DYNAMICS OF MACHINES
Subject Code -10 ME 54

BALANCING OF ROTATING MASSES

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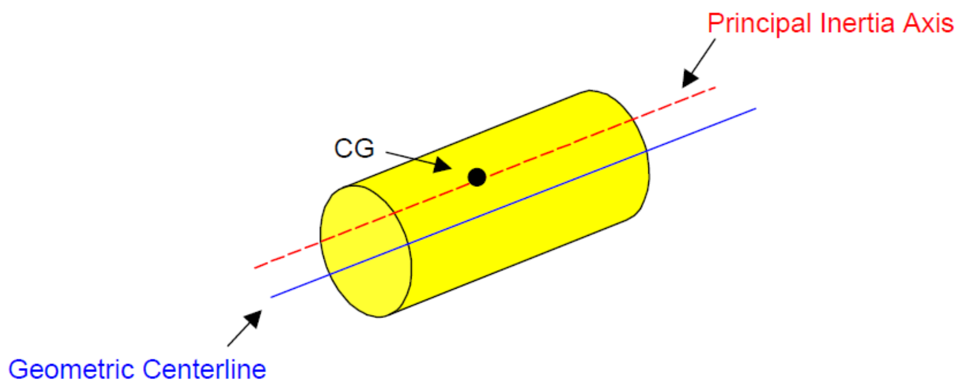
INTRODUCTION:

When man invented the wheel, he very quickly learnt that if it wasn't completely round and if it didn't rotate evenly about it's central axis, then he had a problem!
What the problem he had?
The wheel would vibrate causing damage to itself and it's support mechanism and in severe cases, is unusable.
A method had to be found to minimize the problem. The mass had to be evenly distributed about the rotating centerline so that the resultant vibration was at a minimum.

UNBALANCE:

The condition which exists in a rotor when vibratory force or motion is imparted to its bearings as a result of centrifugal forces is called unbalance or the uneven distribution of mass about a rotor's rotating centerline.



Rotating centerline:
The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

Geometric centerline:
The geometric centerline being the physical centerline of the rotor.

When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

Different types of unbalance can be defined by the relationship between the two centerlines. These include:
Static Unbalance – where the PIA is displaced parallel to the geometric centerline. (Shown above)
Couple Unbalance – where the PIA intersects the geometric centerline at the center of gravity. (CG)
Dynamic Unbalance – where the PIA and the geometric centerline do not coincide or touch.
The most common of these is dynamic unbalance.

Causes of Unbalance:
In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

1. slight variation in the density of the material or
2. inaccuracies in the casting or
3. inaccuracies in machining of the parts.

Why balancing is so important?

1. A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.
2. As machines get bigger and go faster, the effect of the unbalance is much more severe.
3. The force caused by unbalance increases by the square of the speed.
4. If the speed is doubled, the force quadruples; if the speed is tripled the force increases

by a factor of nine!

Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

BALANCING:

Balancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.
The objectives of balancing an engine are to ensure:

1. That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and
2. That the couples involved in acceleration of the different moving parts balance each other.

Types of balancing:

a) Static Balancing:
- i) Static balancing is a balance of forces due to action of gravity.
- ii) A body is said to be in static balance when its centre of gravity is in the axis of rotation.

b) Dynamic balancing:
- i) Dynamic balance is a balance due to the action of inertia forces.
- ii) A body is said to be in dynamic balance when the resultant moments or couples, which involved in the acceleration of different moving parts is equal to zero.
- iii) The conditions of dynamic balance are met, the conditions of static balance are also met.

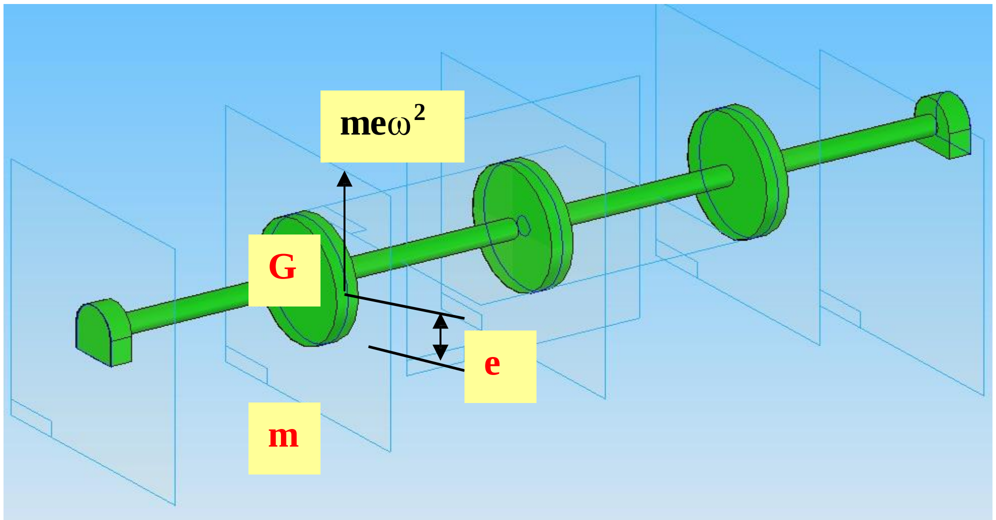
In rotor or reciprocating machines many a times unbalance of forces is produced due to inertia forces associated with the moving masses. If these parts are not properly balanced, the dynamic forces are set up and forces not only increase loads on bearings and stresses in the various components, but also unpleasant and dangerous vibrations.

Balancing is a process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible eliminated entirely.

BALANCING OF ROTATING MASSES

When a mass moves along a circular path, it experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force called centrifugal force acts radially outwards and is a disturbing force on the axis of rotation. The magnitude of this remains constant but the direction changes with the rotation of the mass.

In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of rotor lies on the axis of rotation of the shaft. When this does not happen, there is an eccentricity and an unbalance force is produced. This type of unbalance is common in steam turbine rotors, engine crankshafts, rotors of compressors, centrifugal pumps etc.



The unbalance forces exerted on machine members are time varying, impart vibratory motion and noise, there are human discomfort, performance of the machine deteriorate and detrimental effect on the structural integrity of the machine foundation.

Balancing involves redistributing the mass which may be carried out by addition or removal of mass from various machine members

Balancing of rotating masses can be of

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of several masses rotating in the same plane
4. Balancing of several masses rotating in different planes

STATIC BALANCING:

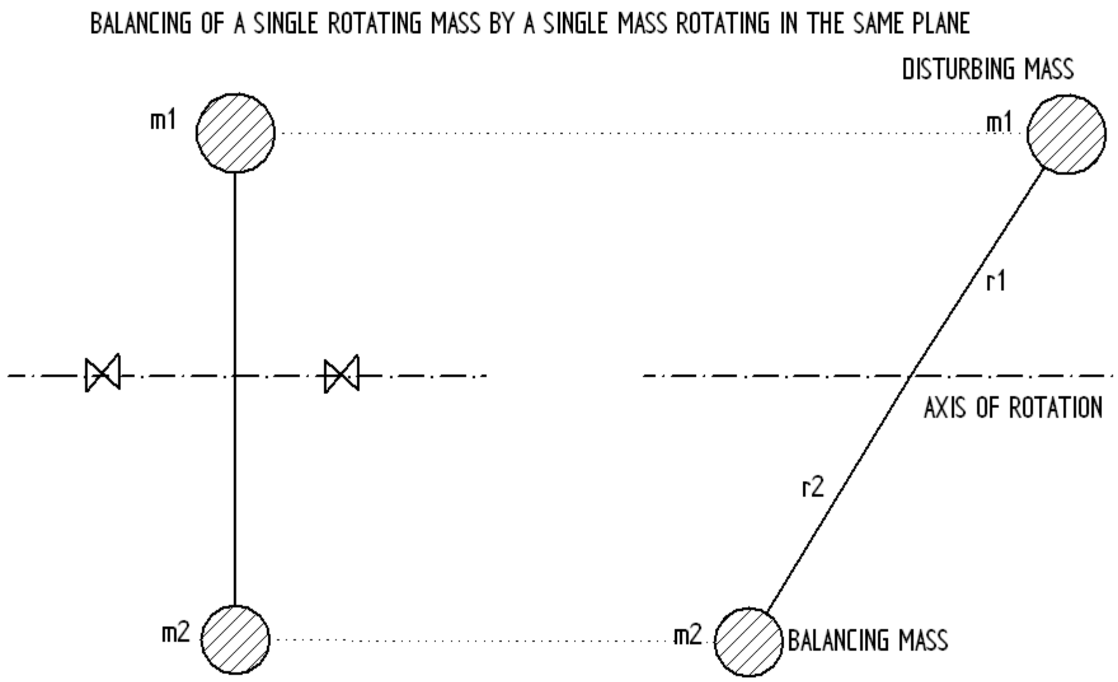
A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation

DYNAMIC BALANCING;

When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

CASE 1.

BALANCING OF A SINGLE ROTATING MASS BY A SINGLE MASS ROTATING IN THE SAME PLANE



Consider a disturbing mass m_1 which is attached to a shaft rotating at ω rad/s.
Let

$$r_1 = \text{radius of rotation of the mass } m_1$$
$$= \text{distance between the axis of rotation of the shaft and the centre of gravity of the mass } m_1$$

The centrifugal force exerted by mass m_1 on the shaft is given by,

$$\mathbf{F_{c1}} = \mathbf{m_1} \omega^2 \mathbf{r_1} ------------------- (1)$$

This force acts radially outwards and produces bending moment on the shaft. In order to counteract the effect of this force F_{c1} , a balancing mass m_2 may be attached in the same plane of rotation of the disturbing mass m_1 such that the centrifugal forces due to the two masses are equal and opposite.

Let,

$$\begin{aligned} r_2 &= \text{radius of rotation of the mass } m_2 \\ &= \text{distance between the axis of rotation of the shaft and} \\ &\quad \text{the centre of gravity of the mass } m_2 \end{aligned}$$

Therefore the centrifugal force due to mass m_2 will be,

$$F_{c2} = m_2\, \omega^2\, r_2 \text{ -----------------}(2)$$

Equating equations (1) and (2), we get

$$F_{c1} = F_{c2}$$

$$m_1\, \omega^2\, r_1 = m_2\, \omega^2\, r_2 \quad \text{or} \quad m_1\, r_1 = m_2\, r_2 \text{ ----------------}(3)$$

The product $\mathbf{m_2\ r_2}$ can be split up in any convenient way. As for as possible the radius of rotation of mass m_2 that is r_2 is generally made large in order to reduce the balancing mass m_2.

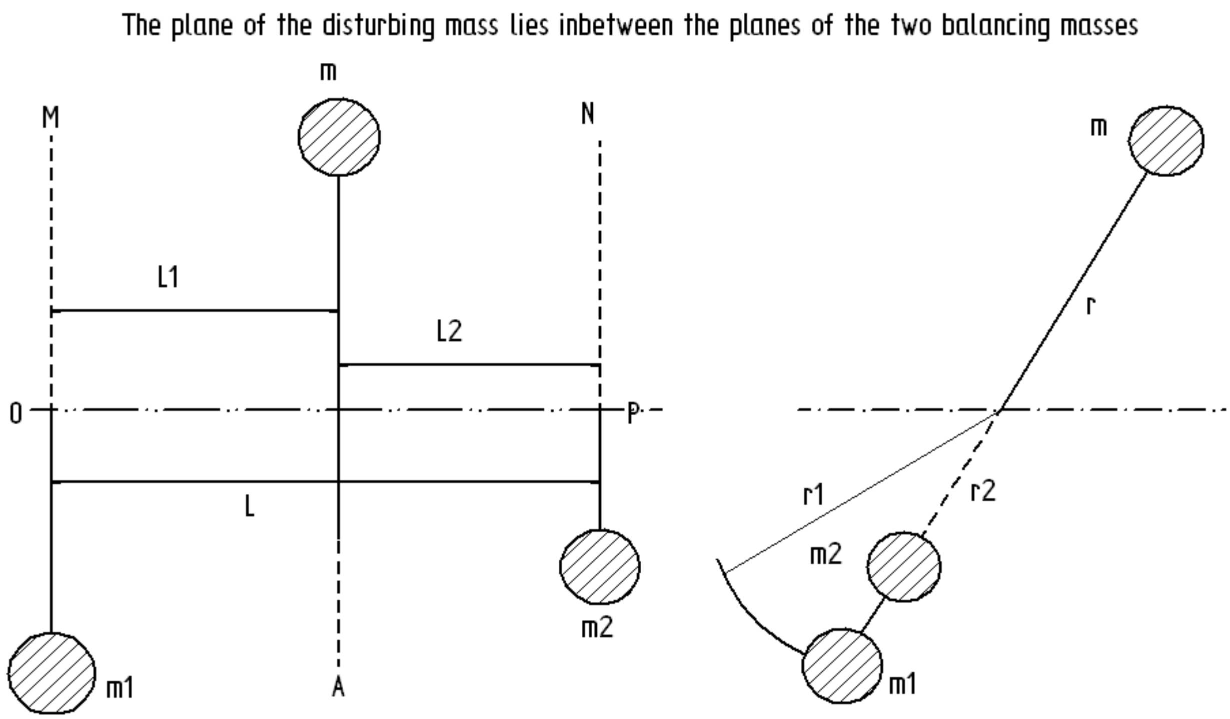
CASE 2:
BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN DIFFERENT PLANES.

There are two possibilities while attaching two balancing masses:
1.The plane of the disturbing mass may be in between the planes of the two balancing masses.
2.The plane of the disturbing mass may be on the left or right side of two planes containing the balancing masses.

In order to balance a single rotating mass by two masses rotating in different planes which are parallel to the plane of rotation of the disturbing mass i) the net dynamic force acting on the shaft must be equal to zero, i.e. the centre of the masses of the system must lie on the axis of rotation and this is the condition for static balancing ii) the net couple due to the dynamic forces acting on the shaft must be equal to zero, i.e. the algebraic sum of the moments about any point in the plane must be zero. The conditions i) and ii) together give dynamic balancing.

CASE 2(I):

THE PLANE OF THE DISTURBING MASS LIES IN BETWEEN THE PLANES OF THE TWO BALANCING MASSES.



Consider the disturbing mass m lying in a plane A which is to be balanced by two rotating masses m_1 and m_2 lying in two different planes M and N which are parallel to the plane A as shown.

Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, M and N respectively.
Let L_1, L_2 and L be the distance between A and M, A and N, and M and N respectively.
Now,
The centrifugal force exerted by the mass m in plane A will be,

$$F_c = m\,\omega^2 r \text{ ------------------ } (1)$$

Similarly,
The centrifugal force exerted by the mass m_1 in plane M will be,

$$F_{c1} = m_1 \omega^2 r_1 \text{ ------------------ } (2)$$

And the centrifugal force exerted by the mass m_2 in plane N will be,

$$F_{c2} = m_2\,\omega^2 r_2 -------------------(3)$$

For the condition of static balancing,

$$F_c = F_{c1} + F_{c2}$$
$$\text{or} \quad m\omega^2 r = m_1\,\omega^2 r_1 + m_2\,\omega^2 r_2$$
$$\text{i.e.} \quad m\,r = m_1\,r_1 + m_2\,r_2 -----------------(4)$$

Now, to determine the magnitude of balancing force in the plane 'M' or the dynamic force at the bearing 'O' of a shaft, take moments about ' P ' which is the point of intersection of the plane N and the axis of rotation.

Therefore,

$$F_{c1}\,x L = F_c\,x L_2$$
$$\text{or} \quad m_1\,\omega^2 r_1\;x\,L = m\omega^2 r\;x L_2$$
$$\text{Therefore,}$$
$$m_1 r_1 L = mrL_2 \quad \text{or} \quad m_1 r_1 = mr\frac{L_2}{L} ---------(5)$$

Similarly, in order to find the balancing force in plane 'N' or the dynamic force at the bearing 'P' of a shaft, take moments about ' O ' which is the point of intersection of the plane M and the axis of rotation.

Therefore,

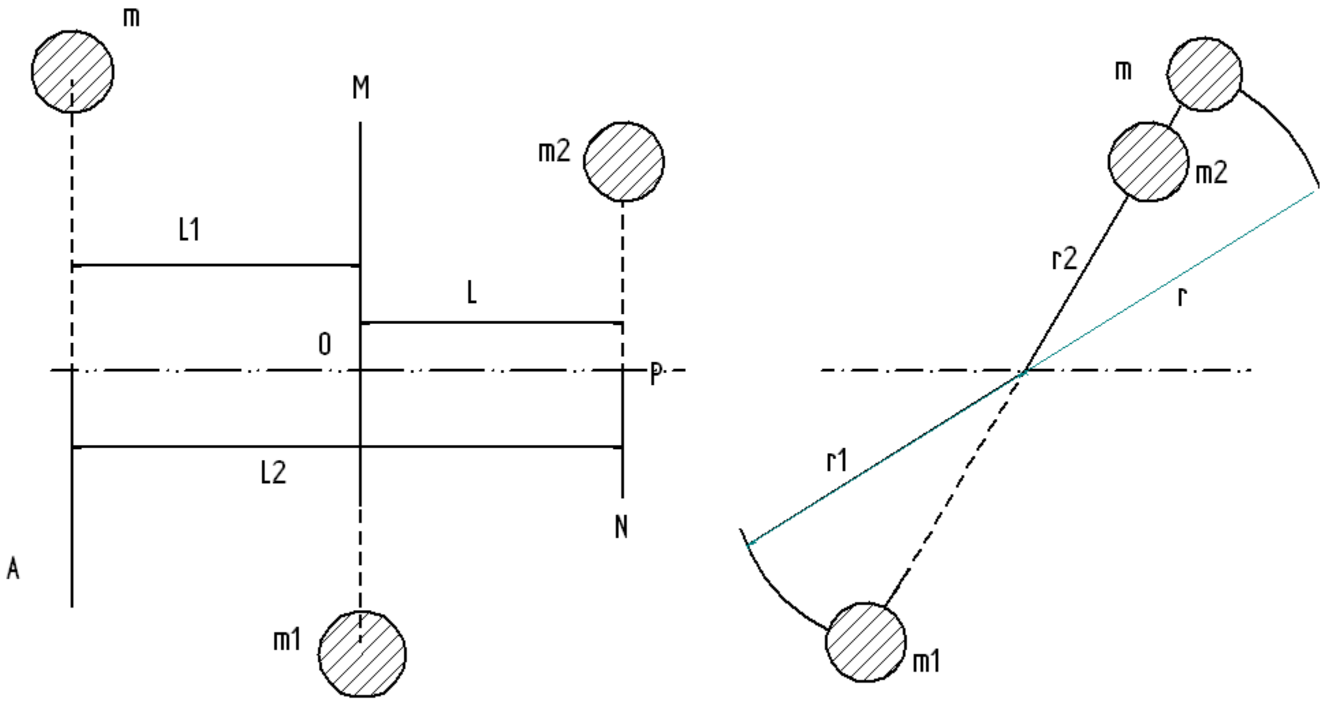
$$F_{c2}\,x L = F_c\,x L_1$$
$$\text{or} \quad m_2\,\omega^2 r_2\;x\,L = m\omega^2 r\;x L_1$$
$$\text{Therefore,}$$
$$m_2 r_2 L = mrL_1 \quad \text{or} \quad m_2 r_2 = mr\frac{L_1}{L} ---------(6)$$

For dynamic balancing equations (5) or (6) must be satisfied along with equation (4).

CASE 2(II):

WHEN THE PLANE OF THE DISTURBING MASS LIES ON ONE END OF THE TWO PLANES CONTAINING THE BALANCING MASSES.

When the plane of the disturbing mass lies on one end of the planes of the balancing masses



For static balancing,

$$F_{c1} = F_c + F_{c2}$$
$$\text{or} \quad m_1 \omega^2 r_1 = m\omega^2 r + m_2 \omega^2 r_2$$
$$\text{i.e.} \quad m_1 r_1 = m r + m_2 r_2 \text{-----------------}(1)$$

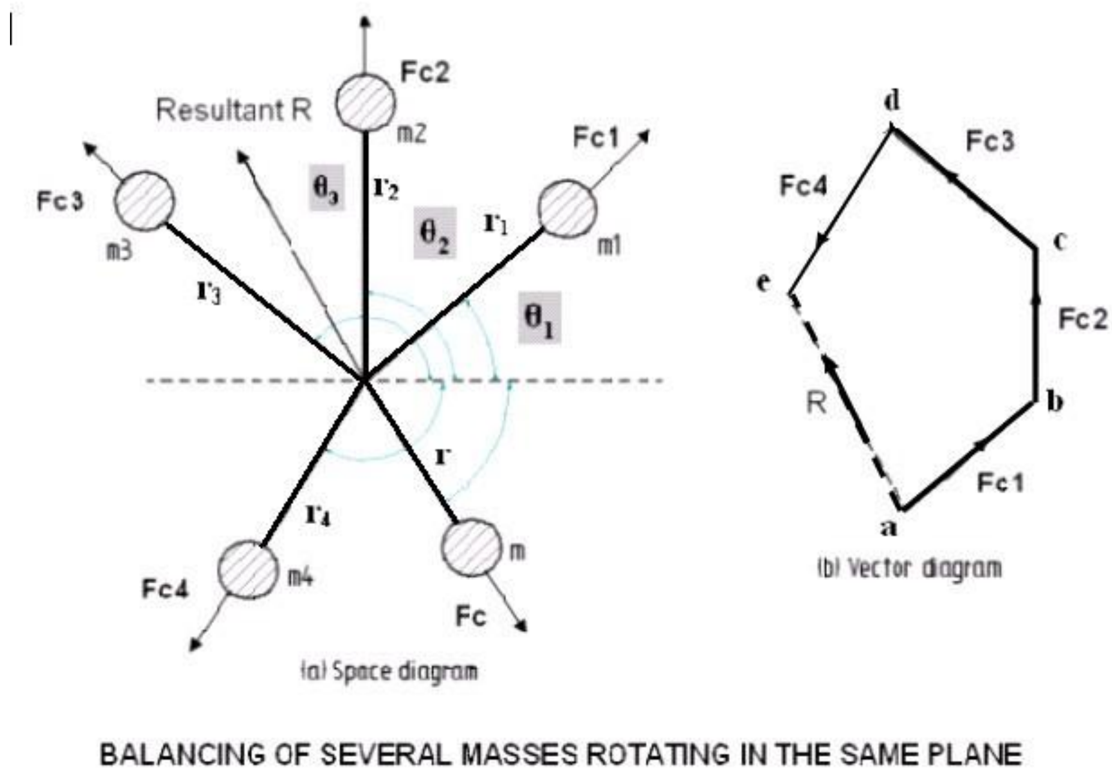
For dynamic balance the net dynamic force acting on the shaft and the net couple due to dynamic forces acting on the shaft is equal to zero.
To find the balancing force in the plane 'M' or the dynamic force at the bearing 'O' of a shaft, take moments about 'P'. i.e.

$$F_{c1} x L = F_c x L_2$$
$$\text{or } m_1 \omega^2 r_1 \, x \, L = m\omega^2 r \, x L_2$$
Therefore,
$$m_1 r_1 L = mrL_2 \quad \text{or} \quad m_1 r_1 = mr\frac{L_2}{L} ---------(2)$$

Similarly, to find the balancing force in the plane 'N' , take moments about 'O', i.e.,

$$F_{c2} x L = F_c x L_1$$
$$\text{or } m_2 \omega^2 r_2 \, x \, L = m\omega^2 r \, x L_1$$
Therefore,
$$m_2 r_2 L = mrL_1 \quad \text{or} \quad m_2 r_2 = mr\frac{L_1}{L} ---------(3)$$

CASE 3:
BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE



BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE

Consider a rigid rotor revolving with a constant angular velocity ω rad/s. A number of masses say, four are depicted by point masses at different radii in the same transverse plane.

If m_1, m_2, m_3 and m_4 are the masses revolving at radii r_1, r_2, r_3 and r_4 respectively in the same plane.
The centrifugal forces exerted by each of the masses are F_{c1}, F_{c2}, F_{c3} and F_{c4} respectively.
Let F be the vector sum of these forces. i.e.

$$F = F_{c1} + F_{c2} + F_{c3} + F_{c4}$$
$$= m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 ---------- (1)$$

The rotor is said to be statically balanced if the vector sum F is zero. If the vector sum F is not zero, i.e. the rotor is unbalanced, then introduce a counterweight (balance weight) of mass 'm' at radius 'r' to balance the rotor so that,

$$m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 + m \ \omega^2 r = 0 ---------- (2)$$

or

$$m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m \ r = 0 ----------------- (3)$$

The magnitude of either 'm' or 'r' may be selected and the other can be calculated.
In general, if $\sum \mathbf{m_i r_i}$ is the vector sum of $\mathbf{m}_1 \mathbf{r}_1$, $\mathbf{m}_2 \mathbf{r}_2$, $\mathbf{m}_3 \mathbf{r}_3$, $\mathbf{m}_4 \mathbf{r}_4$ etc, then,

$$\sum m_i r_i + mr = 0 -------- (4)$$

The above equation can be solved either analytically or graphically.

1. Analytical Method:

Procedure:
Step 1: Find out the centrifugal force or the product of mass and its radius of rotation exerted by each of masses on the rotating shaft, since ω^2 is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.
Step 2: Resolve these forces into their horizontal and vertical components and find their sums. i.e.,

$$\text{Sum of the horizontal components}$$
$$= \sum_{i=1}^{n} m_i r_i \cos \theta_i = m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + --------$$

$$\text{Sum of the vertical components}$$
$$= \sum_{i=1}^{n} m_i r_i \sin \theta_i = m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + --------$$

Step 3: Determine the magnitude of the resultant centrifugal force

$$R=\sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2+\left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

Step 4: If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta=\frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i}$$

Step 5: The balancing force is then equal to the resultant force, but in opposite direction.
Step 6: Now find out the magnitude of the balancing mass, such that

$$R=mr$$

Where, m = balancing mass and r = its radius of rotation

2. Graphical Method:

Step 1:
Draw the space diagram with the positions of the several masses, as shown.

Step 2:
Find out the centrifugal forces or product of the mass and radius of rotation exerted by each mass.

Step 3:
Now draw the vector diagram with the obtained centrifugal forces or product of the masses and radii of rotation. To draw vector diagram take a suitable scale.
Let ab, bc, cd, de represents the forces F_{c1}, F_{c2}, F_{c3} and F_{c4} on the vector diagram.
Draw 'ab' parallel to force F_{c1} of the space diagram, at 'b' draw a line parallel to force F_{c2}. Similarly draw lines cd, de parallel to F_{c3} and F_{c4} respectively.

Step 4:
As per polygon law of forces, the closing side 'ae' represents the resultant force in magnitude and direction as shown in vector diagram.

Step 5:
The balancing force is then , equal and opposite to the resultant force.

Step 6:

Determine the magnitude of the balancing mass (m) at a given radius of rotation (r), such that,

$$F_c = m\omega^2 r$$

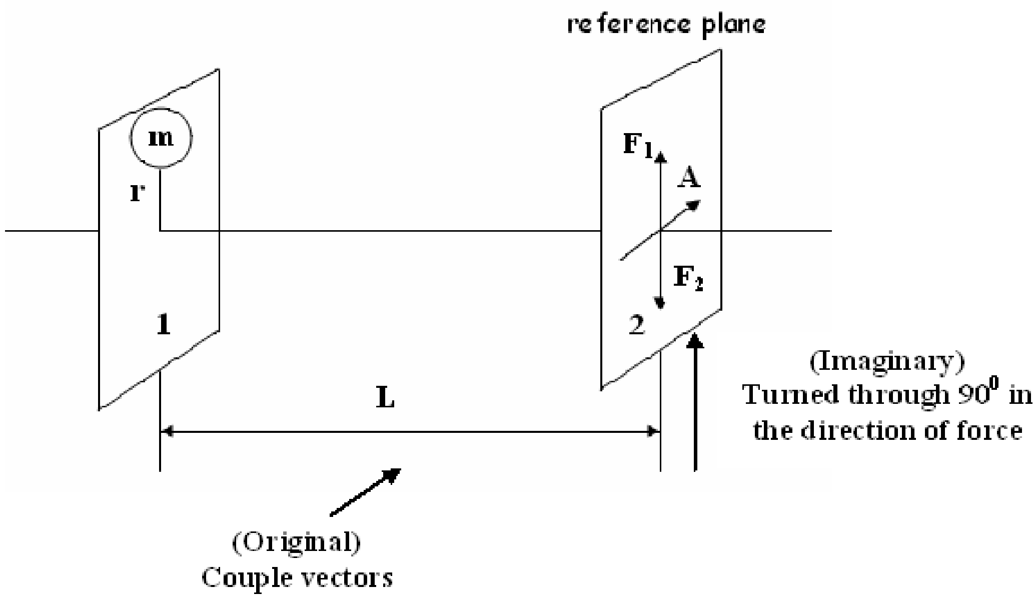
or

$$mr = \text{resultant of } m_1 r_1, m_2 r_2, m_3 r_3 \text{ and } m_4 r_4$$

CASE 4:

BALANCING OF SEVERAL MASSES ROTATING IN DIFFERENT PLANES

When several masses revolve in different planes, they may be transferred to a reference plane and this reference plane is a plane passing through a point on the axis of rotation and perpendicular to it.



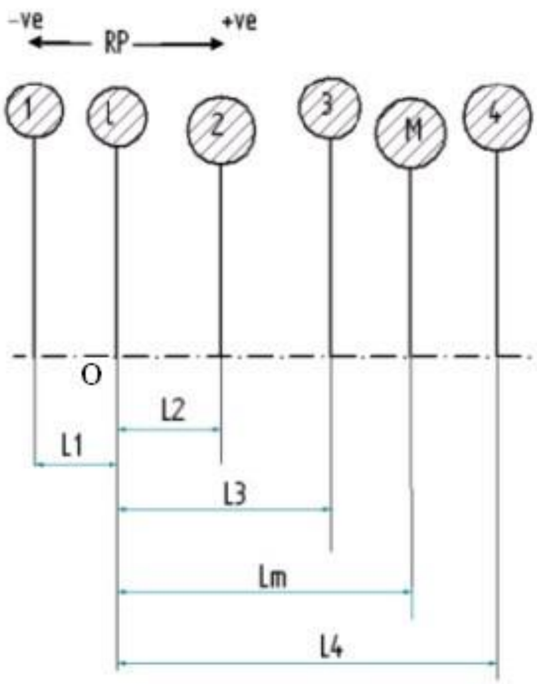
When a revolving mass in one plane is transferred to a reference plane, its effect is to cause a force of same magnitude to the centrifugal force of the revolving mass to act in the reference plane along with a couple of magnitude equal to the product of the force and the distance between the two planes.
In order to have a complete balance of the several revolving masses in different planes,

1. the forces in the reference plane must balance, i.e., the resultant force must be zero and
2. the couples about the reference plane must balance i.e., the resultant couple must be zero.

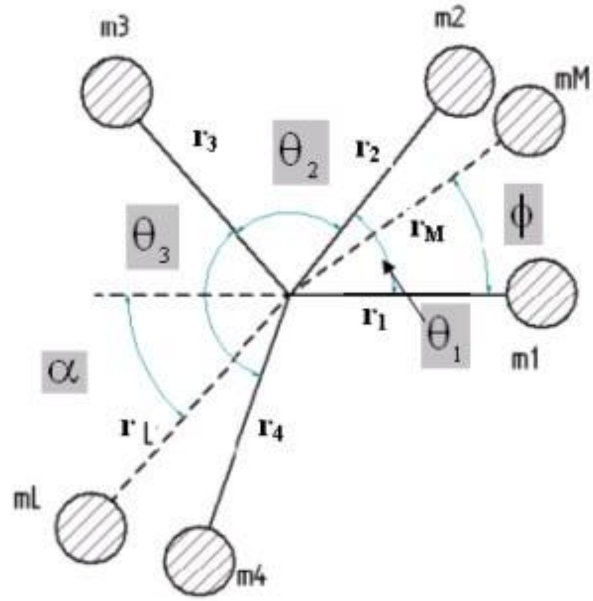
A mass placed in the reference plane may satisfy the first condition but the couple balance is satisfied only by two forces of equal magnitude in different planes. Thus, in general, two planes are needed to balance a system of rotating masses.

Example:

Consider four masses m_1, m_2, m_3 and m_4 attached to the rotor at radii r_1, r_2, r_3 and r_4 respectively. The masses m_1, m_2, m_3 and m_4 rotate in planes 1, 2, 3 and 4 respectively.



(a) position of planes of masses



(b) Angular position of masses

a) Position of planes of masses

Choose a reference plane at 'O' so that the distance of the planes 1, 2, 3 and 4 from 'O' are L_1, L_2 , L_3 and L_4 respectively. The reference plane chosen is plane 'L'. Choose another plane 'M' between plane 3 and 4 as shown.

Plane 'M' is at a distance of L_m from the reference plane 'L'. The distances of all the other planes to the left of 'L' may be taken as negative(-ve) and to the right may be taken as positive (+ve).

The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained by following the steps given below.

Step 1:

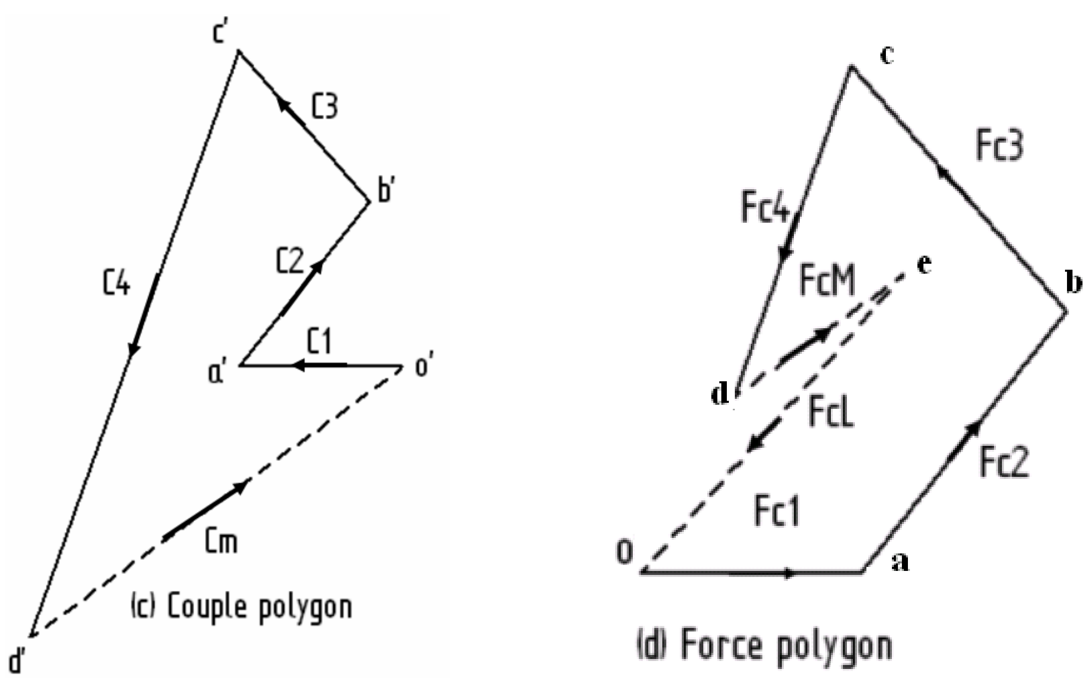
Tabulate the given data as shown after drawing the sketches of position of planes of masses and angular position of masses. The planes are tabulated in the same order in which they occur from left to right.

Plane 1	Mass (m) 2	Radius (r) 3	Centrifugal force/ω^2 (m r) 4	Distance from Ref. plane 'L' (L) 5	Couple/ ω^2 (m r L) 6
1	m_1	r_1	$m_1\, r_1$	$-L_1$	$-m_1\, r_1\, L_1$
L	m_L	r_L	$m_L\, r_L$	0	0
2	m_2	r_2	$m_2\, r_2$	L_2	$m_2\, r_2\, L_2$
3	m_3	r_3	$m_3\, r_3$	L_3	$m_3\, r_3\, L_3$
M	m_M	r_M	$m_M\, r_M$	L_M	$m_M\, r_M\, L_M$
4	m_4	r_4	$m_4\, r_4$	L_4	$m_4\, r_4\, L_4$

Step 2:

Construct the couple polygon first. (The couple polygon can be drawn by taking a convenient scale)

Add the known vectors and considering each vector parallel to the radial line of the mass draw the couple diagram. Then the closing vector will be '$m_M\, r_M\, L_M$'.



(c) Couple polygon

(d) Force polygon

The vector d 'o' on the couple polygon represents the balanced couple. Since the balanced couple C_M is proportional to $m_M\, r_M\, L_M$, therefore,

$$C_M = m_M\, r_M\, L_M = \text{vector } d'o'$$
$$\text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M\, L_M}$$

From this the value of m_M in the plane M can be determined and the angle of inclination ϕ of this mass may be measured from figure (b).

Step 3:
Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors along with 'm_M r_M'. The closing vector will be 'm_L r_L'. This represents the balanced force. Since the balanced force is proportional to 'm_L r_L' ,

$$m_L\, r_L = \text{vector } eo$$
$$\text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this the balancing mass m_L can be obtained in plane 'L' and the angle of inclination of this mass with the horizontal may be measured from figure (b).

Problems and solutions

Problem 1.
Four masses A, B, C and D are attached to a shaft and revolve in the same plane. The masses are 12 kg, 10 kg, 18 kg and 15 kg respectively and their radii of rotations are 40 mm, 50 mm, 60 mm and 30 mm. The angular position of the masses B, C and D are 60^0 , 135^0 and 270^0 from mass A. Find the magnitude and position of the balancing mass at a radius of 100 mm.

Solution:
Given:

Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 12 kg (reference mass)	r_A = 0.04 m	m_Ar_A = 0.48 kg-m	$\theta_A = 0^0$
m_B = 10 kg	r_B = 0.05 m	m_Br_B = 0.50 kg-m	$\theta_B = 60^0$
m_C = 18 kg	r_C = 0.06 m	m_Cr_C = 1.08 kg-m	$\theta_C = 135^0$
m_D = 15 kg	r_D = 0.03 m	m_Dr_D = 0.45 kg-m	$\theta_D = 270^0$

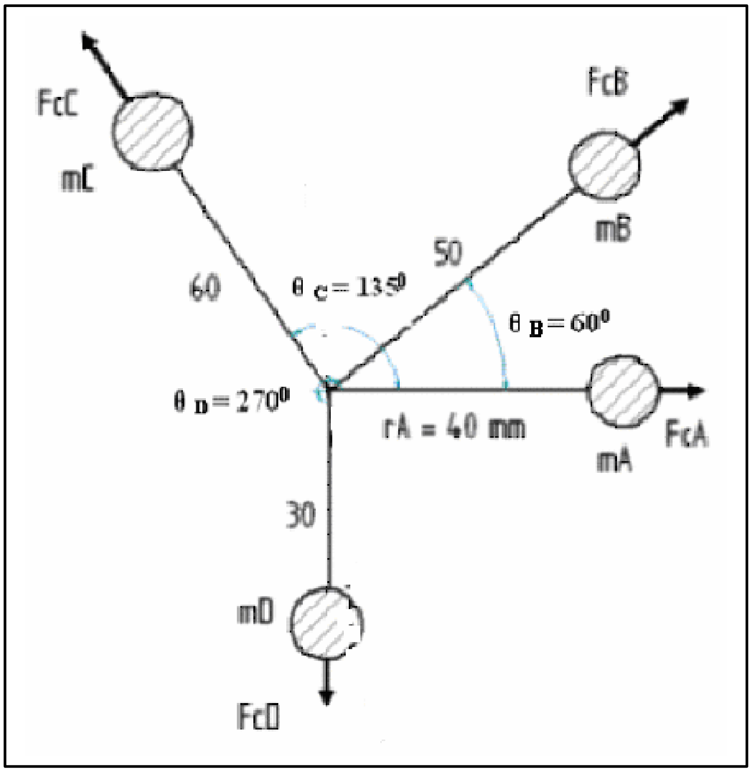
To determine the balancing mass 'm' at a radius of r = 0.1 m.

The problem can be solved by either analytical or graphical method.

Analytical Method:

Step 1:

Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.



Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Step 2:

Resolve the centrifugal forces horizontally and vertically and find their sum.
Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 0.48 \text{ x } \cos 0^0 + 0.50 \text{ x } \cos 60^0 + 1.08 \text{ x } \cos 135^0 + 0.45 \text{ x } \cos 270^0$$

$$= 0.48 + 0.25 + (-0.764) + 0 = -0.034 \text{ kg} - \text{m} \quad ----------(1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 0.48 \text{ x } \sin 0^0 + 0.50 \text{ x } \sin 60^0 + 1.08 \text{ x } \sin 135^0 + 0.45 \text{ x } \sin 270^0$$

$$= 0 + 0.433 + 0.764 + (-0.45) = 0.747 \text{ kg} - \text{m} \quad ----------(2)$$

Step 3:
Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(-0.034)^2 + (0.747)^2} = 0.748 \text{kg} - \text{m}$$

Step 4:
The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = mr = 0.748 \text{kg} - \text{m}$$

$$\text{Therefore,} \quad m = \frac{R}{r} = \frac{0.748}{0.1} = 7.48 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

Step 5:
Determine the position of the balancing mass 'm'.
If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{0.747}{-0.034} = -21.97$$

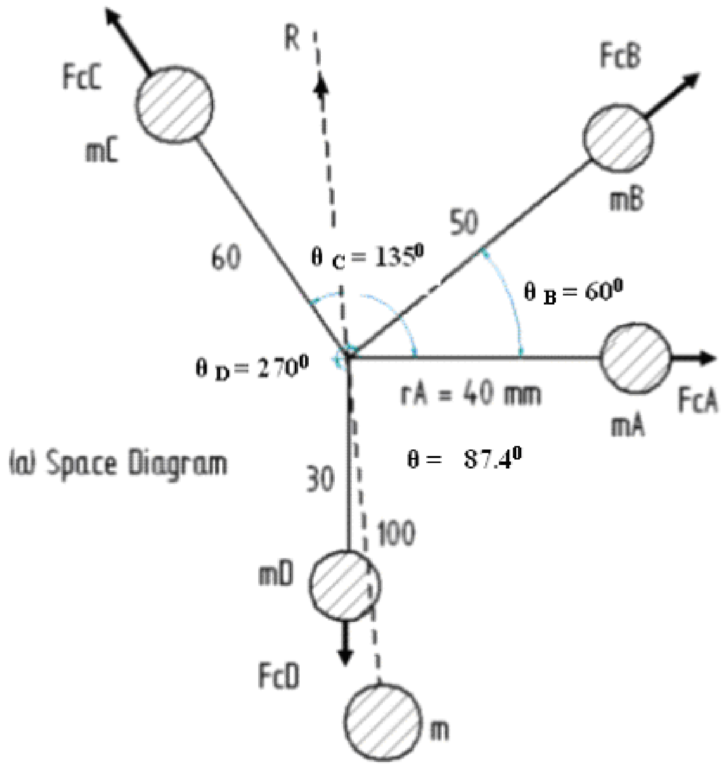
$$\text{and } \theta = -87.4^0 \text{ or } 92.6^0$$

Remember ALL STUDENTS TAKE COPY i.e. in first quadrant all angles **($\sin\theta$, $\cos\theta$ and $\tan\theta$)** are positive, in second quadrant only **$\sin\theta$** is positive, in third quadrant only **$\tan\theta$** is positive and in fourth quadrant only **$\cos\theta$** is positive.

Since numerator is positive and denominator is negative, the resultant force makes with the horizontal, an angle (measured in the counter clockwise direction)

$$\theta = 92.6^0$$

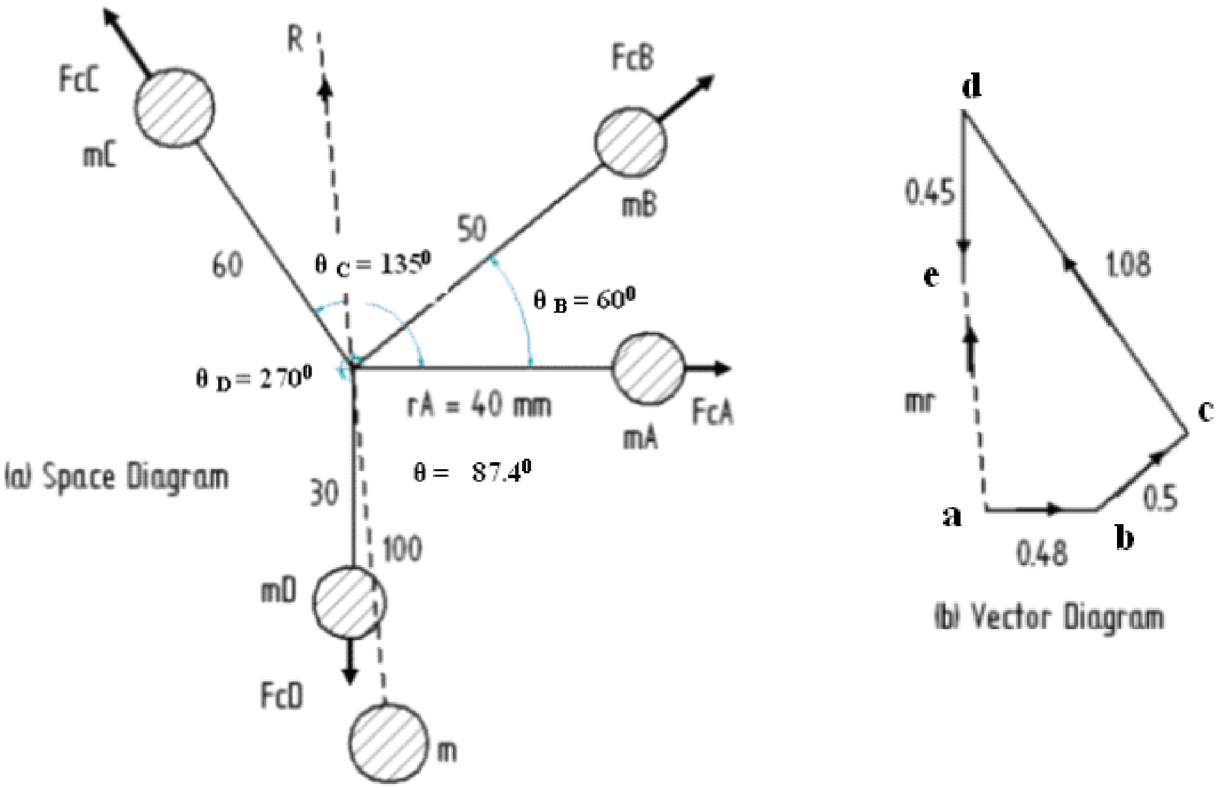
The balancing force is then equal to the resultant force, but in opposite direction.
The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.



Graphical Method:

Step 1:
Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Draw the space diagram or angular position of the masses taking the actual angles(Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$).



Step 2:

Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors as follows.

Draw a line 'ab' parallel to force F_{CA} (or the product m_Ar_A to a proper scale) of the space diagram. At 'b' draw a line 'bc' parallel to F_{CB} (or the product m_Br_B). Similarly draw lines 'cd', 'de' parallel to F_{CC} (or the product m_Cr_C) and F_{CD} (or the product m_Dr_D) respectively. The closing side 'ae' represents the resultant force 'R' in magnitude and direction as shown on the vector diagram.

Step 3:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore,} \quad m = \frac{R}{r} = 7.48 \text{ kg Ans}$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.

Problem 2:

The four masses A, B, C and D are 100 kg, 150 kg, 120 kg and 130 kg attached to a shaft and revolve in the same plane. The corresponding radii of rotations are 22.5 cm, 17.5 cm, 25 cm and 30 cm and the angles measured from A are 45^0, 120^0 and 255^0. Find the position and magnitude of the balancing mass, if the radius of rotation is 60 cm.

Solution:

Analytical Method:

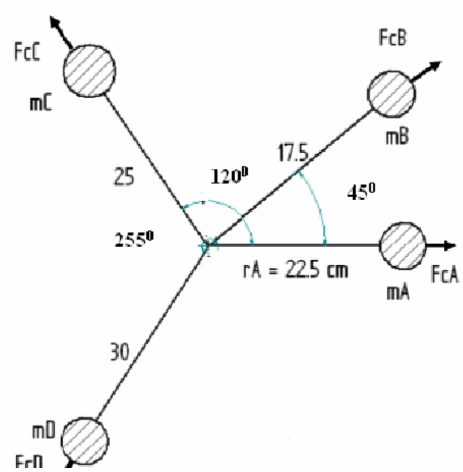
Given:

Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 100 kg (reference mass)	r_A = 0.225 m	$m_A r_A$ = 22.5 kg-m	$\theta_A = 0^0$
m_B = 150 kg	r_B = 0.175 m	$m_B r_B$ = 26.25 kg-m	$\theta_B = 45^0$
m_C = 120 kg	r_C = 0.250 m	$m_C r_C$ = 30 kg-m	$\theta_C = 120^0$
m_D = 130 kg	r_D = 0.300 m	$m_D r_D$ = 39 kg-m	$\theta_D = 255^0$
m =?	r = 0.60		θ = **?**

Step 1:

Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.



Step 2:
Resolve the centrifugal forces horizontally and vertically and find their sum.

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} \mathbf{m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D}$$

$$\mathbf{= 22.5 \ x \ \cos 0^0 + 26.25 \ x \ \cos 45^0 + 30 \ x \ \cos 120^0 + 39 \ x \ \cos 255^0}$$

$$\mathbf{= 22.5 + 18.56 + (-15) + (-10.1) = 15.97 \ kg - m \quad ---------- (1)}$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} \mathbf{m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D}$$

$$\mathbf{= 22.5 \ x \ \sin 0^0 + 26.25 \ x \ \sin 45^0 + 30 \ x \ \sin 120^0 + 39 \ x \ \sin 255^0}$$

$$\mathbf{= 0 + 18.56 + 25.98 + (-37.67) = 6.87 \ kg - m \quad ---------- (2)}$$

Step 3:
Determine the magnitude of the resultant centrifugal force

$$\mathbf{R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}}$$

$$\mathbf{= \sqrt{(15.97)^2 + (6.87)^2} = 17.39 \ kg - m}$$

Step 4:
The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$\mathbf{R = m\,r = 17.39 \ kg - m}$$

$$\mathbf{Therefore, \quad m = \frac{R}{r} = \frac{17.39}{0.60} = 28.98 \ kg \ Ans}$$

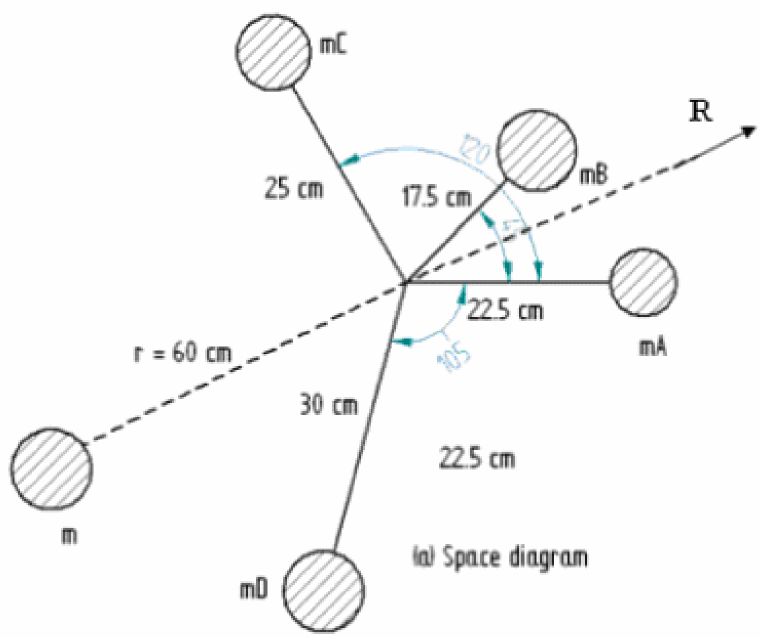
Where, m = balancing mass and r = its radius of rotation

Step 5:
Determine the position of the balancing mass ‘m’.
If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{6.87}{15.97} = 0.4302$$

and $\theta = 23.28^0$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203.28^0$ angle measured in the counter clockwise direction.



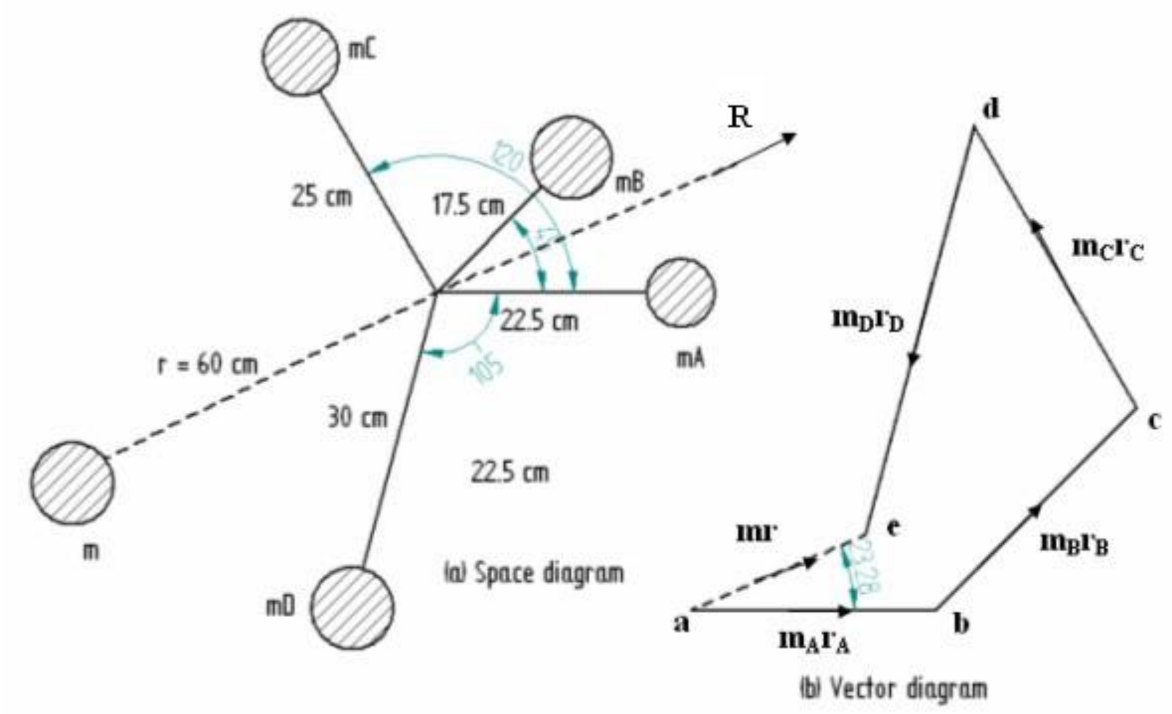
Graphical Method:

Step 1:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Step 2:

Draw the space diagram or angular position of the masses taking the actual angles (Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$).



Draw a line 'ab' parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At 'b' draw a line 'bc' parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines 'cd', 'de' parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side 'ae' represents the resultant force 'R' in magnitude and direction as shown on the vector diagram.

Step 4:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore,} \quad m = \frac{R}{r} = 29 \text{ kg Ans}$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203^0$ angle measured in the counter clockwise direction.

Problem 3:

A rotor has the following properties.

Mass	magnitude	Radius	Angle	Axial distance from first mass
1	9 kg	100 mm	$\theta_A = 0^0$	-
2	7 kg	120 mm	$\theta_B = 60^0$	160 mm
3	8 kg	140 mm	$\theta_C = 135^0$	320 mm
4	6 kg	120 mm	$\theta_D = 270^0$	560 mm

If the shaft is balanced by two counter masses located at 100 mm radii and revolving in planes midway of planes 1 and 2, and midway of 3 and 4, determine the magnitude of the masses and their respective angular positions.

Solution:

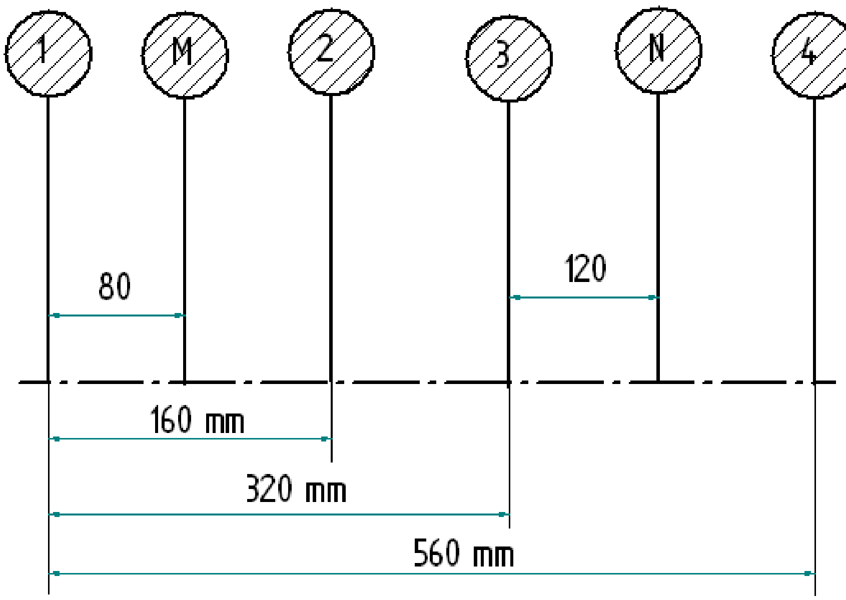
Analytical Method:

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'M' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
1	9.0	0.10	$m_1 r_1 = 0.9$	-0.08	-0.072	0^0
M	m_M = ?	0.10	$m_M r_M = 0.1\ m_M$	0	0	θ_M = **?**
2	7.0	0.12	$m_2 r_2 = 0.84$	0.08	0.0672	60^0
3	8.0	0.14	$m_3 r_3 = 1.12$	0.24	0.2688	135^0
N	m_N = ?	0.10	$m_N r_N = 0.1\ m_N$	0.36	$m_N r_N l_N = 0.036\ m_N$	θ_N = **?**
4	6.0	0.12	$m_4 r_4 = 0.72$	0.48	0.3456	270^0

For dynamic balancing the conditions required are,

$$\sum mr + m_M r_M + m_N r_N = 0 \text{ ----------(I)} \quad \text{for force balance}$$

$$\sum mrl + m_N r_N l_N = 0 \text{ --------------(II)} \quad \text{for couple balance}$$



(a) Position of planes of masses

Step 1:
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_N r_N l_N \cos\theta_N = 0$$

On substitution we get

$$-0.072\cos 0^0 + 0.0672\cos 60^0 + 0.2688\cos 135^0$$
$$+0.3456\cos 270^0 + 0.036\, m_N \cos\theta_N = 0$$
$$\text{i.e. } 0.036\, m_N \cos\theta_N = 0.2285 -----(1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_N r_N l_N \sin\theta_N = 0$$

On substitution we get

$$-0.072\sin 0^0 + 0.0672\sin 60^0 + 0.2688\sin 135^0$$
$$+0.3456\sin 270^0 + 0.036\, m_N \sin\theta_N = 0$$
$$\text{i.e. } 0.036\, m_N \sin\theta_N = 0.09733 -----(2)$$

Squaring and adding (1) and (2), we get

$$m_N r_N l_N = \sqrt{(0.2285)^2 + (0.09733)^2}$$
$$\text{i.e., } 0.036 m_N = 0.2484$$
$$\text{Therefore,} \quad m_N = \frac{0.2484}{0.036} = 6.9\text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_N = \frac{0.09733}{0.2285} \quad \text{and} \quad \theta_N = 23.07^0$$

Step 2:
Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_M r_M \cos\theta_M + m_N r_N \cos\theta_N = 0$$
On substitution we get
$$0.9\cos 0^0 + 0.84\cos 60^0 + 1.12\cos 135^0 + 0.72\cos 270^0$$
$$+ m_M r_M \cos\theta_M + 0.1 \text{x} 6.9 \text{x} \cos 23.07^0 = 0$$
$$\text{i.e. } m_M r_M \cos\theta_M = -1.1629 -----(3)$$

Sum of the vertical components gives,

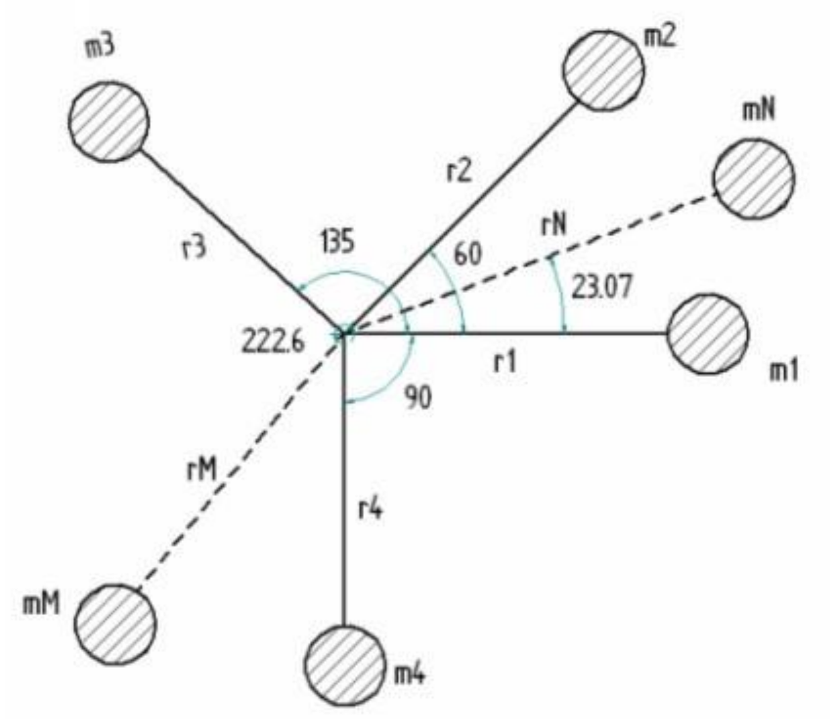
$$\sum mr\sin\theta + m_M r_M \sin\theta_M + m_N r_N \sin\theta_N = 0$$
On substitution we get
$$0.9\sin 0^0 + 0.84\sin 60^0 + 1.12\sin 135^0 + 0.72\sin 270^0$$
$$+ m_M r_M \sin\theta_M + 0.1 \text{x} 6.9 \text{x} \sin 23.07^0 = 0$$
$$\text{i.e. } m_M r_M \sin\theta_M = -1.0698 -----(4)$$

Squaring and adding (3) and (4), we get

$$m_M r_M = \sqrt{(-1.1629)^2 + (-1.0698)^2}$$
$$\text{i.e., } 0.1 m_M = 1.580$$
$$\text{Therefore,} \quad m_M = \frac{1.580}{0.1} = 15.8\text{ kg Ans}$$

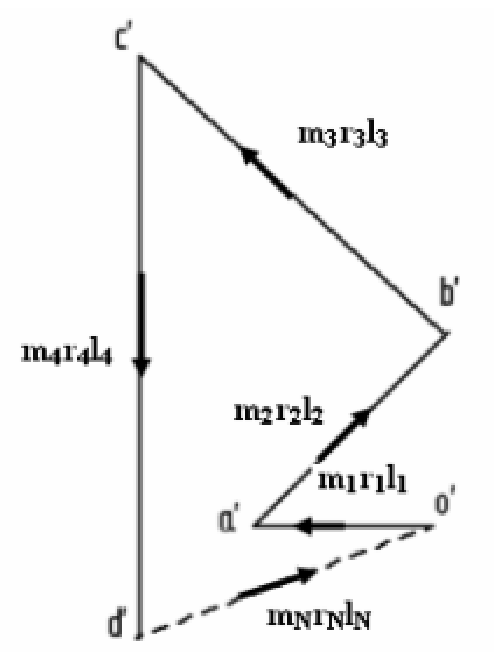
Dividing (4) by (3), we get

$$\tan\theta_M = \frac{-1.0698}{-1.1629} \quad \text{and} \quad \theta_M = 222.61^0 \text{ Ans}$$

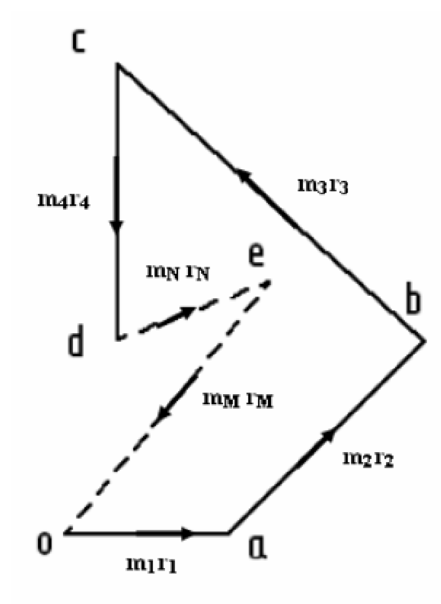


(b) Angular position of masses

Graphical Solution:



(c) Couple polygon



(d) Force polygon

Problem 4:
The system has the following data.

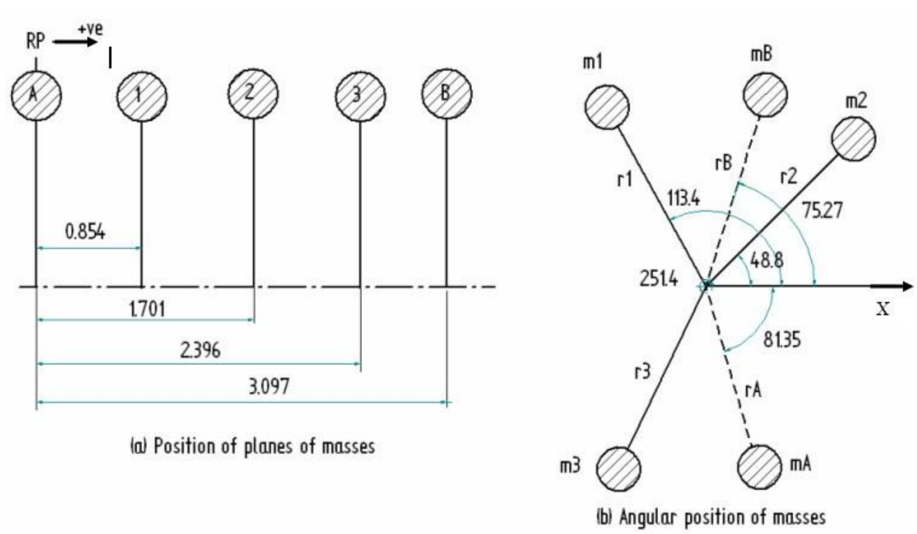
$\mathbf{m_1} = 1.2\ \mathbf{kg}$	$\mathbf{r_1} = 1.135\ \mathbf{m} @ \angle 113.4^0$
$\mathbf{m_1} = 1.8\ \mathbf{kg}$	$\mathbf{r_2} = 0.822\ \mathbf{m} @ \angle 48.8^0$
$\mathbf{m_1} = 2.4\ \mathbf{kg}$	$\mathbf{r_3} = 1.04\ \mathbf{m} @ \angle 251.4^0$

The distances of planes in metres from plane A are:

$$l_1 = 0.854,\ l_2 = 1.701,\ l_3 = 2.396,\ l_B = 3.097$$

Find the mass-radius products and their angular locations needed to dynamically balance the system using the correction planes A and B.

Solution: **Analytical Method**



(a) Position of planes of masses

(b) Angular position of masses

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	m_A	r_A	$m_A r_A$ =?	0	0	$\theta_A = ?$
1	1.2	1.135	1.362	0.854	1.163148	113.4^0
2	1.8	0.822	1.4796	1.701	2.5168	48.8^0
3	2.4	1.04	2.496	2.396	5.9804	251.4^0
B	m_B	r_B	$m_B r_B$ =?	3.097	3.097 $m_B r_B$	$\theta_B = ?$

Step 1:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_B r_B l_B \cos\theta_B = 0$$

On substitution we get

$$1.163148\cos 113.4^0 + 2.5168\cos 48.8^0 + 5.9804\cos 251.4^0$$
$$+3.097\, m_B r_B \cos\theta_B = 0$$

$$\text{i.e. } m_B r_B \cos\theta_B = \frac{0.71166}{3.097} ------(1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_B r_B l_B \sin\theta_B = 0$$

On substitution we get

$$1.163148\sin 113.4^0 + 2.5168\sin 48.8^0 + 5.9804\sin 251.4^0$$
$$+3.097\, m_B r_B \sin\theta_B = 0$$

$$\text{i.e. } m_B r_B \sin\theta_B = \frac{2.7069}{3.097} ------(2)$$

Squaring and adding (1) and (2), we get

$$m_B r_B = \sqrt{\left(\frac{0.71166}{3.097}\right)^2 + \left(\frac{2.7069}{3.097}\right)^2}$$
$$= 0.9037\,\text{kg} - \text{m}$$

Dividing (2) by (1), we get

$$\tan\theta_B = \frac{2.7069}{0.71166} \quad \text{and} \quad \theta_B = 75.27^0 \ \text{Ans}$$

Step 2:

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_A r_A \cos\theta_A + m_B r_B \cos\theta_B = 0$$

On substitution we get

$$1.362\cos 113.4^0 + 1.4796\cos 48.8^0 + 2.496\cos 251.4^0$$
$$+ m_A r_A \cos\theta_A + 0.9037\cos 75.27^0 = 0$$

Therefore

$$m_A r_A \cos\theta_A = 0.13266 ----------(3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_A r_A \sin\theta_A + m_B r_B \sin\theta_B = 0$$

On substitution we get

$$1.362\sin 113.4^0 + 1.4796\sin 48.8^0 + 2.496\sin 251.4^0$$
$$+ m_A r_A \sin\theta_A + 0.9037\sin 75.27^0 = 0$$

Therefore

$$m_A r_A \sin\theta_A = -0.87162 ----------(4)$$

Squaring and adding (3) and (4), we get

$$m_A r_A = \sqrt{(0.13266)^2 + (-0.87162)^2}$$
$$= 0.8817 \text{ kg} - \text{m}$$

Dividing (4) by (3), we get

$$\tan\theta_A = \frac{-0.87162}{0.13266} \quad \text{and} \quad \theta_A = -81.35^0 \text{ Ans}$$

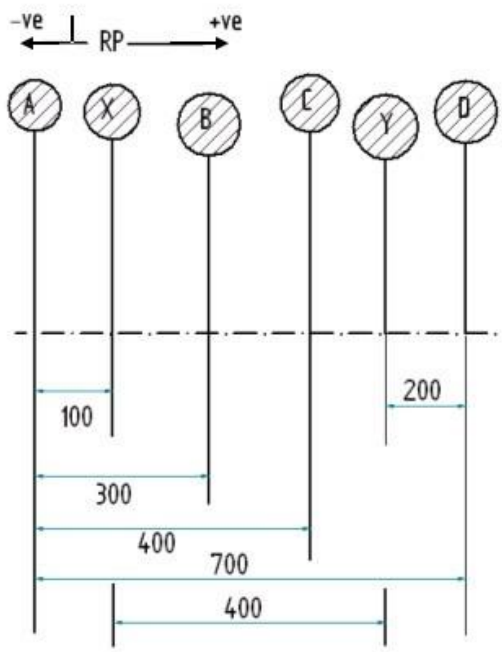
Problem 5:

A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45^0, B to C 70^0 and C to D 120^0. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

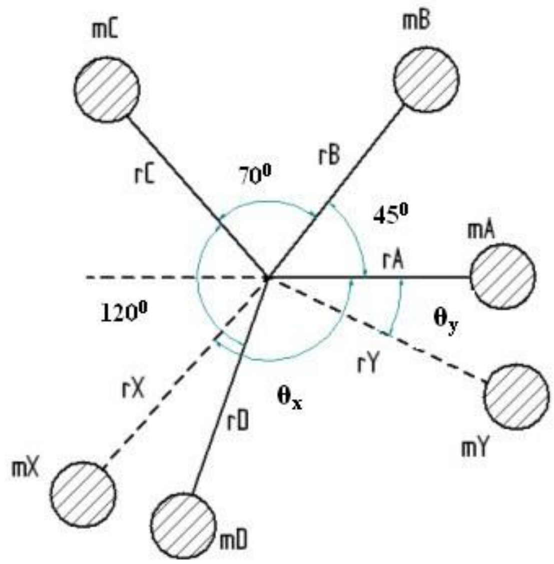
Graphical solution:

Let, m_X be the balancing mass placed in plane X and m_Y be the balancing mass placed in plane Y which are to be determined.

Step 1:
Draw the position of the planes as shown in figure (a).



(a) position of planes of masses



(b) Angular position of masses

Let X be the reference plane (R.P.). The distances of the planes to the right of the plane X are taken as positive (+ve) and the distances of planes to the left of X plane are taken as negative(-ve). The data may be tabulated as shown

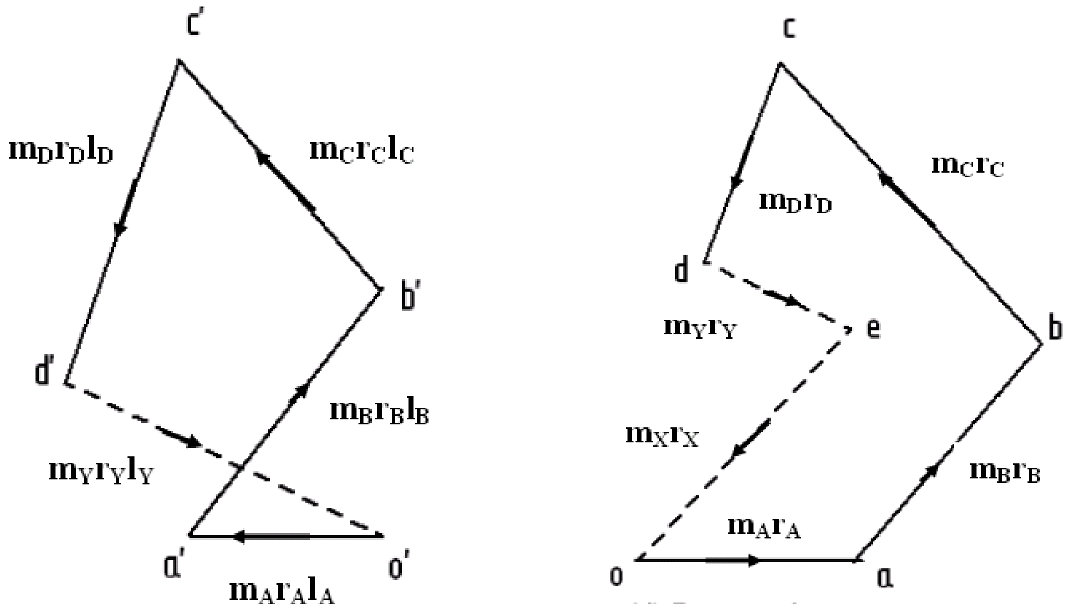
Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product ‘m r’ can be calculated and tabulated. Similarly the magnitude of the couples are proportional to the product of the mass , its radius and the axial distance from the reference plane, the product ‘m r l’ can be calculated and tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'X' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	200	0.08	$m_A r_A = 16$	-0.10	-1.60	-
X	m_X =?	0.10	$m_X r_X = 0.1 m_X$	0	0	θ_x =**?**
B	300	0.07	$m_B r_B = 21$	0.20	4.20	A to B 45^0
C	400	0.06	$m_C r_C = 24$	0.30	7.20	B to C 70^0
Y	m_Y =?	0.10	$m_Y r_Y = 0.1 m_Y$	0.40	$m_Y r_Y l_Y = 0.04 m_Y$	θ_Y =**?**
D	200	0.08	$m_D r_D = 16$	0.60	9.60	C to D 120^0

Step 2:

Assuming the mass A as horizontal draw the sketch of angular position of masses as shown in figure (b).

Step 3:
Draw the couple polygon to some suitable scale by taking the values of 'm r l' (column no. 6) of the table as shown in figure (c).



(c) Couple polygon

(d) Force polygon

Draw line o'a' parallel to the radial line of mass m_A.
At a' draw line a'b' parallel to radial line of mass m_B.
Similarly, draw lines b'c', c'd' parallel to radial lines of masses m_C and m_D respectively.
Now, join d' to o' which gives the balanced couple.

We get, $$0.04\ m_Y = \text{vector } d'o' = 7.3\ kg - m^2$$
$$\text{or} \quad m_Y = 182.5\ kg \ \text{ Ans}$$

Step 4:
To find the angular position of the mass m_Y draw a line om_Y in figure (b) parallel to d'o' of the couple polygon.

By measurement we get $\theta_Y = 12^0$ in the clockwise direction from m_A.

Step 5:
Now draw the force polygon by considering the values of 'm r' (column no. 4) of the table as shown in figure (d).
Follow the similar procedure of step 3. The closing side of the force polygon i.e. 'e o' represents the balanced force.

$$m_x r_x = \text{vector } eo = 35.5\ kg - m$$
$$\text{or} \quad m_x = 355\ kg \ \text{ Ans}$$

Step 6:
The angular position of m_X is determined by drawing a line om_X parallel to the line 'e o' of the force polygon in figure (b). From figure (b) we get,
$\theta_X = 145^0$, measured clockwise from m_A. Ans

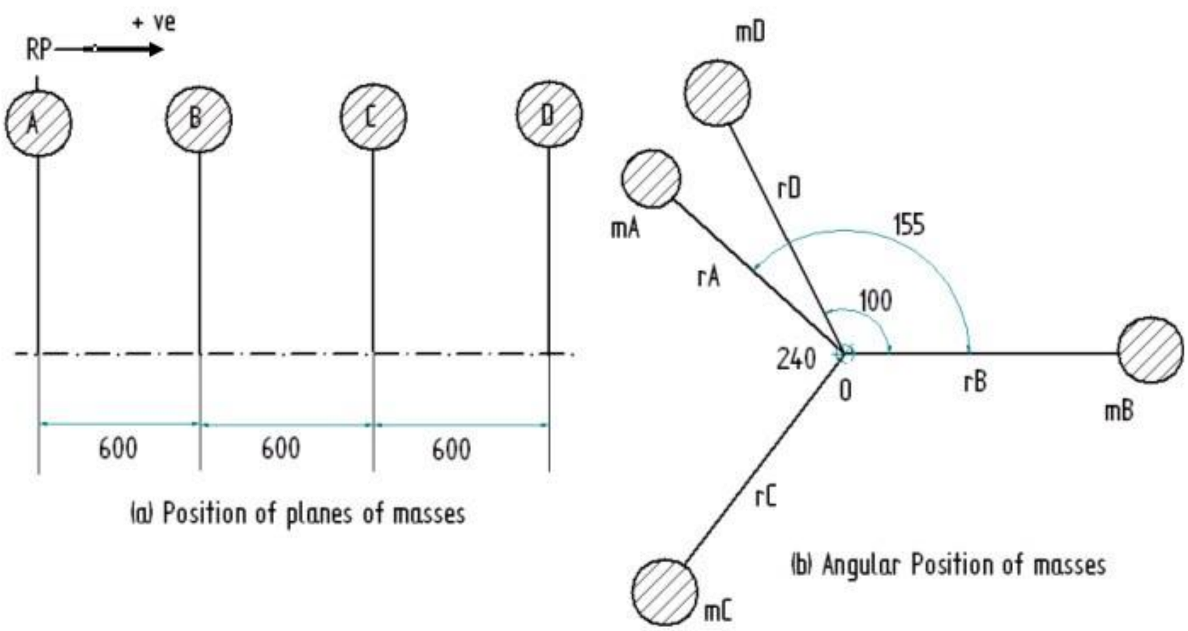
Problem 6:
A, B, C and D are four masses carried by a rotating shaft at radii 100 mm, 125 mm, 200 mm and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg and 4 kg respectively. Find the required mass A and relative angular settings of the four masses so that the shaft shall be in complete balance.
Solution:

Graphical Method:

Step 1:

Let, m_A be the balancing mass placed in plane A which is to be determined along with the relative angular settings of the four masses.
Let A be the reference plane (R.P.).
Assume the mass B as horizontal
Draw the sketch of angular position of mass m_B (line om_B) as shown in figure (b). The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.1	$m_A\ r_A$ = 0.1 m_A	0	0	$\theta_A = ?$
B	10	0.125	$m_B\ r_B$ = 1.25	0.6	0.75	$\theta_B = 0$
C	5	0.2	$m_C\ r_C$ = 1.0	1.2	1.2	$\theta_C = ?$
D	4	0.15	$m_D\ r_D$ = 0.6	1.8	1.08	$\theta_D = ?$



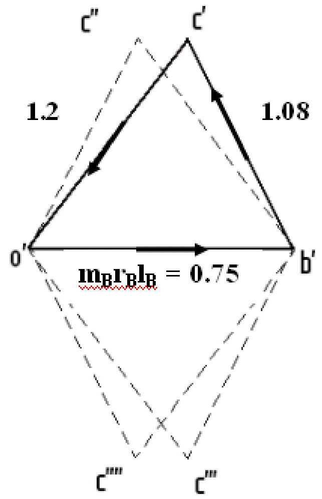
(a) Position of planes of masses

(b) Angular Position of masses

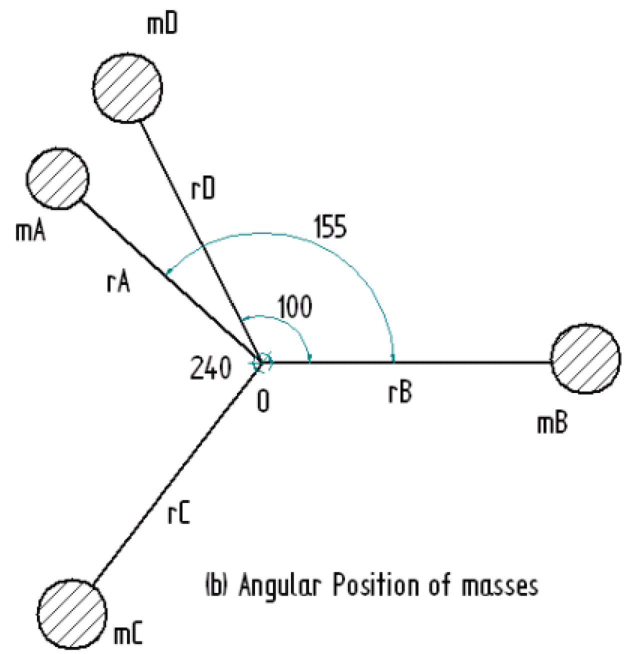
Draw a line o'b' equal to 0.75 kg-m^2 parallel to the line om$_B$. At point o' and b' draw vectors o'c' and b'c' equal to 1.2 kg-m^2 and 1.08 kg-m^2 respectively. These vectors intersect at point c'.

For the construction of force polygon there are four options.

Any one option can be used and relative to that the angular settings of mass C and D are determined.



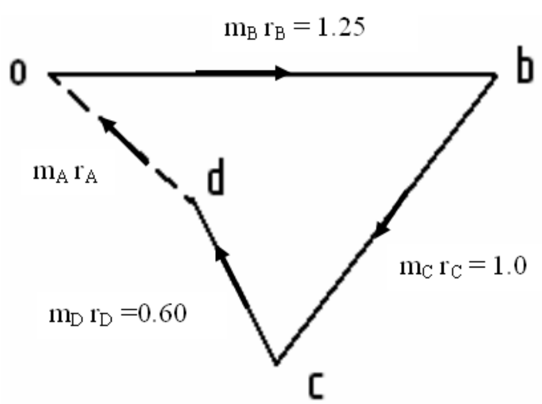
(c) Couple polygon



(b) Angular Position of masses

$$\theta_D = 100^0 \quad \text{and} \quad \theta_C = 240^0 \quad \text{Ans}$$

Step 4:
In order to find m_A and its angular setting draw the force polygon as shown in figure (d).



(d) Force polygon

Closing side of the force polygon od represents the product $m_A\, r_A$. i.e.

$$m_A\ r_A = 0.70\ kg-m$$

$$\text{Therefore,} \qquad m_A = \frac{0.70}{r_A} = 7\ kg \quad \text{Ans}$$

Step 5:

Now draw line om_A parallel to od of the force polygon. By measurement, we get,

$$\theta_A = 155^0 \qquad \text{Ans}$$

Problem 7:
A shaft carries three masses A, B and C. Planes B and C are 60 cm and 120 cm from A. A , B and C are 50 kg, 40 kg and 60 kg respectively at a radius of 2.5 cm. The angular position of mass B and mass C with A are 90^0 and 210^0 respectively. Find the unbalanced force and couple. Also find the position and magnitude of balancing mass required at 10 cm radius in planes L and M midway between A and B, and B and C.

Solution:

Case (i):

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	50	0.025	$m_A\ r_A = 1.25$	0	0	$\theta_A = 0^0$
B	40	0.025	$m_B\ r_B = 1.00$	0.6	0.6	$\theta_B = 90^0$
C	60	0.025	$m_C\ r_C = 1.50$	1.2	1.8	$\theta_C = 210^0$

Analytical Method

Step 1:
Determination of unbalanced couple
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta = 0.6\cos 90^0 + 1.8\cos 210^0 = -1.559 - - - - - (1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta = 0.6\sin 90^0 + 1.8\sin 210^0 = -0.3 - - - - - (2)$$

Squaring and adding (1) and (2), we get

$$C_{unbalanced} = \sqrt{(-1.559)^2 + (-0.3)^2}$$
$$= 1.588 \, kg - m^2$$

Step 2:
Determination of unbalanced force

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta = 1.25\cos 0^0 + 1.0\cos 90^0 + 1.5\cos 210^0$$
$$= 1.25 + 0 + (-1.299) = -0.049 ---------(3)$$

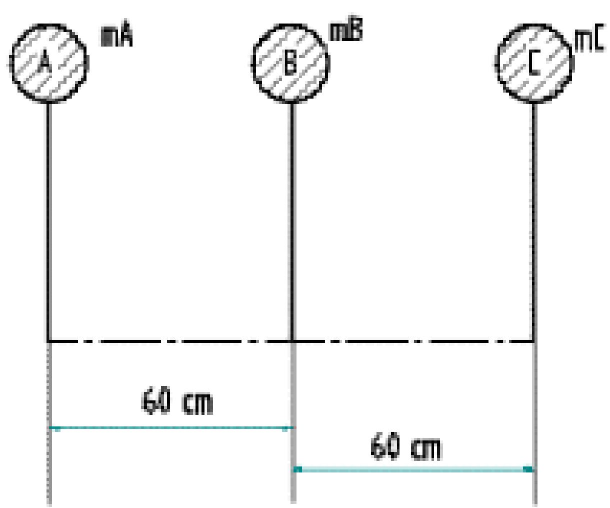
Sum of the vertical components gives,

$$\sum mr\sin\theta = 1.25\sin 0^0 + 1.0\sin 90^0 + 1.5\sin 210^0$$
$$= 0 + 1.0 + (-0.75) = 0.25 ---------(4)$$

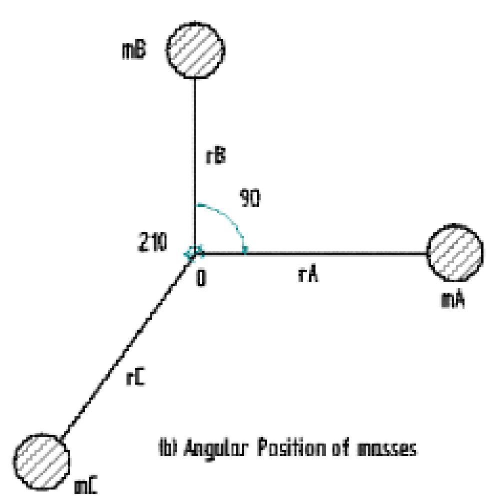
Squaring and adding (3) and (4), we get

$$F_{unbalanced} = \sqrt{(-0.049)^2 + (0.25)^2}$$
$$= 0.2548 \; kg - m$$

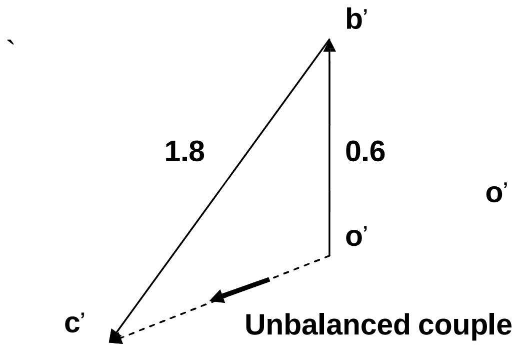
Graphical solution:



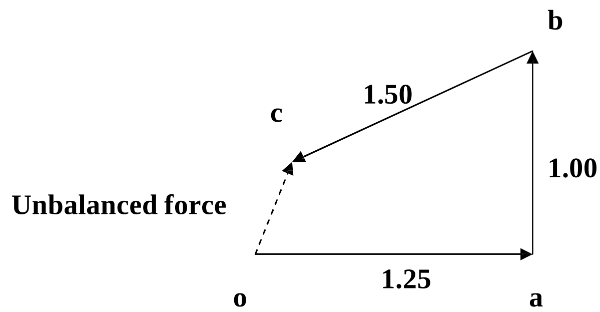
(a) Position of planes of masses



(b) Angular Position of masses

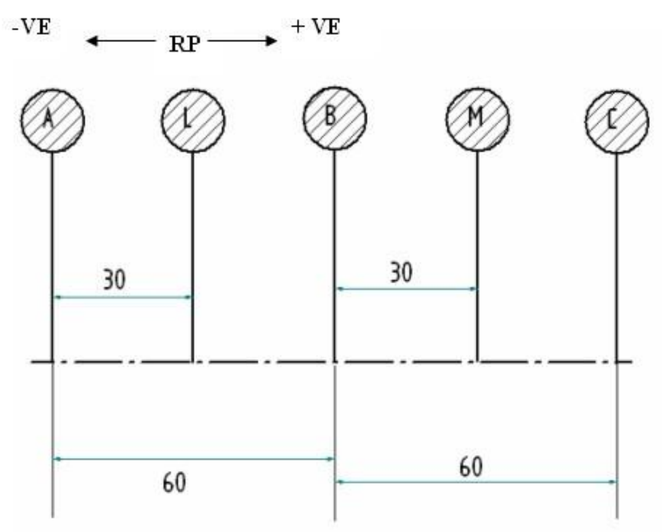


Couple polygon

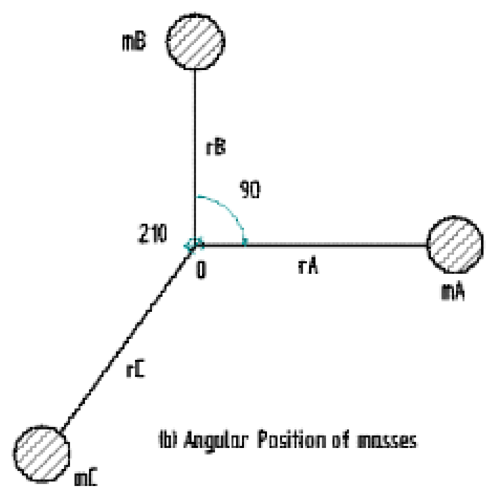


Force polygon

Case (ii):



(a) Position of planes of masses



(b) Angular Position of masses

To determine the magnitude and directions of masses m_M and m_L.

Let, m_L be the balancing mass placed in plane L and m_M be the balancing mass placed in plane M which are to be determined.

The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'L' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	50	0.025	m_A r_A = 1.25	-0.3	-0.375	$\theta_A = 0^0$
L (R.P.)	m_L = ?	0.10	0.1 m_L	0	0	θ_L = ?
B	40	0.025	m_B r_B = 1.00	0.3	0.3	$\theta_B = 90^0$
M	m_M = ?	0.10	0.1 m_M	0.6	0.06 m_M	θ_M = ?
C	60	0.025	m_C r_C = 1.50	0.9	1.35	$\theta_C = 210^0$

Analytical Method:

Step 1:
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum m r l \cos\theta + m_M r_M l_M \cos\theta_M = 0$$

On substitution we get

$$-0.375 \cos 0^0 + 0.3 \cos 90^0 + 0.06\, m_M \cos\theta_M + 1.35 \cos 210^0 = 0$$

i.e. $-0.375 + 0 + 0.06\, m_M \cos\theta_M + (-1.16913) = 0$

$$0.06\, m_M \cos\theta_M = 1.54413$$

$$m_M \cos\theta_M = \frac{1.54413}{0.06} = 25.74 \quad -----(1)$$

Sum of the vertical components gives,

$$\sum m r l \sin\theta + m_M r_M l_M \sin\theta_M = 0$$

On substitution we get

$$-0.375 \sin 0^0 + 0.3 \sin 90^0 + 0.06\, m_M \sin\theta_M + 1.35 \sin 210^0 = 0$$

i.e. $0 + 0.3 + 0.06\, m_M \sin\theta_M + (-0.675) = 0$

$$0.06\, m_M \sin\theta_M = 0.375$$

$$m_M \sin\theta_M = \frac{0.375}{0.06} = 6.25 \quad -----(2)$$

Squaring and adding (1) and (2), we get

$$(m_M \cos\theta_M)^2 + (m_M \sin\theta_M)^2 = (25.74)^2 + (6.25)^2 = 701.61$$

$$\text{i.e. } m_M^2 = 701.61 \quad \text{and} \quad m_M = 26.5 \text{ kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_M = \frac{6.25}{25.74} \quad \text{and} \quad \theta_M = 13.65^0 \text{ Ans}$$

Step 2:
Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_L r_L \cos\theta_L + m_M r_M \cos\theta_M = 0$$

On substitution we get

$$1.25\cos 0^0 + 0.1 m_L \cos\theta_L + 1.0\cos 90^0 + 2.649\cos 13.65^0 + 1.5\cos 210^0 = 0$$

$$1.25 + 0.1 m_L \cos\theta_L + 0 + 2.5741 + (-1.299) = 0$$

Therefore

$$0.1 m_L \cos\theta_L + 2.5251 = 0$$

$$\text{and} \quad m_L \cos\theta_L = \frac{-2.5251}{0.1} = -25.251 \text{----------}(3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_L r_L \sin\theta_L + m_M r_M \sin\theta_M = 0$$

On substitution we get

$$1.25\sin 0^0 + 0.1 m_L \sin\theta_L + 1.0\sin 90^0 + 2.649\sin 13.65^0 + 1.5\sin 210^0 = 0$$

$$0 + 0.1 m_L \sin\theta_L + 1 + 0.6251 + (-0.75) = 0$$

Therefore

$$0.1 m_L \sin\theta_L + 0.8751 = 0$$

$$\text{and} \quad m_L \sin\theta_L = \frac{-0.8751}{0.1} = -8.751 \text{----------}(4)$$

Squaring and adding (3) and (4), we get

$$(m_L \cos\theta_L)^2 + (m_L \sin\theta_L)^2 = (-25.251)^2 + (-8.751)^2 = 714.193$$

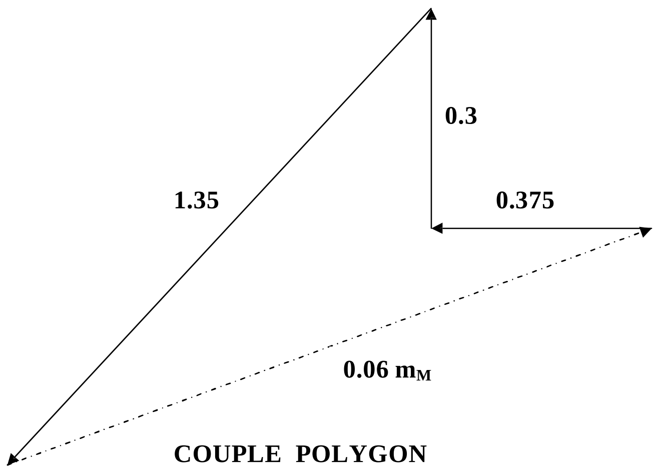
$$\text{i.e. } m_L^2 = 714.193 \quad \text{and} \quad m_L = 26.72 \text{ kg Ans}$$

Dividing (4) by (3), we get

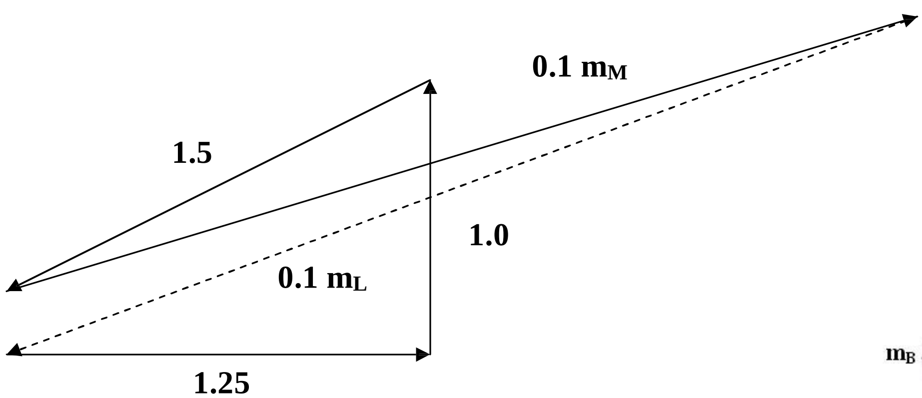
$$\tan\theta_L = \frac{-8.751}{-25.251} \quad \text{and} \quad \theta_L = 19.11^0 \text{ Ans}$$

The balancing mass m_L is at an angle $19.11^0 + 180^0 = 199.11^0$ measured in counter clockwise direction.

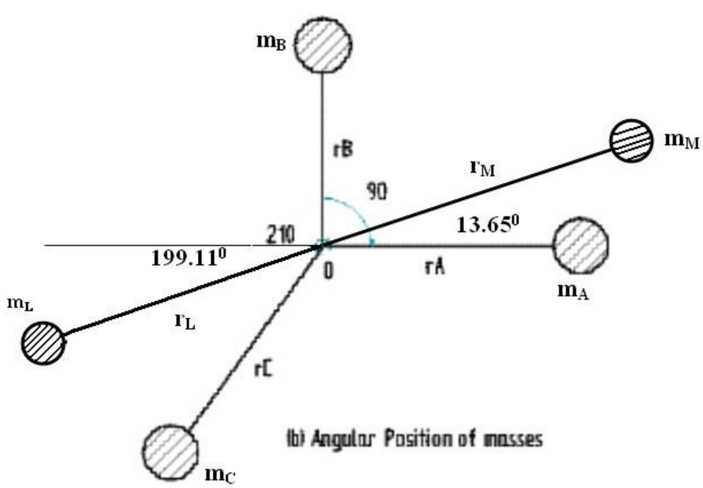
Graphical Method:



COUPLE POLYGON



FORCE POLYGON



Problem 8:

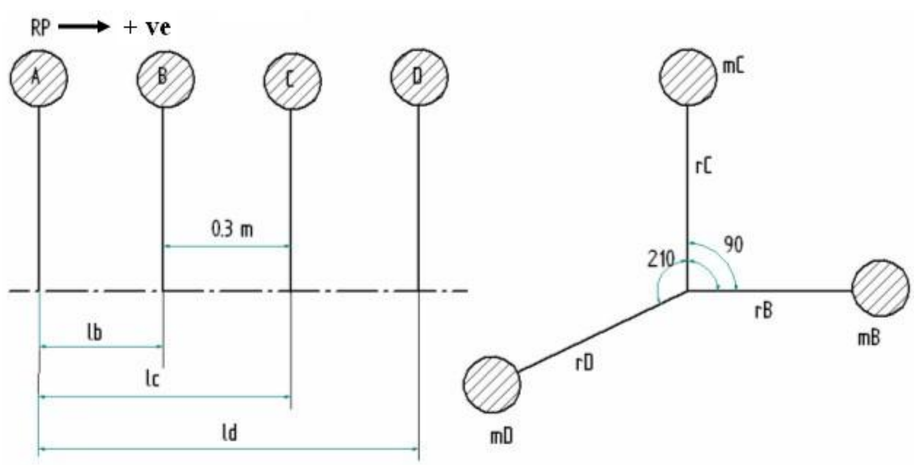
Four masses A, B, C and D are completely balanced. Masses C and D make angles of 90^0 and 210^0 respectively with B in the same sense. The planes containing B and C are 300 mm apart. Masses A, B, C and D can be assumed to be concentrated at radii of 360 mm, 480 mm, 240 mm and 300 mm respectively. The masses B, C and D are 15 kg, 25 kg and 20 kg respectively. Determine i) mass A and its angular position ii) position of planes A and D.

Solution: **Analytical Method**

Step 1:
Draw the space diagram or angular position of the masses. Since the angular position of the masses C and D are given with respect to mass B, take the angular position of mass B as $\theta_B = 0^0$.

Tabulate the given data as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.36	$m_A r_A = 0.36\, m_A$	0	0	$\theta_A =$**?**
B	15	0.48	$m_B r_B = 7.2$	l_B = ?	7.2 l_B	$\theta_B = 0$
C	25	0.24	$m_C r_C = 6.0$	l_C = ?	6.0 l_C	$\theta_C = 90^0$
D	20	0.30	$m_D r_D = 6.0$	l_D = ?	6.0 l_D	$\theta_D = 210^0$



(a) Position of planes of masses (Assumed) (b) Angular position of masses

Step 2:
Mass m_A be the balancing mass placed in plane A which is to be determined along with its angular position.

Refer column 4 of the table. Since m_A is to be determined (which is the only unknown) ,resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum \mathbf{mr\ cos\theta = m_A r_A\ cos\theta_A + m_B r_B\ cos\theta_B + m_C r_C\ cos\theta_C + m_D r_D\ cos\theta_D = 0}$$

On substitution we get

$$\mathbf{0.36\, m_A\ cos\theta_A + 7.2\ cos\, 0^0 + 6.0\ cos\, 90^0 + 6.0\ cos\, 210^0 = 0}$$

Therefore

$$\mathbf{0.36\ m_A\ cos\theta_A = -\,2.004 ---------- (1)}$$

Sum of the vertical components gives,

$$\sum mr\ sin\ \theta = m_A r_A\ sin\theta_A + m_B r_B\ sin\theta_B + m_C r_C\ sin\theta_C + m_D r_D\ sin\theta_D = 0$$

On substitution we get

$$0.36\ m_A\ sin\theta_A + 7.2\ sin\, 0^0 + 6.0\ sin\, 90^0 + 6.0\ sin\, 210^0 = 0$$

Therefore

$$0.36\ m_A\ sin\theta_A = -3.0 ---------- (2)$$

Squaring and adding (1) and (2), we get

$$0.36^2 (m_A)^2 = (-2.004)^2 + (-3.0)^2 = 13.016$$

$$m_A = \sqrt{\frac{13.016}{0.36^2}} = 10.02\ \text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_A = \frac{-3.0}{-2.004} \quad \text{and Resutltant makes an angle} = 56.26^0$$

$$\text{The balancing mass A makes an angle of } \theta_A = 236.26^0 \text{ Ans}$$

Step 3:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum \mathbf{mr\,l\,\cos\theta} = \mathbf{m_A r_A l_A \cos\theta_A + m_B r_B l_B \cos\theta_B + m_C r_C l_C \cos\theta_C + m_D r_D l_D \cos\theta_D = 0}$$

On substitution we get

$$\mathbf{0 + 7.2 l_B \cos 0^0 + 6.0 l_C \cos 90^0 + 6.0 l_D \cos 210^0 = 0}$$

$$\mathbf{7.2 l_B - 5.1962 l_D = 0 \text{ - - - - - - - - - - - } (3)}$$

Sum of the vertical components gives,

$$\sum \mathbf{mr\,l\,\sin\theta} = \mathbf{m_A r_A l_A \sin\theta_A + m_B r_B l_B \sin\theta_B + m_C r_C l_C \sin\theta_C + m_D r_D l_D \sin\theta_D = 0}$$

On substitution we get

$$\mathbf{0 + 7.2 l_B \sin 0^0 + 6.0 l_C \sin 90^0 + 6.0 l_D \sin 210^0 = 0}$$

$$\mathbf{0} + 0 + \mathbf{6.0 l_C} - 3\mathbf{l_D} = \mathbf{0 \text{ - - - - - - - - - - - } (4)}$$

But from figure we have, $\mathbf{l_C} = \mathbf{l_B} + 0.3$

On substituting this in equation (4), we get

$$\mathbf{6.0}(\mathbf{l_B} + 0.3) - 3\mathbf{l_D} = \mathbf{0}$$

$$\textbf{i.e. } \mathbf{6.0 l_B} - 3\mathbf{l_D} = \mathbf{1.8 \text{ - - - - - - - - - - } (5)}$$

Thus we have two equations (3) and (5), and two unknowns $\mathbf{l_B, l_D}$

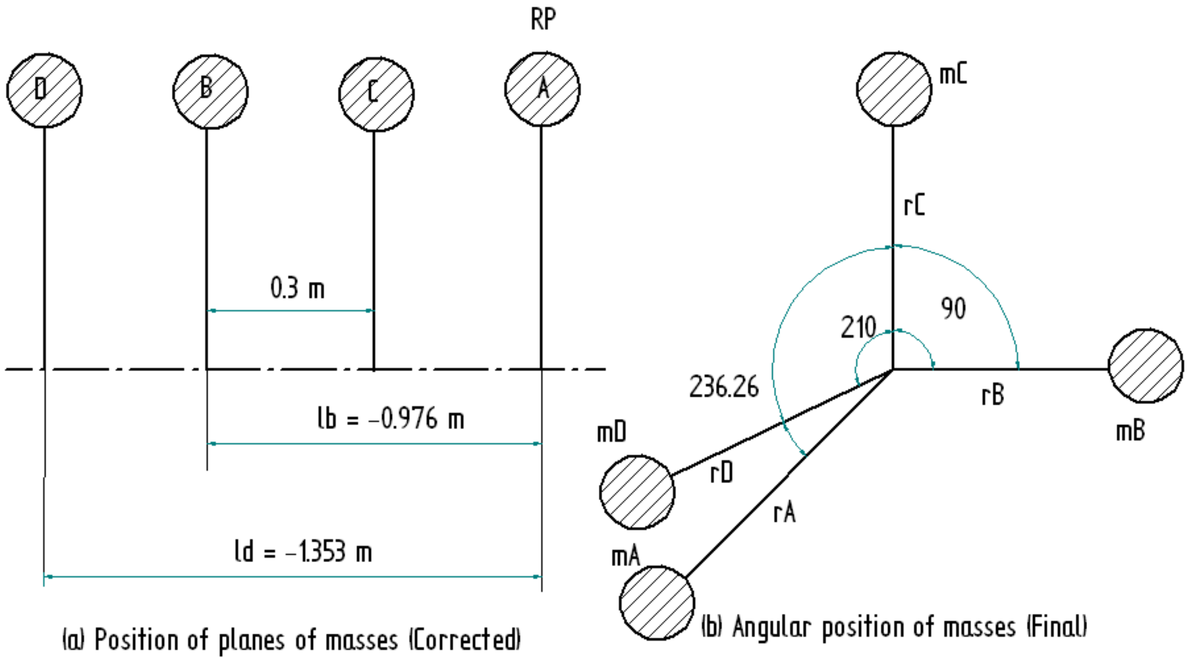
$$\mathbf{7.2 l_B - 5.1962 l_D = 0 \text{ - - - - - - - - - - - } (3)}$$

$$\mathbf{6.0 l_B} - 3\mathbf{l_D} = \mathbf{1.8 \text{ - - - - - - - - - - } (5)}$$

On solving the equations, we get

$\mathbf{l_D} = -1.353\,\text{m}$ and $\mathbf{l_B} = -0.976\,\text{m}$

As per the position of planes of masses assumed the distances shown are positive (+ ve) from the reference plane A. But the calculated values of distances l_B and l_D are negative. The corrected positions of planes of masses is shown below.



(a) Position of planes of masses (Corrected)

(b) Angular position of masses (Final)

References:

1.Theory of Machines by S.S.Rattan, Third Edition, Tata McGraw Hill Education Private Limited.
2.Kinematics and Dynamics of Machinery by R. L. Norton, First Edition in SI units, Tata McGraw Hill Education Private Limited.
3. Primer on Dynamic Balancing "Causes, Corrections and Consequences" By
Jim Lyons International Sales Manager IRD Balancing Div. EntekIRD International