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Backward stochastic differential equations with Young drift

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Abstract We show the well-posedness of backward stochastic differential equations containing an additional drift driven by a path of finite q-variation with $q \in [1, 2)$. In contrast to previous work, we apply a direct fixpoint argument and do not rely on any type of flow decomposition. The resulting object is an effective tool to study semilinear rough partial differential equations via a Feynman–Kac type representation.

Keywords Rough paths theory · Young integration · BSDE · rough PDE

Introduction

Stochastic differential equations (SDEs) driven by Brownian motion W and an additional deterministic path η of low regularity (so called "mixed SDEs") have been well-studied. In (Guerra and Nualart 2008), the well-posedness of such SDEs is established if η has finite q-variation with $q \in [1, 2)$. [1] The integral with respect to the latter is handled via fractional calculus. Independently, in (Diehl 2012) the same problem is studied using Young integration for the integral with respect to η. Interestingly, both approaches need to establish (unique) existence of solutions via the Yamada–Watanabe theorem. A direct proof using a contraction argument is not obvious to implement.

[1] See Section "Appendix - Young integration" for background on the variation norm and Young integration.

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For paths of q-variation with $q \in (2, 3)$, integration has to be dealt with via the theory of rough paths. Motivated by a problem in stochastic filtering, (Dan and et al 2013) gives a formal meaning to the mixed SDE by using a flow decomposition which separates the stochastic integration from the deterministic rough path integration. It is not shown that the resulting object actually satisfies any integral equation.

In (Diehl et al. 2015), well-posedness of the corresponding mixed SDE is established by first constructing a joint rough path "above" W and η. The deterministic theory of rough paths then allows mixed SDEs to be solved. The main difficulty in that work is the proof of exponential integrability of the resulting process, which is needed for applications. In (Diehl et al. 2014), these results have been used to study linear "rough" partial differential equations via Feyman-Kac formulae.

Backward stochastic differential equations (BSDEs) were introduced by Bismut in 1973. In (Bismut 1973), he applied linear BSDEs to stochastic optimal control. In 1990, Pardoux and Peng (Pardoux and Peng 1990) then considered non-linear equations. A solution to a BSDE with driver f and random variable $\xi \in L^2(\mathcal{F}_T)$ is an adapted pair of processes (Y, Z) in suitable spaces, satisfying

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)ds - \int_t^T Z_r dW_r, \quad t \leq T.$$

Under appropriate conditions on f and ξ, they showed the existence of a unique solution to such an equation. One important use for BSDEs is their application to semilinear partial differential equations. This "nonlinear Feynman–Kac" formula is, for example, studied in (Pardoux and Peng 1992).

In this work, we are interested in showing well-posedness of the following equation

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(Y_r)d\eta_r - \int_t^T Z_r dW_r. \tag{1}$$

Here W is a multidimensional Brownian motion, η is a multidimensional (deterministic) path of finite q-variation, $q \in [1, 2)$, and ξ is a bounded random variable, measurable at time T.

Such equations have previously been studied in (Diehl and Friz 2012). In that work η is even allowed to be a rough path, i.e., every $q \geq 1$ is feasible. The drawback of that approach is that no intrinsic meaning is given to the equation. Indeed a solution to (1) is only defined as the limit of smooth approximations, which is shown to exist using a flow decomposition. In the current work we solve (1) directly via a fixpoint argument. The resulting object solves the integral equation, where the integral with respect to η is a pathwise Young integral. In Section "Main result", we state and prove this main result.

It is well-known that BSDEs provide a stochastic representation for solutions to semi-linear parabolic partial differential equations (PDEs), in what is sometimes called the "nonlinear Feynman–Kac formula" (Pardoux and Peng 1992). In Section "Application to rough PDEs", we extend this representation to rough PDEs; thereby giving a novel and short proof for their well-posedness in the Young regime.

In Section "Appendix - Young integration", we recall the notions of p-variation and Young integration.

Main result

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a stochastic basis, where $\mathcal{F}_t$ is the usual filtration of a standard d-dimensional Brownian motion W. Denote by $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ the conditional expectation at time t.

We shall need the following spaces.

Definition 1 *For $p > 2$, define $\mathcal{B}_p$ to be the space of adapted process $Y : \Omega \times [0, T] \to \mathbb{R}$ with*[2]

$$\|Y\|_{p,2} := \operatorname*{ess\,sup}_{t,\omega} \mathbb{E}_t\left[\|Y\|^2_{\text{p-var};[t,T]}\right]^{1/2} + \operatorname*{ess\,sup}_{\omega}|Y_T| < \infty.$$

Denote by BMO *the space of all progressively measurable $Z : \Omega \times [0, T] \to \mathbb{R}^d$ with*

$$\|Z\|_{\mathsf{BMO}} := \operatorname*{ess\,sup}_{t,\omega}\mathbb{E}_t\left[\int_t^T |Z_r|^2 dr\right] < \infty.$$

Theorem 2 *Let $T > 0$, $\xi \in L^\infty(\mathcal{F}_T)$, $q \in [1, 2)$ and $\eta \in C^{0,\text{q-var}}([0, T], \mathbb{R}^e)$. Assume $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, satisfies for some $C_f > 0$, $\mathbb{P}-a.s.$,*

$$\sup_{t\in[0,T]} |f(t, 0, 0)| < C_f$$
$$|f(t, y, z) - f(t, y', z')| \le C_f \left(|y - y'| + |z - z'|\right).$$

Let $g_1, \ldots, g_e \in C^2_b(\mathbb{R})$. Let $p > 2$ such that $1/p + 1/q > 1$.

(i) There exists a unique $Y \in \mathcal{B}_p$, $Z \in$ BMO such that

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(Y_r)d\eta_r - \int_t^T Z_r dW_r, \tag{2}$$

where the $d\eta$ integral is a well-defined (pathwise) Young integral.

(ii) If, for i = 1, 2,

$$Y^i_t = \xi_i + \int_t^T f_i\left(s, Y^i_s, Z^i_s\right) ds + \int_t^T g(Y_s)d\eta_s - \int_t^T Z^i_s dW_s,$$

and $\xi_1 \le \xi_2$, $f_1 \le f_2$, then $Y^1 \le Y^2$.

(iii) The solution mapping

$$L^\infty(\mathcal{F}_T) \times C^{\text{q-var}}([0, T], \mathbb{R}^e) \to \mathcal{B}_p \times \mathsf{BMO}$$
$$(\xi, \eta) \mapsto (Y, Z),$$

is locally uniformly continuous.

[2] The space $C^{0,\text{p-var}}([t, T], \mathbb{R})$ and the norm $||\cdot||_{\text{p-var};[t,T]}$ are reviewed in Section "Appendix - Young integration".

(iv) Fixing f, g there exists for every $M > 0$ a $C(M) > 0$ such that for $\xi, \xi' \in \mathcal{F}_T$ with $||\xi||_\infty, ||\xi'||_\infty, ||\eta||_{\text{q-var};[0,T]} < M$ we have for the corresponding solutions $(Y, Z), (Y', Z')$

$$\left|Y_0 - Y_0'\right| \leq C(M)\mathbb{E}\left[\left|\xi - \xi'\right|^2\right]^{1/2}.$$

Remark 1 *The refined continuity statement in (iv) will be important for our application to rough PDEs in Section "Application to rough PDEs".*

Remark 2 *Note that the coefficient g preceding the Young path is not allowed to depend on Z. This stems from the fact that we want this integral to be a well-defined Young integral, and Z, in general, does not possess enough regularity for this (a priori, it is only known to be predictable and square integrable).*

In special cases, it turns out that the classical BSDE (without the Young integral) is solved with a Z that is a quite regular path in time, and one could hope for something similar for the "rough" BSDE.

Since we do not want to impose such regularity constraints, which would either involve a Markovian setting with smooth coefficients or a study of Malliavin differentiability, we do not pursue this direction.

Remark 3 *The use of the space of essentially bounded processes Y and BMO processes Z (Definition 1) is essential for our proof.*

In classical BSDE theory, these spaces usually only appear when studying equations with a driver f that is quadratic in z and with a bounded terminal condition.

The f we consider is Lipschitz, so our need for these spaces stems from the interplay with the Young integral.

Indeed, the map $Y \mapsto g(Y)$ is only locally Lipschitz in p-variation norm (Lemma 2), which, in general, presents a problem when trying to close estimates involving the expectation of the processes under consideration. Here the fact that we have a bound on the essential supremum of Y comes to the rescue, as it allows us to pull a term out of the expectation, see (7). This explains the norm for Y.

In order to bound the p-variation norm of the stochastic integral, we apply the conditional version of the Burkholder–Davis–Gundy inequality for p-variation, see (5). This explains the use of the BMO norm for Z.

Note that this in stark contrast to the theory of SDEs, where one, in general, does not have a handle on the essential supremum of solutions. Hence, as explained in the introduction, for SDEs with a Young drift, a fixpoint procedure has not yet been established. On the other hand, it is not clear how to treat the BSDEs under consideration here using classical L^2-type theory with possibly unbounded terminal condition.

Interestingly, the flow decomposition used in (Diehl and Friz 2012) *leads to a transformed BSDE that is quadratic in Z. Hence, also there the terminal condition needs to be bounded.*

Proof For $R > 0$ define

$$B(R) := \left\{(Y, Z) : \|Y\|_{p,2} < R, \|Z\|_{\mathsf{BMO}} < R\right\}.$$

For $Y \in \mathcal{B}_p$, $Z \in \mathsf{BMO}$ define $\Phi(Y, Z) := (\tilde{Y}, \tilde{Z})$, where

$$\tilde{Y}_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(Y_r)d\eta_r - \int_t^T \tilde{Z}_r dW_r.$$

This is well-defined, as is standard in the BSDE literature (see, for example, (Pardoux and Peng 1990)), by setting

$$\tilde{Y}_t := \mathbb{E}_t\left[\xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(Y_r)d\eta_r\right],$$

and letting $\tilde{Z}$ be the integrand in the Itô representation of the martingale

$$\tilde{Y}_t + \int_0^t f(r, Y_r, Z_r)dr + \int_0^t g(Y_r)d\eta_r.$$

In what follows, $A \lesssim B$ means there exists a constant $C > 0$ that is independent of η, ξ such that $A \leq CB$. The constant is bounded for $||g||_{C_b^2}$, C_f bounded.

Unique existence on small interval

We first show that for a $T > 0$ small enough, Φ leaves a ball invariant, i.e., for a T small enough and R large enough

$$\Phi(B(R)) \subset B(R).$$

Let $(\tilde{Y}, \tilde{Z}) = \Phi(Y, Z)$, then

$$\begin{aligned}
\left\|\int_t^\cdot f(r, Y_r, Z_r)dr\right\|_{\text{p-var};[t,T]} &\leq \left\|\int_t^\cdot f(r, Y_r, Z_r)dr\right\|_{\text{1-var};[t,T]} \\
&\leq \int_t^T |f(r, Y_r, Z_r)|dr \\
&\lesssim \int_t^T |f(r, 0, 0)|dr + T\|Y\|_{\infty;[t,T]} + \int_t^T |Z_r|dr \\
&\lesssim T + T\|Y\|_{\text{p-var};[t,T]} + T|Y_T| + \int_t^T |Z_r|dr.
\end{aligned} \tag{3}$$

Using the Young estimate (Theorem 5 in the Appendix), we estimate

$$\begin{aligned}
\left\|\int_t^\cdot g(Y_r)d\eta_r\right\|_{\text{p-var};[t,T]} &\leq \left\|\int_t^\cdot g(Y_r)d\eta_r\right\|_{\text{q-var};[t,T]} \\
&\lesssim \left(1 + \|Y\|_{\text{p-var};[t,T]}\right)\|\eta\|_{\text{q-var};[t,T]}.
\end{aligned} \tag{4}$$

The Burkholder–Davis–Gundy inequality for p-variation (Friz and Victoir 2010, Theorem 14.12) gives

$$\mathbb{E}_t\left[\left\|\int_t^\cdot \tilde{Z}_r dW_r\right\|^2_{\text{p-var};[t,T]}\right] \lesssim \mathbb{E}_t\left[\int_t^T |\tilde{Z}_r|^2 dr\right]. \tag{5}$$

Now the $d\eta$ integral satisfies the usual product rule, so with Itô's formula we get

$$\tilde{Y}_t^2 = \xi^2 + 2\int_t^T f(r, Y_r, Z_r)\tilde{Y}_r dr + 2\int_t^T g(Y_r)\tilde{Y}_r d\eta_r - \int_t^T 2\tilde{Y}_r \tilde{Z}_s dW_r - \int_t^T |\tilde{Z}_r|^2 dr.$$

By Lemma 2 (refer again to the Appendix)

$$\begin{aligned}
\|g(Y)\tilde{Y}\|_{\text{p-var};[t,T]} &\leq \|g\|_\infty \|\tilde{Y}\|_{\text{p-var};[t,T]} + \|g(Y)\|_{\text{p-var};[t,T]} \|\tilde{Y}\|_{\infty;[t,T]} \\
&\leq \|g\|_\infty \|\tilde{Y}\|_{\text{p-var};[t,T]} \\
&\quad + \|Dg\|_\infty \|Y\|_{\text{p-var};[t,T]} \left(\|\tilde{Y}\|_{\text{p-var};[t,T]} + |Y_T| \right) \\
&\lesssim \|\tilde{Y}\|_{\text{p-var};[t,T]} + \|\tilde{Y}\|^2_{\text{p-var};[t,T]} + R^2.
\end{aligned}$$

Taking the conditional expectation, we get

$$\begin{aligned}
\mathbb{E}_t\left[\tilde{Y}_t^2\right] + \mathbb{E}_t\left[\int_t^T |\tilde{Z}_s|^2 ds\right] &\lesssim \mathbb{E}_t\left[\xi^2\right] + \mathbb{E}_t\left[\int_t^T (|f(r,0,0)| + |Y_r| + |Z_r|)\,|\tilde{Y}_r| dr\right] \\
&\quad + \|\eta\|_{\text{q-var}}\left(1 + \mathbb{E}_t\left[\|\tilde{Y}\|_{\text{p-var}} + \|\tilde{Y}\|^2_{\text{p-var}}\right] + R^2\right).
\end{aligned} \tag{6}$$

Now

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^T (|f(r,0,0)| + |Y_r| + |Z_r|)\,|\tilde{Y}_r| dr\right] \\
&\lesssim \mathbb{E}_t\left[\int_t^T |f(r,0,0)|^2 + |Y_r|^2 + |Z_r|^2 + |\tilde{Y}_r|^2 dr\right] \\
&\lesssim \mathbb{E}_t\left[\int_t^T |f(r,0,0)|^2 dr\right] + T\mathbb{E}_t\left[\|Y\|^2_\infty\right] + \mathbb{E}_t\left[\int_t^T |Z_r|^2 dr\right] + T\mathbb{E}_t\left[\|\tilde{Y}\|^2_\infty\right] \\
&\lesssim 1 + T\mathbb{E}_t\left[\|Y\|^2_{\text{p-var}} + |Y_T|^2\right] + R + T\mathbb{E}_t\left[\|\tilde{Y}\|^2_{\text{p-var}} + |\tilde{Y}_T|^2\right] \\
&\lesssim 1 + TR^2 + T\mathbb{E}_t\left[\xi^2\right] + R + T\mathbb{E}_t\left[\|\tilde{Y}\|^2_{\text{p-var}}\right] + T\mathbb{E}_t\left[\xi^2\right].
\end{aligned}$$

We trivially estimate

$$\begin{aligned}
\mathbb{E}_t\left[\|\tilde{Y}\|^2_{\text{p-var};[t,T]}\right] &\lesssim \mathbb{E}_t\left[\left\|\int_t^\cdot f(r, Y_r, Z_r) dr\right\|^2_{\text{p-var};[t,T]}\right] \\
&\quad + \mathbb{E}_t\left[\left\|\int_t^\cdot g(Y_r) d\eta_r\right\|^2_{\text{p-var};[t,T]}\right] \\
&\quad + \mathbb{E}_t\left[\left\|\int_t^\cdot \tilde{Z}_r dW_r\right\|^2_{\text{p-var};[t,T]}\right],
\end{aligned}$$

which we can bound, using (3), (4), and (5), by a constant times

$$
\begin{aligned}
&T^2 + T^2\mathbb{E}_t\left[\|Y\|^2_{\text{p-var};[t,T]}\right] + T^2\mathbb{E}_t\left[\xi^2\right] + (1+T)\,\mathbb{E}\left[\int_t^T |\tilde{Z}_r|^2 dr\right] \\
&\quad + \left(1 + \mathbb{E}_t\left[\|Y\|^2_{\text{p-var};[t,T]}\right]\right)\|\eta\|^2_{\text{q-var};[t,T]} \\
&\lesssim T^2 + T^2R^2 + T^2\mathbb{E}_t\left[\xi^2\right] + (1+T)\,\mathbb{E}\left[\int_t^T |\tilde{Z}_r|^2 dr\right] \\
&\quad + \left(1+R^2\right)\|\eta\|^2_{\text{q-var};[t,T]}.
\end{aligned}
$$

Combining with (6), we get

$$
\begin{aligned}
&\mathbb{E}_t\left[\tilde{Y}_t^2\right] + \mathbb{E}_t\left[\int_t^T |\tilde{Z}_s|^2 ds\right] \\
&\lesssim \left(1+T+T^2\right)\mathbb{E}_t[\xi^2] + 1 + TR^2 + R + T\mathbb{E}_t\left[\|\tilde{Y}\|^2_{\text{p-var}}\right] \\
&\quad + \|\eta\|_{\text{q-var}}\left(1 + \mathbb{E}_t\left[\|\tilde{Y}\|_{\text{p-var}} + \|\tilde{Y}\|^2_{\text{p-var}}\right] + R + R^2\right) \\
&\lesssim \left(1+T+T^2\right)\mathbb{E}_t[\xi^2] + 1 + TR^2 + R \\
&\quad + T\left\{T^2 + T^2R^2 + T^2\mathbb{E}_t\left[\xi^2\right] + (1+T)\,\mathbb{E}\left[\int_t^T |\tilde{Z}_r|^2 dr\right]\right. \\
&\quad \left. + \left(1+R^2\right)\|\eta\|^2_{\text{q-var};[t,T]}\right\} + \|\eta\|_{\text{q-var}} \\
&\quad \times \left(1 + \left\{T + TR + T\mathbb{E}_t\left[\xi^2\right]^{1/2} + T^{1/2}\mathbb{E}\left[\int_t^T |\tilde{Z}_r|^2 dr\right]^{1/2} + (1+R)\,\|\eta\|_{\text{q-var};[t,T]}\right.\right. \\
&\quad \left.\left. + T^2 + T^2R^2 + T^2\mathbb{E}_t\left[\xi^2\right] + T\mathbb{E}\left[\int_t^T |\tilde{Z}_r|^2 dr\right] + \left(1+R^2\right)\|\eta\|^2_{\text{q-var};[t,T]}\right\} + R + R^2\right)
\end{aligned}
$$

Using $|a| \leq 1 + |a|^2$ and picking $T > 0$ such that $T + T^2 \leq 1/2$, we get

$$
\mathbb{E}_t\left[\int_t^T |\tilde{Z}_s|^2 ds\right] \leq c\left(1 + F(T)\left(R + R^2\right)\right),
$$

with $F(T) \to 0$, as $T \to 0$ (here we use that $\|\eta\|_{\text{q-var};[0,T]} \to 0$ for $T \to 0$, see (Friz and Victoir 2010, Theorem 5.31)).

Then

$$
\begin{aligned}
\mathbb{E}_t\left[\|\tilde{Y}\|^2_{\text{p-var};[t,T]}\right]^{1/2} &\lesssim T + TR + T\mathbb{E}_t\left[\xi^2\right]^{1/2} + T^{1/2}\left(1 + F(T)\left(R + R^2\right)\right) \\
&\quad + (1+R)\,\|\eta\|_{\text{q-var};[t,T]},
\end{aligned}
$$

which can be made smaller than $R/2$ by first picking R large and then T small. So indeed the ball stays invariant.

We now show that for a T small enough, Φ is a contraction on $B(R)$. So let $(Y, Z), (Y', Z') \in B(R)$ be given. Note that, since $Y_T = Y'_T$, we have for every $t \in [0, T]$

$$
|Y_t - Y'_t| = |(Y_T - Y_t) - (Y'_T - Y'_t)| \leq \|Y - Y'\|_{\text{p-var};[t,T]}.
$$

Hence,

$$|Y_t - Y'_t| = \mathbb{E}_t\left[|Y_t - Y'_t|\right] \le \mathbb{E}_t\left[\|Y - Y'\|_{\text{p-var};[t,T]}\right] \le \mathbb{E}_t\left[\|Y - Y'\|^2_{\text{p-var};[t,T]}\right]^{1/2}.$$

So that

$$\operatorname*{ess\,sup}_{\omega}\|Y(\omega) - Y'(\omega)\|_\infty \le \|Y - Y'\|_{p,2}.$$

Let $(\tilde{Y}, \tilde{Z}) = \Phi(Y, Z), (\tilde{Y}', \tilde{Z}') = \Phi(Y', Z')$. Using the Young estimate (Theorem 5) and Lemma 1 (in the Appendix below), we have for some constant c, that can change from line to line,

$$\begin{aligned}\|\tilde{Y} - \tilde{Y}'\|_{\text{p-var};[t,T]} \le{}& cT\|Y - Y'\|_{\text{p-var};[t,T]} + c\int_t^T |Z_r - Z'_r|dr \\ &+ \|Y - Y'\|_{\text{p-var};[t,T]}\|\eta\|_{\text{q-var}} \\ &+ c\left(1 + \|Y\|_{\text{p-var};[t,T]}\right)\|Y - Y'\|_\infty\|\eta\|_{\text{q-var}} \\ &+ \|M - M'\|_{\text{p-var};[t,T]},\end{aligned}$$

where $M = \int \tilde{Z}dW, M' = \int \tilde{Z}'dW$.

Hence,

$$\begin{aligned}\mathbb{E}_t\left[\|\tilde{Y} - \tilde{Y}'\|^2_{\text{p-var};[t,T]}\right]^{1/2} \le{}& c\mathbb{E}_t\left[\|Y - Y'\|_{\text{p-var};[t,T]}\right]^{1/2}\left(T + \|\eta\|_{\text{q-var}}\right) \\ &+ cT^{1/2}\mathbb{E}_t\left[\int_t^T |Z_r - Z'_r|^2 dr\right]^{1/2} \\ &+ c\left(\mathbb{E}_t\left[\|Y\|^2_{\text{p-var};[t,T]}\right]^{1/2}\right)\sup_\omega\|Y(\omega) - Y'(\omega)\|_\infty\|\eta\|_{\text{q-var}} \\ &+ c\mathbb{E}_t\left[\|M - M'\|^2_{\text{p-var};[t,T]}\right]^{1/2} \\ \le{}& cT^{1/2}\mathbb{E}_t\left[\int_t^T |Z_r - Z'_r|^2 dr\right]^{1/2} + c\left(T + \|\eta\|_{\text{q-var}}\right)\|Y - Y'\|_{p,2} \\ &+ c\,(1 + R)\,\|\eta\|_{\text{q-var}}\|Y - Y'\|_{p,2} + \mathbb{E}_t\left[\int_t^T (\tilde{Z}_s - \tilde{Z}'_s)^2 ds\right]^{1/2}.\end{aligned} \tag{7}$$

So, for a T small enough,

$$\|\tilde{Y} - \tilde{Y}'\|_{p,2} \le \frac{1}{4}\left[\|Y - Y'\|_{p,2} + \|Z - Z'\|_{\mathsf{BMO}}\right] + \|\tilde{Z} - \tilde{Z}'\|_{\mathsf{BMO}}.$$

On the other hand,

$$\begin{aligned}\left(\tilde{Y}_t - \tilde{Y}'_t\right)^2 ={}& 2\int_t^T \left[(f(Y_s, Z_s) - f(Y'_s, Z'_s))(Y_s - Y'_s)\right]ds \\ &+ 2\int_t^T \left[(g(Y_s) - g(Y'_s))(\tilde{Y}_s - \tilde{Y}'_s)\right]d\eta_s \\ &- 2\int_t^T \left[(\tilde{Y}_s - Y'_s)(Z_s - Z'_s)\right]dB_s - \int_t^T |\tilde{Z}_s - \tilde{Z}'_s|^2 ds.\end{aligned}$$

Note that by Lemma 2 and then Lemma 1

$$\begin{aligned}
&\| \left(g(Y) - g(Y')\right) \left(Y - Y'\right) \|_{\text{p-var};[t,T]} \\
&\quad \lesssim \|g(Y) - g(Y')\|_\infty \|Y - Y'\|_{\text{p-var};[t,T]} + \|g(Y) - g(Y')\|_{\text{p-var};[t,T]} \|Y - Y'\|_\infty \\
&\quad \lesssim \|Y - Y'\|^2_{\text{p-var};[t,T]} + (1 + \|Y\|_{\text{p-var};[t,T]}) \|Y - Y'\|^2_{\text{p-var};[t,T]} \\
&\quad \lesssim (1 + R) \|Y - Y'\|^2_{\text{p-var};[t,T]}.
\end{aligned}$$

Hence, the Young integral is bounded by a constant times $\|\eta\|_{\text{q-var};[t,T]}(1 + R)\|Y - Y'\|^2_{\text{p-var};[t,T]}$.

We estimate the Lebesgue integral as

$$\begin{aligned}
\left| \int_t^T \left[(f(Y_s, Z_s) - f(Y'_s, Z'_s))(Y_s - Y'_s) \right] ds \right| &\lesssim \int_t^T \left(|Y_s - Y'_s| + |Z_s - Z'_s| \right) |Y_s - Y'_s| ds \\
&\lesssim T \left(1 + \frac{1}{\lambda} \right) \|Y - Y\|_\infty + \lambda \int_t^T |Z_s - Z'_s|^2 ds.
\end{aligned}$$

So, after taking the conditional expectation,

$$\begin{aligned}
\mathbb{E}_t \left[\int_t^T |\tilde{Z}_s - \tilde{Z}'_s|^2 ds \right]^{1/2} \lesssim T^{1/2} \left(1 + \frac{1}{\lambda} \right)^{1/2} &\mathbb{E}_t \left[\|Y - Y'\|^2_{\text{p-var};[t,T]} \right]^{1/2} \\
&+ \lambda \mathbb{E}_t \left[\int_t^T |Z_r - Z'_r|^2 dr \right]^{1/2} \\
&+ \|\eta\|^{1/2}_{\text{q-var}} (1 + R)^{1/2} \mathbb{E}_t \left[\|\tilde{Y} - \tilde{Y}'\|^2_{\text{p-var};[t,T]} \right]^{1/2}.
\end{aligned}$$

That is

$$\begin{aligned}
\|\tilde{Z} - \tilde{Z}'\|_{\mathsf{BMO}} \lesssim T^{1/2} &\left(1 + \frac{1}{\lambda} \right)^{1/2} \|Y - Y'\|_{p,2} + \lambda \|Z - Z'\|_{\mathsf{BMO}} \\
&+ \|\eta\|_q (1 + R)^{1/2} \|Y - Y'\|_{p,2}
\end{aligned}$$

Picking a small λ and then a small T, we get

$$\|\tilde{Z} - \tilde{Z}'\|_{\mathsf{BMO}} \le \frac{1}{4} \|Y - Y'\|_{p,2} + \frac{1}{4} \|Z - Z'\|_{\mathsf{BMO}}$$

Define the modified norm

$$\|\|Y, Z\|\| := \|Y\|_{p,2} + 2\|Z\|_{\mathsf{BMO}}.$$

Then,

$$\begin{aligned}
&\|\|\tilde{Y} - \tilde{Y}', \tilde{Z} - \tilde{Z}'\|\| \\
&\qquad \le \frac{1}{4} \left[\|Y - Y'\|_{p,2} + \|Z - Z'\|_{\mathsf{BMO}} \right] + \|\tilde{Z} - \tilde{Z}'\|_{\mathsf{BMO}} + \frac{1}{2} \|Y - Y'\|_{p,2} \\
&\qquad + \frac{1}{2} \|Z - Z'\|_{\mathsf{BMO}} = \frac{3}{4} \|Y - Y'\|_{p,2} + \frac{7}{4} \|Z - Z'\|_{\mathsf{BMO}} \\
&\qquad \le \frac{7}{8} \|\|Y - Y', Z - Z'\|\|.
\end{aligned}$$

We therefore have a contraction and thereby existence of a unique solution on small enough time intervals.

Continuity on small time interval

This follows from virtually the same argument as the contraction mapping argument.

Comparison on small time interval

Let $C_B > 0$ be given, and pick $T = T(C_B)$ so small that the BSDE is well-posed for any f, g with $\|g\|_{C_b^2}, C_f < C_B$ and any $\eta \in C^{\text{q-var}}, \xi \in \mathcal{F}_T$ with $\|\eta\|_{\text{q-var};[0,T]}, \|\xi\|_\infty < C_B$.

Let $\xi_1, \xi_2 \in \mathcal{F}_T$ be given with $\|\xi_1\|_\infty < C_B$ and $\eta \in C^{\text{q-var}}$ with $\|\eta\|_{\text{q-var};[0,T]} < C_B$.

Let η^n be a sequence of smooth paths approximating η in q-variation norm, with $\|\eta^n\|_{\text{q-var};[0,T]} < C_B$ for all $n \geq 1$.

Let Y_1^n (resp. Y_2^n) be the classical BSDE solution with driving path η^n and data (ξ_1, f_1, g) (resp. (ξ_2, f_2, g)). Then, by a standard comparison theorem (for example, see (El Karoui et al. 1997)),

$$Y_1^n \leq Y_2^n.$$

By continuity we know that

$$\|Y_1^n - Y_1\|_{p,2} + \|Y_2^n - Y_2\|_{p,2} \to 0.$$

In particular, almost surely,

$$\|Y_1^n - Y_1\|_\infty + \|Y_2^n - Y_2\|_\infty \to 0.$$

Hence $Y_1 \leq Y_2$.

Unique existence on arbitrary time interval

We show existence for arbitrary $T > 0$. Denote

$$\overline{\xi} := \operatorname*{ess\,sup}_\omega \xi, \quad \underline{\xi} := \operatorname*{ess\,inf}_\omega \xi, \quad \overline{f} := \operatorname*{ess\,sup}_\omega f, \quad \underline{f} := \operatorname*{ess\,inf}_\omega f.$$

By assumption

$$|\overline{\xi}| + |\underline{\xi}| + \int_0^T \left[|\overline{f}|^2 + |\underline{f}|^2 \right] (t, 0, 0) dt < \infty.$$

Consider the following Young ODEs:

$$\overline{Y}_t = \overline{\xi} + \int_t^T \overline{f}(s, \overline{Y}_s, 0) + \int_t^T g(\overline{Y}_s) d\eta_s;$$

$$\underline{Y}_t = \underline{\xi} + \int_t^T \underline{f}(s, \overline{Y}_s, 0) + \int_t^T g(\underline{Y}_s) d\eta_s.$$

Note that $(\overline{Y}, 0)$ and $(\underline{Y}, 0)$ solve the following BSDEs respectively:

$$\overline{Y}_t = \overline{\xi} + \int_t^T \overline{f}(s, \overline{Y}_s, \overline{Z}_s) + \int_t^T g(\overline{Y}_s) d\eta_s - \int_t^T \overline{Z}_s dW_s;$$

$$\underline{Y}_t = \underline{\xi} + \int_t^T \underline{f}(s, \underline{Y}_s, \underline{Z}_s) + \int_t^T g(\underline{Y}_s) d\eta_s - \int_t^T \underline{Z}_s dW_s.$$

Choose δ such that the BSDE (2) is well-posed on a time interval of length δ whenever the terminal condition is bounded by $\|\overline{Y}\|_\infty \vee \|\underline{Y}\|_\infty$.

Let $\pi\colon 0 = t_0 < \cdots < t_n = T$ be a partition such that $t_{i+1} - t_i \leq \delta$ for all i. First, by the preceding arguments, BSDE (2) on $[t_{n-1}, t_n]$ with terminal condition ξ is well-posed and we denote the solution by (Y^n, Z^n). By comparison we have $\underline{Y}_{t_{n-1}} \leq Y^n_{t_{n-1}} \leq \overline{Y}_{t_{n-1}}$. Hence, we can again start the BSDE from $Y_{t_{n-1}}$ at time t_{n-1} and solve back to time t_{n-2}.

Repeating the arguments backwards in time we obtain the existence of a (unique) solution on $[0, T]$.

Continuity

Using the previous step, we can use the continuity result on small intervals to get the continuity of the solution map on arbitrary intervals, that is, point (iii) is proven.

We finish by showing the continuity statement (iv). Since the $d\eta$-term is more difficult than the dt-term, we will assume $f \equiv 0$ for ease of presentation.

First note that since $\|\xi\|_\infty, \|\xi'\|_\infty < M$, the local uniform continuity of the solution map in Theorem 2 we get

$$\|Y^n\|_{p,2} \leq C_0(M).$$

Let

$$\alpha_r := \int_0^1 \partial_y g\left(\theta Y_r + (1-\theta)Y'_r\right) d\theta.$$

Note that

$$\|\alpha\|_{\text{p-var};[t,T]} \leq C_1(M)\Big(\|Y\|_{\text{p-var};[t,T]} + \|Y'\|_{\text{p-var};[t,T]}\Big)$$

So that

$$\mathbb{E}_t\left[\|\alpha\|_{\text{p-var};[t,T]}\right] \leq C_2(M),$$

for some constant $C_2(M)$. Let $\Delta Y := Y - Y'$. Then (almost surely)

$$\begin{aligned}\|\Delta Y\|_{\infty;[t,T]} &\leq \|Y\|_{\infty;[t,T]} + \|Y'\|_{\infty;[t,T]}\\ &\leq \|Y\|_{p,2} + \|Y'\|_{p,2}\\ &\leq 2C_0.\end{aligned}$$

Now

$$d\Delta Y_t = -\alpha_t \Delta Y_t d\eta_t + \Delta Z_t dW_t.$$

By Itô's formula, together with the classical product rule for the $d\eta$-term, we get

$$d\left[\exp\left(\int_0^t \alpha_r d\eta_r\right)\Delta Y_t\right] = \exp\left(\int_0^t \alpha_r d\eta_r\right)\Delta Z_t dW_t,$$

so that if the latter is an honest martingale we get

$$|\Delta Y_0| = \left|\mathbb{E}\left[\exp\left(\int_0^T \alpha_r d\eta_r\right)\Delta Y_T\right]\right| \leq \mathbb{E}\left[\exp\left(2\int_0^T \alpha_r d\eta_r\right)\right]^{1/2} \mathbb{E}[(\Delta Y_T)^2]^{1/2}.$$

Let us calculate the conditional moments of $\Gamma_t := \int_t^T \alpha_r d\eta_r$.

First

$$\mathbb{E}_t\left[\|\Gamma\|_{\text{q}-\text{var};[t,T]}\right] \leq c_{Young}\|\eta\|_{\text{q}-\text{var};[t,T]}\mathbb{E}_t\left[\|\alpha\|_{\text{p}-\text{var};[t,T}\right]$$
$$\leq c_{Young}\|\eta\|_{q;[t,T]}C_2.$$

Further, by the product rule,

$$(\Gamma_t)^{m+1} = (m+1)\int_t^T \Gamma_r^m \alpha_r d\eta_r,$$

so that

$$\begin{aligned}
&\mathbb{E}_t\left[\|(\Gamma)^{m+1}\|_{\text{q}-\text{var};[t,T]}\right] \\
&\leq c_{Young}(m+1)\|\eta\|_{\text{q}-\text{var};[t,T]}\mathbb{E}_t\Big[\|\Gamma^m\|_{\text{p}-\text{var};[t,T]}\|\alpha\|_{\infty;[t,T]} \\
&\qquad + \|\Gamma^m\|_{\infty;[t,T]}\|\alpha\|_{\text{p}-\text{var};[t,T]}\Big] \\
&\leq c_{Young}(m+1)\|\eta\|_{\text{q}-\text{var};[t,T]}\Big(\mathbb{E}_t\left[\|\Gamma^m\|_{\text{p}-\text{var};[t,T]}\right]\|g'\|_\infty \\
&\qquad + \sup_{s\in[t,T]}\mathbb{E}_s\left[\|\Gamma^m\|_{\text{p}-\text{var};[s,T]}\right]\mathbb{E}_t[\|\alpha\|_{\text{p}-\text{var};[t,T]}]\Big) \\
&\leq c_{Young}(m+1)\|\eta\|_{\text{q}-\text{var};[t,T]}\Big(\mathbb{E}_t\left[\|\Gamma^m\|_{\text{p}-\text{var};[t,T]}\right]\|g'\|_\infty \\
&\qquad + \sup_{s\in[t,T]}\mathbb{E}_s\left[\|\Gamma^m\|_{\text{p}-\text{var};[s,T]}\right]C_2\Big)
\end{aligned}$$

Iterating, we get that for some $C_3(M) > 0$

$$\sup_{t\leq T}\mathbb{E}_t\left[\|(\Gamma)^m\|_{\text{q}-\text{var};[t,T]}\right] \leq m!C_3(M)^m.$$

In particular, for every $t \leq T$

$$\mathbb{E}[(\Gamma_t)^m] \leq m!C_3(M)^m.$$

So there is $\varepsilon > 0$ such that

$$\mathbb{E}\left[\exp\left(\varepsilon|\int_t^T \alpha_r \Delta Y_r d\eta_r|^2\right)\right] < C_3(M)$$

In particular, for every $c \in \mathbb{R}$

$$\mathbb{E}\left[\exp\left(c|\int_t^T \alpha_r \Delta Y_r d\eta_r|\right)\right] < C_4(c, M).$$

So the statement follows with $C(M) = C_4(2, M)$ if

$$\int \exp(\int_0^t \alpha_r d\eta_r)\Delta Z_t dW_t,$$

is an honest martingale. However, this follows from

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T \exp\left(2\int_0^t \alpha_r d\eta_r\right)|\Delta Z_t|^2 dt\right)^{1/2}\right] &\le \mathbb{E}\left[\left(\sup_{t\le T}\exp\left(2\int_0^t \alpha_r d\eta_r\right)\int_0^T |\Delta Z_t|^2 dt\right)^{1/2}\right]\\ &\le \mathbb{E}\left[\sup_{t\le T}\exp\left(4\int_0^t \alpha_r d\eta_r\right)\right]^{1/2}\mathbb{E}\left[\int_0^T |\Delta Z_t|^2 dt\right]\\ &< \infty.\end{aligned}$$

Here we used

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^t \alpha_r \Delta Y_r d\eta_r\right)\right] &= \mathbb{E}\left[\exp\left(\int_0^T \alpha_r \Delta Y_r d\eta_r\right)\exp\left(-\int_t^T \alpha_r \Delta Y_r d\eta_r\right)\right]\\ &\le \mathbb{E}\left[\exp\left(2\left|\int_0^T \alpha_r \Delta Y_r d\eta_r\right|\right)\right]^{1/2}\mathbb{E}\left[\exp\left(2\left|\int_t^T \alpha_r \Delta Y_r d\eta_r\right|\right)\right]^{1/2}\\ &< \infty.\end{aligned}$$

□

Application to rough PDEs

In this section, we apply BSDEs with Young drift for the stochastic representation for PDEs of the form

$$\partial_t u = \frac{1}{2}\mathrm{Tr}\left[\sigma(x)\sigma^T(x)D^2u^n\right] + b(x)\cdot Du + f(t,u,\sigma(x)^T Du) + g(u)\dot{\eta}_t. \quad (8)$$

Here η has finite q-variation, with $q \in [1,2)$ and the last term is a priori not well-defined. There are several approaches to make sense of such a "rough" PDE (or pathwise SPDEs). We shall employ the solution concept based on smooth approximation of η. Let us mention (Caruana et al. 2011) studying a class of linear equations. The proofs are based on a transformation of the SPDE into a PDE with random coefficients and a study of the latter using PDE methods. In (Diehl et al. 2015), the convergence of solutions corresponding to smooth approximations of η is shown using a linear Feynman–Kac formula. In (Diehl et al. 2014), these results are extended to show that the limit actually solves an integral equation. Semilinear equations like (8) are investigated in (Diehl and Friz 2012). Again, the convergence of solutions corresponding to smooth approximations of η is shown via a transformation of the rough PDE.

Here we shall show convergence, in the semilinear case and when $q \in [1,2)$, using the concept of BSDEs with Young drift we have developed; see Theorem 4 below. We want to point out that this approach leads to a very compact proof.

For $D = [0,T]\times\mathbb{R}^m$ or $D = \mathbb{R}^m$ we shall denote by BUC(D) the space of bounded, uniformly continuous functions on D. Let us recall the nonlinear Feyman–Kac formula for standard PDEs.

Theorem 3 ((Pardoux and Peng 1992, Section 4)). *Let $h \in \mathrm{BUC}(\mathbb{R}^m)$, $f(t, y, z) : [0, T] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ bounded and Lipschitz in y, z uniformly in t, x, $\sigma : \mathbb{R}^m \to L(\mathbb{R}^d, \mathbb{R}^m)$ Lipschitz, $b : \mathbb{R}^m \to \mathbb{R}^m$ Lipschitz and $g_1, \ldots, g_e \in C_b^2(\mathbb{R})$, and let η be a smooth path. For every $s \in [0, T]$, $x \in \mathbb{R}^m$ let $X^{s,x}$ be the solution to the SDE*

$$dX_t^{s,x} = \sigma\left(X_t^{s,x}\right) dW_t + b\left(X_t^{s,x}\right) dt \qquad X_s^{s,x} = x$$

and $Y^{s,x}$ the solution to the BSDE

$$dY_t^{s,x} = f\left(t, Y_t^{s,x}, Z_t^{s,x}\right) dt + g\left(Y_t^{s,x}\right) d\eta_t - Z_t^{s,x} dW_t \qquad Y_T^{s,x} = h\left(X_T^{s,x}\right).$$

Then $u(t, x) := Y_t^{t,x}$ is the unique viscosity solution in $\mathrm{BUC}([0, T] \times \mathbb{R}^m)$ to the PDE

$$\partial_t u = \frac{1}{2}\mathrm{Tr}\left[\sigma(x)\sigma^T(x)D^2u^n\right] + b(x) \cdot Du + f(t, u, \sigma(x)^T Du) + g(u)\dot{\eta}_t$$
$$u|_T = h.$$

The following theorem extends this representation property to BSDEs with Young drift.

Theorem 4 *Let $\eta \in C^{0,\mathrm{q-var}}$, $q \in [1, 2)$ and let η^n smooth be given such that $\eta^n \to \eta$ in $C^{0,\mathrm{q-var}}$.*

Let $f(t, y, z) : [0, T] \times \mathbb{R} \times \mathbb{R}^g \to \mathbb{R}$ bounded and Lipschitz in y, z uniformly in t, x, $\sigma : \mathbb{R}^g \to L(\mathbb{R}^d, \mathbb{R}^m)$ Lipschitz, $b : \mathbb{R}^h \to \mathbb{R}^m$ Lipschitz and $g_1, \ldots, g_e \in C_b^2(\mathbb{R})$.

Let u^n be the unique $\mathrm{BUC}([0, T] \times \mathbb{R}^m)$ viscosity solution to

$$\partial_t u^n = \frac{1}{2}\mathrm{Tr}\left[\sigma(x)\sigma^T(x)D^2u^n\right] + b(x) \cdot Du^n + f(t, u, \sigma(x)^T Du) + g(u)\dot{\eta}_t^n.$$

Then there exists $u \in \mathrm{BUC}([0, T] \times \mathbb{R}^m)$ such that $u^n \to u$ locally uniformly and the limit does not depend on the approximating sequence.

Formally, u solves the PDE

$$\partial_t u = \frac{1}{2}\mathrm{Tr}\left[\sigma(x)\sigma^T(x)D^2u^n\right] + b(x) \cdot Du + f(t, u, \sigma(x)^T Du) + g(u)\dot{\eta}_t.$$

Moreover, $u(t, x) = Y_t^{t,x}$, where $X^{s,x}$ is the solution to the SDE

$$dX_t^{s,x} = \sigma(X_t^{s,x})dW_t + b(X_t^{s,x})dt \qquad X_s^{s,x} = x$$

and $Y^{s,x}$ the solution to the BSDE with Young drift

$$dY_t^{s,x} = ft, Y_t^{s,x}, Z_t^{s,x})dt + g(Y_t^{s,x})d\eta_t - Z_t^{s,x} dW_t \qquad Y_T^{s,x} = h(X_T^{s,x}).$$

Proof By Theorem 3, we can write $u^n(t, x) = Y_t^{n,t,x}$, where

$$dX_t^{s,x} = \sigma\left(X_t^{s,x}\right) dW_t + b\left(X_t^{s,x}\right) dt,\ X_s^{s,x} = x,$$

and $Y^{n,s,x}$ is the solution to the BSDE

$$dY_t^{n,s,x} = f\left(Y_t^{n,s,x}, Z_t^{n,s,x}\right) dt + g\left(Y_t^{n,s,x}\right) d\eta_t^n - Z_t^{n,s,x} dW_t,\ Y_T^{n,s,x} = h\left(X_T^{s,x}\right).$$

By Theorem 2, we have that for fixed s, x, $Y^{n,s,x} \to Y^{s,x}$ in $\mathcal{B}_p$, where $Y^{s,x}$ solves the corresponding BSDE with Young drift. In particular, for the starting point, $Y_s^{n,s,x} \to Y_s^{s,x}$, and hence we get pointwise convergence of u^n.

We now show that u^n is locally uniformly continuous in (t, x) uniformly in n.

By Theorem 2 (iv), uniformly in n,

$$\begin{aligned} |Y_s^{n,s,x} - Y_s^{n,s,x'}| &\le C\mathbb{E}\left[|h(X_T^{s,x}) - h(X_T^{s,x'})|^2\right]^{1/2} \\ &\le C\|Dh\|_\infty \mathbb{E}\left[|X_T^{s,x} - X_T^{s,x'}|^2\right]^{1/2} \\ &\lesssim |x - x'|, \end{aligned}$$

where we used the Lipschitzness of the map $\mathbb{R}^m \ni x \mapsto X_T^{s,x} \in L^2(\Omega)$, see, for example, (Stroock and Karmakar 1982, Theorem 2.2).

Moreover, for any small $\delta > 0$,

$$\begin{aligned} Y_{s+\delta}^{n,s+\delta,x} - Y_s^{n,s,x} &= \mathbb{E}\left[Y_{s+\delta}^{n,s+\delta,x} - Y_{s+\delta}^{n,s,x}\right] + \mathbb{E}\left[Y_{s+\delta}^{n,s,x} - Y_s^{n,s,x}\right] \\ &= \mathbb{E}\left[Y_{s+\delta}^{n,s+\delta,x} - Y_{s+\delta}^{n,s+\delta,X_{s+\delta}^{s,x}}\right] + \mathbb{E}\left[Y_{s+\delta}^{n,s,x} - Y_s^{n,s,x}\right] \\ &\lesssim \mathbb{E}\left[|x - X_{s+\delta}^{s,x}|^2\right]^{1/2} + \mathbb{E}\left[\int_s^{s+\delta} f(Y_r^{n,s,x}, Z_r^{n,s,x})dr \right. \\ &\quad \left. + \int_s^{s+\delta} g(Y_r^{n,s,x})d\eta_r^n\right] \\ &\lesssim \delta^{1/2} + \delta\|f\|_\infty + \|\eta\|_{\text{q-var};[s,s+\delta]}\mathbb{E}\left[\left(1 + \|Y^{n,s,x}\|_{\text{p-var};[s,s+\delta]}\right)\right] \\ &\lesssim \delta^{1/2} + \delta\|f\|_\infty + \|\eta\|_{\text{q-var};[s,s+\delta]}, \end{aligned}$$

where we used the uniform boundedness of $\|Y^n\|_{p,2}$ in the last step (as in the proof of Theorem 2).

It follows that u^n is locally uniformly continuous in (t, x) uniformly in n. Hence u^n converges to u locally uniformly. The claimed stochastic representation of u is immediate. □

Remark 4 *In the vein of* (Diehl et al. 2014)*, one can also, under appropriate assumptions on the coefficients, verify that u solves an integral equation.*

Appendix - Young integration

For $p \ge 1$, V some Banach space, we denote by $C^{\text{p-var}} = C^{\text{p-var}}([0, T], V)$ the space of V-valued continuous paths X with finite p-variation

$$\|X\|_{\text{p-var}} := \|X\|_{\text{p-var};[0,T]} := \left(\sup_\pi \sum_{[u,v]\in\pi} |X_{u,v}|^p\right)^{1/p}.$$

Here the supremum runs over all partitions of the interval $[0, T]$ and $X_{u,v} := X_v - X_u$.

We shall also need the space $C^{0,\text{p-var}} = C^{0,\text{p-var}}([0,T],V)$, defined as the closure of $C^\infty([0,T],V)$ under the norm $\|\cdot\|_{\text{p-var}}$. Obviously, $C^{0,\text{p-var}} \subset C^{\text{p-var}}$, and the inclusion is strict (Friz and Victoir 2010, Section 5.3.3).

The following basic estimates can be found in (Friz and Victoir 2010, Chapter 5)

$$\|Y\|_\infty \leq |Y_T| + \|Y\|_{\text{p-var}}, \qquad \forall p \geq 1$$
$$\|Y\|_{\text{p-var}} \leq \|Y\|_{\text{q-var}}, \qquad \forall 1 \leq q \leq p.$$

The proof of the following result goes back to (Young 1936). A short modern proof can be found in (Friz and Hairer 2014, Chapter 4). In this statement and in what follows, $a \lesssim b$ means that there exists a constant $c > 0$, not dependent on the paths under consideration, such that $a \leq cb$. The constant c can depend on the vector fields under considerations, the dimension, and the time horizon T, but is bounded for T bounded.

Theorem 5 (Young integration). *Let* $X \in C^{\text{p-var}}([0,T],L(V,W))$, $Y \in C^{\text{q-var}}([0,T],W)$ *with* $1/p + 1/q > 1$.

Then

$$\int_0^T X_s dY_s := \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} X_u Y_{u,v}$$

exists, where the limit is taken over partitions of $[0,T]$ *with mesh size approaching 0.*

Moreover,

$$\left\| \int X_s dY_s \right\|_{\text{q-var};[0,T]} \lesssim (|X_0| + \|X\|_{\text{p-var};[0,T]})\|Y\|_{\text{q-var};[0,T]}$$
$$\lesssim (|X_T| + \|X\|_{\text{p-var};[0,T]})\|Y\|_{\text{q-var};[0,T]}.$$

We also need

Lemma 1 *Let* $p \geq 1$, $g \in C_b^2$, $a, a' \in C^{\text{p-var}}$, *then*

$$\|g(a) - g(a')\|_{\text{p-var}} \leq c\|a - a'\|_{\text{p-var}} + \left(\|a\|_{\text{p-var}} + \|a'\|_{\text{p-var}}\right) \|a - a'\|_\infty.$$

Proof This follows from

$$\begin{aligned}
&|g(a_t) - g(a'_t) - g(a_s) - g(a'_s)| \\
&= \left| \int_0^1 Dg(a'_t + \theta(a_t -'_t))d\theta(a_t - a'_t) - \int_0^1 Dg(a'_s + \theta(a_s - a'_s))d\theta(a_s - a'_s) \right| \\
&\leq \left| \int_0^1 Dg(a'_t + \theta(a_t - a'_t)) - Dg(a'_s + \theta(a_s - a'_s))d\theta(a_t - a'_t) \right| \\
&\quad + \left| \int_0^1 Dg(a'_s + \theta(a_s - a'_s))d\theta(a_t - a'_t) - (a_s - a'_s) \right| \\
&\leq \|D^2 g\|_\infty \left(|a'_t - a'_s| + |a_t - a_s|\right) |a_t - a'_t| + \|Dg\|_\infty |(a_t - a'_t) - (a_s - a'_s)|.
\end{aligned}$$

□

Lemma 2 *Let* $p \geq 1$ *and* $a, b \in C^{\mathrm{p-var}}$ *then*

$$\|ab\|_{\mathrm{p-var}} \lesssim \|a\|_{\mathrm{p-var}}\|b\|_{\infty} + \|a\|_{\infty}\|b\|_{\mathrm{p-var}}$$

Proof This follows from

$$|a_t b_t - a_s b_s| \leq |a_t - a_s|\|b\|_{\infty} + \|a\|_{\infty}|b_t - b_s|.$$

□

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Authors' contributions

Both authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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