BARROW AND LEIBNIZ ON THE FUNDAMENTAL THEOREM OF THE CALCULUS

ABSTRACT. In 1693, Gottfried Whilhelm Leibniz published in the *Acta Eruditorum* a *geometrical* proof of the fundamental theorem of the calculus. During his notorious dispute with Isaac Newton on the development of the calculus, Leibniz denied any indebtedness to the work of Isaac Barrow. But it is shown here, that his geometrical proof of this theorem closely resembles Barrow's proof in Proposition 11, Lecture 10, of his *Lectiones Geometricae*, published in 1670.

RÉSUMÉ. En 1693 Gottfried Whilhelm Leibniz a publié dans *l'Acta Eruditorum* une preuve géométrique de la théorie fondamentale du calcul. Pendant cette dispute notoire avec Isaac Newton concernent la developpement du calcul, Leibniz nié n'importe quel endettement au travail de Isaac Barrow. Cependant il s'est montre ici, que la preuve géométrique de son théorème ressemble beaucoup a la preuve de Barrow dans Proposition 11, lecture 10, de son L͡ectiones Geometricae, publies en 1670.

1. INTRODUCTION

At the height of his priority dispute with Newton concerning the invention of the calculus, Leibniz wrote an account, *Historia et Origo Calculi Differentialis*, describing the contributions by seventeenth century mathematicians that led him to his own development of the calculus (Child 1920, 22). In this account, Isaac Barrow is not mentioned at all, and in several occasions Leibniz denied any indebtedness to his work, particularly during his notorious priority dispute with Isaac Newton[1]. But in Barrow's *Lectiones Geometricae* (hereafter cited as *Geometrical Lectures*), which Leibniz had obtained during a visit to London in 1673, the concepts of the differential and integral calculus are discussed in geometrical form, and a rigorous mathematical proof is given of the fundamental theorem

[1]Earlier, however, Leibniz did mention Barrow's work. In his 1686 article in the *Acta Eruditorum*, "On a deeply hidden geometry and the analysis of indivisibles and infinities", Leibniz referred to Barrow in connection with a geometrical theorem that appeared in Barrow's Geometrical Lectures, and proceeded to give a proof of this theorem by his analytic method (Struik 1696, 281). Also, on Nov. 1, 1675, Leibniz wrote: "Most of the theorems of the geometry of indivisibles which are to be found in the works of Cavaliere, Vincent, Wallis, Gregory and Barrow, are immediately evident from the calculus" (Child 1920, 87). I am indebted to an anonymous reviewer for calling my attention to these two references.

of the calculus[2] [3]. A distinguish mathematician, Otto Toeplitz, wrote that " Barrow was in possesion of most of the rules of differentiation, that he could treat many inverse tangent problem (indefinite integrals), and that in 1667 he discovered and gave an admirable proof of the fundamental theorem - that is the relation [of the inverse tangent] to the definite integral (Toeplitz 1963, 128). According to J. M. Child, "a Calculus may be of two kinds:

> i) An *analytic* calculus, properly so called, that is, a set of algebraical working rules (with their proofs), with which differentiations of known functions of a *dependent* variable, of products, of quotients, etc., can be carried out; together with the full recognition that differentiation and integration are inverse operations, to enable integration from first principles to be avoided . . .
>
> ii) A *geometrical* calculus equivalent embodying the same principles and methods; this would be the more perfect if the construction for tangents and areas could be immediately translated into algebraic form, if it where so desired.

Child concludes

> Between these two there is, in my opinion, not a pin to choose theoretically; it is a mere matter of practical utility that set the first type in front of the second; whereas the balance of rigour, without modern considerations, is all on the side of the second (Child 1930, 296).

Leibniz's original work concerned the *analytic* calculus, and he claimed that he had read the relevant sections of Barrow's lectures on the *geometrical* calculus only several years later, after he had independently made his own discoveries. In a letter to Johann Bernoulli written in 1703 , he attributed his initial inspiration to a "characteristic triangle" he had found in Pascal's *Traité des sinus du quart the cercle.*[4] After examining Leibniz's original manuscript, and his copy of Barrow's *Geometrical Lectures*, which were not available to Child, D. Mahnke (Manhke 1926) concluded that Leibniz had read only the beginning of this book, and that his calculus discoveries were made independently of Barrow's work. Later, J. E. Hofmann (Hofmann 1974), based partly on Mahke's work, likewise concluded

[2]This theorem establishes the inverse relation between integration and differentiation. In geometrical terms, it equates the subtangent of a curve α that gives the area enclosed by a given curve β, to the ratio of the ordinates of these two curves (see section I). Introducing Cartesian coordinates x, z for α, and x, y for β, where x is the common abscissa, the subtangent t of α is $t = zdx/dz$, and expressed in analytic form, Barrow's theorem establishes the relation $t = z/y$. Hence, Barrow's theorem is equivalent to the relation $dz/dx = y$ for the fundamental theorem of the calculus.

[3]It should also be pointed out that a geometrical proof of the fundamental theorem of the calculus similar to Barrow's was given by the Scottish mathematician James Gregory, which he published as Prop. VI in his book *Geometriae pars universalis* (Padua 1668) (Baron 1969, 232).

[4]Leibniz said that on the reading of this example in Pascal a light suddenly burst upon him, and that he then realized what Pascal had not - that the determination of a tangent to a curve depend on the ratio of the differences in the ordinates and abscissas, as these became infinitesimally small, and that the quadrature depended upon the sum of ordinates or inifinitely thin rectangles for infinitesimal intervals on the axis. Moreover, the operations of summing and of finding differences where mutually inverse (Boyer 1949, 203).

that this independence is confirmed from Leibniz's early manuscripts and correspondence at the time (Hofmann 1974). But Child has given considerable circumstantial evidence, based on the reproductions of some of these manuscripts, that Barrow's *Geometrical Lectures* influenced Leibniz's work[5] (Child 1920), and he did not concur with Mahnke's conclusion[6]. To give one example, in the first publication of his integral calculus (Leibniz 1686), Leibniz gave an analytic derivation of Barrow's geometrical proof in Prop. 1, Lecture 11, (Child 1916, 125), that the area bounded by a curve with ordinates equal to the subnormals of a *given* curve is equal to half of the square of the final ordinate of the original curve[7], see Appendix A. In this publication, Leibniz admitted that he was familiar with Barrow's theorem[8], and notes found in the margin of his copy of Barrow's *Lecciones Geometricae* indicate that Leibniz had read it sometimes between 1674 and 1676 (Leibniz 2008, 301). Barrow's proposition appears in one of he last lectures in his book, which suggests that by this time he had read also most of the other theorems in this book, particularly Prop. 11, Lecture 10 which is Barrow's geometrical proof of the fundamental theorem of the calculus.

But in a 1694 letter to the Marquis de l'Hospital, Leibniz wrote:

> I recognize that M. Barrow has advanced considerably, but I can assure you, Sir, that I have derived no assistance from him for my methods (pour mes methodes) (Child 1920, 220),

and later, in an intended postscript[9] to a letter from Berlin to Jacob Bernoulli, dated April 1703, he wrote,

> Perhaps you will think it small-minded of me that I should be irritated with you, your brother [Johann Bernoulli], or anyone else, it you should

[5]In the preface to his 1916 translation of Barrow's *Geometrical Lectures*, Child concluded that

> "Isaac Barrow was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673." (Child 1916, 7)

[6]In 1930, in an article written partly in response to Mahnke's article, Child concluded:

> "But if he [Leibniz] had never seen Barrow, I very much doubt if he (or even Newton) would have invented the analytic calculus, and completed it in their lifetimes." (Child 1930, 307)

[7]Let $y(x)$ be the ordinate of a given curve, where $y(0) = 0$. The subnormal is $n(x) = ydy/dx$, and in accordance with Leibniz's rules for integration, $\int_0^x n(x')dx' = \int_0^y y'dy' = (1/2)y^2$. This example illustrates the greater simplicity of Leibniz's analytic notation compared to Barrow's geometrical formulation.

[8]Child calls attention to the fact that the "characteristic triangle", which Leibniz claimed to have learned from Pascal, appears also in Barrow's diagram in a form very similar to that described by Leibniz (Child 1920, 16), and concluded that "such evidence as that would be enough to hang a man, even in an English criminal court".

[9]This postcript was in a draft, but not included in the letter sent to Jacob Bernoulli. Child erroneously stated that this letter was sent to his brother, Johann Bernoulli. I am indebted to a reviewer for this correction.

> have perceived the opportunities for obligation to Barrow, which it was not necessary for me, his contemporary in these discoveries[10] to have obtained from him. (Child 1920, 11)

Apparently this comment was motivated by an article in the *Acta Eruditorum* of January 1691, where Jacob Bernoulli had emphasized the similarities of the methods of Barrow with those of Leibniz, and argued that anyone who was familliar with the former could hardly avoid recognizing the latter,

> Yet to speak frankly, whoever has understood Barrrow's . . . will hardly fail to know the other discoveries of Mr. Leibniz considering that they were based on that earlier discovery, and do not differ from them, except perhaps in the notation of the differentials and in some abridgment of the operation of it.(Feingold 1993, 325)

More recently, M. S. Mahoney, presumably referring to Child's work, remarked that "Barrow seems to have acquired significant historical importance only at the turn of the twentieth century, when historians revived his reputation on two grounds: as a forerunner of the calculus and as a source of Newton's mathematics." But, he continued, "beginning in the 1960's two lines of historical inquiry began to cast doubt on this consensus" , and he concluded that Barrow was " competent and well informed, but not particularly original" (Mahoney 1990, 180, 240). This sentiment echoes Whiteside's assessment that "Barrow remains . . . only a thoroughly competent university don whose real importance lies more in his coordinating of available knowledge for future use rather than in introducing new concepts" (Whiteside 1961, 289). In contrast to these derogatory comments, Toeplitz concluded, in conformance with Child's evaluation, that "in a very large measure Barrow is indeed the real discoverer [of the fundamental theorem of the calculus] - insofar as an individual can ever be given credit within a course of development such as we have tried to trace here" (Toeplitz 1963, 98). Whiteside claimed that Barrow's proof of the fundamental theorem of the calculus is only a "neat amendment of [James] Gregory's generalization of Neil's rectification method," where Barrow "merely replaced an element of arc length by the ordinate of a curve." But M. Feingold has pointed out that Whiteside's claim that Barrow borrowed results from Gregory is not consistent with the fact that Barrow's manuscript was virtually finished and already in the hands of John Collins, who shepherded it trough the publication process by the time that Barrow received Gregory's book[11](Feingold 1993, 333).

This article is confined primarily to Leibniz *geometrical* proof of the fundamental theorem of the calculus as it appeared in the 1693 *Acta Eruditorum*, and to a detailed comparison with Barrow's proofs of Proposition 11, Lecture 10, and Prop. 19, Lecture 11, published in

[10]In this letter Leibniz contradicted himself, because earlier he had claimed that his discoveries had occurred several years after Barrow's book had appeared.

[11]In Prop. 10, Lecture 11, however, Barrow remarks: "This extremely useful theorem is due to that most learned man, Gregory from Abardeen" indicating that he was familiar at least with one of Gregory's geometrical proofs. Interestingly, as Child pointed out (Child 1920, 140), Leibniz also refers to this theorem as, " an elegant theorem due to Gregory".

his *Geometrical Lectures* 23 years earlier. The related work of Isaac Newton on the development of the calculus will not be discussed here, because recently it has been presented in great detail by N. Guicciardini in a book entitled *Isaac Newton on Mathematical Certainty and Method* that also contains a complete set of references to earlier work on this subject (Guicciardini 2009).

In the next section, Barrow's geometrical proofs of these two propositions is discussed, and in Section 3 the corresponding proof by Leibniz is given, demonstrating its close similarity to Barrow's work. This resemblance has not been discussed in any detail before, although it was noticed by Struik (Struik 1969, 284), and more recently by L. Giacardi (Giacardi 1995) . It also supports Child's thesis that some of Leibniz's contributions to the development of the calculus were influenced by Barrow's work. Section 4 gives a description of an ingenious mechanical device invented by Leibniz to obtain the area bounded by a given curve, and Section 5 contains a brief summary and conclusions. In Appendix A, a detailed discussion is given of Barrow's proposition 1, Lecture 11, that illustrates his application of the characteristic triangle, which Leibniz evidently read sometimes between 1674 and 1676. In Appendix B, it is pointed out that significant errors occur in the reproduction of Leibniz's diagram in Gerhard's edition (Leibniz 1693), and in Struik's English translation (Struik 1969, 282) that apparently have remained unnoticed in the past.

2. Barrow's geometrical proof of the fundamental theorem of the calculus

In lectures 10 and 11 of his *Geometrical Lectures*, Barrow gave two related geometrical proofs of the fundamental theorem of the calculus, (Child 1916, 116,135; Struik 1969, 256; Mahoney 1990, 223,232; Folkerts 2001; Guicciardini 2009, 175). Underscoring the value of this theorem, he started his discussion with the remark

> I add one or two theorems which it will be seen are of great generality, and not lightly to be passed over[12]

Referring to his diagram in Prop. 11, Lecture 10, shown[13] in Fig. 1, the area bounded by a given curve $ZGEG'$ is described by another curve $AIFI'$ with a common abscissa $APDP'$. Barrow then proved that the subtangent DT of $AIFI'$ at F is proportional to the ratio of the ordinates of the curves $AIFI'$ and $ZGEG'$, at a corresponding value AD of the abscissa, i.e. $DT = R(DF/DE)$, where R is an arbitrary parameter with dimensions of length[14]. Even by modern standards, Barrow's proof is mathematically rigorous, and it does not depend on any assumptions about differentials or infinitesimals which were not well

[12]Nevertheless, Mahoney asserted that "what in substance becomes part of the fundamental theorem of the calculus is clearly not fundamental to Barrow" (Mahoney 1990, 236).

[13]Prime superscripts have been added to the labels I, K, L, P and G that appeared repeated on the right side of the original diagram

[14]In Cartesian notation $AD = x, DE = y, DF = z$, Barrow's theorem states that the subtangent $DT = t$ of $AIFI'$ at x satisfies the property $t = R(z/y)$. Treating $dx = IL$ and $dz = FL$ as differentials, i.e., setting $KL \approx IL$, by the similarity of triangles TDF and KLF we have $t/z \approx dx/dz$. Hence, setting $R = 1$, Barrows theorem corresponds to the familiar analytic form of the fundamental theorem of the calculus that, in the limit that $dx \to 0$, $dz/dx = y$.

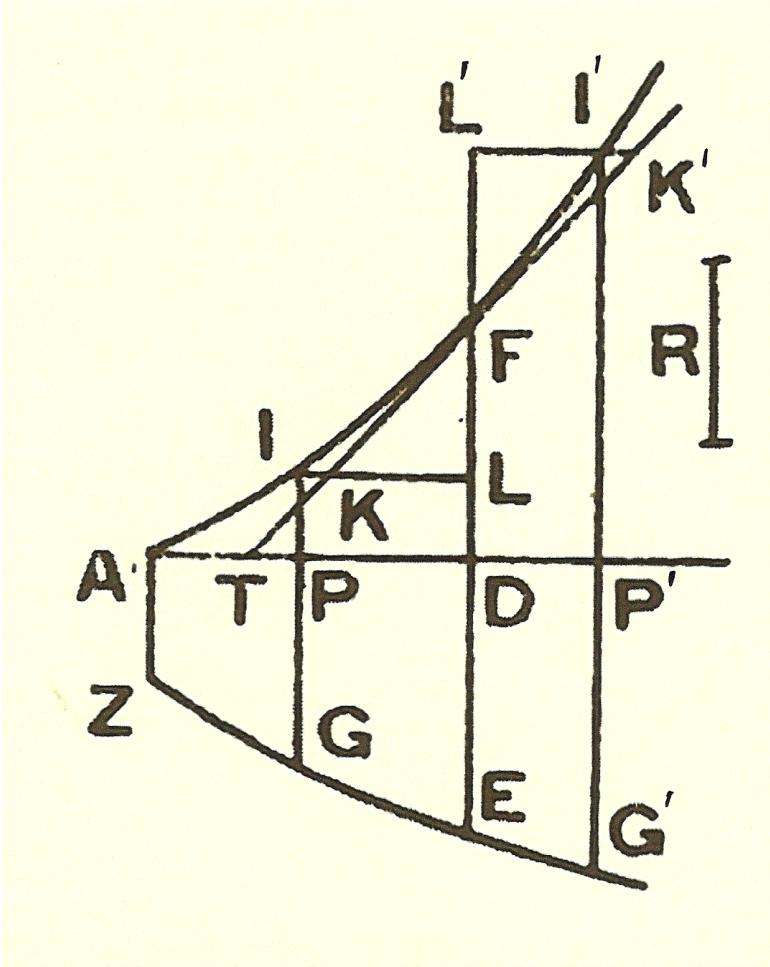

FIGURE 1. Barrow's diagram for Proposition 11, Lecture 10. For clarity, prime superscripts have been added to labels that appear repeated in the original diagram.

understood at the time.[15] In Proposition 19, Lecture 11, Barrow gave a related geometrical proof of his theorem that was based on an expression for the subtangent and for the area of curves in terms of differential quantities that were well known to mathematicians in the seventeenth century (Boyer, 1959).

Barrow started Prop. 11 with a description of his diagram, shown in Fig. 1, as follows:

[15]Child wrote that " . . . it only remains to remark on the fact that the theorem of Prop. 11 is a rigorous proof that differentiation and integration are inverse operations, where integration is defined as a summation." (Child 1916, 124) In Prop. 11, Lecture 10, Barrow did not express the area or integral as a summation, but in Prop. 19, Lecture 11, he expressed integration as a summation.

> Let ZGE be any curve of which the axis is AD, and let ordinates applied to this axis AZ, PG, DE continually increase from the initial coordinate AZ; and also let AIF be a line [another curve] such that, if any line EDF is drawn perpendicular to AD cutting the curves in the points E, F, and AD in D the rectangle contained by DF and a given length R is equal to the intercepted space $ADEZ$;" (Child 1916, 117)

In other word, the ordinate DF that determines the curve $AIFI'$ is equal to the area[16] $ADEZ$ bounded by the curve ZGE, the abcissa AD and the ordinates AZ and DE. Because the quantity associated with an area has the dimensions of length squared, Barrow introduced here an arbitrary parameter R with units of length, and set

$$DF = area(ADEZ)/R. \tag{1}$$

Barrow concluded,

> also let $DE/DF = R/DT$, and join DT. Then TF will touch the curve AIF.(Child, 1916, 117)

which meant that TF is a line tangent to the curve AIF at F. Since it is DE and DF that are determined at a given valued of the abscissa AD, DT is defined by the relation

$$DT = R\frac{DF}{DE}, \tag{2}$$

where $R \cdot DF =$ area $ADEZ$. Hence, Barrow's proof consisted in showing that DT is the subtangent of the curve $AIFI'$ at F.

In Section 3, we shall see that Leibniz named this relation the "tangency relation" without, however, crediting it to Barrow's original work. Before following Barrow's proof further, it is interesting to speculate how he may have discovered his relation for the subtangent DT, Eq. 2. At the end of lecture 10, Barrow gave an argument, originally due to Fermat, that when I and F are close to each other, the triangles TDF and ILF are approximately similar[17], and therefore,[18]

$$\frac{DT}{DF} \approx \frac{IL}{FL}. \tag{3}$$

Since by definition of the curve $AIFI'$,

$$FL = FD - IP = \frac{area(PDEG)}{R}, \tag{4}$$

and

$$area(PDEG) \approx DE \cdot PD \tag{5}$$

[16]Barrow's word for area is "space."

[17]Leibniz named a triangle like ILF the "characteristic triangle", and attributed it to Pascal.

[18]The symbol $\approx$ is introduced to indicate that this relation is only approximately valid for finite differentials IL, FL.

where $PD = IL$, we have

$$FL \approx \frac{DE \cdot IL}{R}. \tag{6}$$

Substituting this approximation for FL in Eq. 3 leads to the relation

$$\frac{DT}{DF} \approx \frac{R}{DE}. \tag{7}$$

In the limit that I approaches F, and IL becomes vanishingly small, the approximate sign $\approx$ in this equation becomes an equality, and this expression becomes Barrow's definition for the subtangent DT, Eq. 2.

In the next step, Barrow gave a rigorous proof for this relation, without appealing to differentials. For the case that DE increases with increasing value of AD, Barrow applied the inequality $DE \cdot PD > area(PDEG)$, to show that for any finite value of IL, $LK < LI$ when I is on the left hand side of F, and when I is on the right hand side F then $LK > LI$. Since K lies on the tangent line, and the curve AIF is convex, this result implies that the line TF, that by definition "touches" the curve at F does not cross it at any other point. Therefore TF is the tangent line at F of the curve AIF.

> For, if any point I is taken in the line AIF (first on the side of F towards A), and if through it IG is drawn parallel to AZ and KL is parallel to AD, cutting the given line as shown in the figure, then
>
> $LF : LK = DF : DT = DE : R$ or $R \cdot LF = LK \cdot DE$
>
> But, from the stated nature of the lines DF, PK, we have $R \cdot LF = area(PDEG)$; therefore $LK \cdot DE = area(PDEG) < PD \cdot DE$; hence $LK < DP = LI$.
>
> Again, if the point I is taken on the other side of F, and the same construction is made as before, plainly it can be easily shown that $LK > DP = LI$.
>
> From which it is quite clear that the whole of the line $TKFK$ lies with or below the curve AIF. (Child 1916,117)

At the end of this presentation, Barrow indicated that when DE decreases with increasing values of AD

> the same conclusion is attained by similar arguments; only one distinction occurs, namely, in this case, contrary to the other, the curve AIF is concave to the axis AD.(Child 1916, 118)

He concluded his proposition with the corollary

$$DE \cdot DT = R \cdot DF = area(ADEZ), \tag{8}$$

which follows from his relation, Eq. 2.

At the end of lecture 10, Barrow considered the application of his geometrical result to the case that the curve $AIFI'$ is determined by an algebraic relation between the abscissa AD and the ordinate DF. Following Fermat, he set $a = IL$ and $e = FL$, and substituting AD for the abscissa, and DF for the ordinate, he showed by a set of three rules how to

obtain an expression for the ratio e/a in terms of AD and DF. These rules amount to keeping only terms that are linear in e and a in a power series expansion of this algebraic relation[19]. It should be pointed out that in his proof, Barrow did not have to specify how to evaluate the area $(ADEZ)$, e.g. by the sum of differential rectangles indicated by Leibniz's notation. Hence, in Prop. 11 Barrow gave a rigorous proof of the fundamental theorem of the calculus which in Cartesian and differential coordinates can be expressed in the following form:

> Given a curve x, y, there exists another curve x, z, where z is the area of the region bounded by the given curve and its coordinate lines, that has the property that its derivative dz/dx is proportional to y.

Along similar lines, in Proposition 19, Lecture 11, Barrow gave a proof of the fundamental theorem of the calculus where he explicitly made use of differentials. For this proposition, Barrow introduced a somewhat different diagram, but since this is not necessary, I will present his discussion by using his diagram, Fig. 1, from Prop. 11, Lecture 10. This procedure will also facilitate comparisons in the next section between Leibniz's and Barrow's proofs, because Leibniz used essentially Barrow's diagram in Prop. 11, Lecture 10. In Prop. 19, Lecture 11, the order in which Barrow constructed the two curves also was altered : he assumed that AFI is the given curve, and obtained the associated curve ZEG by implementing relation, Eq. 2, for the ordinate DE of this curve in terms of the subtangent DT and ordinate DF of the given curveAFI, which previously had been derived quantities. In the next section it will be shown that Leibniz's geometrical proof of the fundamental theorem of the calculus is essentially the same as the proof that Barrow gave in Prop, 19, Lecture 11.

Barrow started Prop.19, Lecture with the following description of his geometrical construction,

> Again, let AFI' be a curve with axis AP and let PG be perpendicular to AP; also let ZEG' be another line [curve] such that, when any point F is taken on the curve AFI' and through it are drawn FT a tangent to the curve AFI' and FDE parallel to PG, cutting ZEG' in E and AP in D, and R is a line of given length, $DT : FD = R : DE$. (Child 1916, 135)

Here the ordinate DE for the curve ZEG' is determined by the Barrow's tangency relation, Eq. 2, previously established in Prop. 11, Lecture 10. Barrow formulated the fundamental theorem as follows:

> Then the space $AP'G'Z$ is equal to the rectangle contained by R and $I'P$. (Child 1916, 135)

Here Barrow's proof made explicit application of differentials:

> For if IF is taken to be an indefinitely small arc of the curve AFI' [and IL is drawn parallel to AP cutting FD at L]. . . then we have

[19]Barrow's rules correspond to the modern definition of the derivative. Let $z' = z - e$ be the ordinate $DF = z$ at a value $x' = AD - a$ of the abscissa $AD = x$. Then $e/a = (z' - z)/a$, and Barrow's rule is to expand $z' - z$ in powers of a and neglect any term on the right hand side of this relation that depends on a.

> $IL : FL := TD : FD = R : DE$; therefore $IL \cdot DE = FL \cdot R$, and $PD \cdot DE = FL \cdot R$. (Child 1916, 135)

Setting $DE = y$, $PD = dx$ and $FL = dz$, Barrow's relation takes the form

$$dx \cdot y = R \cdot dz. \tag{9}$$

Barrow applied the approximate similarity of the differential triangle ILF and the triangle TDF associated with the tangent line of AFI' at F, to equate the area of the differential rectangle $PD \cdot DE$ to the differential change $FL = FD - IP$ in the ordinate of AFI'. Barrow concluded

> Hence, since the sum of such rectangles as $PD \cdot DE$ differs only in the least degree from the space $AP'G'Z$, and the [sum of the] rectangles $FL \cdot R$ from the rectangle $IP \cdot R$, the theorem is quite obvious (Child 1916, 135)

In Leibniz's notation this "sum of rectangles" is expressed in the form

$$\int dx \cdot y = R \cdot \int dz = R \cdot z, \tag{10}$$

corresponding, in modern notation, to the integral relation for the area bounded by a curve.

Although Prop. 19 has been described as the converse of Prop 11, this characterization misses the relevance of this proposition to complete the formulation of the fundamental theorem of the calculus. For finite differentials, the product $PD \cdot DE$ in Barrow's proof is larger than the area of the region $PDEG$; therefore the sum of the areas of these rectangles only gives an upper bound to the area of the region $AP'G'Z$. In the second appendix to Lecture 12, Barrow gave a geometrical proof that the area bounded by a concave curve, is obtained by the " indefinite" sum of either circumscribed or inscribed rectangles. By indicating that these two sums would be the same, Barrow *outlined* a geometrical proof for the existence of the integral of such a curve, but his proof was not completed until 17 years later by Newton, in Section 1, Lemma 2 of his *Principia*. Finally, in 1854 Riemann extended the method of Barrow and Newton to describe the necessary properties of a function for which the concept of an integral as an indefinite sum of rectangles is justified. [20]

[20] In his 1854 Habilitationsschrift, published posthumously by Dedekind, Riemann introduced a bound similar to Barrow's to establish the conditions for the existence of the integral of a function $f(x)$. He wrote:

"First: What is one to understand by $\int_a^b f(x)dx$? In order to fix this relation, we take between a and b a series of values $x_1, x_2, ...x_{n-1}$, and describe the short intervals $x_1 - a$ by δ_1, $x_2 - x_1$ by δ_2, . . ., $b - x_{n-1}$ by δ_n. Hence, the value of the sum $S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + ... + \delta_n f(x_{n-1} + \epsilon_n\delta_n)$ will depend on δ and the magnitude of ϵ [remark: $0 \leq \epsilon \leq 1$]. Given the property that when the $\delta's$ become vanishingly small, the sum approaches a limit A, then this limit corresponds to $\int_a^b f(x)dx$. If it does not have this property, then $\int_a^b f(x)dx$ does not have any meaning.

Under what conditions will a function $f(x)$ permit an integration, and when will it not? Next, we consider the concept of an integral in a narrow sense, that is, we examine the convergence of the sum S, when the various values of δ become vanishingly small. Indicating the largest oscillation of the function $f(x)$ between a and x_1, that is the difference between its largest and smallest value in this interval by D_1, between x_1 and x_2 by D_2 ..., between x_{n-1} and b by D_n; hence the sum $\delta_1 D_1 + \delta_2 D_2 + ... + \delta_n D_n$, with the largest value of δ, must become vanishingly small. Furthermore, we assume that as long as the $\delta's$ remain

3. Leibniz's geometrical proof of the fundamental theorem of the calculus

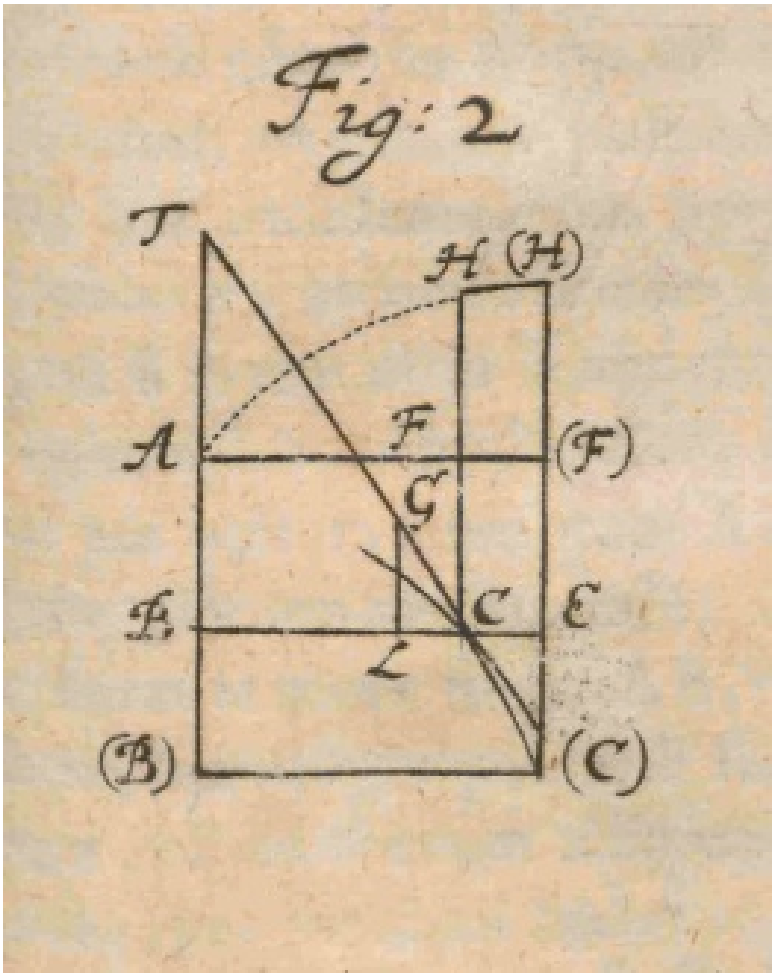

Figure 2. Leibniz's diagram in the 1693 *Acta Eruditorum*. Courtesy of the Gottfried Wilhelm Leibniz Bibliothek-Niedersächsische Landesbibliothek-Hannover.

In this section I will contrast the geometrical proof of the fundamental theorem of the calculus given by Leibniz (Leibniz, 1693), (Kowalewski, 1908), (Struik 1969, 282) with the

smaller than d, then the greatest value of this sum is Δ; Δ will then be a function of d which decreases in magnitude with d and with its size unendlessly decreasing." (Riemann 1854)

Barrow and Newton divided the interval $[a, b]$ into "equal parts", i.e. $\delta_i = d$, and for the special case that the function $f(x)$ decreases monotonically with increasing x, the Riemann's sum is equal to $f(a) \cdot d$.

proof of this theorem given by Barrow discussed in the previous section. Leibniz started his formulation of this theorem with the statement

> I shall now show *that the general problem of quadratures can be reduced to the finding of a line [curve] that has a given law of tangency(declivitas)*, that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line [curve] can be described by a motion that I have invented. (Struik 1969, 282).

For his discussion[21] Leibniz introduced a diagram, Fig. 2, that describes two curves $AH(H)$ and $C(C)$ with a common abscissa AF, and ordinates HF and FC respectively[22]. Leibniz's "general problem of quadratures" is a method to obtain the area bounded by the curve $AH(H)$ and the orthogonal lines AF, HF, by finding the curve $C(C)$ "that has a given law of tangency," such that the ordinate FC of this curve is proportional to this area. Leibniz wrote that he will "show how this line $[C(C)]$ can be described by a motion that I have invented," and the graphical device that he introduced for this purpose will be described in the next section.[23]

The diagram associated with Leibniz's geometrical construction, shown in Fig. 2, is essentially the same as Barrow's diagram for Proposition 11, Lecture 10, shown in Fig. 1, except that Leibniz's diagram is rotated with respect to Barrow's diagram by 180^o around the horizontal axis AF.[24] Thus, Leibniz's curves $AH(H)$ and $C(C)$, Fig. 2, correspond to Barrow's curves ZEG', and AFI', respectively.. Likewise, Leibniz's tangent line TGC to $C(C)$ corresponds to Barrow's tangent line TF to AFI', see Fig. 3. Moreover, I will show that Leibniz's "tangency law", is Barrow's relation for the subtangent of AFI'.

Referring to his diagram, shown in Fig. 2, Leibniz continues

> For this purpose I assume for every curve C(C') *a double characteristic triangle*, one TBC, that is assignable, and one, GLC, that is inassignable, and these two are similar. The inassignable triangle consist of the parts GL, LC, with the elements of the coordinates CF, CB as sides, and GC the element of arc, as the base of the hypotenuse. But the assignable triangle

[21]In modern language, " the general problem of quadratures" is to obtain the integral of a function $y(x)$, described by a curve with Cartesian coordinates x, y. By the fundamental theorem of the calculus, this problem can be solved by finding another function $z(x)$ described by a curve x, z, with the property that its derivative $dz/dx = y$. This property corresponds to Leibniz's "law of tangency" described by "the characteristic triangle", which consist of an infinitesimal right angle triangle with height to base ratio equal to dz/dx, and hypotenuse aligned along the tangent of the curve $z(x)$.

[22]There are two errors, discussed in the Appendix, in the reproduction of this diagram in Leibniz's mathematical papers edited by Gerhard (Leibniz 1693). In the corresponding diagram in Struik's *A Source Book in Mathematics* (Struik, 1969), only one of these errors appeared.

[23]In his English translation of Leibniz's 1693 theorem, Struik mentioned that Leibniz described "an instrument that can perform this construction", but he did not provide this description which is presented in Section 3.

[24]It might be thought that a geometrical proof of this theorem would require the use of similar diagrams, but this is not necessarily the case. As a counter example, compare Barrow's diagram, Fig. 1, with Newton's diagram for his earliest geometrical proof of the fundamental theorem of the calculus (Guicciardini, 2009, 184).

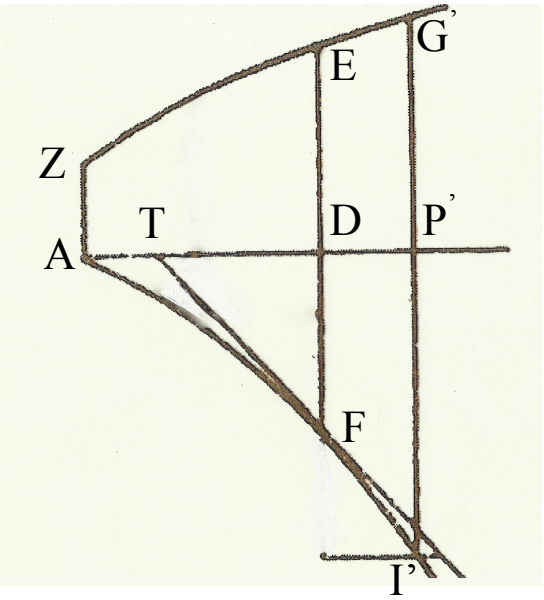

FIGURE 3. Barrow's diagram, shown in Fig. 1, but rotated by 180^0 around the horizonal APD axis with some auxiliary lines deleted

> TBC consists of the axis, the ordinate, and the tangent, and therefore contains the angle between the direction of the curve (or its tangent) and the axis or base, that is, the inclination of the curve at a given point C. (Struik 1969, 283)

The tangent line to the curve $C(C)$ at C is TC, and GL and GC are sides of its "characteristic triangle," GLC, satisfying a "given mutual relation" specified below. But this triangle which Leibniz called *inassignable* [25], does not appear in the proof of Leibniz's

[25]Leibniz chose the Latin word *inassignabilis* for the characteristic triangle, because its sides are differentials which do not have an assignable magnitude.

theorem, while another characteristic triangle, $CE\bar{C}$, where $\bar{C}$ (not indicated in Leibniz's diagram, Fig. 2) is the intersection of the tangent line TC with the extension of $(H)(F)$, turns out to be relevant to Leibniz's proof. Leibniz formulated his "law of tangency" in terms of the sides TB and BC of the similar but assignable triangle TBC. The vertex T of this triangle is the intersection of the tangent TC with a line through the vertex A of the curve $AH(H)$ perpendicular to the abscissa AF, and B is the intersection with this line of a line from C parallel to AF. In Proposition 11, Lecture 10, Barrow gave a proof of this law of tangency which he formulated, instead, in terms of the subtangent $T'F$, where T' (not shown in Fig. 2) is the intersection of the tangent line TC with the abscissa AF.

Up to this point, $C(C)$ has been treated as a *given* curve, but in the next sentence its *construction* is specified by the requirement that its slope conforms to what Leibniz, in his introduction, called a certain "law of tangency."

> Now let $F(H)$, the region of which the area has to be squared, be enclosed between the curve $H(H)$, the parallel lines FH and $(F)(H)$, and the axis $F(F)$, on that axis let A be a fixed point, and let a line AB, the conjugate axis, be drawn through A perpendicular to AF. We assume that point C lies on HF (continued if neccesary); this gives a new curve $C(C)$ with the property that, if from point C to the conjugate axis AB [an axis through A perpendicular to AF] (continued if neccesary) both its ordinate CB (equal to AF) and tangent CT are drawn, the part of the axis between them is to BC as HF to a constant a, or a times BT is equal to the rectangle AFH (circumscribed about the trilinear figure $AFHA$). (Struik 1969, 283)

At this stage the magnitude of FC, the ordinate of $C(C)$, was not specified, but further on Leibniz announced that $FC \cdot a$ is the area of the region $AHFA$, where a is an arbitrary constant with dimensions of length[26]. The law of tangency that the curve $C(C)$ must satisfy is the requirement that at C its subtangent TB satisfy the relation

$$\frac{TB}{BC} = \frac{HF}{a}. \tag{11}$$

At this point, however, Leibniz did not indicate the origin of this relation, but setting $TB/BC = E(C)/EC$ where $E(C)C$ is the associated characteristic triangle, it can be recognized as the fundamental theorem of the calculus in differential form[27], that Barrow had proved[28]in Proposition 11, Lecture 10. But instead of giving a proof of this relation, in the next sentence Leibniz just asserted its validity:

---

[26]The constant of proportionality a is the magnitude of a fixed line which can be chosen arbitrarily, and corresponds to the chosen unit of length.

[27]Setting $AF = BC = y$, $FC = x$, $EC = dy$ and $E(C) = dx$, by similarity of triangles TBC and $CE(C)$, we have $TB/BC = E(C)/EC = dx/dy$. And setting $FH = z$, Eq. 11 becomes $dx/dy = z/a$ which expresses the fundamental theorem of the calculus in differential form.

[28]Formulated in terms of the subtangent $T'F$, where T' (a point not labelled in Leibniz's diagram) is the intersection of the tangent line TGC with the abscissa AF. By similarity of the triangles TBC and $T'FC$ we have $TB/BC = FC/T'F$, which together with Leibniz law of tangency, Eq. 26, gives the relation $T'F = aFC/HF$, corresponding to Barrow's relation, Eq. 2, with the parameter a replaced by R.

> This being established [the law of tangency, Eq. 11], I claim that the rectangle on a and $E(C)$ (we must discriminate between the ordinates FC and $(F)(C)$ of the curve) is equal to the region $F(H)$. (Struik 1969, 254)

By "the region $F(H)$" Leibniz meant the area bounded by the arc $H(H)$ of the curve AFH, the segment $F(F)$ of the abscissa, and the ordinates HF and $(H)(F)$. From the similarity of the triangles $EC\bar{C}$ and TBC, it follows that

$$\frac{E\bar{C}}{CE} = \frac{TB}{BC}, \tag{12}$$

and substituting in this relation Leibniz's form of the law of tangency, Eq. 11, yields

$$a \cdot E\bar{C} = CE \cdot HF. \tag{13}$$

Since $CE = F(F)$, this expression is the area of the rectangle inscribed in the region $F(H)$, which is smaller that the area of this region. Hence

$$E\bar{C} < E(C), \tag{14}$$

as is indicated in Leibniz's diagram, Fig. 2, and the validity of Leibniz's claim, quoted above, is established. Then the approximation that $E(C)$ is the differential change in the ordinate of $C(C)$ for a change $F(F)$ in the abscissa, and the right hand side of Eq. 13 is the differential area of the region $F(H)$, leads to the differential form of the fundamental theorem of the calculus along the same lines described by Barrow in Prop. 19, Lecture 11. As Leibniz explained it,

> This follows immediately from our calculus. Let $AF = y, FH = z, BT = t$, and $FC = x$; then $t = zy/a$, according to our assumption [Barrow's law of tangency in Leibniz's coordinates]: on the other hand, $t = ydx/dy$ because of the property of the tangents expressed in our calculus. Hence $adx = zdy$ and therefore $ax = \int zdy = AFHA$.(Struik 1969, 284)

Here $dx = E(C)$ and $dy = F(F) = CE$. Therefore, $dx/dy = E(C)/EC$, $t/y = BT/AF$, and since triangles TBC and $(C)EC$ are similar, $TB/AF = E(C)/EC$, which corresponds to $t = ydx/dy$. But Leibniz invoked here this relation for the subtangent t as a "property of tangents expressed in our calculus," that can be seen to follow from similarity relations between triangles in his diagram.

In summary, Leibniz based his 1693 geometrical proof of the fundamental theorem of the calculus on a "tangency law", which he claimed to have invented, but that also had been derived by Barrow and published in his *Geometrical Lectures*, Prop. 11, Lecture 10, some 23 years earlier. He then proceed to demonstrate that a certain curve $C(C)$, constructed according to this law, gives the area bounded by a related curve $AH(H)$, along the same lines of Barrow's proof of this theorem in Prop. 19, Lecture 11, using a diagram that turns out to be identical to Barrow's diagram in Prop. 11, Lecture 10, after a 180^0 rotation around the horizontal axis (see Fig. 3)

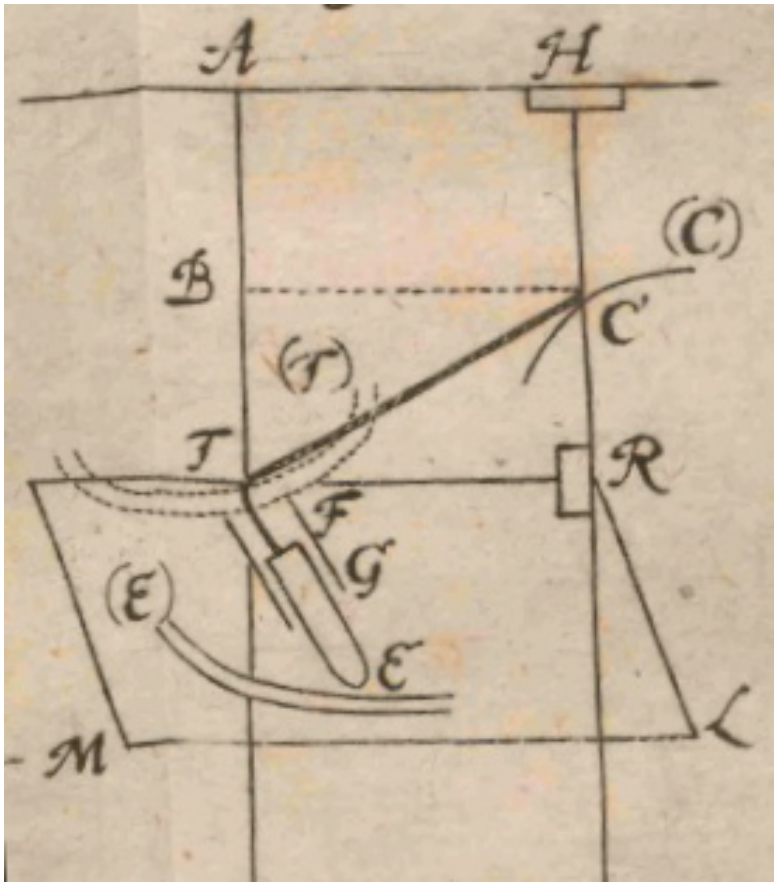

FIGURE 4. Leibniz's diagram for a device to obtain graphically the area bounded by a given curve (1693 Acta Eruditorum). Courtesy of the Gottfried Wilhelm Leibniz Bibliothek-Niedersächsische Landesbibliothek- Hannover.

4. Leibniz graphical device to perform integrations

A novel feature of Leibniz's 1693 discussion of the fundamental theorem of the calculus is his description of an ingenious graphical device to draw a curve $C(C)$ when its slope at each point is given. In his introduction, Leibniz mentioned that this curve "can be described by a motion I have invented" [29]. The motivation for this device appears to have been a problem that was brought to his attention in 1676 by a physician, Claude Perrault, namely, to find the curve traced by a pocket watch pulled along a straight line by its chain Leibniz solved the problem, but he did not divulge his solution until about 20 years later (Bos 1988, 9).

[29]This description is missing in Struik's English translation of Leibniz's 1693 geometrical proof of the fundamental theorem of the calculus

In this section we provide an English translation of Leibniz's description of his device, shown in Fig. 3, obtained from the German translation given in the Oswald's Klassiker edition of the exact sciences (Kowalewski, 1908, 32), and we also give a brief discussion of the principles guiding its operation by an analytic description of the curves involved.

> In Fig. 4, the right angle TAH is fixed and lies on a horizontal plane. A vertical hollow cylinder TG, projecting out of this plane, can move along the side AH. On this cylinder another massive cylinder FE slides upwards and downwards with a string FTC attached at the tip F, in such a way that part FT of the string lies inside the hollow cylinder and part TC lies on the above mentioned horizontal plane. At the end C of the string TC there is a point (of a pen) that is lightly pressed against this plane to describes the curve $C(C)$. The movement comes from the hollow cylinder TG that, while it is guided from A along AT it tightens C. The described point or pin C pushes now HR in the same horizontal plane at right angle to AH (the other side of the fixed right angle TAH) towards A in a progressive manner. This push does not prevent that the point C is moved only by the pull of the string, and therefore follows its direction by this motion. There is also available a board RLM, that advances straight with the point R perpendicularly along the staff HR, after all continuously driven by the hollow cylinder, so that $ATHR$ is a rectangle. Finally, there is a curve $E(E)$ described on this board (if you like in the form of a border) in which the massive cylinder by means of a cut that one can image at the end E, continuously intervenes; in this manner as R moves towards T, the cylinder FE moves upwards (along AT). Since the length $ET + TC$ is given (namely composed of the massive cylinder EF and the entire string FTC), and given the relation between TC and R or BC (from the given inclination [tangency] law), one obtains also the relation between $ET$and TR, the ordinate and abscissa of the curve $E(E)$ whose nature and description on the board LRM can be obtained through ordinary geometry; one obtains also the description of the curve $C(C)$ through this available device. Now, however, it is in the nature of our motion, that TC always is tangent to the curve $C(C)$ so the curve $C(C)$ is described with the given inclination law or the relation of the sides of the characteristic triangle TRC or TBC. Since this curve is the squaring figure corresponding to the quadrature, as was shown a short while before, one has obtained the desired quadrature or measurement. (Kowalewski, 1908)

Given the slope of the curve $C(C)$, what is required to operate Leibniz's device is the relation between the ordinate ET and the abscissa TR of the curve $E(E)$. Leibniz states, that this relation can be obtained "through ordinary geometry," leaving it as an exercise for the reader. Given CR and TR, the length TC of the segment of the string lying on the plane $ATHR$ is

$$TC = \sqrt{TR^2 + CR^2}, \tag{15}$$

and since the total length $TC + ET$ of the string has a fixed value U,

$$ET = U - \sqrt{TR^2 + CR^2}. \tag{16}$$

The operation of Leibniz's integration device can also be understood by expressing its variable components, string length TC and inclination or slope CR/TR, in analytic form. Introducing Cartesian coordinates x, y, z along the mutually orthogonal axes T_oR, T_oE, T_oA with origin at T_o, the initial location of the end point T of the hollow cylinder, the curve $C(C)$ is described by the coordinates $x = TR$, $z = T_oB$, and the curve $E(E)$ by $x = TR$, $y = ET$. Then, by construction,

$$\frac{dz}{dx} = \frac{CR}{TR}, \tag{17}$$

$$TC = x\sqrt{1 + (\frac{dz}{dx})^2}, \tag{18}$$

and

$$y = U - x\sqrt{1 + (\frac{dz}{dx})^2}. \tag{19}$$

For the special case that ET is a constant,

$$\frac{dz}{dx} = -\frac{\sqrt{a^2 - x^2}}{x} \tag{20}$$

where $a = ET - U$, and the curve $C(C)$ is know as the *tractrix*.

5. Concluding remarks

In his translation of the early mathematical manuscripts of Leibniz, J. M. Child called attention to several instances where Leibniz's discussion and diagrams are similar to those found in Barrow's *Geometrical Lectures* (Child 1920). An example is discussed in Appendix A. But the relation of Leibniz's 1693 *geometrical* proof of the fundamental theorem of the calculus to the corresponding proof of Barrow has not been analyzed before.[30]

In his *Geometrical Lectures*, published in 1669, Barrow gave two elegant geometrical proofs of the fundamental theorem of the calculus discussed here in detail. In Proposition 11, Lecture 10, he established this theorem by rigorous geometrical bounds, while in Proposition 19, Lecture 11, he gave an alternate proof based on the application of infinitesimals along the lines practiced by mathematicians in the 17-th century. The eminent historian of science D.T. Whiteside, however, belittled Barrow's proofs calling them "the work of a competent university don" (Whiteside 1961, 289), but this remark is contradicted by the evidence presented here. Likewise, another historian of science, M. S. Mahoney, concluded that Barrow "was not particularly original" (Mahoney 1990, 180,240), without, however,

[30]For example, in his translation of Leibniz's 1693 article on the fundamental theorem of the calculus, D.J. Struik wrote that Leibniz's "expresses by means of a figure the inverse relation of integration and differentiation " (Struik 1969, 282), without commenting on the close similarity of this figure to Barrow's diagram in his proof of this theorem (Struik 1969, 256).

providing any evidence that Barrow's proofs had been given earlier by anyone else. Actually, Barrow was among the foremost mathematicians of his time, whose misfortune was to be eclipsed by his former protégé, Isaac Newton. In contrast to Whiteside's and Mahoney's remarks, a distinguished mathematician, Otto Toeplitz, concluded that "In a very large measure Barrow is indeed the real discoverer [of the fundamental theorem of the calculus] - insofar as an individual can ever be given credit within a course of development such as we have tried to trace here" (Toeplitz 1963, 98) After reviewing the historical record, M. Feingold also disagreed with Whiteside's and Mahoney's views (Feingold, 1990). Recently, Guicciardini discussed some of Barrow's geometrical proofs showing that these proofs also had had a greater impact on Newton's own contributions to the development of the calculus than had been realized in the past (Guicciardini, 2009). Toeplitz commented that "what Newton absorbed from the beginning remained foreign to Barrow throught his life: the turn from the geometrical to the computational function concept - the turn from the confines of the Greek art of proof to the easy flexibility of the indivisibles. On one page of his work Barrow alluded briefly to these matters, but quickly, as though in horror, he dropped them again.(Toeplitz 1963, 130)

Twenty three years after Barrow published his work, Leibniz presented a very similar geometrical proof of the fundamental theorem of the calculus theorem based on a "law of tangency" that Leibniz claimed to be his own invention. But this law corresponds to a theorem that Barrow had proved in Proposition 11, Lecture 10. Moreover, as we have shown here, Leibniz's diagram, Fig. 2, is essentially the same, apart from orientation[31] as Barrow's diagram, Fig.1, given in this proposition, see also Fig. 3, and Leibniz's arguments, which were based on differential quantities, are the same as those given by Barrow in Prop. 19, Lecture 11. Since Leibniz had obtained a copy of Barrow's *Geometrical Lecture* in 1673, it is implausible that in the intervening twenty years he had never encountered Barrow's two propositions, and the fact that his diagram and geometrical proof of the fundamental theorem of the calculus are virtually the same as Barrow's is unlikely to be a coincidence. In fact, Leibniz's marginal annotations in his copy of Barrow's book indicate that at least by 1676 he had studied one of Barrow's propositions contained in one of his last lectures, Prop. 1, Lecture 11 (Child 1920, 16) (Leibniz 2008, 301), see Appendix A. Child wrote that "as far as the actual invention of the calculus as he understood the term is concerned, Leibniz received no help from Newton or Barrow; but for the ideas that underlay it, he obtained from Barrow a great deal more than he acknowledged, and a very great deal less than he would have like to have got, or in fact would have got if only he would have been more fond of the geometry he disliked. For, although the Leibnizian calculus was at the time of this essay far superior to that of Barrow on the question of useful application, it was far inferior in the matter of completness." (Child 1920, 136). Also Feingold commented

[31]An anonymous referee of an earlier version of this manuscript suggested that the similarity between Leibniz's and Barrow's diagram follows from a common tradition, originating with H. van Heuraet (Heuraet 1637), to represent geometrically the area bounded by one curve by a second curve. But such a diagram can be drawn in many different ways as can be seen, for example, in one of Newton's diagram, which is quite different from Barrow's, illustrating his first geometrical proof of the fundamental theorem of the calculus. (Guicciardini 2008, 184). I thank N. Guicciardini for first calling van Heuraet's diagram to my attention.

that "I find it difficult to accept that historians can argue categorically that books that a person own for years went unread simply because he or she failed failed to find dated notes from these books - especially when the figure in question is Leibniz, who was truly a voracious reader. And how can one determine with certainty what a genius like Leibniz was capable of comprehending from various books and letters he encounter or discussion he participated in, however confused their context appears to us today? Such reasoning , it seems to me, subtitutes preconceived notion for constructive historical knowledge" (Feingold 1990, 331). Finally, it should not be forgotten that Leibniz's, when composing his own version of planetary motion, *Tentamen de motuum coelestium causis* (Nauenberg 2010, 281), also denied having read Newton's *Principia*, but his denial has been shown to be false (Bertoloni-Meli, 1993). Hence, Leibniz's persisten claim, particularly during his priority controversy with Newton, of not having any indebtedness to Barrow in his development of the calculus, must be taken *cum grano salis*.

To his credit, in his 1693 article in the *Acta Eruditorum*, Leibniz also called attention to the usefulness of the fundamental theorem of the calculus for the evaluation of integrals, and for this purpose he designed a device to evaluate integrals graphically, see Section 4. Moreover, his work stimulated the applications of the calculus by his celebrated contemporaries, the Bernoulli brothers, Jacob Herman, and Pierre Varignon to the solution of problems in mechanics (Nauenberg, 2009). In his own development of the fundamental theorem of the calculus, Newton also realized the great usefulness of this theorem for integration, and for this purpose he created extensive tables of integrals. But he kept these results to himself, and he did not publish them until 1704 when he appended them to his *Opticks* his *Two Treatises on the Species and Magnitudes of Curvilinear Figures* (Whiteside 1981, 131)

Finally, it should be emphasized that until the 19th century, when the *analytic* calculus was established on proper mathematical foundations, Barrow's Prop. 10 Book 2 was already a rigorous proof, based on sound geometrical principles, of this fundamental theorem. At about the same time, a similar proof was given by James Gregory (Baron, 1969, 233). Today, partly due to the dismissive remarks about Barrow by historians of science like Whiteside and Mahoney, (Whiteside 1961;Mahoney, 1990), it is Leibniz and Newton who get most of the credit for the development of the calculus. But, to quote Rosenberger,

> Like all great advances in the sciences, the analysis of the infinitesimals did not suddenly arise, like Pallas Athena out of the head of Zeus, from the genius of a single author, but instead it was carefully prepared and slowly grown, and finally after laborious trials by the strength of genius, its general significance and long range meaning was brought to light" (Folkerts 2001, 299).

Appendix A. Area of a curve of subnormals

In Proposition 1, Lecture 11, Barrow presented an ingenious geometrical construction to obtain the area bounded by a curve $\phi Z \psi$ with ordinates equal to the subnormals of a *given* curve VEH, with common abscissa VD, shown in Fig. 5. For clarity, we have added

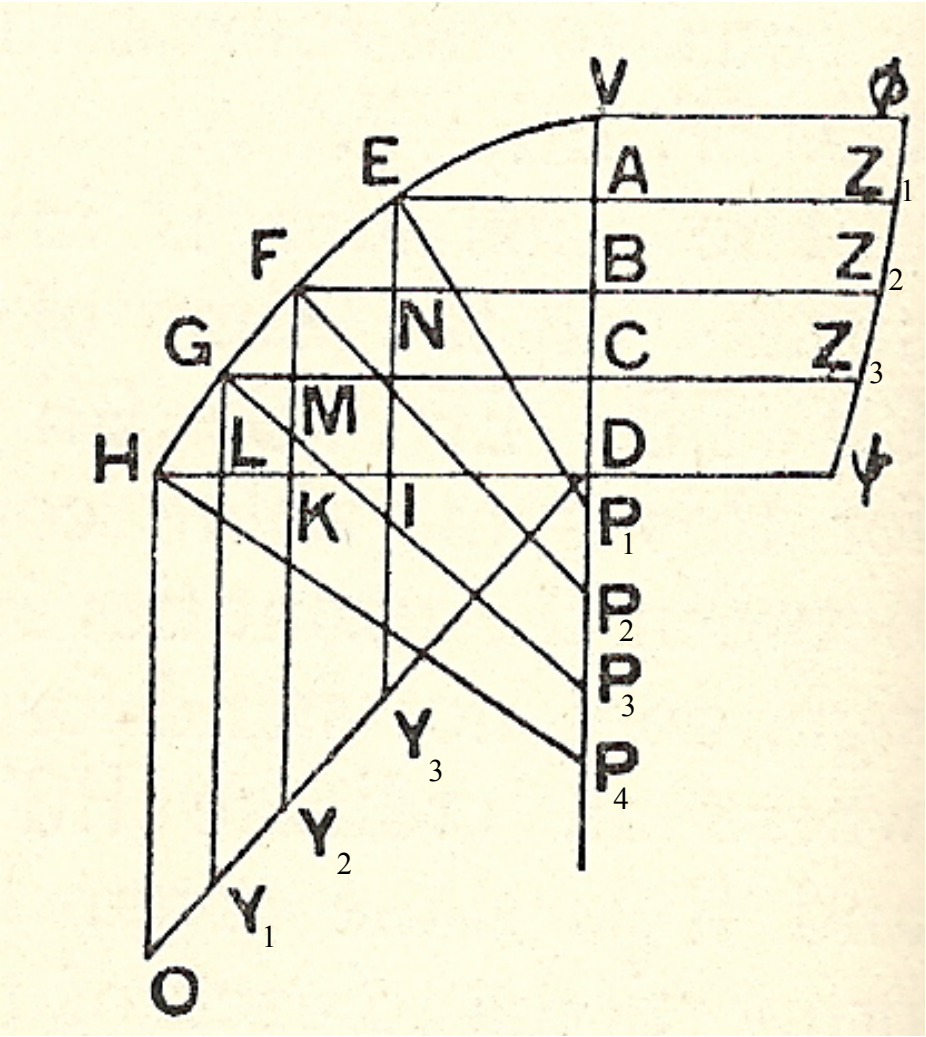

FIGURE 5. Barrow's diagram for the integration of a curve of subnormals

numerical subscript to the symbols P, Z and Y that appear repeatedly in Barrow's original diagram. Barrow gave a proof that this area is equal to the area of a right angle isosceles triangle HDO with sides equal to HD shown at the bottom of his diagram, where HD is

the largest ordinate of VEH. The horizontal and vertical lines in Barrow's diagram, Fig.5, illustrate his approximation by rectangles of the required areas, while the diagonal lines are the normals to VEA at the chosen values of the abscissa of this curve. For example, at A the subnormal AP_1 is obtained by finding the intersection at P_1 of the normal to VEH at E with the axis VD, where EA is the ordinate A. The corresponding ordinate of $\phi Z\psi$ is constructed by setting $AZ_1 = AP_1$, where AZ_1 is taken along the extension of EA. When E approaches V, the arc EV can be approximated by a straight line, and the "characteristic" triangle VAE, assumed to be infinitesimal, becomes similar to the triangle EAP. This construction is then repeated at the equally space points B, C and D along the abscissa VD, e.g. at the next point B on the abscissa, the normal is BP_2, where this second value of P is obtained by the intersection of the normal to VED at F with VD. In this case the characteristic triangle is ENF which is similar to FBP, and $BZ_2 = BP_2$. Then, in Barrow's words,

> the space $VD\psi\phi$ differs in the least degree only from the sum of the rectangles $DC \cdot D\psi + CB \cdot CZ_3 + BA \cdot BZ_2 + AV \cdot AZ_1$

Setting the intervals AV, BA, CB, and DC that appear to be equal as Δx, Barrow's approximation to the area $VD\psi\phi$ is given by the area of the sum of rectangles, $\Delta x \cdot (AP_1 + BP_2 + CP_3 + DP_4)$. Then, in the limit that Δx becomes vanishingly small, and the number of rectangles increases indefinitely, this sum becomes equal to the area $VD\psi\phi$ The similarity of the characteristic triangles with the triangles associated with the subnormals implies that $EA/VA = AP_1/EA, FN/EN = BP_2/FB, GM/FM = CP_3/CG$ and $HL/GL = DP_4/HD$. Hence, the sum $\Delta x \cdot (AP_1 + BP_2 + CP_3 + DP_4) = EA \cdot EA + FN \cdot FB + GM \cdot GC + HL \cdot HD$. According to Barrow's construction, Fig. 5, the intervals $EA = ID, FN = KI, GM = LK$ and HL, are unequal, and $LY_1 = GC, KY_2 = FB$ and $IY_3 = EA$. Hence, the above sum is equal to $HL \cdot HO + LK \cdot LY_1 + KI \cdot KY_2 + ID \cdot IY_3$ which corresponds to a sum of rectangles, giving an upper bound to the area of the right angle isosceles triangle HDO. In the limit that the interval Δx become vanishingly small, and the number of rectangles that bound both areas increases indefinitely this sum gives the area of the triangle HDO, $(1/2)HD^2$, leading to Barrow's conclusion that

$$area\ VD\psi\phi = (1/2)HD^2 \tag{21}$$

By always expressing position of points on his diagrams by letters, sometimes repeating the same letter for different points, Barrow lacked a suitable notation to describe his sums, particularly in the limit $n \to \infty$. Moreover, for this reason the relations that he had obtained geometrically between two finite sums was not evident algebraically [32]

Although it lacked the rigorous mathematical justification of Barrow's analysis, the power of Leibniz's useful notation is that by the substitution of his relation $dx \cdot n(x) = dy \cdot y$

[32] By setting x_i, y_i for the abscissa and ordinate of a point, $\Delta x_i = x_{i+1} - x_i$, $\Delta y_i = y_{i+1} - y_i$, and $n_i = y_i \cdot (\Delta y_i)/\Delta x_i)$ for the subnormal at this point, where $i = 1, n$, the relation between Barrow's two sums becomes obvious without his geometrical analysis:

$$\sum_{i=1}^{i=n} \Delta x \cdot n_i = \sum_{i=1}^{1=n} \Delta x_i \cdot y_i \cdot (\Delta y_i)/\Delta x_i = \sum_{i=1}^{i=n} \Delta y_i \cdot y_i, \tag{22}$$

between the differentials dx and dy, it reduces Barrow's lengthy geometrical construction and derivation to a one line analytic relation between two integrals

$$\int_0^x dx'\ n(x') = \int_0^y dy'\ y', \tag{25}$$

We have shown that Leibniz's relation is based on the characteristic triangle discussed by Barrow, that gave rise to the relation $n_i/y_i \approx \Delta y_i/\Delta x$, for $i = 1, n$. Leibniz claimed that he first learned about the characteristic triangle from Pascal, but evidently he must have recognized it also when he examined Barrow's diagram (Child 1920, 16).

Like Barrow, Leibniz also labelled points on his geometrical diagram with letters, but in the case that the same letter appeared repeated, he added a number of parenthesis corresponding to the number of times this letter was repeated, e.g. $C, (C), ((C))$ etc. But later, he also distinguished repeated letters by adding a numerical subscript in front of these letter, e.g. ${}_1C,\ {}_2C,\ {}_3C$ (Child 1920,137; Leibniz 2008, 573; Bertoloni Meli 1993, 109,135)

Appendix B. Errors in the reproduction of Leibniz's diagram

It should be pointed out that Leibniz 1693 diagram, Fig. 2, for his geometrical proof of the fundamental theorem of the calculus, has been repeatedly reproduced incorrectly. In Gerhardt's edition of Leibniz's mathematical papers, this reproduction, shown in Fig. 5, contains two errors: 1) the curve labelled $C(C)$ touches tangentially the line TC below the intersection C of this line with BE, and 2) the extension of TC ends at its intersection with the extension of $(F)(H)$, labelled incorrectly (C). But (C) is the end point of the curve $C(C)$, and the line $E(C)$ is proportional to the the area $FH(H)(F)$ which is greater than the distance between E and the intersection of the extension of TC that was labelled $\bar{C}$ in Section 2. In Struik's reproduction (Struik 1969, 283), shown in Fig. 6, Leibniz's curve labelled $C(C')$ is drawn correctly, but the intersection (C) is again shown incorrectly as the extension of the tangent TC intersecting the extension, of $(F)(H)$. Thus, when Struik translates Leibniz's text

> This being established, I claim that the rectangle of a and $E(C)$ . . . is equal to the region $F(H)$.

---

where

$$\sum_{i=1}^{i=n} \Delta x \cdot n_i \approx area\, VD\psi\phi, \tag{23}$$

and

$$\sum_{i=1}^{i=n} \Delta y_i \cdot y_i \approx (1/2)HD^2, \tag{24}$$

In this proposition Barrow only hinted at the limit $n \to \infty$ with the intriguing remark,

> A lengthier indirect argument may be used but what advantage is there?

But in an appendix to lecture 12, he discussed more carefully the upper and lower bound of the area of a curve, referring to an *indefinite* number of rectangles. Later on, Isaac Newton improved Barrow's discussion, and included it as Lemma 2, Book 1, in the *Principia* (Guicciardini 2009, 178, 221)

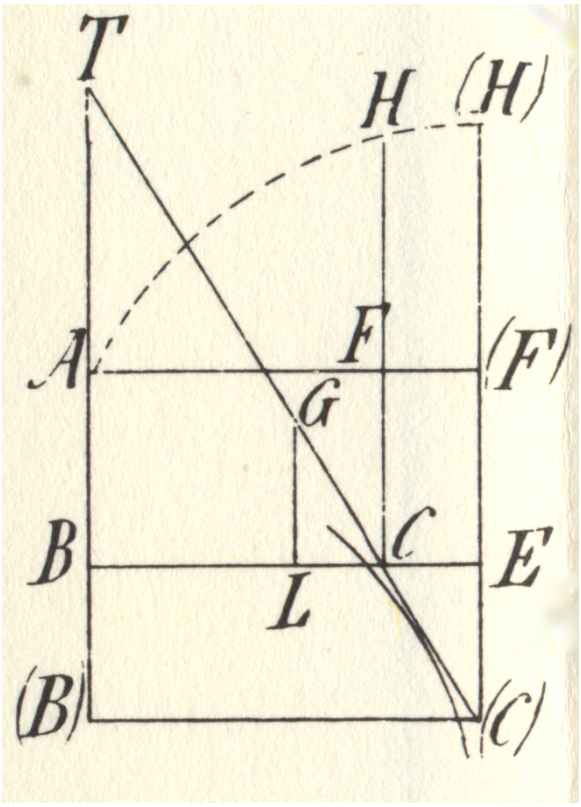

FIGURE 6. Reproduction of Leibniz's 1693 diagram in Gerhardt's edition of Leibniz's mathematical papers (Leibniz, 1693)

it appears as if Leibniz had made a mistake here, but this is due to reference to Struik's incorrect diagram. Such errors make Leibniz's text difficult to comprehend.

6. Acknowlegdments

I would like to thank Niccolò Guicciardini for many helpful comments and discussions, and Eberhard Knobloch, and two anonymous referees for helpful criticisms of an earlier version of this manuscript, and for providing some relevant references. I also thank Mordechai Feingold for some helpful suggestions.

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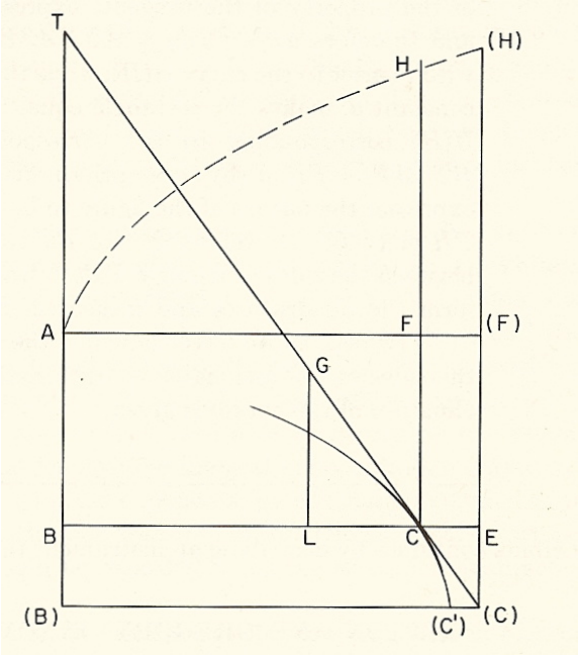

FIGURE 7. Reproduction of Leibniz's 1693 diagram in Struik's translation (Struik 1696, 283)

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