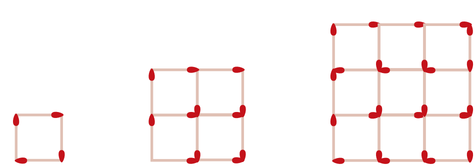
4 a Draw the next item in this sequence:



b Complete the table.

Position	1st	2nd	3rd	4th	5th
Number of dots	1	4			

5 a Matchsticks were used to make the sequence of shapes shown. Draw the next shape in this sequence:



b Complete the table.

Position	1st	2nd	3rd	4th	5th
Number of matchsticks	4	12			

8.2 Arithmetic sequences

The common difference, *d*

In an **arithmetic sequence**, each new term is made by adding to the previous term a fixed number called the **common difference**, ***d***.

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Adding a negative number is the same as subtracting a fixed amount each time. This repeating (recurring) process gives a sequence generated by **recursion**.

Common difference, *d*

In an **arithmetic sequence**, the fixed number added to (or subtracted from) each term to make the next term is called the **common difference, *d***.

$$\begin{aligned}\text{Common difference, } d &= \text{any term} - \text{its previous term}\\ &= t_2 - t_1\\ &= t_3 - t_2\\ &= t_4 - t_3 \quad \text{and so on.}\end{aligned}$$

To identify a sequence as arithmetic, it is necessary to find a common difference between successive terms.

Example 2 Identifying an arithmetic sequence

Which of the following sequences is arithmetic?

a 21, 28, 35, 42, . . . **b** 2, 6, 18, 54, . . .

Solution

a 21, 28, 35, 42, . . .

1 Find the difference between each term and its previous term.
Differences: $28 - 21 = 7$, $35 - 28 = 7$, $42 - 35 = 7$

2 Check that all the differences are the same.
The common difference is 7.

3 Write your conclusion.
The sequence is arithmetic.

b 2, 6, 18, 54, . . .

1 Find the differences by getting each term and subtracting the term just before it.
Differences: $6 - 2 = 4$, $18 - 6 = 12$, $54 - 18 = 36$

2 Are the differences constant?
The differences are not constant.

3 Write your conclusion.
The sequence is not arithmetic.

Graphs of arithmetic sequences

Example 3 Common difference $d > 0$

Consider the arithmetic sequence: 4, 7, 10, . . .

a Find the next term.

b Show the positions and values of the first four terms in a table.

c Use the table to plot the graph.

d Describe the graph.

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Solution

a

1 Find the common difference using $d = t_2 - t_1$.

Difference: $d = t_2 - t_1$
$= 7 - 4 = 3$

2 Check that this difference can be used to make all the terms of the sequence.

Each new term is made by adding 3 to the previous term.

$+3 \quad +3 \quad +3$
$4, \quad 7, \quad 10, \quad 13, \quad \ldots$

3 Add 3 to 10 to make the next term, 13.

4 Write your answer.

The next term is 13.

b

1 Number the positions along the top row of the table.

2 Write the terms in the bottom row.

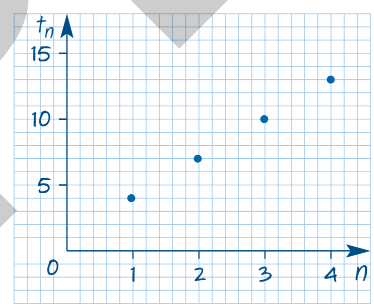
Position, n	1	2	3	4
Term, t_n	4	7	10	13

c

1 Use the horizontal axis, n, for the position of each term.

Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.



d

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (**positive** slope) or falling (**negative** slope)?

The points of an arithmetic sequence with $d = 3$ lie along a rising straight line.
The line has a positive slope.

Example 4 Common difference $d < 0$

Consider the arithmetic sequence: 9, 7, 5, . . .

a Find the next term.

b Show the positions and values of the first four terms in a table.

c Use the table to plot the graph.

d Describe the graph.

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Solution

a

1 Find the common difference using $d = t_2 - t_1$.

Difference: $d = t_2 - t_1$
$= 7 - 9 = -2$

2 Check that this difference can be used to make all the terms of the sequence.
Each new term is made by subtracting 2 from the previous term

$-2 \quad -2 \quad -2$
$9, \quad 7, \quad 5, \quad 3, \quad \ldots$

3 Subtract 2 from 5 to make the next term, 3.

4 Write your answer.

The next term is 3.

b

1 Number the positions along the top row of the table.

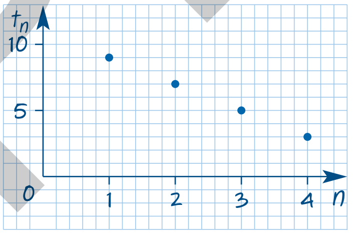
2 Write the terms in the bottom row.

Position, n	1	2	3	4
Term, t_n	9	7	5	3

c

1 Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot the points from the table.



d

1 Are the points along a straight line or a curve?

The points of an arithmetic sequence with $d = -2$ lie along a falling straight line.

2 Is the line of the points rising or falling?

The line has a negative slope.

Graphs of arithmetic sequences

Graphs of arithmetic sequences are:

- points along a line with **positive slope**, when $\mathbf{d} > \mathbf{0}$
- points along a line with **negative slope**, when $\mathbf{d} < \mathbf{0}$

A line with **positive** slope rises from left to right. A line with **negative** slope falls from left to right.

Arithmetic sequences move further and further away from the value of their starting term, so they are called **divergent**.

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Using a graphics calculator

How to use recursion to generate the terms of an arithmetic sequence with the TI-Nspire CAS

Generate the first six terms of the arithmetic sequence: 2, 7, 12, 17, 22, . . .

Steps

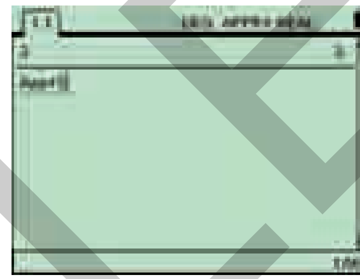
1 Start a new document: press ctrl + N.

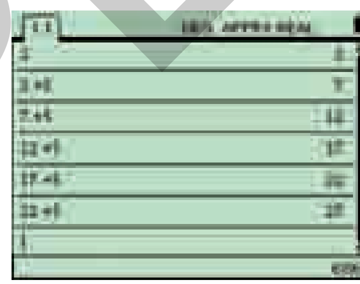
2 Select **1:Add Calculator**.
Enter the value of the first term, **2**. Press enter. The calculator stores the value 2 as **Answer**. (You can't see this yet.)

3 The common difference for the sequence is 5. So, type in +**5**.

4 Press enter. The second term in the sequence, **7**, is generated.

5 Pressing enter again generates the next term, **12**. Keep pressing enter until the desired number of terms is generated.





How to use recursion to generate the terms of an arithmetic sequence with the ClassPad

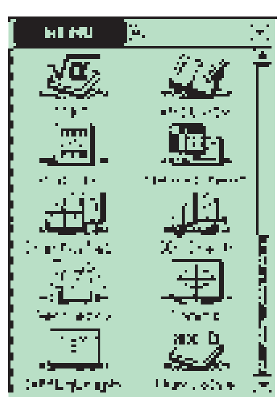
Generate the first six terms of the arithmetic sequence: 2, 7, 12, 17, 22, . . .

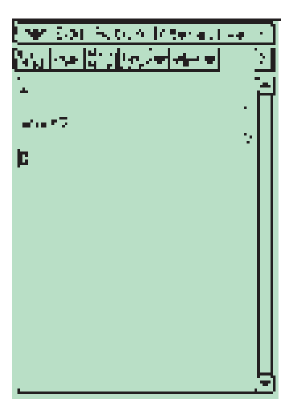
Steps

1 From the application menu screen, locate the built-in **Main** application. Tap Main to open, giving the screen shown opposite.

2 Starting with a clean screen, enter the value of the first term, **2**. Press EXE.
The calculator stores the value 2 as answer. (You can't see this yet.)

3 The common difference for this sequence is 5. So, type +**5**. Then press EXE. The second term in the sequence, **7**, is displayed.





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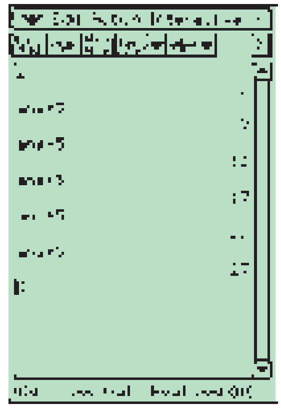
4 Pressing (EXE) again generates the next term, **12**.
Keep pressing (EXE) until the required number of terms is generated.

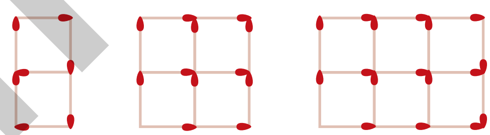


Exercise 8B

1 For the sequence of toothpick structures shown:

a Write the sequence for the number of toothpicks used in each structure.
b How many toothpicks would be used in the next two structures?

2 Matchsticks have been used to make the shapes below:



a Give the sequence for the number of matchsticks used in each shape.
b Is the sequence for the number of matchsticks used arithmetic? If it is arithmetic, give the common difference.
c How many matchsticks would be used in each of the next two shapes?

3 Find out which of the sequences below are arithmetic. Give the common difference for each sequence that is arithmetic.

a 8, 11, 14, 17, . . . **b** 7, 15, 22, 30, . . . **c** 11, 7, 3, −1, . . .
d 12, 9, 6, 3, . . . **e** 16, 8, 4, 2, . . . **f** 8, 17, 26, 35, . . .
g a, b, c, d, e, . . . **h** 2, 0, −2, −4, . . . **i** 1, 1, 1, 1, . . .
j 2, 6, 18, 54, . . . **k** −5, 4, 13, 22, . . . **l** −1, 0, 1, 2, . . .

4 For each of these arithmetic sequences, find the common difference and the next term.

a 5, 11, 17, 23, . . . **b** 17, 13, 9, 5, . . . **c** 11, 15, 19, 23, . . .
d 8, 4, 0, −4, . . . **e** 35, 30, 25, 20, . . . **f** 1.5, 2, 2.5, 3, . . .
g 0.5, 2.5, 4.5, 6.5, . . . **h** 2.0, 2.1, 2.2, 2.3, . . . **i** 6.4, 6.7, 7.0, 7.3, . . .
j 27, 43, 59, 75, . . . **k** 43, 24, 5, −14, . . . **l** 108, 131, 154, 177, . . .
m 24, 17, 10, 3, . . . **n** $\frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \ldots$ **o** $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots$

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5 Give the next two terms in each of these arithmetic sequences.

a 17, 23, 29, 35, . . . **b** 14, 11, 8, 5, . . . **c** $2, 1\frac{1}{2}, 1, \frac{1}{2}, \ldots$
d 27, 35, 43, 51, . . . **e** 33, 21, 9, −3, . . . **f** 0.8, 1.1, 1.4, 1.7, . . .
g 9.8, 9.3, 8.8, 8.3, . . . **h** 19, 23, 27, 31, . . . **i** 27, 23, 19, 15, . . .
j 5, 10, 15, 20, . . . **k** $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$ **l** $\frac{1}{2}, 2, 3\frac{1}{2}, 5, \ldots$

6 Find the missing terms in the following arithmetic sequences.

a 8, 13, 18, 23, □, □, . . . **b** 14, 8, 2, −4, □, □, . . . **c** 6, 15, 24, □, □, 51, . . .
d 23, 18, □, □, 3, −2, . . . **e** 3, □, □, 27, 35, 43, . . . **f** □, □, 29, 37, 45, 53, . . .
g □, □, 7, −4, −15, −26, . . . **h** 36, □, 22, □, 8, 1, . . . **i** 15, □, 31, □, 47, □, . . .

7 For each of these arithmetic sequences, use your graphics calculator to find the required term.

a 3, 8, 13, 18, . . . Find t_{10}. **b** 16, 9, 2, −5, . . . Find t_{12}.
c 1.6, 3.9, 6.2, 8.5, . . . Find t_{14}. **d** 8.7, 5.6, 2.5, −0.6, . . . Find t_{11}.
e 293, 226, 159, 92, . . . Find t_{20}. **f** −143, −171, −199, −227, . . . Find t_{25}.

8 Consider each of the arithmetic sequences below.

i Find the next term. **ii** Show the first four terms in a table.
iii Use the table to plot a graph. **iv** Describe the graph.

a 3, 5, 7, . . . **b** 11, 8, 5, . . .

8.3 Arithmetic sequence applications

Finding the *n*th term in an arithmetic sequence

Repeated addition can be used to make each new term in an arithmetic sequence. However, this would be a very tedious process if we wanted to know t_{50}, the term in the 50th position. We would have to make every term until we got to the 50th term.

We need a rule that will help us calculate any term t_n using n, its position number, together with the values of the first term and the common difference.

In an arithmetic sequence, a is the first term and d is the fixed number added each time to make the next term.

$$\begin{array}{ccccccccccccc} & +d & & +d & & +d & & +d & & +d \ \ldots & +d & & +d \ \ldots & +d & \\ a, & & a+d, & & a+2d, & & a+3d, & & a+4d, & \ldots\ \ldots & a+9d, & \ldots\ \ldots & a+(n-1)d \\ t_1 & & t_2 & & t_3 & & t_4 & & t_5 & & t_{10} & & t_n \end{array}$$

Notice that to make the 5th term, t_5, took four jumps. That is, four lots of d were added to the first term, a.

So $t_5 = a + 4d$

Similarly, $t_{10} = a + 9d$

The number of times d must be added is always 1 less than the position number of the term we want to make.

Rule for finding the *n*th term of an arithmetic sequence

In an **arithmetic sequence**, the rule for the term in the nth position is

$$t_n = a + (n - 1)d$$

where $a = t_1 =$ first term

$d =$ common difference

$n =$ position number of the term.

Example 5 Finding the *n*th term of an arithmetic sequence

Find t_{30}, the 30th term in the arithmetic sequence: 21, 18, 15, 12, ...

Strategy: To use $t_n = a + (n - 1)d$, we need to know a, d and n.

Solution

1 The first term is 21. $a = 21$

2 The term t_{30} is in the 30th position. $n = 30$

3 Find the difference in the first two terms. $d = t_2 - t_1$
$= 18 - 21 = -3$

4 Check that this difference generates the sequence.
$\quad -3 \quad -3 \quad -3$
21, 18, 15, 12, ...

5 Substitute the values of a, d and n into $t_n = a + (n - 1)d$.
$t_n = a + (n - 1)d$
$t_{30} = 21 + (30 - 1)(-3)$

6 Evaluate.
$= 21 + 29(-3)$
$= -66$

7 Write your answer. The 30th term, t_{30}, is -66.

Example 6 Application of an arithmetic sequence

The hire of a car costs \$180 for the first day and \$150 for each following day.

a How much would it cost to hire the car for 7 days?

b Find a rule for the cost of hiring the car for n days.

c For how many days can the car be hired using \$1530?

Solution

a

1 Identify values for a, n and d. $a = 180, n = 7, d = 150$

2 Substitute the values for a, n and d into $t_n = a + (n - 1)d$.
$t_n = a + (n - 1)d$
$t_7 = 180 + (7 - 1)(150)$

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3 Evaluate.

$= 180 + (6)(150)$

$= 1080$

4 Write your answer.

It would cost \$1080 to hire the car for 7 days.

b Substitute $a = 180$ and $d = 150$ into $t_n = a + (n - 1)d$.

$t_n = a + (n - 1)d$

$t_n = 180 + (n - 1)(150)$

Note: This rule is saying that it costs \$180 for the first day and the $(n - 1)$ days left each cost \$150.

c

1 Substitute $t_n = 1530$ into $t_n = 180 + (n - 1)150$ and solve for n.

$1539 = 180 + (n - 1)150$

$1350 = (n - 1)150$

$n - 1 = \dfrac{1350}{150} = 9$

$n = 10$

This can be done by hand, as shown opposite or you can use your calculator.

On the calculation screen, type (or tap) the following:
solve(1530 = 180 + (*n* − 1) × 150, *n*)
and press enter (or EXE).

solve(1530=180+(*n*–1).150,*n*)
n = 10

2 Write your answer.

Using \$ 1530, a car can be hired for 10 days.

Exercise 8C

1 Give the value of a and d in each of the following arithmetic sequences.

a 7, 11, 15, 19, . . . **b** 8, 5, 2, −1, . . . **c** 14, 23, 32, 41, . . .
d 62, 35, 8, −19, . . . **e** −9, −4, 1, 6, . . . **f** −13, −19, −25, −31, . . .
g 2.4, 2.7, 3, 3.3, . . . **h** 8.1, 7.2, 6.3, 5.4, . . . **i** 28, 45, 62, 79, . . .

2 Find the twentieth term, t_{20}, in each of these arithmetic sequences.

a 5, 11, 17, 23, . . . **b** 19, 16, 13, 10, . . . **c** −37, −33, −29, −25, . . .
d −8, −15, −22, −29, . . . **e** 0.1, 0.2, 0.3, 0.4, . . . **f** 16, 15.5, 15, 14.5, . . .

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3 Find the required term in each of these arithmetic sequences.

a 18, 21, 24, 27, . . . Find t_{35}.
b −14, −6, 2, 10, . . . Find t_{41}.
c 27, 14, 1, −12, . . . Find t_{37}.
d 16, 31, 46, 61, . . . Find t_{29}.
e −19, −23, −27, −31, . . . Find t_{26}.
f 0.8, 1.5, 2.2, 2.9, . . . Find t_{36}.
g 82, 68, 54, 40, . . . Find t_{21}.
h 9.4, 8.8, 8.2, 7.6, . . . Find t_{29}.

4 Find the 40th term in an arithmetic sequence that starts at 11 and has a common difference of 8.

5 The first term in an arithmetic sequence is 27 and the common difference is 19. Find the 100th term, t_{100}.

6 A sequence started at 100 and had 7 subtracted each time to make new terms. Find the 20th term, t_{20}.

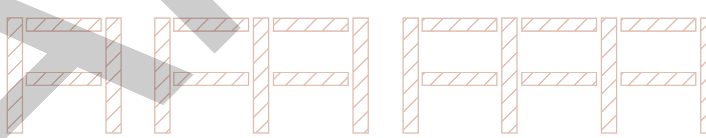
7 A canoe hire shop charges \$15 for the first hour and \$12 for each extra hour. How much would it cost to hire a canoe for 10 hours?

8 At the end of its first year after planting, a tree was 2.50 m high. It grew 0.75 m in each following year. How high was it 18 years after it was planted?

9 Bronwen swam her first race of 50 m in 68.4 seconds. She hopes to reduce her time by 0.3 seconds each time she races. Give her times for the first four races if she succeeds.

10 Tristan had \$250 on the first day of his holidays. If he spent \$23 on each of the following days, how much did he have left after the 10th day of his holidays?

11 A single section of fencing is made from four logs. Two sections use seven logs. Examples of one-, two- and three-section fences are shown below.



a How many logs are needed for a fence with three sections?
b How many logs would be needed to build a fence with 20 sections?

12 Consider the arithmetic sequence: 7, 13, 19, 25, 31, 37, 43, 49, 55, 61, . . .

a State the position of the term 37.
b What is the value of the term in the 8th position?

13 Consider the arithmetic sequence: 16, 33, 50, 67, 84, 101, . . .

a State the position of the term 84.
b Find the position number of the term 373.

14 Fumbles restaurant bought 320 wine glasses for its first week of business. If 15 glasses were broken in each of the following weeks, after how many weeks were there only 35 glasses left?

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15 Elizabeth stored 100 songs on her music storage device in the first month. In each month that followed she stored seven more songs.

a List the number of songs she had stored after each of the first 4 months.

b Find the rule for the number of songs stored by the end of the nth month.

c How many songs were stored by the end of the 24th month?

d After which month did Elizabeth have 345 songs stored?

16 Toby was given \$200 in the first week of the year. He saves \$35 from his part-time job in each of the following weeks.

a How much money will he have after 20 weeks?

b When will he have more than \$1000?

17 Jessica started on a salary of \$45 000 in her first year working in an advertising agency. Her salary is subsequently increased by \$3200 each year.

a What will be her salary in her tenth year? **b** When will her salary be \$61 000?

c When will her salary be more than \$80 000?

8.4 Arithmetic series

A **series** is the sum of the terms in a sequence. So an **arithmetic series** is just the sum of the terms in an arithmetic sequence. Below we see a series in which the first five terms in an arithmetic sequence have been added:

$$8 + 11 + 14 + 17 + 20 = 70$$

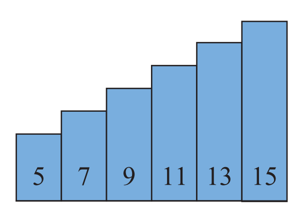
Adding the terms of an arithmetic sequence

To discover a rule for adding the terms in an arithmetic sequence, consider the following example.

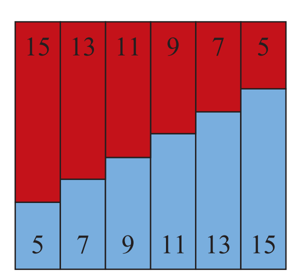
The sum of the arithmetic sequence:

$$5 + 7 + 9 + 11 + 13 + 15$$

can be represented by the sum of the lengths of the blue wooden rods with lengths equal to the terms in the series.



When a set of red rods of the same lengths is turned upside down, it fits neatly onto the original set of blue rods.



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The total lengths of the two sets of rods is equal to twice the original sum.

There are six pairs of rods, each equal to the total of the first and last rods. Each pair is equal to $5 + 15 = 20$.

$$
\begin{aligned}
2 \times (\text{sum of the sequence}) &= 6 \times (5 + 15) \\
\text{Sum of the sequence} &= \frac{6}{2}(5 + 15) \\
&= 60
\end{aligned}
$$

From the second-last line on the previous page, we see that, in general:

$$\text{Sum of an arithmetic sequence} = \frac{n}{2}(a + l)$$

where n = number of terms, a = first term, l = last term.

The last term, l (or t_n), is given by $\quad l = a + (n - 1)d$

Substituting the rule for l into our sum of arithmetic sequence rule, we get:

$$
\begin{aligned}
\text{Sum of an arithmetic sequence} &= \frac{n}{2}[a + a + (n - 1)d] \\
S_n &= \frac{n}{2}[2a + (n - 1)d]
\end{aligned}
$$

The sum of the first n terms of a sequence is represented by the symbol $\boldsymbol{S_n}$.

Rule for finding the sum of the first *n* terms of an arithmetic sequence

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a + l)$$

where n = number of terms, a = first term and l = last term (or nth) term.

When the last term is not given, use

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

The **method of pairing numbers** can be used to add up any arithmetic sequence. For example, to find the sum of all the whole numbers from 1 to 20, pair the first and last terms and keep forming pairs as you work in from both ends.

$$1 + 2 + 3 + 4 + \ldots \; \ldots \; \ldots \; \ldots + 17 + 18 + 19 + 20$$

$4 + 17 = 21$

$3 + 18 = 21$

$2 + 19 = 21$

$1 + 20 = 21$

The first four pairs formed this way are shown. The process continues until all 10 pairs are made.

As there are 10 pairs, each adding to 21, the sum of the sequence is equal to $10 \times 21 = 210$.

In an arithmetic sequence of n terms there are $\frac{n}{2}$ pairs, each adding to $(a + l)$.

So the sum of an arithmetic sequence is again given by

$$S_n = \frac{n}{2}(a + l)$$

Example 7 Adding the terms of an arithmetic sequence

Consider the arithmetic series: $7 + 13 + 19 + 25 + \ldots$
Find the sum of the first thirty terms, S_{30}.

Strategy: We are not given the last term. So we need to use $S_n = \frac{n}{2}[2a + (n - 1)d]$.

Solution

1 The first term a is 7. The common difference d is 6. The number of terms n is 30.

$a = 7, d = 6, n = 30$

2 Substitute the values of a, d and n into $S_n = \frac{n}{2}[2a + (n - 1)d$.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

3 Evaluate. Use a calculator if you wish.

$$\begin{aligned} S_{30} &= \frac{30}{2}[2 \times 7 + (30 - 1) \times 6] \\ &= 15[14 + (29) \times 6] \\ &= 15[188] \\ &= 2820 \end{aligned}$$

4 Write your answer.

The sum of the first 30 terms is 2820.

Example 8 Application of an arithmetic series

Ali began his job as manager of a cafe on a salary of \$53 000 and was given a fixed increase in each following year. In his fifteenth year of employment he was paid \$95 000.
What was the total amount of his income over the 15 years?

Strategy: We are given the values of a, l and n. So we can use $S_n = \frac{n}{2}(a + l)$.

Solution

1 He was paid \$53 000 in the first year.

$a = 53\,000$

2 The last term is his salary in the 15th year.

$t_{15} = 95\,000$
So $l = 95\,000$ and $n = 15$

3 Substitute the values of a, l and n into $S_n = \frac{n}{2}(a + l)$.

$$S_n = \frac{n}{2}(a + l)$$

$$S_{15} = \frac{15}{2}(53\,000 + 95\,000)$$

4 Evaluate. Use a calculator if you wish.

$$= 7.5(148\,000)$$
$$= 1\,110\,000$$

5 Write your answer.

His total income over 15 years was \$1 110 000.

Finding the number of terms in a finite arithmetic series

If the number of terms, n, in a finite arithmetic series is known, either of the formulas

$$S_n = \frac{n}{2}(a + l) \quad \text{or} \quad S_n = \frac{n}{2}[2a + (n - 1)d]$$

can be used to find the sum of the whole series.

However, if the last term in the series is given, but the number of terms in the series is *not* known, it is necessary to find the number of terms before the sum of the whole series can be calculated.

A simple way to do this is to divide the difference between the last and first terms by the common difference (this counts the number of additions or subtractions in the series), then add 1 (to count the first term).

Rule to find the number of terms in a finite arithmetic series

$$n = \frac{l - a}{d} + 1$$

Alternatively, a graphics calculator can be used to find the number of terms.

Example 9 **Finding the number of terms in an arithmetic series and its sum**

Consider the arithmetic series: $7 + 11 + 15 + 19 + \ldots + 87$

a Find the number of terms, n. **b** Calculate the sum of the terms.

Solution

a

1 The first and last terms are given. $a = 7, l = 87$

2 Find the common difference. $d = 11 - 7 = 4$

3 Substitute the values for a, l and d into $n = \dfrac{l - a}{d} + 1$.

$$n = \frac{l - a}{d} + 1 = \frac{87 - 7}{4} + 1$$

4 Evaluate. $= 21$

5 Write your answer. The number of terms is 21.

b

1 Substitute the values for a, n and l into $S_n = \dfrac{n}{2}(a + l)$.

$$S_n = \frac{n}{2}(a + l)$$

$$S_{21} = \frac{21}{2}(7 + 87)$$

2 Evaluate. $= 987$

3 Write your answer. The sum of the terms is 987.

Note: As we also know that $d = 4$, we could have used $S_n = \dfrac{n}{2}[2a + (n - 1)d]$.

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Exercise 8D

1 A popular story in the history of mathematics tells us that, as a young student, the mathematician Gauss was talkative and restless in class, so his teacher decided to keep him busy by telling him to add up all the whole numbers to 100. The teacher was shocked when the young Gauss produced the correct answer within a minute.
Gauss may have used the method of pairing the first and last numbers and making more pairs as he worked in from both ends. Complete the question parts below and see whether you can quickly find the sum of all the whole numbers from 1 to 100. (See page 312 if you want more details on this method.)

a How many pairs will be formed? **b** What will be the value of the sum in each pair?
c Use the number of pairs and the value of each pair to find the answer for the sum of all the numbers from 1 to 100.

2 Consider the following arithmetic series for the sum of all the whole numbers from 1 to 100:

$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100$

a State the values of a, n and l.
b Use the rule $S_n = \frac{n}{2}(a + l)$ to find the sum of the series.

3 Find the required sum for each of these arithmetic series.

a $7 + 13 + 19 + \ldots$ Find S_{18}. **b** $100 + 110 + 120 + \ldots$ Find S_{24}.
c $28 + 23 + 18 + \ldots$ Find S_{16}. **d** $24 + 21 + 18 + \ldots$ Find S_7.
e $3 + 3\frac{1}{2} + 4 + \ldots$ Find S_{21}. **f** $8.6 + 8.4 + 8.2 + \ldots$ Find S_{30}.
g $2.3 + 2.7 + 3.1 + \ldots$ Find S_{34}. **h** $12 + 5 - 2 - \ldots$ Find S_{28}.

4 The sum of the first 50 even numbers is given by the arithmetic series:
$2 + 4 + 6 + 8 + \ldots + 100$

a State the values of n, a and l. **b** Find the sum of the first 50 even numbers.

5 An arithmetic sequence of 25 terms starts at 7 and ends at 79. Find the sum of the terms.

6 Aaron trained by riding his racing bicycle for 17 km on the first day and increased the distance by 2 km on each of the following days.

a How far did he ride on the twentieth day?
b What was the total distance he rode over all of the 20 days?

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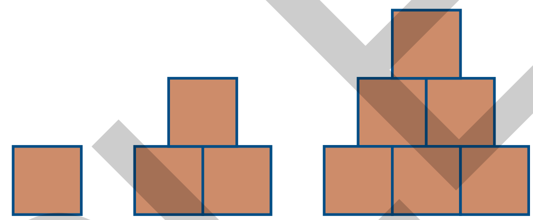
7 Consider the arithmetic sequence: 15, 22, 29, 36, . . .

a Find the 25th term, t_{25} **b** Find the sum of the first 25 terms, S_{25}.

8 A person starts a job on an annual salary of \$38 000 and receives annual increases of \$4000.

a Calculate the salary in the fortieth year of employment.
b Find the person's total earnings after 40 years of employment.

9 Grant stacks boxes in his warehouse in a special way to be sure that they are stable. The patterns used for stacks that are one, two and three rows high are shown:

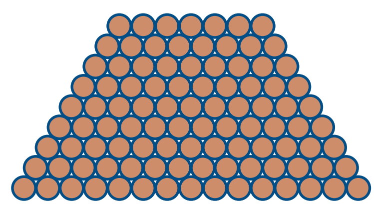


a The series for the number of boxes in a stack three rows high is: $1 + 2 + 3$.

i Write the series for the number of boxes in a stack four rows high.
ii Write the series for the number of boxes in a stack 12 rows high.

b In the series for boxes in a stack 12 rows high, give the values of a, d and n.
c How many boxes are in a stack 12 rows high?

10 The logs in a wood heap are stacked so that there are 15 logs in the bottom row and one less log in each new row above. There are nine rows of logs.



a Write out the series for the sum of the logs in all the rows of logs.
b For the series for the sum of the logs in all the rows:

i state the first term, a **ii** find the number of terms, n
iii find the common difference, d.

c Find the total number of logs.

11 Karen and Barry had to pay local council rates of \$4000 in the first year they moved into their new house. If the council rates increase by \$300 each year, find the total amount of the rates they will have paid after living in the house for 10 years.

12 For each arithmetic series below:

i find the number of terms, n **ii** calculate the sum of the terms.

a $5 + 9 + 13 + \ldots + 49$ **b** $47 + 44 + 41 + \ldots + 5$
c $2 + 4 + 6 + \ldots + 100$ **d** $3 + 6 + 9 + \ldots + 99$
e $1 + 3 + 5 + \ldots + 99$ **f** $6.4 + 8.1 + 9.8 + \ldots + 37$

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13 A person gambling in a casino put \$100 on their first bet. Each subsequent bet was an increase of \$50 on the previous bet. The person eventually made a final bet of \$1000.

a What are the values of a, d, and l? **b** How many bets did the person make?

c Find the total amount that the person spent.

8.5 Geometric sequences

The common ratio, *r*

In a **geometric sequence**, each new term is made by multiplying the previous term by a fixed number called the **common ratio**, ***r***. This repeating or recurring process is another example of a sequence generated by **recursion**.

In the sequence:

$$2, \xrightarrow{\times 3} 6, \xrightarrow{\times 3} 18, \xrightarrow{\times 3} 54, \xrightarrow{\times 3} \ldots$$

each new term is made by multiplying the previous term by 3. The common ratio is 3.

In the sequence:

$$48, \xrightarrow{\times \frac{1}{2}} 24, \xrightarrow{\times \frac{1}{2}} 12, \xrightarrow{\times \frac{1}{2}} 6, \xrightarrow{\times \frac{1}{2}} \ldots$$

each new term is made by halving the previous term. In this sequence we are multiplying each term by $\times\ \frac{1}{2}$, which is equivalent to dividing by 2. The common ratio is $\frac{1}{2}$.

New terms in a geometric sequence $t_1,\ t_2,\ t_3,\ t_4, \ldots$ are made by multiplying the previous term by the common ratio, r.

$$t_1, \xrightarrow{\times r} t_2, \xrightarrow{\times r} t_3, \xrightarrow{\times r} t_4, \xrightarrow{\times r} \ldots$$

So $\quad t_1 \times r = t_2 \quad$ and $\quad t_2 \times r = t_3 \quad$ and so on.

$$r = \frac{t_2}{t_1} \qquad\qquad r = \frac{t_3}{t_2}$$

Common ratio, *r*

In a **geometric sequence**, the **common ratio**, ***r***, is found by dividing a term by the previous term.

$$\text{Common ratio, } r = \frac{\text{any term}}{\text{the previous term}} = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots$$

Example 10 **Finding the common ratio *r***

Find the common ratio in each of the following geometric sequences:

a $3,\ 12,\ 48,\ 192, \ldots$ **b** $162,\ 54,\ 18,\ 6, \ldots$

Solution

a $3,\ 12,\ 48,\ 192, \ldots$

1 The common ratio is equal to any term divided by its previous term.

$$\text{Common ratio, } r = \frac{t_2}{t_1} = \frac{12}{3} = 4$$

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2 Check that multiplying by 4 makes each new term.

$$3, \xrightarrow{\times 4} 12, \xrightarrow{\times 4} 48, \xrightarrow{\times 4} 192, \xrightarrow{\times 4} \ldots$$

3 Write your answer.

The common ratio is 4.

b 162, 54, 18, 6, ...

1 Find the common ratio, r.

Common ratio, $r = \frac{t_2}{t_1} = \frac{54}{162} = \frac{1}{3}$

2 Check that multiplying by $\frac{1}{3}$ makes each new term.

$$162, \xrightarrow{\times \frac{1}{3}} 54, \xrightarrow{\times \frac{1}{3}} 18, \xrightarrow{\times \frac{1}{3}} 6, \xrightarrow{\times \frac{1}{3}} \ldots$$

3 Write your answer.

The common ratio is $\frac{1}{3}$.

To identify a sequence as geometric, it is necessary to find a common ratio between successive terms.

Example 11 Identifying a geometric sequence

Which of the following is a geometric sequence?

a 16, 24, 36, 54, ... **b** 243, 162, 108, 81, ...

Solution

a 16, 24, 36, 54, ...

1 Find the ratio (multiplier) between successive terms.

$r = \frac{t_2}{t_1} = \frac{24}{16} = \frac{3}{2}$ $\quad r = \frac{t_3}{t_2} = \frac{36}{24} = \frac{3}{2}$ $\quad r = \frac{t_4}{t_3} = \frac{54}{36} = \frac{3}{2}$

2 Check that the ratios are the same.

The common ratio is $\frac{3}{2}$.

3 Write your conclusion.

The sequence is geometric.

b 243, 162, 108, 81, ...

1 Find the ratios of successive terms.

$r = \frac{t_2}{t_1} = \frac{162}{243} = \frac{2}{3}$ $\quad r = \frac{t_3}{t_2} = \frac{108}{162} = \frac{2}{3}$ $\quad r = \frac{t_4}{t_3} = \frac{81}{108} = \frac{3}{4}$

2 Are the ratios the same?

The ratios are not the same.

3 Write your conclusion.

The sequence is not geometric.

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Graphs of geometric sequences

Example 12 **Graphing a diverging geometric sequence**

Consider the geometric sequence: 2, 6, 18, . . .

a Find the next term. **b** Show the positions and values of the first four terms in a table.
c Use the table to plot the graph. **d** Describe the graph.

Solution

a

1 Find the common ratio using $r = \frac{t_2}{t_1}$.

Common ratio: $r = \frac{t_2}{t_1} = \frac{6}{2} = 3$

2 Check that this ratio makes the given terms.

$$2, \xrightarrow{\times 3} 6, \xrightarrow{\times 3} 18, \xrightarrow{\times 3} 54, \ldots$$

3 Multiply 18 by 3 to make the next term, 54.

4 Write your answer.

The next term is 54.

b

1 Number the positions along the top row of the table.

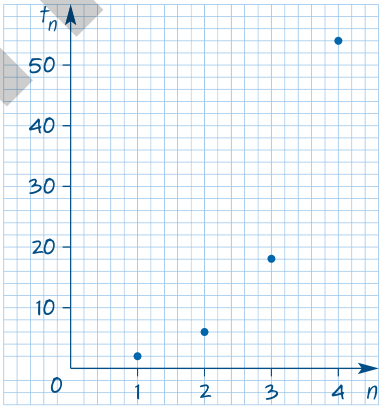
2 Write the terms in the bottom row.

Position, n	1	2	3	4
Term, t_n	2	6	18	54

c

1 Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.



d

1 Do the points lie on a curve or a straight line?

The points lie on a curve that becomes steeper and steeper.

2 Are the points moving further away from the starting value (**diverging**) or are they getting closer and closer to a value (**converging**)?

The graph moves further and further away (diverges) from the starting value, 2.

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Example 13 Graphing a converging and oscillating geometric sequence

Consider the geometric sequence: 8, −4, 2, . . .

a Find the next term.

b Show the positions and values of the first four terms in a table.

c Use the table to plot the graph. **d** Describe the graph.

Solution

a

1 Find the common ratio using $r = \frac{t_2}{t_1}$

common ratio: $r = \frac{t_2}{t_1} = \frac{-4}{8} = -\frac{1}{2}$

2 Check that this ratio makes the given terms.

$8, \xrightarrow{\times -\frac{1}{2}} -4, \xrightarrow{\times -\frac{1}{2}} 2, \xrightarrow{\times -\frac{1}{2}} -1, \ldots$

3 Multiply 2 by $-\frac{1}{2}$ to make the next term, n −1.

4 Write your answer.

The next term is −1.

b

1 Number the positions along the top row of the table.

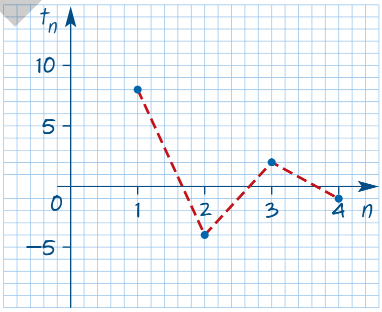
2 Write the terms in the bottom row.

Position, n	1	2	3	4
Term, t_n	8	−4	2	−1

c

1 Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.



d

1 Do the points lie on a curve or a straight line?

The points do not lie on a curve or a straight line.

2 Do the values diverge, converge or **oscillate**?

The graph oscillates on either side of the n axis as its values converge toward zero.

Graphs of geometric sequences

Graphs of geometric sequences:

- **diverge** when $r < -1$ or $r > 1$
- **converge** towards zero when $-1 < r < 1$
- **oscillate** on either side of zero when $r < 0$
- are points on a **curve** when $r > 0$.

Using a graphics calculator

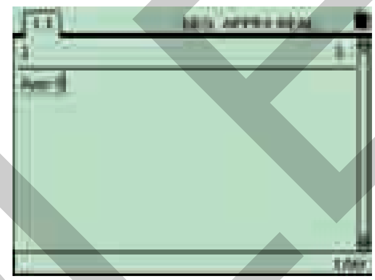
How to use recursion to generate the terms of a geometric sequence with the TI-Nspire CAS

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, ...

Steps

1 Start a new document: press ctrl + N.

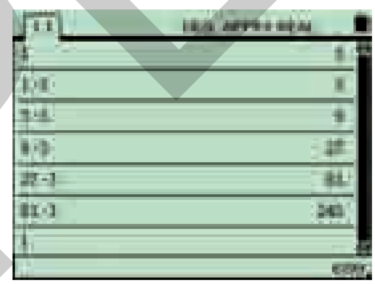
2 Select **1:Add Calculator**.
Enter the value of the first term, **1**. Press enter.
The calculator stores the value 2 as **Ans**wer.
(You can't see this yet.)



3 The common ratio for the sequence is 3. So, type ×3.

4 Press enter. The second term in the sequence, **3**, is generated.

5 Pressing enter again generates the next term, **9**.
Keep pressing enter until the desired number of terms is generated.

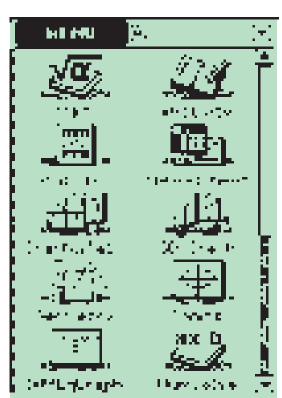


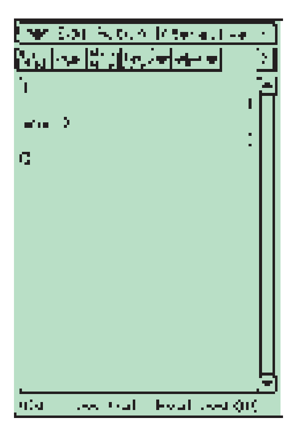
How to use recursion to generate the terms of a geometric sequence with the ClassPad

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, . . .

Steps

1 From the application menu screen, locate the **Main** application. Tap Main to open, giving the screen shown opposite.





2 Starting with a clean screen, enter the value of the first term, **1**. Press EXE.
The calculator stores the value 1 as **answer**. (You can't see this yet.)

3 The common ratio for this sequence is 3. So, type ×3. Then press EXE. The second term in the sequence (i.e. **3**) is displayed.

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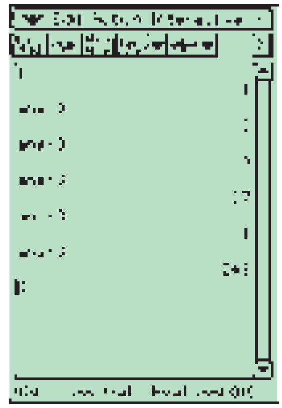
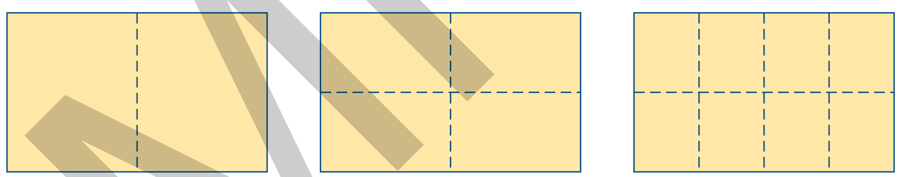
4 Pressing (EXE) again generates the next term, **9**.
Keep pressing (EXE) until the required number of terms is generated.



Exercise 8E

1 A sheet of A4 paper is repeatedly folded in half. The first fold creates two areas separated by the fold crease. The areas created by the first, second and third folds after the paper has been flattened out again are shown:



1 fold 2 folds 3 folds

a Make up to five folds using a sheet of A4 paper and complete the table.

Number of folds	1	2	3	4	5
Number of areas	2	. . .	. . .	. . .	. . .

b For the sequence of areas, give the value of the first term a and the common ratio r.

c How many areas would be made if the paper could be folded seven times?

2 Find the common ratio for each of the following geometric sequences.

a 3, 6, 12, 24, . . .
b 64, 16, 4, 1, . . .
c 6, 30, 150, 750, . . .
d 2, −6, 18, −54, . . .
e 32, 16, 8, 4, . . .
f 108, 36, 12, 4, . . .
g 3, −6, 12, −24, . . .
h 3, 21, 147, 1029, . . .
i 81, 54, 36, 24, . . .

3 Find out which of the following sequences are geometric. Give the common ratio for each sequence that is geometric.

a 4, 8, 16, 32, . . .
b 5, −10, 20, −40, . . .
c 5, 10, 15, 20, . . .
d 5, 15, 45, 135, . . .
e 3, −9, 27, −81, . . .
f 3, −9, −21, −33, . . .
g 4, 8, 12, 16, . . .
h 64, 32, 16, 8, . . .
i 2, 4, 8, 12, . . .

4 Find the next two terms in each of these geometric sequences.

a 7, 14, 28, □, □, ... **b** 162, 54, 18, □, □, ... **c** 5, −15, 45, □, □, ...
d 48, −24, 12, □, □, ... **e** 100, 30, 9, □, □, ... **f** 10, 17, 28.9, □, □, ...
g 1, −2.1, 4.41, □, □, ... **h** $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}$, □, □, ... **i** $\frac{8}{9}, \frac{2}{3}, \frac{1}{2}$, □, □, ...

5 Use your graphics calculator to generate each sequence and find the required term.

a 2, 10, 50, 250, ... Find t_{10}. **b** 5, 10, 20, 40, ... Find t_{11}.
c 2, −4, 8, −16, ... Find t_{10}. **d** 512, 256, 128, 64, ... Find t_{12}.
e 6561, 4374, 2916, ... Find t_9. **f** 256, 192, 144, ... Find t_5.
g 7776, 6480, 5400, ... Find t_6. **h** $\frac{3}{4}, 1\frac{1}{2}$, 3, 6, ... Find t_{12}.

6 Find the missing terms in the following geometric sequences.

a 3, 15, 75, □, □, ... **b** 6, 12, 24, □, □, ... **c** 486, 324, □, 144, □, ...
d 64, 48, □, 27, □, ... **e** 4, □, □, −32, 64, ... **f** 2, □, □, −54, 162, ...
g □, □, 12, −24, 48, ... **h** 4, □, 36, □, 324, 972, ... **i** 729, □, 324, □, 144, 96, ...

7 Consider each of the geometric sequences below.

i Find the next term. **ii** Show the first four terms in a table.
iii Use the table to plot a graph. **iv** Is the graph a straight line, a curve or neither?
v Does the graph diverge, converge or oscillate?

a 3, 6, 12, ... **b** 8, 4, 2, ... **c** 1, −2, 4, ... **d** 4, −2, 1, ...

8.6 Geometric sequence applications

Finding the *n*th term in a geometric sequence

Repeated multiplication can be used to make each new term in a geometric sequence. However, to find the 50th term, we would first have to make all the previous terms. In this section we will derive a rule to find any term t_n by just using its position number, n, the common ratio, r, and the first term, a, in the sequence.

In a geometric sequence, a is the first term and r is the fixed number (common ratio) that multiplies each term to make the next term.

$$\begin{array}{cccccccccccc} & \times r & \times r & \times r & \times r & \times r & \dots & \times r & \times r & \dots & \times r & \\ a, & a \times r, & a \times r^2, & a \times r^3, & a \times r^4, & \dots & \dots & a \times r^9, & \dots & \dots & a \times r^{n-1}, \\ t_1 & t_2 & t_3 & t_4 & t_5 & & & t_{10} & & & t_n \end{array}$$

Notice that to make the 5th term, t_5, took four jumps. That is, a was multiplied by r four times.

So $\quad t_5 = a \times r \times r \times r \times r$

$\quad\quad = a \times r^4$

Similarly, $t_{10} = a \times r^9$

The number of times r multiplies is always 1 less than the position number of the term we want to make.

Rule for finding the *n*th term of a geometric sequence

In a **geometric sequence**, the rule for the term in the nth position is

$$t_n = ar^{n-1}$$

where $a = t_1$ = first term, r = common ratio and n = position number of the term.

Example 14 **Finding the *n*th term of a geometric sequence**

Find t_{15}, the 15th term in the geometric sequence: 3, 6, 12, 24, . . .

Strategy: To use $t_n = ar^{n-1}$, we need to know a, r and n.

Solution

1 The first term is 3. $\quad a = 3$

2 The term t_{15} is in the 15th position. $\quad n = 15$

3 Find the ratio of the first two terms. $\quad r = \frac{t_2}{t_1} = \frac{6}{3} = 2$

4 Check that this ratio generates the sequence.

$\times 2 \quad \times 2 \quad \times 2$

3, 6, 12, 24, . . .

5 Substitute the values of a, r and n into $t_n = ar^{n-1}$.

$t_n = ar^{n-1}$

$t_{15} = 3 \times 2^{15-1}$

6 Use a calculator to find t_{15}.

$= 3 \times 2^{14}$

$= 49\,152$

7 Write your answer. $\quad$ The 15th term, t_{15}, is 49 152.

Percentage change

Let's investigate the sequence generated when each term is 10% less than the previous term. This sequence would arise if someone had \$100 and decided that on each new day they would spend 10% of whatever money they had left.

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They start with $100.	$t_1 = 100$
On the second day they spent 10% of $100, which is $10, leaving $90.	$t_2 = 100 - 10\% \text{ of } 100$ $= 100 - 10$
The second term is $90.	$= 90$
On the third day they spent 10% of $90, which is $9, leaving $81.	$t_3 = 90 - 10\% \text{ of } 90$ $= 90 - 9$
The third term is $81.	$= 81$
The sequence is 100, 90, 81, . . .	
To find the ratio, use $r = \frac{t_2}{t_1}$	$r = \frac{t_2}{t_1} = \frac{90}{100} = 0.90$

We can make more terms in the sequence by multiplying by 0.90 to make each new term.

$$100, \xrightarrow{\times 0.90} 90, \xrightarrow{\times 0.90} 81, \xrightarrow{\times 0.90} 72.90, \xrightarrow{\times 0.90} 65.61, \ldots$$

So on the fifth day the person had $65.61 left.

Tip: A 10% reduction means that there will only be 90% left. In other words, each new term will be 90% of the previous term.

So $r = 90\% = 0.90$

In general, an R% reduction means that each new term will be $(100\% - R\%)$ or $1 - \frac{R}{100}$ of the previous term.

Rule for finding the common ratio *r*, given the percentage change

For an R% **reduction** when making each new term, use $r = 1 - \frac{R}{100}$

For an R% **increase** when making each new term, use $r = 1 + \frac{R}{100}$

Example 15 Calculating the common ratio from percentage change

State, correct to 2 decimal places, the first four terms in each of these geometric sequences for the changes given.

a Starts at 200 and each new term is 4% *less* than the previous term.

b Starts at 500 and each new term is 12% *more* than the previous term.

Solution

a

1 The first term is 200. $a = 200$

2 Use $r = 1 - \frac{R}{100}$ with $R = 4$.

or

As each new term is 4% *less* than the previous term, the new term is 96% of the previous term. So $r = 96\% = 0.96$.

$r = 1 - \frac{R}{100}$ Put $R = 4$.

$= 1 - \frac{4}{100} = 0.96$

or use:

$r = 96\% = 0.96$

3 Starting at 200, multiply by 0.96 to make each new term.

$\times 0.96 \quad \times 0.96 \quad \times 0.96$

200, 192, 184.32, 176.9472, ...

4 Write your answer correct to 2 decimal places.

The first four terms are:

200, 192, 184.32, 176.95.

b

1 The first term is 500.

$a = 500$

2 Use $r = 1 + \frac{R}{100}$ with $R = 12$.

or

As each new term is 12% *more* than the previous term, the new term is 112% of the previous term. So $r = 112\% = 1.12$.

$r = 1 + \frac{R}{100}$ Put $R = 12$.

$= 1 + \frac{12}{100} = 1.12$

or use:

$r = 112\% = 1.12$

3 Starting at 500, multiply by 1.12 to make each new term.

$\times 1.12 \quad \times 1.12 \quad \times 1.12$

500, 560, 627.2, 702.464, ...

4 Write your answer correct to 2 decimal places.

The first four terms are:

500, 560, 627.2, 702.46

Example 16 Application of a geometric sequence

As a park ranger, Ayleisha has been working on a project to increase the number of rare native orchids in Wilsons Promontory National Park.

At the start of the project, a survey found 200 of the orchids in the park. It is assumed from similar projects that the number of orchids will increase by about 18% each year.

a State the first term a, and the common ratio r, for the geometric sequence for the number of orchids each year.

b Find a rule for the number of orchids at the start of the nth year.

c How many orchids are predicted in 10 years time?

d It is believed that a viable population of orchids will be established when the numbers reach 1000. When will this happen?

Solution

a

1 The number of orchids starts at 200.

$a = 200$

2 Use $r = 1 + \frac{R}{100}$ with $R = 18$.

or

An 18% increase means each year there are 118% of the previous year. So $r = 1.18$.

$r = 1 + \frac{R}{100}$ Put $R = 18$.

$= 1 + \frac{18}{100}$

$= 1.18$

b Substitute $a = 200$ and $r = 1.18$ into $t_n = ar^{n-1}$.

$t_n = ar^{n-1}$

$t_n = 200 \times 1.18^{n-1}$

c

1 Substitute $n = 10$ into the rule for t_n.

$t_n = 200 \times 1.18^{n-1}$

$t_{10} = 200 \times 1.18^{10-1}$

2 Evaluate.

$= 200 \times 1.18^9$

≈ 887

3 Write your answer.

There will be about 887 orchids in 10 years time.

d Substitute $t_n = 1000$ into $t_n = 200 \times 1.18^{n-1}$ and solve for n. With a CAS calculator this is best done using the **solve** command.

TI-nspire CAS

On the calculation screen, type **solve(1000 = 200 × 1.18^{n-1}, *n*)** and press (enter).

Hint: Use the (^) key after typing **1.18** to raise the cursor so that ***n* − 1** can be entered as a power. Use the ▼ arrow to lower the cursor.

ClassPad

On the calculation screen, tap in **solve(1000 = 200 × 1.18(*n* − 1), *n*)** and press (EXE).

TI-Nspire CAS
solve(1000=200.(1.18)$^{n-1}$, n) n = 10.723…

ClassPad
solve(1000=200×1.18^(n–1), n)
{n = 10.723…}

Write your answer.

Over 1000 orchids are predicted in the eleventh year.

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Exercise 8F

1 Consider the geometric sequence: 6, −12, 24, −48, 96, −192, 384, −768, ...

a State the term in the 7th position. **b** Name the position of the term 96.
c Which term is 6th? **d** When does the term 24 occur?

2 Give the value of the first term, a, and common ratio, r, in each of the following geometric sequences.

a 12, 24, 48, ... **b** 6, 18, 54, ... **c** 2, 8, 32, ...
d 56, 28, 14, ... **e** 36, 12, 4, ... **f** 3, −9, 27, ...
g 5, −10, 20, ... **h** 100, 10, 1, ... **i** 27, 18, 12, ...

3 Find the tenth term, t_{10}, in each of these geometric sequences.

a 4, 12, 36, ... **b** 3, 6, 12, ... **c** 2, −10, 50, ...
d −3, −12, −48, ... **e** 10, 30, 90, ... **f** 512, 256, 128, ...

4 State the values of the first term, a, and the common ratio, r, for the geometric sequences with the rules given.

a $t_n = 3 \times 4^{n-1}$ **b** $t_n = 5 \times 2^{n-1}$ **c** $t_n = -3 \times 7^{n-1}$
d $t_n = 200 \times 1.10^{n-1}$ **e** $t_n = 6 \times (-3)^{n-1}$ **f** $t_n = -4 \times (-5)^{n-1}$

5 Find the required term in each of these geometric sequences.

a 64, 32, 16, ... Find t_7. **b** 9, 18, 36, ... Find t_8.
c 1, 2, 4, ... Find t_{20}. **d** 1, 3, 9, ... Find t_{13}.
e 729, 243, 81, ... Find t_8. **f** 100 000, 10 000, 1000, ... Find t_{10}.
g 256, 384, 576, ... Find t_9. **h** 1024, −512, 256, ... Find t_{14}.

6 Find the 10th term, t_{10}, in the geometric sequence that starts at 6 and has a common ratio of 2.

7 The first term in a geometric sequence is 200 and the common ratio is 1.10. Find the 30th term, t_{30}, correct to 2 decimal places.

8 A sequence started at 1000 and each term was multiplied by 0.90 to make the next term. Find the 50th term, t_{50}, correct to 2 decimal places.

9 Give the rule for t_n for each of these geometric sequences.

a 9, 18, 36, 72, ... **b** 54, 18, 6, 2, ... **c** 7, −21, 63, −189, ...
d −3, 12, −48, 192, ... **e** 1, 1.1, 1.21, 1.331, ... **f** 1, 0.8, 0.64, 0.512, ...

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10 The rule for a geometric sequence is $t_n = 5 \times (3)^{n-1}$.

a Find t_8. **b** Find the position number of the term 98 415.

11 The zoom feature on a photocopy machine was set to 300%, making the new image 3 times larger than the original. A small photo, only 1 cm wide, of a person's face was photocopied. The new image was then photocopied to make a further enlargement, and the process was repeated.
Start with the original width and then list the widths of the next three images produced.

12 State, correct to 2 decimal places, the first four terms in each of these geometric sequences for the changes given.

a Starts at 100 and decreases by 5% **b** Starts at 100 and increases by 20%
c Starts at 5000 and increases by 3% **d** Starts at 7000 and decreases by 4%

13 Nick purchased his car for \$30 000 and expects it to depreciate in value by 15% each year. What will be the value of his car after 12 years?

14 Kathy invested \$60 000 so that it grows by 8% each year.

a What will her investment be worth at the start of the seventh year?
b When will the value of her investment have doubled?

15 The population of 2000 in a small country town is expected to decrease by 10% each year.

a State the first term, a, and the common ratio, r, for the geometric sequence of the population.
b Give the rule for the population at the start of each year.
c What will be the population at the start of the tenth year?
d After how many years will the population be less than 500?

16 Consider each sequence **a** to **f** below.

i State whether the sequence is arithmetic or geometric.
ii Decide whether the sequence is convergent, divergent or oscillating.
iii Match the sequence with its graph.

a 1, 2, 4, 8, . . . **b** 2, 4, 6, 8, . . . **c** 8, 4, 2, 1, . . .
d 8, 6, 4, 2, . . . **e** 1, −2, 4, −8, . . . **f** 8, −4, 2, −1, . . .

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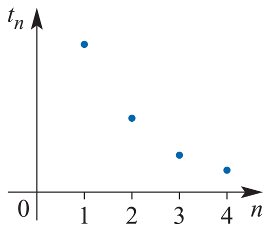
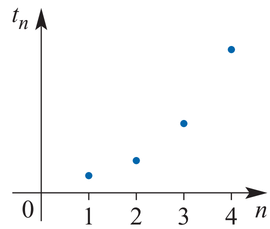
A



B



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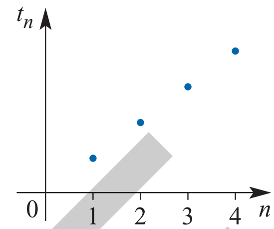

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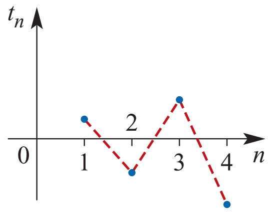
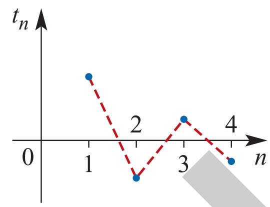
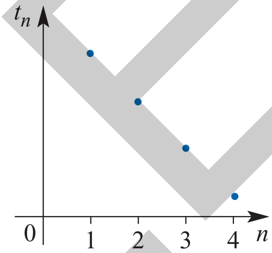


E



F



8.7 Geometric series

A **geometric series** is the sum of the terms in a geometric sequence. Here is a series in which the first five terms of a geometric sequence have been added:

$$3 + 6 + 12 + 24 + 48 = 93$$

Adding the terms of a geometric sequence

The sum of the first n terms of a geometric sequence can be written as:

$$S_n = a + ar + ar^2 + \ldots\ \ldots + ar^{n-3} + an^{n-2} + ar^{n-1} \qquad ①$$

Multiplying both sides of equation ① by r duplicates all but the first of the original terms.

$$rS_n = ar + ar^2 + ar^3 + \ldots\ \ldots + ar^{n-2} + ar^{n-1} + ar^n \qquad ②$$

From equation ① subtract equation ②. Notice that this eliminates the common terms, ar through to ar^{n-1}.

$$\begin{array}{rcl}
S_n &=& a + ar + ar^2 + \ldots \quad \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \qquad ① \\
rS_n &=& \quad ar + ar^2 + ar^3 + \ldots \quad \ldots \quad + ar^{n-2} + ar^{n-1} + ar^n \qquad ② \\
\hline
S_n - rS_n &=& a + 0 \ + 0 \ + 0 \ + \ldots + 0 \ + 0 \ + 0 \ - ar^n \qquad ① - ②
\end{array}$$

$$S_n(1 - r) = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

An alternative form of this rule can be made by multiplying the numerator and denominator by -1.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Some students prefer to use this form when $r > 1$ to avoid negative numbers in their calculation.

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Rule for finding the sum of the first n terms of a geometric sequence

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{when } r < 1$$

or

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{when } r > 1$$

where n = number of terms, a = first term and r = common ratio.

Check that the rule works by using it to find the sum of the terms in the geometric sequence given at the start of this section: $3 + 6 + 12 + 24 + 48$. Use $n = 5$, $a = 3$ and $r = 2$.

Here is an example where it is worthwhile to have a useful rule.

Example 17 Adding the terms of a geometric sequence

Find the sum of the first 20 terms of the geometric sequence: 2, 6, 18, 54, . . .

Strategy: We need to use the rule for S_n for $r > 1$.

Solution

1 The first term a is 2.
The number of terms to be added, n, is 20.
The common ratio r is 3.

$a = 2, \quad n = 20, \quad r = 3$

2 Substitute the values of a, n and r into the rule $S_n = \dfrac{a(r^{n-1})}{r-1}$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{20} = \frac{2(3^{20} - 1)}{3 - 1}$$

3 Use a calculator to evaluate S_{20}

$$= \frac{2(3^{20} - 1)}{2}$$

$$= 3^{20} - 1$$

$$= 3\,486\,784$$

$$\approx 3.487 \times 10^9$$

4 Write your answer.

The sum of the first 20 terms is 3 486 784 400 or about 3.487×10^9.

Example 18 Application of percentage change

Mahmoud started on a salary of $45 000 and receives a 5% increase each year.

a Calculate his salary in his fortieth year of employment.

b Find the total of his earnings over the 40-year period.

Solution

a

Strategy: We need to use the rule for t_n, the nth term of a geometric sequence.

1 The first term is $45 000. $a = 45\,000$

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2 A 5% increase means that each new term is 105% of the previous term. So $r = 1.05$.

$r = 1.05$

3 We need to find the 40th term.

$n = 40$

4 Substitute the values of a, r and n into the rule $t_n = ar^{n-1}$.

$t_n = ar^{n-1}$

$t_{40} = 45\,000 \times 1.05^{40-1}$

5 Use a calculator to evaluate t_{40}.

$= 45\,000 \times 1.05^{39}$

$= 301\,713.80$

6 Write your answer.

In his 40th year his salary will be \$301 713.80.

b

Strategy: We need to use the rule for S_n, the sum of the first n terms of a geometric sequence, with $r > 1$.

1 From part **a** we have the values of a, r and n.

$a = 45\,000, \quad r = 1.05, \quad n = 40$

2 Substitute the values of a, r and n into the rule $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{40} = \frac{45\,000(1.05^{40} - 1)}{1.05 - 1}$$

3 Use a calculator to evaluate S_{40}.

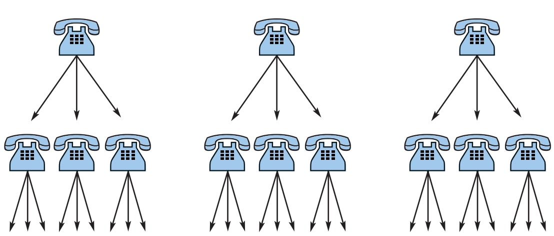
$= 5\,435\,989.84$

4 Write your answer.

His total earnings for the 40 years will be \$5 435 989.84.

Example 19 Application of a geometric series

On Monday, Rebecca rang three friends and invited them to her party. Each was told to invite three friends the next day and to tell them to continue the process. Telephone invitations for Monday and Tuesday are shown. Sunday was the last day on which people were invited.



a Write the sequence for the number of people invited on each day from Monday to Sunday.
b What is the total number of people invited to the party by Sunday?
c If the process continued for 2 weeks, how many would have been invited to the party?

Solution

a

1 This is a geometric sequence with $a = 3, r = 3$ and $n = 7$. — $a = 3,\ r = 3,\ n = 7$

2 Write the sequence. — 3, 9, 27, 81, 243, 729, 2187

b

1 The values of a, r and n are the same as in part **a**. — $a = 3,\ r = 3,\ n = 7$

2 Use the rule for the sum of the terms of a geometric sequence, for $r > 1$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_7 = \frac{3(3^7 - 1)}{3 - 1}$$

3 Evaluate S_7.

$$= 3279$$

4 Write your answer. — By Sunday, 3279 people were invited.

c

1 This time, a and r are the same but $n = 14$. — $a = 3, r = 3, n = 14$

2 Use the rule for the sum of the terms of a geometric sequence, for $r > 1$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{14} = \frac{3(3^{14} - 1)}{3 - 1}$$

3 Evaluate S_{14}.

$$= 7\,174\,452$$

4 Write your answer. — In 2 weeks, 7 174 452 people would be invited.

Exercise 8G

1 Consider the geometric sequence: 3, 12, 48, 192, . . .

 a State the values of a and r.

 b Find the sum of the first 10 terms using the appropriate rule for S_n.

2 Find S_{20}, the sum of the first 20 terms, of the geometric sequence. 6, 12, 24, 48, . . .

3 Find the sum of the required number of terms in each geometric sequence. Give decimal answers correct to 2 decimal places.

a 2, 6, 18, 54, . . .	Find S_{10}.	**b** 3, 6, 12, 24, . . .	Find S_{16}.
c 5, −10, 20, −40, . . .	Find S_{10}.	**d** 1, −3, 9, −27, . . .	Find S_{11}.
e 64, 32, 16, 8, . . .	Find S_{12}.	**f** 81, 54, 36, 24, . . .	Find S_{10}.
g 16, 24, 36, 54, . . .	Find S_{14}.	**h** 32, −48, 72, −108, . . .	Find S_{10}.
i 1, 1.1, 1.21, 1.331, . . .	Find S_{20}.	**j** 2, 1.8, 1.62, 1.458, . . .	Find S_{30}.

4 In her first year as a surveyor, Lee was paid \$50 000 and guaranteed a 6% increase each year.

 a How much would she be paid in her twentieth year of employment?

 b Find the total amount she could earn over a period of 20 years.

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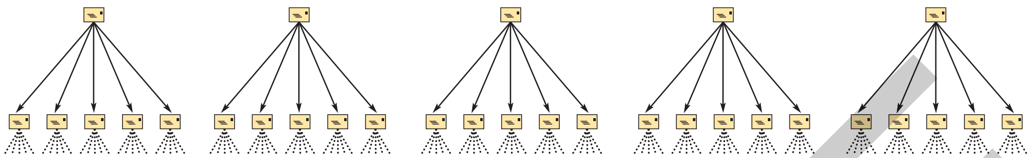
5 On the first day of the month, Dorian posted five letters to friends. Each was told to post 5 letters to their friends, with instructions to repeat the process.



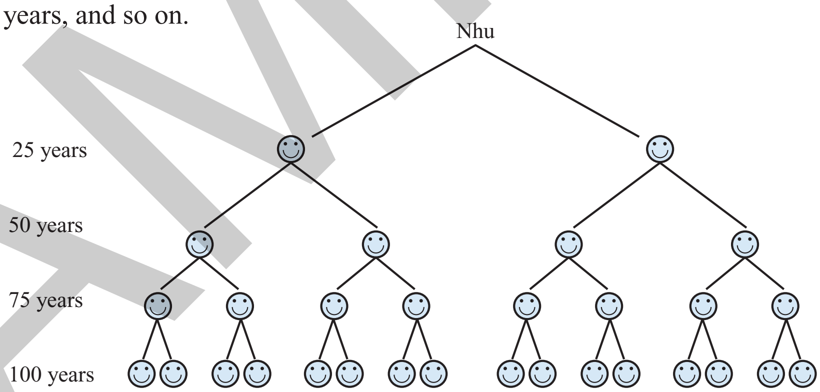
Assume that each letter posted is delivered the next day, and that each recipient posts their five letters immediately.

a Write the sequence for the number of letters posted on each of the first 4 days.
b How many letters were posted on the tenth day?
c What was the total number of letters posted over all of the 10 days?

6 A small company sold $200 000 worth of stock in its first year under a new management team. If the marketing manager predicts an annual growth in sales of 7% each year, calculate:

a the expected value of sales in the eighth year
b the total value of all sales in the first 8 years.

7 Suppose that Nhu has two children in the next 25 years. Assume that they each have two children in the following 25 years, and that all their descendants each have two children in the next 25 years, and so on.



a Write out the sequence for the number of new descendants in each 25 years for the first 100 years.
b What will be the total number of Nhu's descendants over the first 200 years?

8 A gambler's strategy is to place a first bet of $50 and to double the bet each time he loses.

a Write out the sequence of the bets he has made after four losses.
b What is the total of all his losses if he loses nine bets in a row?

9 There is a legend that the inventor of chess was asked by his king to name his own reward. The inventor asked for the amount of rice needed to put one grain on the first square of a chessboard, two grains on the next square, then four grains, and so on. The number of grains

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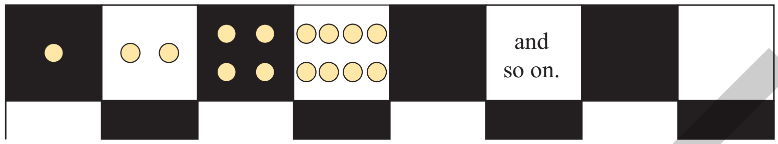
on each square was to be double the number on the previous square. There are 64 squares on a chessboard.



How many grains of rice did the inventor of chess ask for?

8.8 Infinite geometric series

Sum of an infinite geometric series

We have seen that the sum of the terms of a geometric sequence is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

However, when the size of r is less than 1, the r^n part of the rule tends towards zero as n becomes very large. For example, if $r = \frac{1}{2}$ and we wanted to find S_{30},

$$r^n = \left(\tfrac{1}{2}\right)^{30} \approx 0.000\,000\,000\,931.$$

This is very close to zero. If the series continued forever, we would call it an **infinite series**. In that case the number of terms, n, is infinitely large and it is reasonable to let the r^n part of the rule equal zero.

$$S_\infty = \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}$$

Rule for finding the sum of an infinite geometric series

The sum of an infinite geometric series is given by

$$\boldsymbol{S_\infty = \frac{a}{1 - r}} \qquad \text{for } |r| < 1$$

where a = first term and r = common ratio.

The symbol S_∞ means the sum of infinitely many terms: in other words, the sum of a series that goes on forever. Another way of saying that the size of r is less than 1 is to write $|r| < 1$ or $-1 < r < 1$.

Example 20 **Finding the sum of an infinite series**

Consider the series: $24 + 12 + 6 + \ldots$

a Find the sum of the first eight terms. **b** Find the infinite sum of the series.

Solution

a

1 The first term is 24. We are adding 8 terms. $a = 24, n = 8$

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2 Find the common ratio, r.

$$r = \frac{t_2}{t_1} = \frac{12}{24} = \frac{1}{2} \text{ or } 0.5$$

3 Substitute the values of a, n and r into the rule $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_8 = \frac{24(1 - 0.5^8)}{1 - 0.5}$$

4 Use a calculator to evaluate S_8.

$$= 47.8125$$

5 Write your answer.

The sum of the first eight terms is 47.8125.

b

1 We know the values of a and r from part **a**.

$$a = 24, r = 0.5$$

2 Substitute the values of a and r into the rule $S_\infty = \dfrac{a}{1 - r}$.

$$S_\infty = \frac{a}{1 - r}$$

$$= \frac{24}{1 - 0.5}$$

3 Evaluate S_∞. Use a calculator if you wish.

$$= 48$$

4 Write your answer.

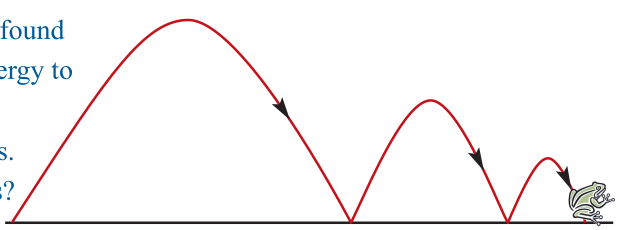
The infinite sum of the series is 48.

From the last example we see that the sum of the first eight terms of the series was 47.8125. However, if the addition of terms was carried on forever, the result would only increase to 48. The infinitely many terms become smaller and smaller such that they cannot make the total exceed 48.

The example illustrates the surprising mathematical truth that the sum of infinitely many finite quantities can be a **finite amount**. This seemingly paradoxical result underlies the famous **Zeno paradoxes**, which you may like to investigate further.

Example 21 Application of an infinite geometric series

A frog made a first jump of 81 cm, but found after each jump that he only had the energy to jump $\frac{2}{3}$ of the previous jump.



a List the lengths of his first five jumps.

b What is the sum of his first 10 jumps?

c If he could keep jumping forever, what would be the total for all of his jumps?

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Solution

a

1 The first jump was 81 cm.

$a = 81$

2 Multiply each jump by $\frac{2}{3}$ to make the size of the next jump.

$$81 \xrightarrow{\times\frac{2}{3}} 54 \xrightarrow{\times\frac{2}{3}} 36 \xrightarrow{\times\frac{2}{3}} 24 \xrightarrow{\times\frac{2}{3}} 16, \ldots$$

3 Write your answer.

The first five jumps are: 81, 54, 36, 24, 16 cm.

b

1 We know the values of a, r and n.

$a = 81,\ r = \frac{2}{3},\ n = 10$

2 Substitute the values of a, r and n into the rule $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{10} = \frac{81\left(1 - \left(\frac{2}{3}\right)^{10}\right)}{1 - \frac{2}{3}}$$

3 Use a calculator to evaluate S_{10}.
Remember: Put the divisor, $\left(1 - \frac{2}{3}\right)$, within brackets when using your calculator.

$$= 238.786\ldots$$

4 Write your answer.

He travelled about 239 cm in 10 jumps.

c

1 Write the values of a and r.

$a = 81,\ r = \frac{2}{3}$

2 Substitute the values of a and r into the rule $S_\infty = \dfrac{a}{1 - r}$.

$$S_\infty = \frac{a}{1 - r} = \frac{81}{1 - \frac{2}{3}}$$

3 Evaluate S_∞. Use a calculator if you wish.
Remember: Put the divisor, $\left(1 - \frac{2}{3}\right)$, within brackets When using your calculator.

$$= 243$$

4 Write your answer.

If he jumped forever, the total of his jumps would approach 243 cm.

Exercise 8H

1 Consider the geometric series: $54 + 18 + 6 + 2 + \ldots$

a State the values of a and r.

b Find the sum of the first seven terms, correct to 2 decimal places.

c Find the sum of the infinite geometric series.

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2 Find the sum of each of the following infinite geometric series. Write decimal answers correct to 2 decimal places.

a $72 + 36 + 18 + \ldots$ **b** $81 + 27 + 9 + \ldots$ **c** $27 + 18 + 12 + \ldots$
d $48 + 24 + 12 + \ldots$ **e** $320 + 80 + 20 + \ldots$ **f** $100 + 70 + 49 + \ldots$
g $400 + 360 + 324 + \ldots$ **h** $125 + 25 + 5 + \ldots$ **i** $321 + 32.1 + 3.21 + \ldots$

3 For the geometric series: $64 + 16 + 4 + 1 + \ldots$, find:

a S_5 **b** S_{10} **c** S_∞

4 A lava flow advanced 360 m in the first day, 240 m in the second day and 160 m in the third day. If the trend continued, would a house 1000 m from the start of the flow be safe? Give a reason for you answer.

5 When a stone is thrown so that it skips across a lake, each skip is 70% of the previous skip. Find the total distance of all the skips if the first skip was 3 m.

6 In the year after his fifteenth birthday, Omar grew 12 cm, then 6 cm in the following year and 3 cm in the next year.

a If the pattern continues, what is the maximum total amount Omar can expect to grow in the years after his fifteenth birthday?
b If Omar was 160 cm tall on his fifteenth birthday, what is the maximum that his height will approach?

7 A rubber tree yields 5 L of latex in its first year of harvesting. In each following year it only yields 90% of the previous year's harvest. Answer the following correct to 2 decimal places where necessary.

a List its yields for each of the first 4 years.
b What is the total yield over the first 4 years?
c Find the total yield for the first 20 years.
d If the rubber tree was harvested forever, what would be its total yield?
e After how many years will the annual yield be less than half the yield of the first year?

8.9 Difference equations

Difference equations provide instructions that can be used repeatedly to generate a sequence. This is another way of using **recursion**, the repeated use of an instruction, to generate a sequence.

For example, the arithmetic sequence: 8, 13, 18, 23, ... could be generated using the instructions:

Start at 8. Then repeatedly use, 'To make the next term, add 5 to the present term'.

The term $\boldsymbol{t_n}$ is used to stand for any term in the sequence. Its position depends on the number we use for n. So the term after t_n, in the $(n + 1)$th position, is called $\boldsymbol{t_{n+1}}$.

The instruction 'To make the next term, add 5 to the present term' can be written as $t_{n+1} = t_n + 5$.

A **difference equation** defines a sequence. It must include the equation showing how to make the next term, t_{n+1}, and give the first term, t_1.

The difference equation for the arithmetic sequence in this example is

$$t_{n+1} = t_n + 5, \text{ where } t_1 = 8$$

The geometric sequence: 4, 8, 16, 32, . . . results from the difference equation

$$t_{n+1} = 2t_n, \quad t_1 = 4$$

In the first example of this chapter we saw that the Fibonacci sequence: 1, 1, 2, 3, 5, . . . makes each new term by adding the previous two terms. Its difference equation is

$$t_{n+1} = t_n + t_{n-1}, \quad t_1 = 1, \quad t_2 = 1.$$

Some sequences are made by using a combination of operations, such as multiplication, addition or squaring.

Example 22 **Using a difference equation to generate a sequence**

Generate the first four terms of the sequence with the difference equation:

$$t_{n+1} = 2t_n + 3, \quad t_1 = 5$$

Solution

Method 1: Understanding how the difference equation tells us to make each new term

Strategy: The difference equation, $t_{n+1} = 2t_n + 3$, tells us that the next term is made by multiplying the previous term by 2 and then adding 3.

1 The first term is 5. — $t_1 = 5$

2 Make each new term by multiplying the previous term by 2 and then adding 3.

$$5, \xrightarrow{(2\times\ldots)+3} 13, \xrightarrow{(2\times\ldots)+3} 29, \xrightarrow{(2\times\ldots)+3} 61, \ \ldots$$

3 Write the first four terms. — The first four terms are: 5, 13, 29, 61.

Method 2: Substituting *n* values into the difference equation

1 We were given the first term. — $t_1 = 5$

2 Using $t_{n+1} = 2t_n + 3$, put $n = 1$. — $t_2 = 2t_1 + 3$
$= 2 \times 5 + 3 = 13$

3 Using $t_{n+1} = 2t_n + 3$, put $n = 2$. — $t_3 = 2t_2 + 3$
$= 2 \times 13 + 3 = 29$

4 Using $t_{n+1} = 2t_n + 3$, put $n = 3$. — $t_4 = 2t_3 + 3$
$= 2 \times 29 + 3 = 61$

5 Write the first four terms. — The first four terms are: 5, 13, 29, 61.

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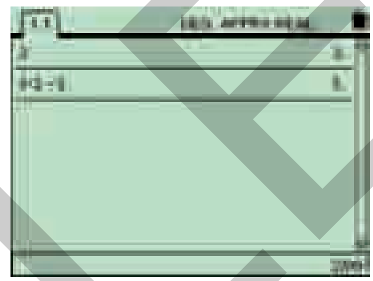
How to generate a sequence defined by a difference equation using the TI-Nspire CAS

Generate the first five terms of the sequence defined by the difference equation $t_{n+1} = 3t - 1$ where $t_1 = 2$.

Steps

1 Start a new document: Press ctrl + N.



2 Select **1:Add Calculator**.
Type in **2**, the value of the first term. Press enter. The calculator stores the value 2 as **Ans**wer. (You can't see this yet.)

3 Now type **3 * ans – 1** then press enter.
Keystrokes: 3 × ctrl (–) – 1 ,.
The second term in the sequence is **5**. This value is now stored as **Ans**.

4 Press enter. The second term in the sequence is **5**. This value is now stored as the new **Ans**wer.
Continue pressing enter until the required number of terms is generated.

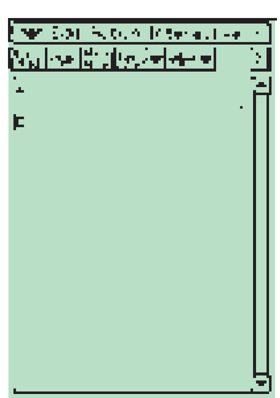
How to generate a sequence defined by a difference equation using the ClassPad

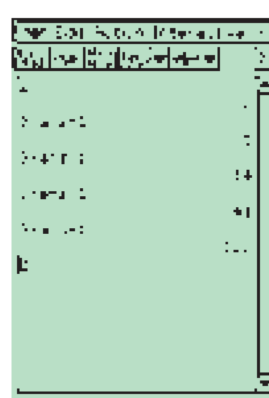
Generate the first five terms of the sequence defined by the difference equation $t_{n+1} = 3t - 1$ where $t_1 = 2$.

Steps

1 From the application menu screen, open the **Main** application, Main.

2 Starting with a clean screen, enter the value of the first term, **2**, and press EXE.
The calculator stores the value 2 as **ans**wer. (You can't see this yet.)





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3 Press Keyboard and locate the button ans. Enter **3 × ans − 1**. (See opposite.) **Then press** EXE.
The second term in the sequence, **5**, is generated and stored as the new **ans**wer.

4 Pressing EXE again will generate the next term in the sequence, **14**. Keep pressing EXE until the required number of terms is generated.

Example 23 Application of a difference equation

Gwen started with three plants of a particular species, and each following year she took one cutting from each plant to grow additional plants. To provide genetic diversity she also bought four similar plants from a nursery in the second and subsequent years. Assume that no plants die.

a Write the difference equation for generating the number of plants in each year.

b List the sequence for the number of plants in the first 4 years.

Solution

a

1 In the first year Gwen had 3 plants.	$t_1 = 3$
2 Taking a cutting from each plant to make a new plant doubles the number of plants.	So far, $t_{n+1} = 2t_n$ But . . .
3 Gwen also buys 4 new plants each year. So add 4.	Each year she buys another four plants. So, $t_{n+1} = 2t_n + 4$
4 Write the difference equation. Don't forget to state the first term, t_1.	$t_{n+1} = 2t_n + 4,\ t_1 = 3$

b

1 Start at 3.

$t_1 = 3$

2 To make the next term, double the previous term and add 4.

$(2 \times \ldots) + 4 \quad (2 \times \ldots) + 4 \quad (2 \times \ldots) + 4$

3, 10, 24, 52, ...

3 List the first four terms.

The first four terms are:

3, 10, 24, 52.

Exercise 8I

1 Use each difference equation to generate the first four terms of an arithmetic sequence.

a $t_{n+1} = t_n + 3$, where $t_1 = 7$
b $t_{n+1} = t_n - 6$, where $t_1 = 28$
c $t_{n+1} = t_n + 8$, where $t_1 = 19$
d $t_{n+1} = t_n - 13$, where $t_1 = 67$
e $t_{n+1} = t_n + 0.4$, where $t_1 = 9.8$
f $t_{n+1} = t_n - 1.7$, where $t_1 = 14.6$
g $t_{n+1} = t_n - \frac{1}{4}$, where $t_1 = \frac{1}{2}$
h $t_{n+1} = t_n + \frac{1}{3}$, where $t_1 = \frac{1}{6}$
i $t_{n+1} = t_n - 37$, where $t_1 = 106$
j $t_{n+1} = t_n + 1.5$, where $t_1 = -8$

2 Use each difference equation to generate the first four terms of a geometric sequence.

a $t_{n+1} = 3 \times t_n$, where $t_1 = 4$
b $t_{n+1} = -2 \times t_n$, where $t_1 = 3$
c $t_{n+1} = 2 \times t_n$, where $t_1 = -6$
d $t_{n+1} = 5 \times t_n$, where $t_1 = -1$
e $t_{n+1} = -3 \times t_n$, where $t_1 = 1$
f $t_{n+1} = \frac{1}{2} \times t_n$, where $t_1 = 24$
g $t_{n+1} = -\frac{1}{3} \times t_n$, where $t_1 = 54$
h $t_{n+1} = 0.2 \times t_n$, where $t_1 = 125$

3 Use the difference equations given below to find the first four terms of each sequence. Give decimal answers correct to 2 decimal places where necessary.

a $t_{n+1} = t_n + 3, \quad t_1 = 4$
b $t_{n+1} = 5t_n, \quad t_1 = 2$
c $t_{n+1} = 2t_n + 7, \quad t_1 = 6$
d $t_{n+1} = 3t_n - 1, \quad t_1 = 8$
e $t_{n+1} = t_n - 8, \quad t_1 = 100$
f $t_{n+1} = -2t_n, \quad t_1 = 1$
g $t_{n+1} = 1.10t_n - 300, \quad t_1 = 2000$
h $t_{n+1} = 1.05t_n + 500, \quad t_1 = 3000$
i $t_{n+1} = t_n^2, \quad t_1 = 3$
j $t_{n+1} = \sqrt{t_n}, \quad t_1 = 256$
k $t_{n+1} = 2t_n + 6, \quad t_1 = 4$
l $t_{n+1} = 2(t_n + 3), \quad t_1 = 4$
m $t_{n+1} = 5t_n + 10, \quad t_1 = 1$
n $t_{n+1} = 5(t_n + 2), \quad t_1 = 1$
o $t_{n+1} = t_n(t_n + 1), \quad t_1 = 1$
p $t_{n+1} = t_n(t_n - 1), \quad t_1 = 3$

4 Write the difference equation for each of the following instructions.

a Double the previous term, then add five. The first term is 3.
b Triple the previous term, then add seven. The first term is 4.
c Multiply the previous term by five, then add two. The first term is 1.
d Multiply the previous term by six, then subtract nine. The first term is 2.
e Square the previous term, then add two. The first term is 1.

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5 Find the first four terms for each of the sequences described in Question **4**.

6 Use your graphics calculator to find the required term for each sequence with the difference equation given. Give answers correct to 2 decimal places where necessary.

Hint: See 'How to use difference equations to generate recursive sequences on a graphics calculator' (page 340).

a $t_{n+1} = t_n + 5, t_1 = 2$ Find t_{12}.
b $t_{n+1} = t_n + 7, t_1 = 4,$ Find t_{10}.
c $t_{n+1} = 4t_n, t_1 = 3$ Find t_9.
d $t_{n+1} = 3t_n, t_1 = 1$ Find t_{11}.
e $t_{n+1} = 2t_n - 5, t_1 = 1$ Find t_{10}.
f $t_{n+1} = 5t_n + 4, t_1 = 2$ Find t_8.
g $t_{n+1} = 1.06t_n + 200, t_1 = 1000$ Find t_{12}.
h $t_{n+1} = 1.08t_n - 150, t_1 = 5000$ Find t_{10}.
i $t_{n+1} = t_n(t_n + 1), t_1 = 2$ Find t_5.
j $t_{n+1} = 2t_n(t_n - 10), t_1 = 11$ Find t_4.

7 Ken purchases 10 trout for his dam. He estimates that the number of trout will triple during each following year. During the second and each following year, he hopes to catch seven of the fish.

a Give the difference equation for the expected number of fish in the dam at the end of each year.
b List the sequence for the fish numbers for the first 5 years.

8 Steven invested \$10 000 at 5% interest per year. At the start of each following year the previous amount in the account was multiplied by 1.05, to allow for the 5% increase because of the interest paid, and he deposited an extra \$2000.

a Give the difference equation for the sequence showing the amount in his account each year.
b List the amounts for the first 4 years.
c Use your graphics calculator to find the amount in his account at the end of the tenth year.

9 When Aida retires she plans to have \$800 000 invested at 6% per year interest. She intends to withdraw \$70 000 each year after yearly interest has been added to her account. The money withdrawn will be used for living expenses during the year.

Hint: The previous amount in her account is multiplied by 1.06 to allow for the 6% increase due to the interest payment, then \$70 000 is subtracted.

a Give the difference equation for the amount in Aida's account at the start of each year.
b List the amounts in Aida's account at the start of each of the first 4 years.
c How much will be in Aida's account in the twentieth year?

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Review

Key ideas and chapter summary

Sequence

A sequence is a list of numbers in a particular order.

Arithmetic sequence

A **recursive** sequence is generated by the repeated application of a rule. In an **arithmetic sequence**, each new term is made by adding a fixed number, called the common difference, d, to the previous term.

Example: 3, 5, 7, 9, . . . is made by adding 2 to each term.

The **common difference**, d is found by getting any term and subtracting its previous term, e.g. $t_2 - t_1$.

In our example above, $d = 5 - 3 = 2$.

Difference equation for an arithmetic sequence:

$$t_{n+1} = t_n + d, \quad t_1 = a$$

where d = common difference and a = first term.

In our example: $t_{n+1} = t_n + 2$, $t_1 = 3$

Rule for finding t_n, **the nth term** in an arithmetic sequence:

$$t_n = a + (n - 1)d$$

To find t_{10} in our example: put $n = 10$, $a = 3$, $d = 2$

$$\begin{aligned} t_{10} &= 3 + (10 - 1) \times 2 \\ &= 3 + (9) \times 2 = 21 \end{aligned}$$

The graph of an arithmetic sequence:

- is a straight line
- diverges away from its starting value
- has a **positive slope** (rises) when $d > 0$
- has a **negative slope** (falls) when $d < 0$.

Arithmetic series

An arithmetic series is the sum of the terms in an arithmetic sequence.

Example: $3 + 5 + 7 + 9$ is the sum of the first four terms of an arithmetic sequence

Rule for finding S_n, **the sum of the first n terms** in an arithmetic sequence:

$$S_n = \frac{n}{2}[2a + (n - 1)d] \text{ when the last term is not given}$$

or

$$S_n = \frac{n}{2}(a + l)$$

where n = number of terms, a = first term and l = last term (or nth) term.

Using the S_n rule to find the sum of the first ten terms in our example:

$$n = 10, a = 3, d = 2$$
$$S_n = \frac{10}{2}[2 \times 3 + (10 - 1) \times 2]$$
$$S_n = 5[6 + 9 \times 2] = 120$$

Rule for finding n, the number of terms in an arithmetic series, given the last term:

$$n = \frac{l - a}{d} + 1$$

where l = last term, a = first term and d = common difference.

The number of terms in a finite arithmetic sequence can also be found by generating the sequence using a graphics calculator.

Geometric sequence

In a **geometric sequence**, each term is made by multiplying the previous term by a fixed number, called the common ratio, r.

Example: 5, 20, 80, 320, . . . is made by multiplying each term by 4

The **common ratio**, r, is found by dividing any term by its previous term, e.g. $\frac{t_2}{t_1}$.

In our example: $r = \frac{20}{5} = 4$

Difference equation for a geometric sequence:

$$t_{n+1} = r \times t_n, \; t_1 = a$$

where r = common ratio and a = first term.

In our example: $t_{n+1} = 4 \times t_n, \; t_1 = 5$

Rule for finding $\boldsymbol{t_n}$**, the** $\boldsymbol{n}$**th term** in a geometric sequence:

$$t_n = ar^{n-1}$$

where a = first term and r = common ratio.

To find t_7 in our example: put $n = 7, a = 5, r = 4$

$$t_7 = 5 \times (4)^{7-1}$$
$$= 5 \times (4)^6 = 20\,480$$

The graph of a geometric sequence:

- diverges when $r < -1$ or $r > 1$
- converges when $-1 < r < 1$
- oscillates when $r < 0$
- curves when $r > 0$.

Recurring **percentage change** is an application of a geometric sequence.

Geometric series

A geometric series is the sum of the terms in a geometric sequence.

Example: $5 + 20 + 80 + 320$ is the sum of the first four terms of a geometric sequence.

Rule for finding S_n, **the sum of the first n terms** in a geometric sequence:

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{when } r < 1$$

or

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{when } r > 1$$

where n = number of terms, a = first term and r = common ratio.

Using our example to find the sum of the first 6 terms:

$$n = 6, a = 5, r = 4$$
$$S_6 = \frac{5(4^6 - 1)}{4 - 1} = 6825$$

Infinite geometric series

Rule for finding S_∞, **the sum of a never-ending geometric sequence** with the size of r less than 1:

$$S_\infty = \frac{a}{1 - r} \quad \text{for } |r| < 1$$

where a = first term and r = common ratio.

Example:

$$8 + 4 + 2 + 1 + \ldots$$
$$a = 8, r = \tfrac{1}{2}$$
$$S_\infty = \frac{8}{1 - \frac{1}{2}} = 16$$

Convergent sequence

The terms get closer and closer to a particular value.

For example: 1, 0.1, 0.01, 0.001, 0.0001, ... gets closer to zero.

Divergent sequence

The terms get further and further away from the first term.

For example: 1, 2, 4, 8, ...

Oscillating sequence The terms oscillate or bounce on either side of the value they are approaching.
For example: 8, −4, 2, −1, . . . oscillates either side of zero.

Difference equation A difference equation defines a sequence. It must include the equation showing how to make the next term, t_{n+1}, and the first term, t_1.
Example: $t_{n+1} = 5t_n + 2$, $t_1 = 4$ tells us to multiply each term by 5 then add 2 to make each new term. Start at 4.

4, 22, 112, 562, . . .

Skills check

Having completed the current chapter you should be able to:

- decide whether a sequence is arithmetic, geometric or neither
- give the first term, a, and find the common difference, d, of an arithmetic sequence
- give the first term, a, and find the common ratio, r, of a geometric sequence
- generate a sequence using its difference equation
- find the nth term of an arithmetic or geometric sequence when given a few terms in the sequence
- find the sum of the first n terms in an arithmetic or geometric sequence
- find the infinite sum of a geometric sequence when the size of the common ratio r is less than 1
- generate the terms of a sequence using a graphics calculator
- calculate the common ratio, r, required for a given percentage change
- solve application problems involving sequences and series.

Multiple-choice questions

1 In the sequence: 1, 4, 7, 10, 13, . . ., the 4th term is:
A 4 **B** 10 **C** 3 **D** 1 **E** 2

2 Which of the following is an arithmetic sequence?
A 2, 4, 8, . . . **B** 2, 6, 18, . . . **C** 2, 4, 6, . . . **D** 2, 3, 5, . . . **E** 2, 4, 7, . . .

3 In the sequence: 27, 19, 11, 3, . . ., the value of the common difference, d, is:
A 8 **B** 3 **C** −8 **D** −24 **E** −5

4 In the sequence: 63, 56, 49, 42, . . ., the 15th term is:
A −35 **B** 161 **C** 15 **D** 168 **E** −42

5 Using the difference equation $t_{n+1} = t_n + 6$, $t_1 = 5$, the 7th term would be:
A 11 **B** 41 **C** 18 **D** 47 **E** 13

6 The sum of the first 30 terms of the sequence: 7, 11, 15, 19, . . . is:
A 390 **B** 555 **C** 3900 **D** 1950 **E** 2010

7 The common ratio, r, of the sequence: 27, 18, 12, 8, . . . is:
A $\frac{3}{2}$ **B** $\frac{2}{3}$ **C** −9 **D** $\frac{3}{4}$ **E** $\frac{1}{3}$

8 Which of the following sequences is geometric?
A 2, 6, 10, 14, . . . **B** 2, 6, 12, 24, . . . **C** 54, 18, 6, 2, . . .
D 54, 27, 9, 3, . . . **E** 1, 3, 9, 18, . . .

9 The difference equation for the sequence: 3, 6, 12, 24, . . . is:
A $t_{n+1} = t_n + 3$, $t_1 = 3$ **B** $t_{n+1} = 3t_n$, $t_1 = 3$ **C** $t_{n+1} = 2t_n$, $t_1 = 3$
D $t_{n+1} = 3t_n$, $t_1 = 2$ **E** $t_{n+1} = 3t_n + 6$, $t_1 = 2$

10 The 20th term, t_{20}, of the sequence: 3, −6, 12, −24, . . . is:
A −177 **B** −168 **C** −3 145 728
D 3 145 728 **E** −1 572 864

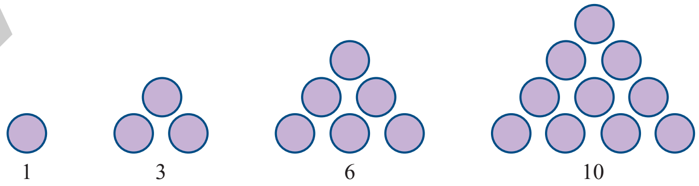
11 A 7% increase is made by using a common ratio of:
A 7 **B** 0.07 **C** 1.7 **D** 1.07 **E** 107

12 Which of the following sequences is convergent and oscillating?
A 9, 3, 1, $\frac{1}{3}$, . . . **B** $\frac{1}{3}$, 1, 3, 9, . . . **C** 9, 3, −3, −9, . . .
D 9, −3, 1, $-\frac{1}{3}$, **E** −9, −3, 3, 9, . . .

13 For the sequence: 3, 6, 12, 24, . . ., the sum of the first 15 terms is:
A −98 301 **B** 98 301 **C** −255 **D** 255 **E** 270

14 The sum of the infinite geometric sequence: 64 + 48 + 36 + 27 + . . . is:
A 128 **B** 256 **C** 512 **D** 1 024 **E** 2 048

15 Triangular numbers are based on the pattern:



The first four triangular numbers are 1, 3, 6, 10. The sixth triangular number is:
A 21 **B** 18 **C** 15 **D** 36 **E** 14

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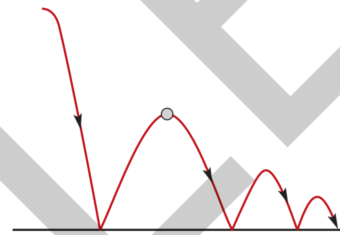
Short-answer questions

1 Find t_{20}, the 20th term in the sequence: 7, 11, 15, 19, . . .

2 Microwave cooking instructions for heating muffins say to heat one muffin for 45 seconds and to allow another 30 seconds for each extra muffin. State the heating times for 1, 2, 3 and 4 muffins.

3 Find t_{10}, the 10th term in the sequence: 3, 6, 12, . . .

4 A basketball was dropped from a height of 48 m. After each bounce it reached only half of the previous height.
Starting with the height of 48 m, list the next three heights that it reached after each bounce.



5 Find the sum of the first 12 terms in the sequence: 100, 90, 81, . . . Answer correct to 2 decimal places.

6 Find the sum of the first 20 terms in an arithmetic sequence with a first term of 6 and a last term of 82.

7 Find t_5, the 5th term, of the sequence generated by the difference equation
$t_{n+1} = 2t_n, \quad t_1 = 7.$

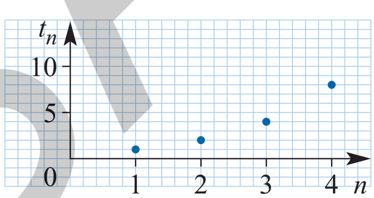
8 The terms of a sequence start at 1000 and each new term is 8% less than the previous term. Find the 6th term, correct to 2 decimal places.

9 Find the sum of the infinite geometric sequence: 24, −12, 6, −3, . . .

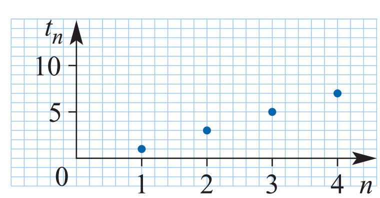
10 Match the descriptions given in **a** to **d** with the graphs of the sequences shown.

a Arithmetic with $d = 2$ **b** Arithmetic with $d = -2$
c Geometric with $r = 2$ **d** Geometric with $r = -2$
e Geometric with $r = \frac{1}{2}$ **f** Geometric with $r = -\frac{1}{2}$

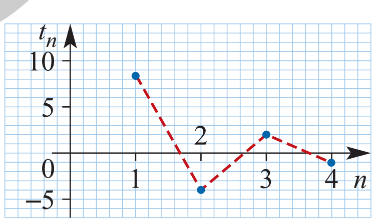
A



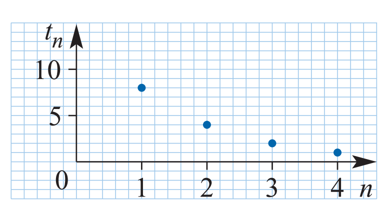
B



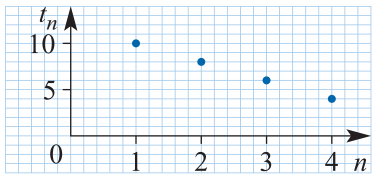
C



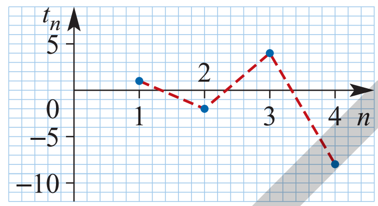
D



A



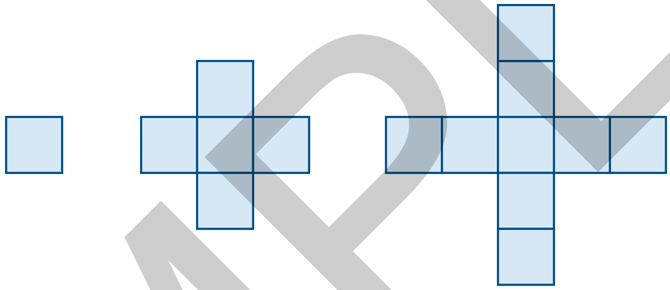
B



Extended-response questions

1 **a** How many squares are in the tenth shape in the sequence below?

b What is the position in the sequence for the shape with 81 squares?



2 The number of bacteria in a colony doubles each day. On the first day there were 300 bacteria.

a State the value of the first term, a, and the common ratio, r.

b How many bacteria will there be on the eighth day?

c On which day will the number of bacteria be 614 400?

d When will the number of bacteria exceed 100 million?

3 Margaret started working for a company on an annual salary of \$65 000 with a guaranteed increase of 9% each year.

a Give the value of the first term, a, and the common ratio, r, for the sequence of her salaries each year.

b Find Margaret's salary for the twenty-fifth year working with the company.

c What is the total amount she will have earned after working for the company for 25 years?

d In which year will her salary be \$100 010.56?

e Find the year when her salary will exceed \$200 000.

4 A snail, starting from the bottom of a drainpipe 250 cm tall, climbs 64 cm during the first day, 48 cm the next day and 36 cm the following day. Assuming that this pattern continues, answer the following.

a Find the common ratio for the sequence of distances travelled.

b How far will the snail have climbed after 6 days?

c Will the snail ever reach the top of the drainpipe? Give a reason for your answer.

5 A skyrocket was fired vertically. It climbed 53.9 m in the first second, 44.1 m in the next second and 34.3 m in the third second. Using this as the basis for the sequence of distances climbed each second, answer the following.

a How far did the skyrocket travel in the fifth second?

b What was the height of the skyrocket after 5 seconds?

c Make up a table for the height of the skyrocket after each second.

d What was the greatest height reached by the skyrocket?

e When did the skyrocket reach its greatest height?

f When did the skyrocket hit the ground?

Review