ALGEBRA and personal technology

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Until recently, the school algebra curriculum was rarely influenced by either computers or calculators. There are now many powerful, interesting and enticing technologies presently available for school mathematics in general, and algebra in particular. These include educational computer software such as *Derive* and *Cabri Geometry*, powerful new developments in telecommunications such as multimedia pages accessible via Internet web browsers, and mixtures of the two such as *LiveMath*. Exciting and important as such developments are, they share the common weakness of requiring relatively expensive computer and telecommunications facilities, so that it will be some time before they can be genuinely regarded as accessible to pupils at large in a particular state or country.

In this paper, the focus is on *personal technology*, here regarded as technology that is likely to be available at the personal level to essentially all pupils involved in studying mathematics (Kissane, 1995). The significance of such technology is that it is unreasonable and unrealistic to expect the official and classroom-realised mathematics curriculum to routinely use technology that is not personal in this sense. Indeed, although great strides have been made in technology generally over the past generation, the influence of these developments have not been felt in school curricula because the technology has not been personally accessible to enough pupils to be taken seriously by curriculum developers and examination boards, as has been argued elsewhere (e.g. Kissane, 2000).

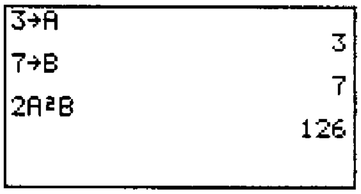
The term 'algebra' has a range of meanings within mathematics. But for the present purpose, focussing on secondary school mathematics, three aspects of algebra seem to be of key significance: representing situations and objects using algebraic symbols, dealing with functions and their graphs, and solving equations and inequalities. These same three aspects were highlighted in the analysis of algebra provided by *A national statement on mathematics for Australian schools* (Australian Education Council, 1990), where the first of these was described as 'expressing generality'.

Algebra is of particular interest in the consideration of technology and the curriculum for a number of reasons. One of these concerns the development over the last decade of powerful computer software with algebraic capabilities exceeding those of most professional mathematicians, such as *Mathematica* and *Maple*. A second concerns the central role algebra has long played in school mathematics, beautifully described recently by Kennedy (1995) using the metaphor of a very large and difficult to climb tree trunk. A third reason concerns the fact that the best examples of personal technology at present, graphics calculators, contain several capabilities of particular importance to algebra, some of which will be briefly described and considered in this paper.

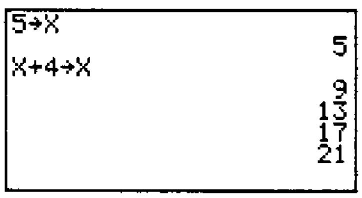
This paper focuses on the teaching of learning of algebra under the assumption that all students have access to a graphics calculator whenever they might wish to use one, including situations of teaching, learning and assessment. Such an assumption demands some reconsideration of what aspects of algebra are of key importance. Without claiming to treat this question in full, the paper offers some perspectives on different aspects of the algebra curriculum and their significance for the next few years.

Symbolising

An important aspect of algebra concerns representing situations and objects symbolically. A graphics calculator has an inbuilt notion of a variable as a placeholder in the form of storage memories, which are named alphabetically in the same way that variables usually are.

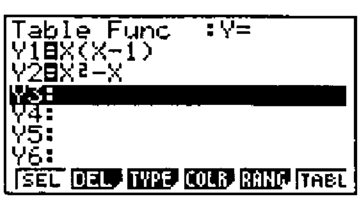


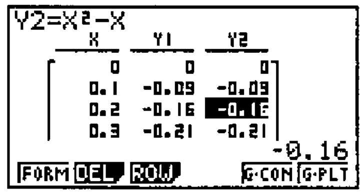
The syntax of the Casio cfx-9850GB PLUS calculator, reflected in the screen above, follows conventional mathematical syntax, so that pupils can interact with variable quantities on their calculator in the same way that they do on the printed page. The calculator screen shows $A = 3$ and $B = 7$, two particular values of the variables A and B. For these values, the calculator determines that $2A^2B = 126$. Notice that there is no multiplication sign after the 2 or between the variable expressions.

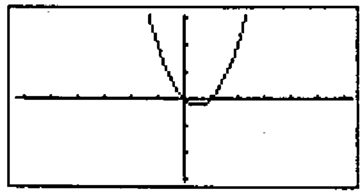


The screen above shows how an important idea of a recursive relationship can be represented symbolically and powerfully on the calculator; each press of the calculator's EXE key executes the recursive step $x + 4 \rightarrow x$ another time, with immediate and understandable results, in this case providing successive terms of an arithmetic progression.

In addition to symbolic representations of these kinds, a graphics calculator offers ways for pupils to make sense of some symbolic statements. For example, the algebraic equivalence of different expressions such as $x(x - 1)$ and $x^2 - x$ can be experienced by considering a table of values or a graph related to each.



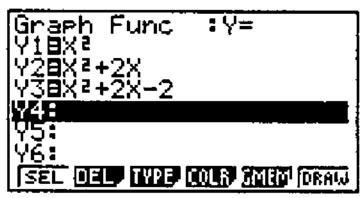


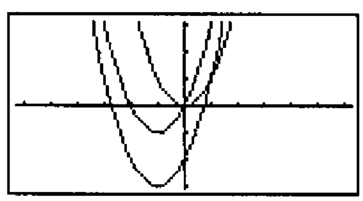


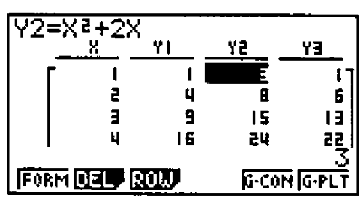
The table suggests that the two expressions always have the same value and the graph of the two expressions is seen to be a single graph, unexpectedly so for many young pupils. These ideas, along with others, are explored for pupils by Kissane & Harradine (2000).

Functions and graphs

It is clear that a graphics calculator has capabilities for efficiently representing functions in three ways: symbolically, graphically and numerically. The screens below show that the symbolic representation uses (mostly) conventional mathematical syntax. For example, a multiplication sign between numerals and variables is unnecessary.

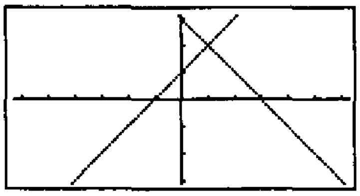


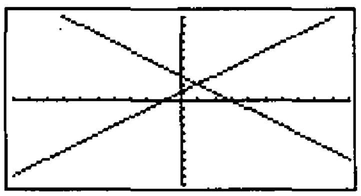




Once represented symbolically, graphs can be drawn on a suitable part of the coordinate plane, called a *window*, and can then be explored by pupils in various ways.

Such possibilities make it imperative that pupils understand aspects of the *scaling* of graphs, previously given less attention in elementary algebra. Unless care is taken (and suitable expertise developed), a graph may be drawn by pupils on axes with different scales, resulting in a kind of distortion. For example, the graphs drawn below each show the two linear functions, $f(x) = x + 1$ and $g(x) = 3 - x$. In the first screen the graphs are drawn on axes with equal scales, while those below are drawn on axes without equal scales (but on a common calculator default screen of $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$). While each graph shows a line, within the screen limitations, only those in the first screen are perpendicular, with each line inclined at 45 degrees to the x-axis, as expected.





A major advantage of these automatic functional representations is that pupils can easily generate and study many examples of functions from a particular family. The experience provided by this activity, if carefully structured by the teacher, can provide pupils with important insights about functions and their graphical representations. For example, translation properties of graphs are readily experienced: the graph of $f(x - k)$ is congruent to that of $f(x)$ moved k units to the right. Prior to the availability of personal technology, it was usually difficult to provide pupils with adequate experience of the various families of functions. Since a great deal of time was needed for plotting individual points in order to see a graph, correspondingly less time was available to think about the results of these labours, or even about graphs as representations at all.

Apart from representing functions graphically, a graphics calculator allows pupils to generate many numerical values for a function. Prior to the availability of technology, pupils needed to do this for themselves (in order to produce a graph). Now, they can use the table of values to reinforce and understand the nature of the function represented graphically. For example, a table of values for a linear function shows a constant increase in the y-values associated with equally spaced x-values, a reflection of the critical idea of a gradient of the function — and of a *linear* function in particular; quadratic and exponential functions have different patterns of change. This powerful possibility of thinking about a function symbolically, graphically and numerically is sometimes referred to as the *rule of three*.

Having ready access to a graph of a function demands a good answer to the reasonable question of what purpose is served by drawing a graph. In my experience, such a question was rarely asked prior to the availability of graphics calculators; indeed, drawing (or 'plotting' or 'sketching') a graph of a function frequently has seemed to be an end in itself in school algebra. Once a decent graph is available, however, pupils can be expected to *use* it for some purpose, such as to seek maxima or minima of a function on an interval, or relate it to the solutions of associated equations. The first of these tasks was until recently regarded as a major purpose of the calculus (except for the particular case of quadratic functions), while the second of these was until recently not regarded as a practical purpose of graphing at all. It has become necessary to reconsider these views of algebra in the light of an accessible technology.

Equations

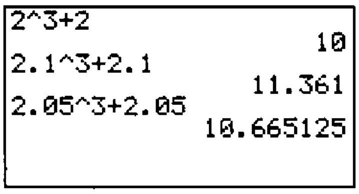
The idea of solving an equation is absolutely fundamental to school algebra, and arguably provided its strongest justification for inclusion in the common curriculum. A generation ago, prior to the so-called 'New Mathematics', the use of symbols in algebra was mostly for the purpose of representing and solving equations. Indeed, generations of pupils learned to think of x as 'the unknown' and of algebra as a ritual of some kind for 'finding' the unknown. The formal study of functions and their representations in school algebra seems rather more recent.

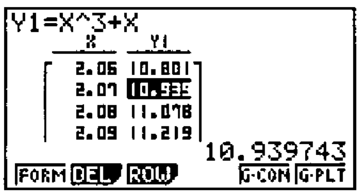
With the benefit of hindsight, it is now perhaps surprising to realise how limited was the repertoire of equations that pupils encountered for which solutions were possible. Indeed, most pupils never advanced beyond the solution of linear and quadratic equations, all of which could be solved, and the few particular species of indicial equations and polynomial equations for which exact solutions were possible. The reason for these limitations, of course, was that the only technology for solving equations was that associated with exact solutions. In the senior secondary years, some exponential equations could be solved (exactly), but required a knowledge of logarithms.

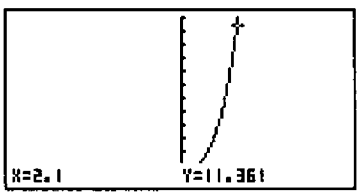
The availability of personal technology has changed this situation considerably. A calculator such as Casio's cfx-9850GB PLUS offers users a variety of ways of finding numerical approximations to solutions of elementary equations, and thereby provides an alternative mechanism to deal with the problems that appeared to motivate the study of equations in the first place. For example, pupils can readily find an approximate solution to almost any equation understandable to them using a process of numerical 'trial and adjustment' on a graphics calculator.

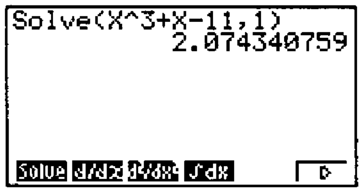
a function, using tracing and zooming as a mechanism to find, and improve, an approximation. Although a little tedious, such methods offer a conceptual understanding of the nature of the solution to an equation that was frequently missing in standard, exact, symbolic routines.

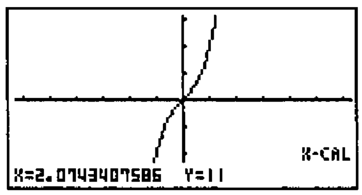
It is important to distinguish strategies like these from more efficient versions that might be used after pupils have a confident sense of the concepts involved. In the case of the Casio cfx-9850GB PLUS calculator, automated versions of these kinds of approximation procedures are available. Importantly, there are a variety of alternatives available, requiring pupils to exercise discretion and thoughtfulness when solving an equation, rather than using the one, standard, guaranteed method of the past. The screens below illustrate three of the alternatives in this case.

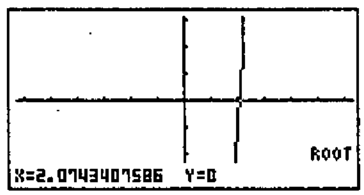








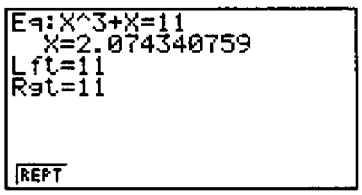




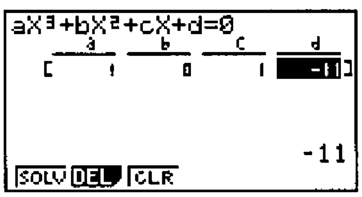
Three of the many possible paths towards solving on this calculator are suggested by the screens above. The first of these mechanisms illustrates the ease with which pupils can 'guess, check and improve' a potential solution to the equation $x^3 + x = 11$ using a calculator with a screen that allows them to see their previous work. The powerful strategy of 'guess, check and improve' is one of the first introduced into the *Access to Algebra* course (Lowe et al., 1993–4). The second screen illustrates how a tabulation of a suitable function allows for easy identification of good approximations to a solution, which can be followed by some tracing and zooming to improve an approximation. The third screen represents the relationship between an equation and a corresponding graph of

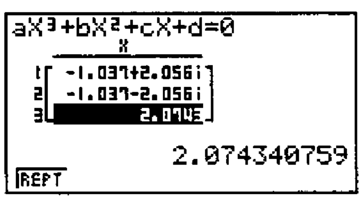
The first screen shows a solve command used in the calculator's home computational screen (in which it is essentially being used as a scientific calculator). To use such a command, pupils need to interpret the calculator's syntax requirements, and understand the equivalence of solving the equation $x^3 + x = 11$ and finding roots of the function $f(x) = x^3 + x - 11$. The second screen essentially automates the earlier procedures involving repeated tracing and zooming. The calculator's *G-Solve* mechanism performs the automation automatically, to locate the x-value for which the function $f(x) = x^3 + x$ has a value of 11. The third screen indicates that there are variations on the graphical approximation theme. The one illustrated here concerns finding the root of the function $f(x) = x^3 + x - 11$, which requires a similar transformation to that needed for the numerical solution. Still other possibilities exist, such as looking for points of intersection of the graphs of the function $f(x) = x^3$ and the function $g(x) = 11 - x$.

As if these possibilities were not enough, the calculator also allows for automated solving by entering the equation into the Solver, as shown on the next screen. The screen suggests how the calculator reinforces the important conceptual fact that the value obtained makes the left and right sides of the equation equivalent (at least within the limits of accuracy of the calculator).



Alternatively, as the equation used to illustrate these ideas here is a cubic, it can also be tackled using the *Polynomial Solver* on this calculator, providing the additional information that there are two complex roots as well as the real root. The two screens below show this process.





Before the advent of personal technology of this kind, pupils could solve a restricted range of equations exactly in essentially two ways. Successive transformations using the field properties of equality are called *do the same thing to both sides* in Lowe et al. (1993–94); alternatively, solutions relied on the multiplication property of zero, useful for solving polynomial equations for which linear factors could be found. Important as these two techniques are for exact solutions of equations, their limitations have become clearer in the light of calculators offering a range of alternatives in addition. Access to technology demands a reconsideration of the balance.

Symbolic manipulation

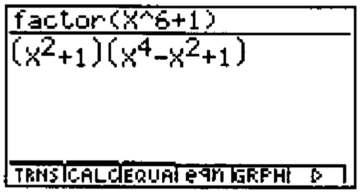
For many pupils in the past, algebra has been seen as mainly concerned with symbolic manipulation, generally devoid of a context, which seems to have been an end in itself. At least, this is the impression created when looking at typical pupil texts and typical pupil assessment.

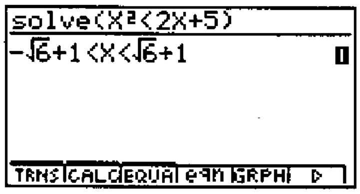
A good deal of symbolic manipulation in schools has been necessary in order to deal with the limited mechanisms available for solving equations. The main practical justification for the very considerable emphasis on techniques such as expanding, simplifying, collecting like terms and factorising has been that these provided the sole mechanisms for the exact solution of equations and the theoretical verification of the solutions. As noted above, some of the capabilities of graphics calculators offer opportunities to reconsider the emphasis on exact solution methods, at least at the early stages of algebraic education, since viable alternatives exist, provided all pupils have reasonable access to them.

Additionally, recent manifestations of personal technology have introduced the new dimension of performing symbolic manipulation with the support of a graphics calculator. The resulting devices have come to be called *algebraic calculators*, and are essentially graphics calculators with the addition of symbolic manipulation software of various kinds. They are, and are intended to be, examples of personal technology in the sense that the term has been used in this paper. Indeed, it seems likely that such devices will become the standard for secondary school mathematics education before much longer, especially if the affordability issues are addressed (as they seem to have been in some countries already).

The significance of such devices for the algebra curriculum is still being determined internationally, even though they have been available now for at least six or seven years. A brief description of a continuum of possible responses is provided in Kissane (1999), in which parallels are drawn with the case of the arithmetic calculator in the mathematics curriculum of the primary school. It seems entirely unreasonable to expect, however, that the algebra curriculum will be impervious to such developments. Whilst it might be argued that a graphics calculator is limited algebraically because it does not offer exact solutions and the cosy reassurance of symbolic mathematical reasoning and justification, it is much harder to resist the algebraic calculator on the same grounds.

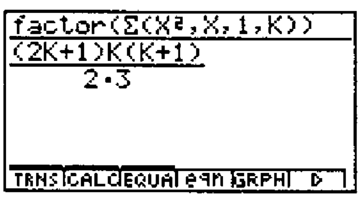
There is space here to give only a few examples of the substantial capacity of one current algebraic calculator, Casio's Algebra FX 2.0, arguably the first model developed specifically for secondary school pupils. (Earlier algebraic calculators were principally developed with the needs of early undergraduate mathematics in mind.)



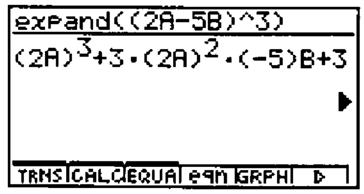


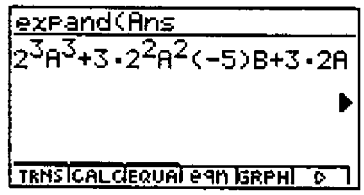
As readers know, both factorising polynomials and solving quadratic inequalities usually require a series of symbolic manipulations. These have many steps and are thus quite error-prone, requiring lengthy time periods for pupils to develop expertise. On this calculator, both operations are performed with a single command, located provocatively in a menu labelled 'transformation'. Indeed, both factorising and solving require chains of successive transformations from one expression to an equivalent one.

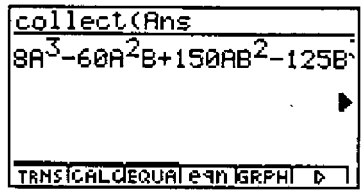
The next example involves a pair of operations (summation, followed by factorisation) in order to find an expression for the sum of the first k squares. Although such an algebraic task is beyond most of secondary school algebra, it is readily accessible using this hand-held device.



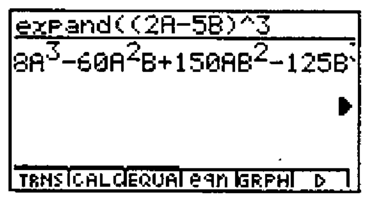
A distinctive difference between this calculator and others of similar species is that it offers pupils both a full-fledged computer algebra system (CAS) as well as a more cumbersome, slower, but more completely understandable alternative for some kinds of symbolic manipulation. To illustrate this difference, the next three screens show an elementary example in the calculator's Algebra mode. Decisions about which algebraic transformations to employ, and in what sequence, are left to the pupil, with the mechanics of the manipulation left to the machine.







This is in stark contrast to the same task in the CAS mode of the calculator, where the result is produced immediately without any need for interaction by the pupil, as the next screen shows.



Whilst the CAS is convenient in practice, it will worry some of us that such a machine may become a black box for producing answers in a mysterious way without any need for pupils to understand what is happening in the process. Of course, as for the earlier examples of slower and faster means of solving an equation, it will require educational expertise and care to avoid the thoughtless use of the more efficient CAS, without some understanding of the kind engendered by the less efficient, but more developmental, procedures. But that is why we need mathematics teachers in classrooms, not just pieces of technology. A calculator like this, with very powerful CAS and algebraic capabilities, would seem to demand some kind of reconsideration of the status of paper-and-pencil symbolic manipulation. It is indeed ironic that the same calculator also contains within it a Tutor mode, the express purpose of which is to develop and practice traditional pupil skill at such manipulation.

The development of algebraic calculators may allow us to see clearly that extensive skill in symbolic manipulation is necessarily of limited value, since a machine can be programmed to perform it fairly well. The task of the algebra curriculum is partly to make sure that the results of such manipulations are sensible to pupils and, most critically, that they develop the necessary expertise to decide what to do. In the past, our focus has necessarily been upon *how* to do things such as factorise expressions and solve equations, since there was no alternative. We might now be able to pay more attention to helping pupils to decide *when* and *why* it might be a good idea to factorise an expression and when it might be inappropriate or just plain unhelpful. Similarly, we may be able to give more attention to constructing, inventing, abstracting and interpreting equations, as well as thoughtfully considering the meaning(s) of any solutions. While such mathematical thinking has always been central to the algebra curriculum, it has frequently been neglected in practice because so much time has been needed to develop the relevant symbolic manipulation skills.

Making transitions of these kinds is unlikely to be an easy task, especially for a generation of us who have learned the very skills that an algebraic calculator can perform with ease, and thus naturally assume that our pupils ought follow the same paths. While it is clear that pupils need to know a good deal about algebra before they can exploit the symbolic capabilities of an algebraic calculator to their fullest, it is less clear now exactly what things are important, and precisely when they are needed.

A concluding comment

The significance of a personal technology is that it may be reasonable to construct an algebra curriculum that takes it into account, rather than more adventurous curricula reliant on more powerful technologies (which too easily become exercises in optimism in the real world of most classrooms and education systems), or the algebra curricula of the 1950's (for which such technologies were never imagined). This is especially so in countries such as Australia for which equity and social justice are important guiding principles, so that curricula are likely to accommodate only technologies that are genuinely available to the masses.

In this paper, some of the connections between the personal technology of the graphics calculator and the algebra curriculum, in urgent need of revitalisation, have been made. Around Australia, the use of graphics calculators of various makes and levels of sophistication has increased substantially over the past few years, although there have not been many corresponding substantial changes in official curricula. Exploratory work of these kinds has been patchy, too, both within and between states. The next few years should allow us to accumulate more experience and more evidence across a range of settings, in order to understand better the nature of our algebra curriculum, located within the technology of the day, despite the rapid changes in the latter.

Note

An earlier version of this paper was presented to the *New Ideas in Mathematics Education* conference, Palm Cove, Queensland, 2001.

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