Bartle - Introduction to Real Analysis - Chapter 6 Solutions

Section 6.2

Problem 6.2-4. *Let $a_1, a_2, \ldots, a_n$ be real numbers and let f be defined on $\mathbb{R}$ by*

$$f(x) = \sum_{i=0}^{n} (a_i - x)^2 \ for\ x \in \mathbb{R}.$$

Find the unique point of relative minimum for f.

Solution: The first derivative of f is:

$$f'(x) = -2 \sum_{i=1}^{n} (a_i - x).$$

Equating f' to zero, we find the relative extrema $c \in \mathbb{R}$ as follows:

$$f'(c) = -2 \sum_{i=1}^{n} (a_i - c) = -2 \left[-nc + \sum_{i=1}^{n} a_i \right] = 0.$$

Accordingly, the only relative extremum of f on $\mathbb{R}$ is:

$$c = \frac{1}{n} \sum_{i=1}^{n} a_i,$$

or more compactly the mean of the constants a_i. By the Interior-Extremum Theorem, this extremum is unique (and therefore global) because there are no other points in $\mathbb{R}$ where f' vanishes and the interval on which f is unbounded.

We can use to First Derivative Test for Extrema to determine if f is at a minimum or maximum at c. For $\delta > 0$, we have for for $x = c + \delta > c$:

$$f'(x) = f(c + \delta) = -2 \sum_{i=1}^{n} (a_i - c - \delta) = f'(c) - 2 \sum_{i=1}^{n} (-\delta) = 2\delta > 0,$$

where we have relied on $f'(c) = 0$. For $x = c - \delta < c$:

$$f'(x) = f(c - \delta) = -2 \sum_{i=1}^{n} (a_i - c + \delta) = f'(c) - 2 \sum_{i=1}^{n} \delta = -2\delta < 0.$$

By the First Derivative Test for Extrema, f is at a minimum at c.

Problem 6.2-6. *Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$.*

Solution: Let $f(x) = \sin x$, so $f'(x) = \cos x$. For all $x \in \mathbb{R}$, we have $-1 \leq \cos x \leq 1$. Let $x, y \in \mathbb{R}$. We may assume without loss of generality that $x < y$. By the Mean Value Theorem, there is a $c \in (x, y)$ such that:

$$\sin x - \sin y = (x - y) f'(c) = (x - y) \cos c.$$

Because $|\cos c| \leq 1$, it follows that $|\sin x - \sin y| \leq |x - y|$.

Problem 6.2-9. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := 2x^4 + x^4 \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$. Show that f has an absolute minimum at $x = 0$, but that its derivative has both positive and negative values in every neighborhood of 0.*

Solution: By the definition of the derivative:

$$f'(0) = \lim_{x \to 0} \frac{2x^4 + x^4 \sin(1/x)}{x - 0} = \lim_{x \to 0} [2x^3 + x^3 \sin(1/x)] = 0.$$

By the Interior Extremum Theorem, there is a relative extremum at $x = 0$. Because $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, it follows that $f(x) = x^4[2 + \sin(1/x)] \geq x^4(2-1) \geq 0$. Therefore, $x = 0$ is an absolute minimum on $\mathbb{R}$.

For $x \neq 0$, we have:

$$f'(x) = 8x^3 + 3x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right).$$

Let $\delta > 0$ and $V_\delta(0)$ be a δ-neighborhood of zero. By the Archimedean Property, there is an $n' \in \mathbb{N}$ such that $0 < 1/n' < 2\pi\delta$. It follows that there is an $n \in \mathbb{N}$ where $n = n' + 1$ such that $0 < 1/n < 2\pi\delta$. Accordingly, letting $x' = 1/(2\pi n)$, we see that $x' \in V_\delta(0)$ and:

$$f'(x') = \frac{8}{8\pi^3 n^3} + 0 - \frac{1}{4\pi^2 n^2} = \frac{4 - \pi n}{4\pi^3 n^3} < 0,$$

where, in the last step, we relied upon $n = n' + 1 > 1$.

By a similar argument, there is an $n \in \mathbb{N}$ such that $0 < 1/(n + 1/4) < 1/n < 2\pi\delta$. Letting $x'' = 1/[2\pi(n + 1/4)]$, it follows that $x'' \in V_\delta(0)$ and:

$$f'(x'') = \frac{8}{8\pi^3(n+1/4)^3} + \frac{3}{8\pi^3(n+1/4)^3} - 0 = \frac{11}{8\pi^3(n+1/4)^3} > 0.$$

Since δ is arbitrary, we conclude that f' on any neighborhood of zero attains both positive and negative values.

Problem 6.2-11. *Give an example of a uniformly continuous function on $[0, 1]$ that is differentiable on $(0, 1)$ but whose derivative is not bounded on $(0, 1)$.*

Solution: Suppose $f : [0, 1] \to \mathbb{R}$ and $f(x) = \sqrt{x}$. If $c \in [0, 1]$, then $\lim_{x \to c} \sqrt{x} = \sqrt{c} = f(c)$; therefore, f is continuous on $[0, 1]$. By the Uniform Continuity Theorem, f is uniformly continuous on $[0, 1]$.

On $(0, 1)$, the derivative of f is $f'(x) = 1/(2\sqrt{x})$. Assume that f' is bounded by $M > 0$ on $(0, 1)$. Then $|1/(2\sqrt{x})| \leq M$ for any $x \in (0, 1)$. However, if $x < 1/(4M^2)$, then $|1/(2\sqrt{x})| > |1/(2\sqrt{(1/4M^2)})| = M$, contradicting our assumption that f' is bounded. It follows that f' is unbounded on $(0, 1)$.

Problem 6.2-13. *Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I.*

Solution: Suppose $f'(x) > 0$ for all $x \in I$. Let $x_0, x_1 \in I$ where $x_0 < x_1$. Because f is differentiable on I by hypothesis (and therefore is continuous on I by Theorem 6.1.2), it follows from the Mean Value Theorem that there is a $c \in I$ such that:

$$f(x_1) - f(x_0) = f'(c)(x_1 - x_0)$$

Because $f'(c) > 0$ and $x_1 - x_0 > 0$, the difference $f(x_1) - f(x_0)$ must be positive. Since $x_0 < x_1$ are arbitrary points in I, the function f is strictly increasing on I.

Problem 6.2-14. *Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is never 0 on I, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.*

Solution: Suppose $f'(x) \neq 0$ for all $x \in I$. Now assume that there are $x_0, x_1 \in I$ where $f'(x_0) < 0$ and $f'(x_1) > 0$. Because $f'(x_0) < 0 < f'(x_1)$, by Darboux's Theorem, there is a $c \in I$ such $f'(c) = 0$, which contradicts our hypothesis that f' is never zero on I. Therefore, either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

Problem 6.2-16. *Let $f : [0, \infty) \to \mathbb{R}$ be differentiable on $(0, \infty)$ and assume $f'(x) \to b$ as $x \to \infty$. (a) Show that for any $h > 0$, we have $lim_{x \to \infty}(f(x + h) - f(x))/h = b$. (b) Show that if $f(x) \to a$ and $x \to \infty$, then $b = 0$. (c) Show that $\lim_{x \to \infty}(f(x)/x) = b$.*

Solution: Part (a) Because f is differentiable on $(0, \infty)$, it is continuous on $(0, \infty)$. For $x, y \, in(0, \infty)$ where $y = x + h$ for any $h > 0$, the Mean Value Theorem ensures that there is a $c_x \in (x, x + h)$ such that:

$$f(x + h) - f(x) = f'(c_x)(x + h - x) = hf'(c_x).$$

Accordingly, $(f(x + h) - f(x))/h = f'(c_x)$.

By hypothesis, $\lim_{x\to\infty} f'(x) = b$. Therefore, for any $\epsilon > 0$, there is a $K(\epsilon) > 0$ such that if $x > K(\epsilon)$, then $|f'(x) - b| < \epsilon$. For any such x, it follows that $c_x > x > K(\epsilon)$. Consequently, for any $x > K(\epsilon)$:

$$|f'(c_x) - b| = \left|\frac{f(x + h) - f(x)}{h} - b\right| < \epsilon,$$

from which it follows that $\lim_{x\to\infty}(f(x + h) - f(x))/h) = b$.

Part (b) If $\lim_{x\to\infty} f(x) = a$, then for any $\epsilon > 0$, there is a $K'(\epsilon)$ such that if $x > K'(\epsilon)$, then $|f(x)-a| < \epsilon$. Because $x+h > x$, it follows that $|f(x+h)-a| < \epsilon$, and therefore, $\lim_{x\to\infty} f(x+h) = a$. Consequently, $\lim_{x\to\infty}[f(x+h)-f(x)] = a - a = 0$. It is then obvious that:

$$b = \lim_{x\to\infty} \frac{f(x + h) - f(x)}{h} = \frac{1}{h}(0) = 0.$$

Part (c) [**IGNORE, NOT VALID SOLUTION***]** It suffices to show that for $\epsilon > 0$, there is a $K''(\epsilon) > 0$ such that if $x > \sup\{0, K''(\epsilon)\}$, then:

$$|f(2x) - b| < \epsilon, \tag{1}$$

where the argument of f is permissible because $x > 0$ ensures that $2x > x$.

From part (a) with $h = x$, we infer that for $0 < \epsilon < b$, there is a $K(\epsilon) > 0$ such that if $x > \sup\{0, K(\epsilon)\}$, then:

$$\left|\frac{f(2x) - f(x)}{x} - b\right| < \epsilon.$$

Note that the upper bound on ϵ is acceptable because we are only concerned about small ϵ. As a result:

$$-\epsilon < \frac{f(2x) - f(x)}{x} - b < \epsilon,$$

from which we have:

$$\frac{f(x)}{x} - b < \frac{f(x)}{x} - \epsilon < \frac{f(2x)}{x} - b.$$

*****THIS STEP IS ERRONEOUS***** Substituting this inequality into equation (1) and letting $K''(\epsilon) = \sup\{0, 1/2K(\epsilon)\}$, we have, for $\epsilon > 0$ and $x > K''(\epsilon)$:

$$\left|\frac{f(x)}{x} - b\right| < \left|\frac{f(2x)}{x} - b\right| < \epsilon.$$

Consequently, $\lim_{x\to\infty} f(x)/x = b$, as we sought to prove.

Problem 6.2-17. *Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.*

Solution: Let $\phi(x) = g(x) - f(x)$. Therefore, $\phi'(x) = g'(x) - f'(x)$. Because $f'(x) \leq g'(x)$ for all $x \geq 0$, it follows that $\phi'(x) \geq 0$ for all $x \geq 0$. By Theorem 6.2.7, ϕ' is increasing on $[0, \infty)$. Therefore, if $x_0, x_1 \in [0, \infty)$ and $x_0 < x_1$, then $\phi(x_0) \leq \phi(x_1)$. Since $\phi(0) = 0$, it follows that if $x \geq 0$, then $\phi(x) = g(x) - f(x) \geq 0$ and therefore $f(x) \leq g(x)$.

Problem 6.2-18. *Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - y| < \delta$ and $a \leq x \leq c \leq y \leq b$, then*

$$\left|\frac{f(x) - f(y)}{x - y} - f'(c)\right| < \epsilon$$

Solution: First note that $x \neq y$ because, by hypothesis, $|x-y| > 0$. Because f is differentiable at c, for $\epsilon/2 > 0$, we have $\delta' > 0$ where for $x \in [a, c)$ and $0 < |x-c| < \delta'$:

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \frac{\epsilon}{2}, \tag{2}$$

and, for $y \in (c, b]$ where $0 < |y-c| < \delta'$:

$$\left|\frac{f(y)-f(c)}{y-c} - f'(c)\right| < \frac{\epsilon}{2}. \tag{3}$$

Because $x \leq c$ and $y \geq c$ and $x \neq y$ by hypothesis, it follows that $|x-y| > |x-c|$ and $|x-y| > |y-c|$. Consequently, for equation (2), we have:

$$\frac{\epsilon}{2} > \frac{1}{|x-c|}\left|f(x)-f(c)-(x-c)f'(c)\right| > \frac{1}{|x-y|}\left|f(c)-f(x)+(x-c)f'(c)\right|. \tag{4}$$

Similarly, for equation (3), we have:

$$\frac{\epsilon}{2} > \frac{1}{|y-c|}\left|f(y)-f(c)-(y-c)f'(c)\right| = \frac{1}{|x-y|}\left|f(y)-f(c)-(y-c)f'(c)\right|. \tag{5}$$

Taking the sum of equations (4) and (5) and using the Triangle Inequality, we have:

$$\begin{aligned}
\epsilon &> \frac{1}{|x-y|}\left(|f(y)-f(c)-(y-c)f'(c)| + |f(c)-f(x)+(x-c)f'(c)\right) \\
&= \frac{1}{|x-y|}\left|f(y)-f(x)-f(c)+f(c)-yf'(c)+xf'(c)+cf'(c)-cf'(c)\right| \\
&= \frac{1}{|x-y|}\left|f(y)-f(x)+f'(c)(x-y)\right| = \left|\frac{f(x)-f(y)}{x-y} - f'(c)\right|,
\end{aligned} \tag{6}$$

which is true for $0 < |x-c| < \delta'$ and $0 < |y-c| < \delta'$. From these relationships, we infer that $0 < |x-y| < 2\delta'$. If we let $\delta = 2\delta'$, we conclude that for any $\epsilon > 0$, the relationship in (6) holds for any $0 < |x-y| < \delta$ where $a \leq x \leq c \leq y \leq b$.

Section 6.3

Problem 6.3-4. *Let $f(x) := x^2$ for x rational and let $f(x) := 0$ for x irrational, and let $g(x) = \sin x$ for $x \in \mathbb{R}$. Use Theorem 6.3.1 to show that $lim_{x\to 0} f(x)/g(x) = 0$. Explain why Theorem 6.3.3 cannot be used.*

Solution: The desired limit is in $0/0$ indeterminate form. Theorem 6.3.1 requires both f and g to be differentiable at 0. The function g is obviously differentiable at 0, where $g'(0) = \cos 0 = 1$. By the definition of the derivative of f at 0:

$$\left|\frac{f(x)-f(0)}{x-0} - 0\right| = \left|\frac{f(x)}{x}\right| = \begin{cases} \left|\frac{x^2}{x}\right| = |x| & \text{for} \quad x \in \mathbb{Q} \\ \left|\frac{0}{x}\right| = 0 & \text{for} \quad x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

For any $\epsilon > 0$, let $\delta(\epsilon) < \epsilon$. If $0 < |x-0| = |x| < \delta(\epsilon)$, then $|f(x)/x| \leq |x| < \epsilon$. It follows that $f'(0) = 0$.

Because $f(0) = g(0) = 0$ and $g(x) \neq 0$ for $x \neq 0$, we can use Theorem 6.3.1 to find the limit:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = \frac{0}{1} = 0.$$

We cannot avail ourselves of Theorem 6.3.3 because f is discontinuous (and therefore is not differentiable) everywhere on $\mathbb{R}$ except at 0. Let $c \in \mathbb{R}$ where $c \neq 0$. Suppose (x_n) is a sequence that converges to c where $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$. It follows that $\lim f(x_n) = \lim x_n^2 = c^2 \neq 0$. Now suppose (y_n) is a sequence that converges to c where $y_n \in \mathbb{R}\setminus\mathbb{Q}$ for all $n \in \mathbb{N}$. We see that $\lim f(y_n) = 0$. If c is rational, $(f(y_n))$ does not converge to $f(c) = c^2 \neq 0$. On the other hand, if c is irrational, $(f(x_n))$ does not converge to $f(c) = 0$. By the Discontinuity Criterion (5.1.4), f is not continuous at any $c \in \mathbb{R}$ where $c \neq 0$.

Problem 6.3-13. *Show that if $c > 0$, then $\lim_{c\to c}\frac{x^c - c^x}{x^x - c^c} = \frac{1-\ln c}{1+\ln c}$.*

Solution: Because $c > 0$, let:

$$f(x) = x^c - c^x = x^c - e^{x \ln c},$$

and

$$g(x) = x^x - c^c = e^{x \ln x} - c^c.$$

Since $f(c) = g(c) = 0$, the desired limit is in $0/0$ indeterminate form. The derivatives of f and g are:

$$f'(x) = cx^{c-1} - (\ln c)e^{x \ln c},$$

and

$$g'(x) = e^{x \ln x}(1 + \ln c).$$

For $x > 0$, the derivatives of f and g exist and $g'(x) \neq 0$. Accordingly, we may apply L'Hôpital's Rule to find the desired limit of f/g at $c > 0$. We then have:

$$\begin{aligned}\lim_{x\to c} \frac{x^c - c^x}{x^x - c^c} = \lim_{x\to c} \frac{f'(x)}{g'(x)} &= \lim_{x\to c} \frac{cx^{c-1} - (\ln c)e^{x \ln c}}{e^{x \ln x}(1 + \ln c)} \\ &= \frac{cc^{c-1} - (\ln c)e^{c \ln c}}{e^{c \ln c}(1 + \ln c)} = \frac{c^c - c^c \ln c}{c^c + c^c \ln c} \\ &= \frac{1 - \ln c}{1 + \ln c},\end{aligned}$$

as we set out to show.

Section 6.4

Problem 6.4-3. *Use induction to prove Leibniz's rule for the nth derivative of a product:*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Solution: The base case of $n = 1$ is trivial. Using the product rule, we have:

$$(fg)' = f'g + fg' = \sum_{k=0}^{1} \binom{1}{k} f^{(1-k)} g^{(k)}.$$

Now we assume the inductive hypothesis holds for $n \geq 1$, and we will show that the inductive hypothesis also holds for $n + 1$. From the inductive hypothesis and the product rule, we have:

$$\begin{aligned}(fg)^{(n+1)} = [(fg)^{(n)}]' &= \left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right]' \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n+1-k)} g^{(k)} + f^{(n-k)} g^{(k+1)}\right]. \\ &= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)}.\end{aligned}$$

By changing the index variable of the second sum to $l = k + 1$, we then have:

$$\begin{aligned}(fg)^{(n+1)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{l=1}^{n+1} \binom{n}{l-1} f^{(n+1-l)} g^{(l)}. \\ &= f^{(n+1)}g + fg^{(n+1)} + \sum_{k=1}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{l=1}^{n} \binom{n}{l-1} f^{(n+1-l)} g^{(l)}\end{aligned}$$

$$= f^{(n+1)}g + fg^{(n+1)} + \sum_{k=1}^{n} \left[\binom{n}{k} f^{(n+1-k)}g^{(k)} + \binom{n}{k-1} f^{(n+1-k)}g^{(k)} \right].$$

Pascal's Rule states that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. Applying this to the sum above, we find:

$$(fg)^{(n+1)} = f^{(n+1)}g + fg^{(n+1)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)}g^{(k)}$$
$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}g^{(k)},$$

which is what we sought to show for $n+1$. Therefore, by the principle of mathematical induction, the inductive hypothesis holds for all $n \in \mathbb{N}$.

Problem 6.4-8. *If $f(x) = e^x$, show that the remainder term in Taylor's Theorem coverges to zero as $n \to \infty$ for each fixed x_0 and x.*

Solution: For any x_0 and x, the remainder term of the *n*th Taylor polynomial is:

$$R_n(x, x_0) = \frac{1}{(n+1)!} e^c (x - x_0)^{n+1},$$

for some c between x and x_0. Let $y_n = R_n(x, x_0)$. For $x \neq x_0$, the quotient of y_{n+1} and y_n is:

$$\frac{y_{n+1}}{y_n} = \frac{\frac{1}{(n+2)!} e^c (x - x_0)^{n+2}}{\frac{1}{(n+1)!} e^c (x - x_0)^{n+1}} = \frac{x - x_0}{n+2}.$$

For all $n \in \mathbb{N}$, we have $0 < 1/(n+2) < 1/n$. By the Squeeze Theorem, $\lim 1/(n+2) = 0 = (x - x_0) \lim 1/(n+2)$. Because this limit is less than one, it follows from Theorem 3.2.1 that $\lim y_n = 0$. Therefore, as $n \to \infty$, the remainder term $R_n(x, x_0) \to 0$. This shows that the Taylor polynomial of e^x with an infinite number of terms equals e^x.

Problem 6.4-10. *If $x \in [0, 1]$ and $n \in \mathbb{N}$, show that:*

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

Use this to approximate $\ln 1.5$ with an error less than 0.01. Less than 0.001.

Solution: Let $f(x) = \ln(1+x)$. First, we will prove by induction that:

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

For the base case $n = 1$, we have $f^{(1)}(x) = 1/(1+x) = ((-1)^0 0!)/(1+x)^1$. Now we will assume the inductive hypothesis is true for $n \geq 1$. For $n+1$, we have:

$$f^{(n+1)}(x) = (f^{(n)})' = (-1)^{n-1}(n-1)! \frac{(-1)n}{(1+x)^{n+1}} = \frac{(-1)^n n!}{(1+x)^{n+1}}.$$

Therefore, the inductive hypothesis is true for all $n \in \mathbb{N}$.

The *n*th Taylor polynomial $P_n(x)$ for $x_0 = 0$ is:

$$P_n(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \cdots + \frac{1}{n!} f^{(n)}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n}.$$

The remainder term $R_n(x)$ is:

$$R_n(x) = (-1)^n \frac{x^{n+1}}{(n+1)(1+c)^{n+1}},$$

for some c is between x and x_0. Because $x \in [0,1]$ by hypothesis and $x_0 = 0$, it follows that $c \in (0,1)$ and:

$$|R_n(x)| = \frac{x^{n+1}}{(n+1)(1+c)^{n+1}}.$$

By Taylor's Theorem, $\ln(1+x) = P_n(x) + R_n(x)$, from which it follows that $\ln(1+x) - P_n(x) = R_n(x)$. Taking the absolute value of both sides, we have:

$$\left|\ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| = \frac{x^{n+1}}{(n+1)(1+c)^{n+1}}.$$

Because $c \in (0,1)$, there is an upper bound on $R_n(x)$ of $x^{n+1}/(n+1)$. We infer:

$$\left|\ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| < \frac{x^{n+1}}{(n+1)},$$

as we sought to show.

We will now estimate $\ln 1.5$. We see that $\ln 1.5 = f(0.5)$. For an estimation error of less than 0.01, we must have $R_n(0.5) = 0.5^{n+1}/(n+1) < 0.01$. By trial and error, we find that $R_3(0.5) = 0.015625$ and $R_4(0.5) = 0.00625$. Consequently, we need at least a fourth-order Taylor polynomial to meet the estimation error. We find that $P_4(0.5) \approx 0.40104$, which has an error within the sought tolerance of approximately 0.004425.

We will now refine the estimate with an error less than 0.001. Again by trial and error, we find that $R_6(0.5) = 0.00116$ and $R_7(0.5) = 0.000458$. A seventh-order Taylor polynomial will meet the estimation error. We see that $P_7(0.5) \approx 0.4058$, which has an error of approximately 0.0003349.

Problem 6.4-17. *Suppose that $I \subseteq \mathbb{R}$ is an open interval and that $f''(x) \geq 0$ for all $x \in I$. If $c \in I$, show that the part of the graph of f on I is never below to tangent line to the graph at $(c, f(c))$.*

Solution: Because $f'(c)$ is the slope of the tangent line to f at c, we can express the tangent line as $g(x) = f(c) + f'(c)(x-c)$.

The 1st Taylor polynomial $P_1(x)$ is:

$$P_1(x) = f(c) + f'(c)(x-c),$$

which is equal to g. The remainder term is:

$$R_1(x) = \frac{1}{2}f''(d)(x-c)^2,$$

for some d between c and x. Because $f''(d) \geq 0$ by hypothesis, $R_1(x) \geq 0$ for all $x \in I$.

From Taylor's Theorem, if $x \in I$, then $f(x) = P_1(x) + R_1(x) = g(x) + R_1(x)$. Since $R_1(x) \geq 0$, necessarily $f(x) \geq g(x)$, from which it follows that no point on I is below the tangent line to f at c.

As an aside, note that f is convex on I by Theorem 6.4.6.

Problem 6.4-15. *Let f be continuous on $[a,b]$ and assume the second derivative f'' exists on (a,b). Suppose that the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a,b)$ such that $f''(c) = 0$.*

Solution: Let g be the line segment between $p_a = (a, f(a))$ and $p_b = (b, f(b))$, where $g(x) = g(a) + m(x-a)$ and $m = (g(b) - g(a))/(b-a)$. Because $p_{x0} = (x_0, f(x_0))$ is on the line segment given by g, the line segment between p_a and p_{x0} and the line segment between p_{x0} and p_b each have slope m. By the Intermediate Value Theorem, there is a point $d_1 \in (a, x_0)$ and a point $d_2 \in (x_0, b)$ such that:

$$f'(d_1) = \frac{f(x_0) - f(a)}{x_0 - a} = m,$$

and

$$f'(d_2) = \frac{f(b) - f(x_0)}{b - x_0} = m.$$

Let $\phi : (a,b) \to \mathbb{R}$ where $\phi(x) = f'(x) - m$. Because f'' exists everywhere on (a,b), by Theorem 6.1.2 f' (and therefore ϕ) is necessarily continuous and differentiable on (a,b). Given that $\phi(d_1) = \phi(d_2) = 0$, by Rolle's Theorem there is a point $c \in (d_1, d_2)$ such that $\phi'(c) = 0$. Since $\phi'(x) = f''(x)$, it follows that $f''(c) = \phi'(c) = 0$. Because $(d_1, d_2) \subset (a,b)$, there is a point c on (a,b) where $f''(c) = 0$.

Problem 6.4-16. *Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on I, and suppose $f''(a)$ exists at $a \in I$. Show that*

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Give an example where this limit exists but the function does not have a second derivative at a.

Solution: The second-order Taylor polynomial for f at $x_0 = a$ is:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Applying Taylor's Theorem for $f(a+h)$ and $f(a-h)$, we have:

$$f(a+h) = P_2(a+h) + R_2(a+h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 - \frac{1}{6}f'''(d_1)h^3, \tag{7}$$

for some $d_1 \in (a, a+h)$ and:

$$f(a-h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(d_2)h^3, \tag{8}$$

for some $d_2 \in (a-h, a)$.

Adding (7) and (8), we have:

$$f(a+h) + f(a-h) = 2f(a) + f''(a)h^2 - \frac{1}{6}f'''(d_1)h^3 + \frac{1}{6}f'''(d_2)h^3.$$

from which we have:

$$f''(a) - \frac{1}{6}f'''(d_1)h + \frac{1}{6}f'''(d_2)h = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Taking the limit of both sides as $h \to 0$ achieves the desired result:

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Now we will turn to an example where this limit exists for a function, but that function has no second derivative. Specifically, this limit exists for $f(x) = \text{signum}(x)$ at $a = 0$, where:

$$f''(0) = \lim_{h \to 0} \frac{\text{signum}(0+h) - 2(0) + \text{signum}(0-h)}{h^2} = \lim_{h \to 0} \frac{0}{h^2} = 0.$$

Clearly the limit exists, but we know that f is not differentiable at $x = 0$ because it is not continuous at that point. Therefore f cannot have a second derivative.

Problem 6.4-18. *Let $I \subseteq \mathbb{R}$ be an interval and let $c \in I$. Suppose that f and g are defined on I and that the derivatives $f^{(n)}, g^{(n)}$ exist and are continuous on I. If $f^{(k)}(c) = 0$ and $g^{(k)}(c) = 0$ for $k = 0, 1, ..., n-1$, but $g^{(n)}(c) \neq 0$, show that*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

Solution: There is a straightforward proof involving Taylor's Theorem, using the remainder of f and g to show the limit. To get some practice with proving the existence of neighborhoods with certain properties, I will prove the proposition using L'Hôpital's Rule.

Because $f(c) = g(c) = 0$, the desired limit is in 0/0 indeterminate form. Applying L'Hôpital's Rule iteratively—if we can—will clearly result in the desired solution. To avail ourselves of the rule, however, we must show that for each $g^{(k)}$, there is a neighborhood of c on which $g^{(k)}$ exists and is zero only at c.

First, we will prove by induction for $l \in \mathbb{N}$ where $l < n$, if $g^{(n-l)}(c) = 0$ (which is the case by hypothesis), there is a neighborhood U_l of c on which $g^{(n-1)}$ is differentiable and, if $x \in U_l$ and $x \neq c$, then $g^{(n-1)}(x) \neq 0$. Let $L = g^{(n)}(c)$, which must be non-zero by hypothesis. For $l = 1$, suppose $\epsilon < |L|$. Because $g^{(n)}$ exists at c, there is a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - c| < \delta_1$, then:

$$\left| \frac{g^{(n-1)}(x) - g^{(n-1)}(c)}{x - c} - L \right| = \left| \frac{g^{(n-1)}(x)}{x - c} - L \right| < \epsilon.$$

If we assume $g^{(n-1)}(x) = 0$, then

$$\left| \frac{g^{(n-1)}(x)}{x - c} - L \right| = | - L| = |L| > \epsilon,$$

which contradicts the requirement that this value must be less than ϵ. Therefore, $g^{(n-1)}(x) \neq 0$. We conclude that there is a δ_1-neighborhood U_1 of c such that if $x \in U_1 \backslash \{c\}$, then $g^{(n-1)}(x) \neq 0$.

Assume that the inductive hypothesis holds for l. If $l + 1 < n$, then by the inductive hypothesis, there is a U_l of c such that if $x \in U_l \backslash \{c\}$, then $g^{(n-l)}(x) \neq 0$. Because $g^{(n-l)}$ is the derivative of $g^{(n-l-1)}$, we infer that $g^{(n-l-1)}$ is differentiable and therefore continuous on U_l. Let $G = g^{(n-l-1)}$ and $G' = g^{(n-l)}$. Then suppose $x_1 \in U_l$ where $x_1 < c$. By the Mean Value Theorem, there is a $d_1 \in (x_1, c)$ such that $G(c) - G(x_1) = -G(x_1) = G'(d_1)(c - x_1)$. Clearly, $d_1 \neq c$ and $d_1 \in U_l$; therefore, $G'(d_1) \neq 0$ and $c - x_1 \neq 0$. It follows that $G(x_1) \neq 0$. By a similar argument, we see that if $x_2 \in U_l$ where $x_2 > c$, then $G(x_2) \neq 0$. Because this is true for all $x \in U_l \backslash \{c\}$, there is a neighborhood $U_{l+1} = U_l$ of c where if $x \in U_{l+1} \backslash \{c\}$, then $g^{(n-l-1)}(x) \neq 0$.

Accordingly, by the principle of mathematical induction, the inductive hypothesis holds for all $l \in \mathbb{N}$ where $l < n$.

It is now a trivial matter to solve the limit. By hypothesis and the proof above, if $1 \leq k < n$, then $g^{(k)}(c) = 0$ and $g^{(k)}(x) \neq 0$ for $x \in U_{n-k} \backslash \{c\}$. L'Hôpital's Rule therefore applies for $1 \leq k < n$. Because $g^{(n)}(c) \neq 0$ and f and g are continuous at c, the desired result follows:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$