

Basics of Fluid Mechanics

Genick Bar–Meir, Ph. D.
1107 16th Ave S. E.
Minneapolis, MN 55414-2411
email:barmeir@gmail.com

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‘We are like dwarfs sitting on the shoulders of giants’

from The Metalogicon by John in 1159

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Dan Olsen	xxiv
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NOMENCLATURE

\bar{R}	Universal gas constant, see equation (2.26), page 38
ℓ	Units length., see equation (2.1), page 33
μ	viscosity at input temperature T, see equation (1.17), page 12
μ_0	reference viscosity at reference temperature, T_{i0} , see equation (1.17), page 12
Ξ	Martinelli parameter, see equation (5.43), page 129
A	The area of surface, see equation (4.117), page 85
a	The acceleration of object or system, see equation (4.0), page 55
B_f	Body force, see equation (2.9), page 35
C_p	Specific pressure heat, see equation (2.23), page 37
C_v	Specific volume heat, see equation (2.22), page 37
E_U	Internal energy, see equation (2.3), page 34
E_u	Internal Energy per unit mass, see equation (2.6), page 34
E_i	System energy at state i, see equation (2.2), page 34
G	The gravitation constant, see equation (4.62), page 70
g_G	general Body force, see equation (4.0), page 55
H	Enthalpy, see equation (2.18), page 36
h	Specific enthalpy, see equation (2.18), page 36

k	the ratio of the specific heats, see equation (2.24), page 37
L	Angular momentum, see equation (3.38), page 51
P_{atmos}	Atmospheric Pressure, see equation (4.85), page 77
q	Energy per unit mass, see equation (2.6), page 34
Q_{12}	The energy transferred to the system between state 1 and state 2, see equation (2.2), page 34
R	Specific gas constant, see equation (2.27), page 38
S	Entropy of the system, see equation (2.13), page 36
S_{uth}	Suth is Sutherland's constant and it is presented in the Table 1.1, see equation (1.17), page 12
T_{τ}	Torque, see equation (3.40), page 51
T_{i0}	reference temperature in degrees Kelvin, see equation (1.17), page 12
T_{in}	input temperature in degrees Kelvin, see equation (1.17), page 12
U	velocity, see equation (2.4), page 34
w	Work per unit mass, see equation (2.6), page 34
W_{12}	The work done by the system between state 1 and state 2, see equation (2.2), page 34
z	the coordinate in z direction, see equation (4.14), page 58

The Book Change Log

Version 0.1.8

Aug 6, 2008 (2.6 M 189 pages)

- Add the chapter on introduction to multi-phase flow
- Again additional improvement to the index (thanks to Irene).
- Add the Rayleigh–Taylor instability.
- Improve the doChap scrip to break up the book to chapters.

Version 0.1.6

Jun 30, 2008 (1.3 M 151 pages)

- Fix the English in the introduction chapter, (thanks to Touser).
- Improve the Index (thanks to Irene).
- Remove the multiphase chapter (it is not for public consumption yet).

Version 0.1.5a

Jun 11, 2008 (1.4 M 155 pages)

- Add the constant table list for the introduction chapter.
- Fix minor issues (English) in the introduction chapter.

*Version 0.1.5***Jun 5, 2008 (1.4 M 149 pages)**

- Add the introduction, viscosity and other properties of fluid.
- Fix very minor issues (English) in the static chapter.

*Version 0.1.1***May 8, 2008 (1.1 M 111 pages)**

- Major English corrections for the three chapters.
- Add the product of inertia to mechanics chapter.
- Minor corrections for all three chapters.

Version 0.1a April 23, 2008

*Version 0.1a***April 23, 2008**

- The Thermodynamics chapter was released.
- The mechanics chapter was released.
- The static chapter was released (the most extensive and detailed chapter).

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- **Date(s) of contribution(s):** 1999 to present
- **Nature of contribution:** Original author.
- **Contact at:** barmeir at gmail.com

Steven from artofproblemsolving.com

- **Date(s) of contribution(s):** June 2005

- **Nature of contribution:** LaTeX formatting, help on building the useful equation and important equation macros.

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- **Nature of contribution:** Some discussions about chapter on mechanics and correction of English.

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- **Date(s) of contribution(s):** April 2008
- **Nature of contribution:** Some discussions about chapter on mechanics and correction of English.

Tousher Yang

- **Date(s) of contribution(s):** April 2008
- **Nature of contribution:** Major review of statics and thermo chapters.

Your name here

- **Date(s) of contribution(s):** Month and year of contribution
- **Nature of contribution:** Insert text here, describing how you contributed to the book.
- **Contact at:** my_email@provider.net

Typo corrections and other "minor" contributions

- **R. Gupta**, January 2008, help with the original **img** macro and other (LaTeX issues).

About This Author

Genick Bar-Meir holds a Ph.D. in Mechanical Engineering from University of Minnesota and a Master in Fluid Mechanics from Tel Aviv University. Dr. Bar-Meir was the last student of the late Dr. R.G.E. Eckert. Much of his time has been spend doing research in the field of heat and mass transfer (related to renewal energy issues) and this includes fluid mechanics related to manufacturing processes and design. Currently, he spends time writing books (there are already two very popular books) and softwares for the POTTO project (see Potto Prologue). The author enjoys to encourage his students to understand the material beyond the basic requirements of exams.

In his early part of his professional life, Bar-Meir was mainly interested in elegant models whether they have or not a practical applicability. Now, this author's views had changed and the virtue of the practical part of any model becomes the essential part of his ideas, books and software.

He developed models for Mass Transfer in high concentration that became a building blocks for many other models. These models are based on analytical solution to a family of equations¹. As the change in the view occurred, Bar-Meir developed models that explained several manufacturing processes such the rapid evacuation of gas from containers, the critical piston velocity in a partially filled chamber (related to hydraulic jump), application of supply and demand to rapid change power system and etc. All the models have practical applicability. These models have been extended by several research groups (needless to say with large research grants). For example, the Spanish Comision Interministerial provides grants TAP97-0489 and PB98-0007, and the CICYT and the European Commission provides 1FD97-2333 grants for minor aspects of that models. Moreover, the author's models were used in numerical works, in GM, British industry, Spain, and Canada.

¹Where the mathematicians were able only to prove that the solution exists.

In the area of compressible flow, it was commonly believed and taught that there is only weak and strong shock and it is continue by Prandtl–Meyer function. Bar–Meir discover the analytical solution for oblique shock and show that there is three solutions, and there is a gap between the oblique shock and Prandtl–Meyer. He also build analytical solution to several moving shock cases. He described and categorized the filling and evacuating of chamber by compressible fluid in which he also found analytical solutions to cases where the working fluid was ideal gas. The common explanation to Prandtl–Meyer function shows that flow can turn in a sharp corner. Engineers have constructed design that based on this conclusion. Bar-Meir demonstrated that common Prandtl–Meyer explanation violates the conservation of mass and therefor the turn must be around a finite radius. The author’s explanations on missing diameter and other issues in fanno flow and “naughty professor’s question” are used in the industry.

In his book “Basics of Fluid Mechanics”, Bar-Meir demonstrated that fluids must have wavy surface when the materials flow together. All the previous models for the flooding phenomenon did not have a physical explanation to the dryness. He built a model to explain the flooding problem (two phase flow) based on the physics. He also build many categories for two flow regimes.

The author lives with his wife and three children. A past project of his was building a four stories house, practically from scratch. While he writes his programs and does other computer chores, he often feels clueless about computers and programing. While he is known to look like he knows about many things, the author just know to learn quickly. The author spent years working on the sea (ships) as a engine sea officer but now the author prefers to remain on solid ground.

Prologue For The POTTO Project

This books series was born out of frustrations in two respects. The first issue is the enormous price of college textbooks. It is unacceptable that the price of the college books will be over \$150 per book (over 10 hours of work for an average student in The United States).

The second issue that prompted the writing of this book is the fact that we as the public have to deal with a corrupted judicial system. As individuals we have to obey the law, particularly the copyright law with the “infinite²” time with the copyright holders. However, when applied to “small” individuals who are not able to hire a large legal firm, judges simply manufacture facts to make the little guy lose and pay for the defense of his work. On one hand, the corrupted court system defends the “big” guys and on the other hand, punishes the small “entrepreneur” who tries to defend his or her work. It has become very clear to the author and founder of the POTTO Project that this situation must be stopped. Hence, the creation of the POTTO Project. As R. Kook, one of this author’s sages, said instead of whining about arrogance and incorrectness, one should increase wisdom. This project is to increase wisdom and humility.

The POTTO Project has far greater goals than simply correcting an abusive Judicial system or simply exposing abusive judges. It is apparent that writing textbooks especially for college students as a cooperation, like an open source, is a new idea³. Writing a book in the technical field is not the same as writing a novel. The writing of a technical book is really a collection of information and practice. There is always someone who can add to the book. The study of technical

²After the last decision of the Supreme Court in the case of Eldred v. Ashcroft (see <http://cyber.law.harvard.edu/openlaw/eldredvashcroft> for more information) copyrights practically remain indefinitely with the holder (not the creator).

³In some sense one can view the encyclopedia Wikipedia as an open content project (see http://en.wikipedia.org/wiki/Main_Page). The wikipedia is an excellent collection of articles which are written by various individuals.

material isn't only done by having to memorize the material, but also by coming to understand and be able to solve related problems. The author has not found any technique that is more useful for this purpose than practicing the solving of problems and exercises. One can be successful when one solves as many problems as possible. To reach this possibility the collective book idea was created/adapted. While one can be as creative as possible, there are always others who can see new aspects of or add to the material. The collective material is much richer than any single person can create by himself.

The following example explains this point: The army ant is a kind of carnivorous ant that lives and hunts in the tropics, hunting animals that are even up to a hundred kilograms in weight. The secret of the ants' power lies in their collective intelligence. While a single ant is not intelligent enough to attack and hunt large prey, the collective power of their networking creates an extremely powerful intelligence to carry out this attack (see for information <http://www.ex.ac.uk/bugclub/raiders.html>)⁴. When an insect which is blind can be so powerful by networking, So can we in creating textbooks by this powerful tool.

Why would someone volunteer to be an author or organizer of such a book? This is the first question the undersigned was asked. The answer varies from individual to individual. It is hoped that because of the open nature of these books, they will become the most popular books and the most read books in their respected field. In a way, the popularity of the books should be one of the incentives for potential contributors. The desire to be an author of a well-known book (at least in his/her profession) will convince some to put forth the effort. For some authors, the reason is the pure fun of writing and organizing educational material. Experience has shown that in explaining to others any given subject, one also begins to better understand the material. Thus, contributing to these books will help one to understand the material better. For others, the writing of or contributing to this kind of books will serve as a social function. The social function can have at least two components. One component is to come to know and socialize with many in the profession. For others the social part is as simple as a desire to reduce the price of college textbooks, especially for family members or relatives and those students lacking funds. For some contributors/authors, in the course of their teaching they have found that the textbook they were using contains sections that can be improved or that are not as good as their own notes. In these cases, they now have an opportunity to put their notes to use for others. Whatever the reasons, the undersigned believes that personal intentions are appropriate and are the author's/organizer's private affair.

If a contributor of a section in such a book can be easily identified, then that contributor will be the copyright holder of that specific section (even within question/answer sections). The book's contributor's names could be written by their sections. It is not just for experts to contribute, but also students who happened to be doing their homework. The student's contributions can be done by

⁴see also in Franks, Nigel R.; "Army Ants: A Collective Intelligence," *American Scientist*, 77:139, 1989

adding a question and perhaps the solution. Thus, this method is expected to accelerate the creation of these high quality books.

These books are written in a similar manner to the open source software process. Someone has to write the skeleton and hopefully others will add “flesh and skin.” In this process, chapters or sections can be added after the skeleton has been written. It is also hoped that others will contribute to the question and answer sections in the book. But more than that, other books contain data⁵ which can be typeset in \LaTeX . These data (tables, graphs and etc.) can be redone by anyone who has the time to do it. Thus, the contributions to books can be done by many who are not experts. Additionally, contributions can be made from any part of the world by those who wish to translate the book.

It is hoped that the books will be error-free. Nevertheless, some errors are possible and expected. Even if not complete, better discussions or better explanations are all welcome to these books. These books are intended to be “continuous” in the sense that there will be someone who will maintain and improve the books with time (the organizer(s)).

These books should be considered more as a project than to fit the traditional definition of “plain” books. Thus, the traditional role of author will be replaced by an organizer who will be the one to compile the book. The organizer of the book in some instances will be the main author of the work, while in other cases only the gate keeper. This may merely be the person who decides what will go into the book and what will not (gate keeper). Unlike a regular book, these works will have a version number because they are alive and continuously evolving.

The undersigned of this document intends to be the organizer–author–coordinator of the projects in the following areas:

Table -1: Books under development in Potto project.

Project Name	Progress	Remarks	Version	Availability for Public Download
Compressible Flow	beta		0.4.8.2	✓
Die Casting	alpha		0.0.3	✓
Dynamics	NSY		0.0.0	✗
Fluid Mechanics	alpha		0.1.8	✓
Heat Transfer	NSY	Based on Eckert	0.0.0	✗
Mechanics	NSY		0.0.0	✗

⁵ Data are not copyrighted.

Table -1: Books under development in Potto project. (continue)

Project Name	Progress	Remarks	Version	Availability for Public Download
Open Channel Flow	NSY		0.0.0	✘
Statics	early alpha	first chapter	0.0.1	✘
Strength of Material	NSY		0.0.0	✘
Thermodynamics	early alpha		0.0.1	✘
Two/Multi phases flow	NSY	Tel-Aviv'notes	0.0.0	✘

NSY = Not Started Yet

The meaning of the progress is as:

- The Alpha Stage is when some of the chapters are already in a rough draft;
- in Beta Stage is when all or almost all of the chapters have been written and are at least in a draft stage;
- in Gamma Stage is when all the chapters are written and some of the chapters are in a mature form; and
- the Advanced Stage is when all of the basic material is written and all that is left are aspects that are active, advanced topics, and special cases.

The mature stage of a chapter is when all or nearly all the sections are in a mature stage and have a mature bibliography as well as numerous examples for every section. The mature stage of a section is when all of the topics in the section are written, and all of the examples and data (tables, figures, etc.) are already presented. While some terms are defined in a relatively clear fashion, other definitions give merely a hint on the status. But such a thing is hard to define and should be enough for this stage.

The idea that a book can be created as a project has mushroomed from the open source software concept, but it has roots in the way science progresses. However, traditionally books have been improved by the same author(s), a process in which books have a new version every a few years. There are book(s) that have continued after their author passed away, i.e., the *Boundary Layer Theory* originated⁶ by Hermann Schlichting but continues to this day. However, projects

⁶Originally authored by Dr. Schlichting, who passed way some years ago. A new version is created every several years.

such as the Linux Documentation project demonstrated that books can be written as the cooperative effort of many individuals, many of whom volunteered to help.

Writing a textbook is comprised of many aspects, which include the actual writing of the text, writing examples, creating diagrams and figures, and writing the \LaTeX macros⁷ which will put the text into an attractive format. These chores can be done independently from each other and by more than one individual. Again, because of the open nature of this project, pieces of material and data can be used by different books.

⁷One can only expect that open source and readable format will be used for this project. But more than that, only \LaTeX , and perhaps troff, have the ability to produce the quality that one expects for these writings. The text processes, especially \LaTeX , are the only ones which have a cross platform ability to produce macros and a uniform feel and quality. Word processors, such as OpenOffice, Abiword, and Microsoft Word software, are not appropriate for these projects. Further, any text that is produced by Microsoft and kept in "Microsoft" format are against the spirit of this project In that they force spending money on Microsoft software.

Prologue For This Book

Version 0.1.8 August 6, 2008

pages 189 size 2.6M

When this author was an undergraduate student, he spend time to study the wave phenomenon at the interface of open channel flow. This issue is related to renewal energy of extracting energy from brine solution (think about the Dead Sea, so much energy). The common explanation to the wave existence was that there is always a disturbance which causes instability. This author was bothered by this explanation. Now, in this version, it was proven that this wavy interface is created due to the need to satisfy the continuous velocity and shear stress at the interface and not a disturbance.

Potto project books are characterized by high quality which marked by presentation of the new developments and clear explanations. This explanation (on the wavy interface) demonstrates this characteristic of Potto project books. The introduction to multi-phase is another example to this quality. While it is a hard work to discover and develop and bring this information to the students, it is very satisfying for the author. The number of downloads of this book results from this quality. Even in this early development stage, number of downloads per month is about 5000 copies.

Version 0.1 April 22, 2008

pages 151 size 1.3M

The topic of fluid mechanics is common to several disciplines: mechanical engineering, aerospace engineering, chemical engineering, and civil engineering. In fact, it is also related to disciplines like industrial engineering, and electrical en-

gineering. While the emphasis is somewhat different in this book, the common material is presented and hopefully can be used by all. One can only admire the wonderful advances done by the previous geniuses who work in this field. In this book it is hoped to insert, what and when a certain model is suitable than other models.

One of the difference in this book is the insertion of the introduction to multiphase flow. Clearly, multiphase is an advance topic. However, some minimal familiarity can be helpful for many engineers who have to deal with non pure single phase fluid.

This book is the third book in the series of POTTO project books. POTTO project books are open content textbooks so everyone are welcome to joint in. The topic of fluid mechanics was chosen just to fill the introduction chapter to compressible flow. During the writing it became apparent that it should be a book in its own right. In writing the chapter on fluid statics, there was a realization that it is the best chapter written on this topic. It is hoped that the other chapters will be as good this one.

This book is written in the spirit of my adviser and mentor E.R.G. Eckert. Eckert, aside from his research activity, wrote the book that brought a revolution in the education of the heat transfer. Up to Egret's book, the study of heat transfer was without any dimensional analysis. He wrote his book because he realized that the dimensional analysis utilized by him and his adviser (for the post doc), Ernst Schmidt, and their colleagues, must be taught in engineering classes. His book met strong criticism in which some called to "burn" his book. Today, however, there is no known place in world that does not teach according to Eckert's doctrine. It is assumed that the same kind of individual(s) who criticized Eckert's work will criticize this work. Indeed, the previous book, on compressible flow, met its opposition. For example, anonymous Wikipedia user name EMBaero claimed that the material in the book is plagiarizing, he just doesn't know from where and what. Maybe that was the reason that he felt that is okay to plagiarize the book on Wikipedia. These criticisms will not change the future or the success of the ideas in this work. As a wise person says "don't tell me that it is wrong, show me what is wrong"; this is the only reply. With all the above, it must be emphasized that this book is not expected to revolutionize the field but change some of the way things are taught.

The book is organized into several chapters which, as a traditional textbook, deals with a basic introduction to the fluid properties and concepts (under construction). The second chapter deals with Thermodynamics. The third book chapter is a review of mechanics. The next topic is statics. When the Static Chapter was written, this author did not realize that so many new ideas will be inserted into this topic. As traditional texts in this field, ideal flow will be presented with the issues of added mass and added forces (under construction). The classic issue of turbulence (and stability) will be presented. An introduction to multi-phase flow, not a traditional topic, will be presented next (again under construction). The next two chapters will deals with open channel flow and gas dynamics. At this stage, dimensional analysis will be present (again under construction).

How This Book Was Written

This book started because I needed an introduction to the compressible flow book. After a while it seems that is easier to write a whole book than the two original planned chapters. In writing this book, it was assumed that introductory book on fluid mechanics should not contained many new ideas but should be modern in the material presentation. There are numerous books on fluid mechanics but none of which is open content. The approach adapted in this book is practical, and more hands-on approach. This statement really meant that the book is intent to be used by students to solve their exams and also used by practitioners when they search for solutions for practical problems. So, issue of proofs so and so are here only either to explain a point or have a solution of exams. Otherwise, this book avoids this kind of issues.

The structure of Hansen, Streeter and Wylie, and Shames books were adapted and used as a scaffolding for this book. This author was influenced by Streeter and Wylie book which was his undergrad textbooks. The chapters are not written in order. The first 4 chapters were written first because they were supposed to be modified and used as fluid mechanics introduction in “Fundamentals of Compressible Flow.” Later, multi-phase flow chapter was written.

The presentation of some of the chapters is slightly different from other books because the usability of the computers. The book does not provide the old style graphical solution methods yet provides the graphical explanation of things.

Of course, this book was written on Linux (Micro\$oftLess book). This book was written using the vim editor for editing (sorry never was able to be comfortable with emacs). The graphics were done by TGIF, the best graphic program that this author experienced so far. The figures were done by gle. The spell checking was done by ispell, and hope to find a way to use gaspell, a program that currently cannot be used on new Linux systems. The figure in cover page was created by Genick Bar-Meir, and is copyleft by him.

Preface

*"In the beginning, the POTTO project was without form, and void; and emptiness was upon the face of the bits and files. And the Fingers of the Author moved upon the face of the keyboard. And the Author said, Let there be words, and there were words."*⁸.

This book, Basics of Fluid Mechanics, describes the fundamentals of fluid mechanics phenomena for engineers and others. This book is designed to replace all introductory textbook(s) or instructor's notes for the fluid mechanics in undergraduate classes for engineering/science students but also for technical peoples. It is hoped that the book could be used as a reference book for people who have at least some basics knowledge of science areas such as calculus, physics, etc.

The structure of this book is such that many of the chapters could be usable independently. For example, if you need information about, say, statics' equations, you can read just chapter (4). I hope this makes the book easier to use as a reference manual. However, this manuscript is first and foremost a textbook, and secondly a reference manual only as a lucky coincidence.

I have tried to describe why the theories are the way they are, rather than just listing "seven easy steps" for each task. This means that a lot of information is presented which is not necessary for everyone. These explanations have been marked as such and can be skipped.⁹ Reading everything will, naturally, increase your understanding of the many aspects of fluid mechanics.

This book is written and maintained on a volunteer basis. Like all volunteer work, there is a limit on how much effort I was able to put into the book and its organization. Moreover, due to the fact that English is my third language and time limitations, the explanations are not as good as if I had a few years to perfect them. Nevertheless, I believe professionals working in many engineering fields will benefit from this information. This book contains many worked examples, which can be very useful for many.

⁸To the power and glory of the mighty God. This book is only to explain his power.

⁹At the present, the book is not well organized. You have to remember that this book is a work in progress.

I have left some issues which have unsatisfactory explanations in the book, marked with a Mata mark. I hope to improve or to add to these areas in the near future. Furthermore, I hope that many others will participate of this project and will contribute to this book (even small contributions such as providing examples or editing mistakes are needed).

I have tried to make this text of the highest quality possible and am interested in your comments and ideas on how to make it better. Incorrect language, errors, ideas for new areas to cover, rewritten sections, more fundamental material, more mathematics (or less mathematics); I am interested in it all. I am particularly interested in the best arrangement of the book. If you want to be involved in the editing, graphic design, or proofreading, please drop me a line. You may contact me via Email at "barmeir@gmail.com".

Naturally, this book contains material that never was published before (sorry cannot avoid it). This material never went through a close content review. While close content peer review and publication in a professional publication is excellent idea in theory. In practice, this process leaves a large room to blockage of novel ideas and plagiarism. If you would like be "peer reviews" or critic to my new ideas please send me your comment(s). Even reaction/comments from individuals like David Marshall¹⁰.

Several people have helped me with this book, directly or indirectly. I would like to especially thank to my adviser, Dr. E. R. G. Eckert, whose work was the inspiration for this book. I also would like to thank to Jannie McRotien (Open Channel Flow chapter) and Touser Yang for their advices, ideas, and assistance.

The symbol META was added to provide typographical conventions to blurb as needed. This is mostly for the author's purposes and also for your amusement. There are also notes in the margin, but those are solely for the author's purposes, ignore them please. They will be removed gradually as the version number advances.

I encourage anyone with a penchant for writing, editing, graphic ability, \LaTeX knowledge, and material knowledge and a desire to provide open content textbooks and to improve them to join me in this project. If you have Internet e-mail access, you can contact me at "barmeir@gmail.com".

¹⁰Dr. Marshall wrote to this author that the author should review other people work before he write any thing new (well, literature review is always good, isn't it?). Over ten individuals wrote me about this letter. I am asking from everyone to assume that his reaction was innocent one. While his comment looks like unpleasant reaction, it brought or cause the expansion of the explanation for the oblique shock. However, other email that imply that someone will take care of this author aren't appreciated.

To Do List and Road Map

This book isn't complete and probably never will be completed. There will always new problems to add or to polish the explanations or include more new materials. Also issues that associated with the book like the software has to be improved. It is hoped the changes in T_EX and L^AT_EX related to this book in future will be minimal and minor. It is hoped that the style file will be converged to the final form rapidly. Nevertheless, there are specific issues which are on the "table" and they are described herein.

At this stage, many chapters are missing. Specific missing parts from every chapters are discussed below. These omissions, mistakes, approach problems are sometime appears in the book under the Meta simple like this

Meta

sample this part.

Meta End

You are always welcome to add a new material: problem, question, illustration or photo of experiment. Material can be further illuminate. Additional material can be provided to give a different angle on the issue at hand.

Properties

The chapter isn't in development stage yet.

Open Channel Flow

The chapter isn't in the development stage yet. Some parts were taken from Fundamentals of Die Casting Design book and are in a process of improvement.

CHAPTER 1

Introduction

1.1 *What is Fluid Mechanics?*

Fluid mechanics deals with the study of all fluids under static and dynamic situations. Fluid mechanics is a branch of continuous mechanics which deals with a relationship between forces, motions, and statical conditions in continuous material. This study area deals with many and diversified problems such as surface tension, fluid statics, flow in enclosed bodies, or flow round bodies (solid or otherwise), flow stability, etc. In fact, almost any action a person is doing involves some kind of a fluid mechanics problem. Furthermore, the boundary between the solid mechanics and fluid mechanics is some kind of gray shed and not a sharp distinction (see Figure (1.1) for the complex relationships between the different branches which only part of it should be drawn in the same time.). For example, glass appears as a solid material, but a closer look reveals that the glass is a liquid with a large viscosity. A proof of the glass “liquidity” is the change of the glass thickness in high windows in European Churches after hundred years. The bottom part of the glass is thicker than the top part. Materials like sand (some call it quick sand) and grains should be treated as liquids. It is known that these materials have the ability to drown people. Even material such as aluminum just below the mushy zone also behaves as a liquid similarly to butter. After it was established that the boundaries of fluid mechanics aren’t sharp, the discussion in this book is limited to simple and (mostly) Newtonian (sometimes power fluids) fluids which will be defined later.

The fluid mechanics study involve many fields that have no clear boundary between them. Researchers distinguish between orderly flow and chaotic flow as the laminar flow and the turbulent flow. The fluid mechanics can also be distinguish between a single phase flow and multiphase flow (flow made more than one phase or single distinguishable material). The last boundary (as all the boundaries in fluid

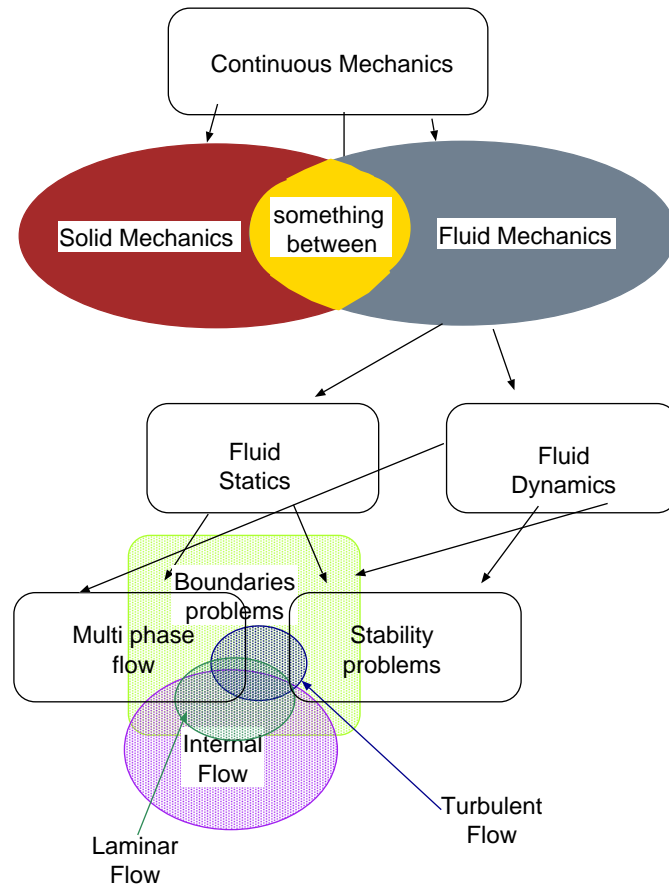


Fig. -1.1: Diagram to explain part of relationships of fluid mechanics branches.

mechanics) isn't sharp because fluid can go through a phase change (condensation or evaporation) in the middle or during the flow and switch from a single phase flow to a multi phase flow. Moreover, flow with two phases (or materials) can be treated as a single phase (for example, air with dust particle).

After it was made clear that the boundaries of fluid mechanics aren't sharp, the study must make arbitrary boundaries between fields. Then the dimensional analysis will be used explain why in certain cases one distinguish area/principle is more relevant than the other and some effects can be neglected. Or, when a general model is need because more parameters are effecting the situation. It is this author's personal experience that the knowledge and ability to know in what area the situation lay is one of the main problems. For example, engineers in software company (EKK Inc, <http://ekkin.com/HTML>) analyzed a flow of a complete still

liquid assuming a complex turbulent flow model. Such absurd analysis are common among engineers who do not know which model can be applied. Thus, one of the main goals of this book is to explain what model should be applied. Before dealing with the boundaries, the simplified private cases must be explained.

There are two main approaches of presenting an introduction of fluid mechanics materials. The first approach introduces the fluid kinematic and then the basic governing equations, to be followed by stability, turbulence, boundary layer and internal and external flow. The second approach deals with the Integral Analysis to be followed with Differential Analysis, and continue with Empirical Analysis. These two approaches pose a dilemma to anyone who writes an introductory book for the fluid mechanics. These two approaches have justifications and positive points. Reviewing many books on fluid mechanics made it clear, there isn't a clear winner. This book attempts to find a hybrid approach in which the kinematic is presented first (aside to standard initial four chapters) follow by Integral analysis and continued by Differential analysis. The ideal flow (frictionless flow) should be expanded compared to the regular treatment. This book is unique in providing chapter on multiphase flow. Naturally, chapters on open channel flow (as a sub class of the multiphase flow) and compressible flow (with the latest developments) are provided.

1.2 Brief History

The need to have some understanding of fluid mechanics started with the need to obtain water supply. For example, people realized that wells have to be dug and crude pumping devices need to be constructed. Later, a large population created a need to solve waste (sewage) and some basic understanding was created. At some point, people realized that water can be used to move things and provide power. When cities increased to a larger size, aqueducts were constructed. These aqueducts reached their greatest size and grandeur in those of the City of Rome and China.

Yet, almost all knowledge of the ancients can be summarized as application of instincts, with the exception Archimedes (250 B.C.) on the principles of buoyancy. For example, larger tunnels built for a larger water supply, etc. There were no calculations even with the great need for water supply and transportation. The first progress in fluid mechanics was made by Leonardo Da Vinci (1452-1519) who built the first chambered canal lock near Milan. He also made several attempts to study the flight (birds) and developed some concepts on the origin of the forces. After his initial work, the knowledge of fluid mechanics (hydraulic) increasingly gained speed by the contributions of Galileo, Torricelli, Euler, Newton, Bernoulli family, and D'Alembert. At that stage theory and experiments had some discrepancy. This fact was acknowledged by D'Alembert who stated that, "The theory of fluids must necessarily be based upon experiment." For example the concept of ideal liquid that leads to motion with no resistance, conflicts with the reality.

This discrepancy between theory and practice is called the “D’Alembert paradox” and serves to demonstrate the limitations of theory alone in solving fluid problems. As in thermodynamics, two different schools of thought were created: the first believed that the solution will come from theoretical aspect alone, and the second believed that solution is the pure practical (experimental) aspect of fluid mechanics. On the theoretical side, considerable contributions were made by Euler, La Grange, Helmholtz, Kirchhoff, Rayleigh, Rankine, and Kelvin. On the “experimental” side, mainly in pipes and open channels area, were Brahm, Bossut, Chezy, Dubuat, Fabre, Coulomb, Dupuit, d’Aubisson, Hagen, and Poiseuille.

In the middle of the nineteenth century, first Navier in the molecular level and later Stokes from continuous point of view succeeded in creating governing equations for real fluid motion. Thus, creating a matching between the two schools of thought: experimental and theoretical. But, as in thermodynamics, people cannot relinquish control. As results it created today “strange” names: Hydrodynamics, Hydraulics, Gas Dynamics, and Aeronautics.

The Navier-Stokes equations, which describes the flow (or even Euler equations), were considered unsolvable during the mid nineteenth century because of the high complexity. This problem led to two consequences. Theoreticians tried to simplify the equations and arrive at approximated solutions representing specific cases. Examples of such work are Hermann von Helmholtz’s concept of vortices (1858), Lanchester’s concept of circulatory flow (1894), and the Kutta-Joukowski circulation theory of lift (1906). The experimentalists, at the same time proposed many correlations to many fluid mechanics problems, for example, resistance by Darcy, Weisbach, Fanning, Ganguillet, and Manning. The obvious happened without theoretical guidance, the empirical formulas generated by fitting curves to experimental data (even sometime merely presenting the results in tabular form) resulting in formulas that the relationship between the physics and properties made very little sense.

At the end of the twenty century, the demand for vigorous scientific knowledge that can be applied to various liquids as opposed to formula for every fluid was created by the expansion of many industries. This demand coupled with new several novel concepts like the theoretical and experimental researches of Reynolds, the development of dimensional analysis by Rayleigh, and Froude’s idea of the use of models change the science of the fluid mechanics. Perhaps the most radical concept that effects the fluid mechanics is of Prandtl’s idea of boundary layer which is a combination of the modeling and dimensional analysis that leads to modern fluid mechanics. Therefore, many call Prandtl as the father of modern fluid mechanics. This concept leads to mathematical basis for many approximations. Thus, Prandtl and his students Blasius, von Karman, Meyer, and Blasius and several other individuals as Nikuradse, Rose, Taylor, Buckingham, Stanton, and many others, transformed the fluid mechanics to modern science that we have known today.

While the understanding of the fundamentals did not change much, after World War Two, the way how it was calculated changed. The introduction of

the computers during the 60s and much more powerful personal computer has changed the field. There are many open source programs that can analyze many fluid mechanics situations. Today many problems can be analyzed by using the numerical tools and provide reasonable results. These programs in many cases can capture all the appropriate parameters and adequately provide a reasonable description of the physics. However, there are many other cases that numerical analysis cannot provide any meaningful result (trends). For example, no weather prediction program can produce good engineering quality results (where the snow will fall within 50 kilometers accuracy. Building a car with this accuracy is a disaster). In the best scenario, these programs are as good as the input provided. Thus, assuming turbulent flow for still flow simply provides erroneous results (see for example, EKK, Inc).

1.3 Kinds of Fluids

Some differentiate fluid from solid by the reaction to shear stress. It is a known fact said that the fluid continuously and permanently deformed under shear stress while solid exhibits a finite deformation which does not change with time. It is also said that liquid cannot return to their original state after the deformation. This differentiation leads to three groups of materials: solids and liquids. This test creates a new material group that shows dual behaviors; under certain limits; it behaves like solid and under others it behaves like liquid (see Figure 1.1). The study of this kind of material called rheology and it will (almost) not be discussed in this book. It is evident from this discussion that when a liquid is at rest, no shear stress is applied.

The fluid is mainly divided into two categories: liquids and gases. The main difference between the liquids and gases state is that gas will occupy the whole volume while liquids has an almost fix volume. This difference can be, for most practical purposes considered, sharp even though in reality this difference isn't sharp. The difference between a gas phase to a liquid phase above the critical point are practically minor. But below the critical point, the change of water pressure by 1000% only change the volume by less than 1 percent. For example, a change in the volume by more 5% will required tens of thousands percent change of the pressure. So, if the change of pressure is significantly less than that, then the change of volume is at best 5%. Hence, the pressure will not affect the volume. In gaseous phase, any change in pressure directly affects the volume. The gas fills the volume and liquid cannot. Gas has no free interface/surface (since it does fill the entire volume).

There are several quantities that have to be addressed in this discussion. The first is **force** which was reviewed in physics. The unit used to measure is [N]. It must be remember that force is a vector, e.g it has a direction. The second quantity discussed here is the area. This quantity was discussed in physics class but here it has an additional meaning, and it is referred to the direction of the area. The direction of area is perpendicular to the area. The area is measured in [m^2]. Area

of three-dimensional object has no single direction. Thus, these kinds of areas should be addressed infinitesimally and locally.

The traditional quantity, which is force per area has a new meaning. This is a result of division of a vector by a vector and it is referred to as tensor. In this book, the emphasis is on the physics, so at this stage the tensor will have to be broken into its components. Later, the discussion on the mathematic meaning will be presented (later version). For the discussion here, the pressure has three components, one in the area direction and two perpendicular to the area. The pressure component in the area direction is called pressure (great way to confuse, isn't it?). The other two components are referred as the shear stresses. The units used for the pressure components is $[N/m^2]$.

The density is a property which requires that liquid to be continuous. The density can be changed and it is a function of time and space (location) but must have a continuous property. It doesn't mean that a sharp and abrupt change in the density cannot occur. It referred to density that is independent of the sampling size. Figure 1.2 shows the density as a function of the sample size. After certain sample size, the density remains constant. Thus, the density is defined as

$$\rho = \lim_{\Delta V \rightarrow \epsilon} \frac{\Delta m}{\Delta V} \quad (1.1)$$

It must be noted that ϵ is chosen so that the continuous assumption is not broken, that is, it did not reach/reduced to the size where the atoms or molecular statistical calculations are significant (see Figure 1.2 for point where the green lines converge to constant density). When this assumption is broken, then, the principles of statistical mechanics must be utilized.

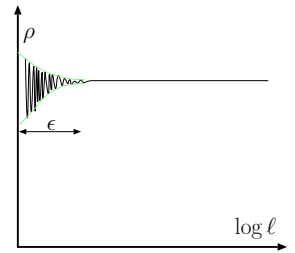


Fig. -1.2: Density as a function of the size of sample.

1.4 Shear Stress

The shear stress is part of the pressure tensor. However, here it will be treated as a separate issue. In solid mechanics, the shear stress is considered as the ratio of the force acting on area in the direction of the forces perpendicular to area. Different from solid, fluid cannot pull directly but through a solid surface. Consider liquid that undergoes a shear stress between a short distance of two plates as shown in Figure (1.3).

The upper plate velocity generally will be

$$U = f(A, F, h) \quad (1.2)$$

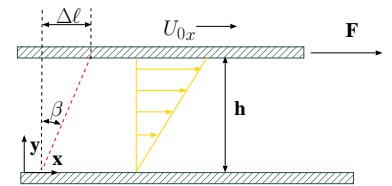


Fig. -1.3: Schematics to describe the shear stress in fluid mechanics.

Where A is the area, the F denotes the force, h is the distance between the plates. From solid mechanics study, it was shown that when the force per area increases, the velocity of the plate increases also. Experiments show that the increase of height will increase the velocity up to a certain range. Consider moving the plate with a zero lubricant ($h \sim 0$) (results in large force) or a large amount of lubricant (smaller force). In this discussion, the aim is to develop differential equation, thus the small distance analysis is applicable.

For cases where the dependency is linear, the following can be written

$$U \propto \frac{h F}{A} \tag{1.3}$$

Equations (1.3) can be rearranged to be

$$\frac{U}{h} \propto \frac{F}{A} \tag{1.4}$$

Shear stress was defined as

$$\tau_{xy} = \frac{F}{A} \tag{1.5}$$

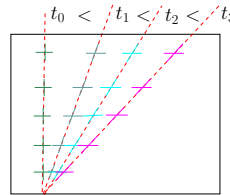
From equations (1.4) and (1.5) it follows that ratio of the velocity to height is proportional to shear stress. Hence, applying the coefficient to obtain a new equality as

$$\tau_{xy} = \mu \frac{U}{h} \tag{1.6}$$

Where μ is called the absolute viscosity or dynamic viscosity which will be discussed later in this chapter in great length.

In steady state, the distance the upper plate moves after small amount of time, δt is

$$d\ell = U \delta t \tag{1.7}$$



From figure (1.4) it can be noticed that for a small angle, the regular approximation provides

Fig. -1.4: The deformation of fluid due to shear stress as progression of time.

$$d\ell = U \delta t = \overbrace{h \delta\beta}^{\text{geometry}} \tag{1.8}$$

From equation (1.8) it follows that

$$U = h \frac{\delta\beta}{\delta t} \tag{1.9}$$

Combining equation (1.9) with equation (1.6) yields

$$\tau_{xy} = \mu \frac{\delta\beta}{\delta t} \quad (1.10)$$

If the velocity profile is linear between the plate (it will be shown later that it is consistent with derivations of velocity), then it can be written for small a angel that

$$\frac{\delta\beta}{\delta t} = \frac{dU}{dy} \quad (1.11)$$

Materials which obey equation (1.10) referred to as Newtonian fluid. For this kind of substance

$$\tau_{xy} = \mu \frac{dU}{dy} \quad (1.12)$$

Newtonian fluids are fluids which the ratio is constant. Many fluids fall into this category such as air, water etc. This approximation is appropriate for many other fluids but only within some ranges.

Equation (1.9) can be interpreted as momentum in the x direction transferred into the y direction. Thus, the viscosity is the resistance to the flow (flux) or the movement. The property of viscosity, which is exhibited by all fluids, is due to the existence of cohesion and interaction between fluid molecules. These cohesion and interactions hamper the flux in y -direction. Some referred to shear stress as viscous flux of x -momentum in the y -direction. The units of shear stress are the same as flux per time as following

$$\frac{F}{A} \left[\frac{kg \ m}{sec^2 \ m^2} \right] = \frac{\dot{m}U}{A} \left[\frac{kg}{sec} \frac{m}{sec} \frac{1}{m^2} \right]$$

Thus, the notation of τ_{xy} is easier to understand and visualize. In fact, this interpretation is more suitable to explain the molecular mechanism of the viscosity. The units of absolute viscosity are $[N \ sec/m^2]$.

Example 1.1:

A space of 1 [cm] width between two large plane surfaces is filled with glycerine. Calculate the force that is required to drag a very thin plate of 1 [m²] at a speed of 0.5 m/sec. It can be assumed that the plates remains in equidistance from each other and steady state is achieved instantly.

SOLUTION

Assuming Newtonian flow, the following can be written (see equation (1.6))

$$F = \frac{A\mu U}{h} \sim \frac{1 \times 1.069 \times 0.5}{0.01} = 53.45[N]$$

Example 1.2:

Castor oil at 25°C fills the space between two concentric cylinders of 0.2[m] and 0.1[m] diameters with height of 0.1 [m]. Calculate the torque required to rotate the inner cylinder at 12 rpm, when the outer cylinder remains stationary. Assume steady state conditions.

SOLUTION

The velocity is

$$U = r \dot{\theta} = 2 \pi r_i \text{ rps} = 2 \times \pi \times 0.1 \times \overbrace{12/60}^{\text{rps}} = 0.4 \pi r_i$$

Where rps is revolution per second.

The same way as in example (1.1), the moment can be calculated as the force times the distance as

$$M = F \ell = \frac{\overbrace{r_i}^{\ell} \overbrace{2 \pi r_i h}^A \mu U}{r_o - r_i}$$

In this case $r_o - r_i = h$ thus,

$$M = \frac{2 \pi^2 \overbrace{0.1^3}^{r_i} \overbrace{0.986}^{\mu} 0.4}{h} \sim .0078 [N m]$$

1.5 Viscosity

1.5.1 General

Viscosity varies widely with temperature. However, temperature variation has an opposite effect on the viscosities of liquids and gases. The difference is due to their fundamentally different mechanism creating viscosity characteristics. In gases, molecules are sparse and cohesion is negligible, while in the liquids, the molecules are more compact and cohesion is more dominant. Thus, in gases, the exchange of momentum between layers brought as a result of molecular movement normal to the general direction of flow, and it resists the

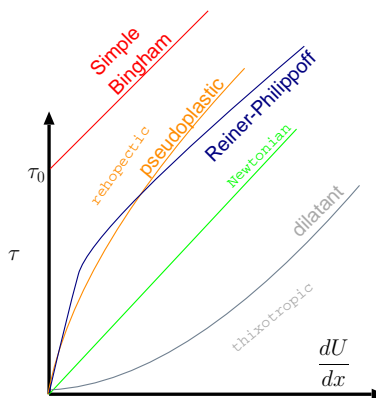


Fig. -1.5: the difference of power fluids

flow. This molecular activity is known to increase with temperature, thus, the viscosity of gases will increase with temperature. This reasoning is a result of the considerations of the kinetic theory. This theory indicates that gas viscosities vary directly with the square root of temperature. In liquids, the momentum exchange due to molecular movement is small compared to the cohesive forces between the molecules. Thus, the viscosity is primarily dependent on the magnitude of these cohesive forces. Since these forces decrease rapidly with increases of temperature, liquid viscosities decrease as temperature increases.

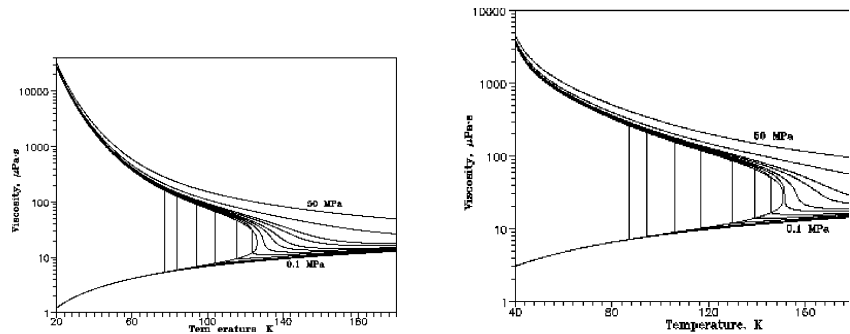


Fig. -1.6: Nitrogen (left) and Argon (right) viscosity as a function of the temperature and pressure after Lemmon and Jacobsen.

Figure 1.6 demonstrates that viscosity increases slightly with pressure, but this variation is negligible for most engineering problems. Well above the critical point, both materials are only a function of the temperature. On the liquid side below the critical point, the pressure has minor effect on the viscosity. It must be stress that the viscosity in the dome is meaningless. There is no such a thing of viscosity at 30% liquid. It simply depends on the structure of the flow as will be discussed in the chapter on multi phase flow. The lines in the above diagrams are only to show constant pressure lines. Oils have the greatest increase of viscosity with pressure which is a good thing for many engineering purposes.

1.5.2 Non-Newtonian Fluids

In equation (1.5), the relationship between the velocity and the shear stress was assumed to be linear. Not all the materials obey this relationship. There is a large class of materials which shows a non-linear relationship with velocity for any shear stress. This class of materials can be approximated by a single polynomial term that is $a = bx^n$. From the physical point of view, the coefficient depends on the velocity gradient. This relationship is referred to as power relationship and it can

be written as

$$\tau = K \overbrace{\left(\frac{dU}{dx}\right)^{n-1}}^{\text{viscosity}} \left(\frac{dU}{dx}\right) \tag{1.13}$$

The new coefficients (n, K) in equation (1.13) are constant. When $n = 1$ equation represent Newtonian fluid and K becomes the familiar μ . The viscosity coefficient is always positive. When n , is above one, the liquid is dilatant. When n is below one, the fluid is pseudoplastic. The liquids which satisfy equation (1.13) are referred to as purely viscous fluids. Many fluids satisfy the above equation. Fluids that show increase in the viscosity (with increase of the shear) referred to as thixotropic and those that show decrease are called reoplectic fluids (see Figure 1.5).

Materials which behave up to a certain shear stress as a solid and above it as a liquid are referred as Bingham liquids. In the simple case, the “liquid side” is like Newtonian fluid for large shear stress. The general relationship for simple Bingham flow is

$$\tau_{xy} = -\mu \pm \tau_0 \quad \text{if } |\tau_{yx}| > \tau_0 \tag{1.14}$$

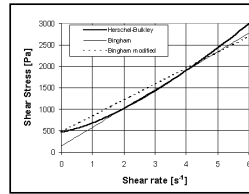


Fig. -1.7: The shear stress as a function of the shear rate

$$\frac{dU_x}{dy} = 0 \quad \text{if } |\tau_{yx}| < \tau_0 \tag{1.15}$$

There are materials that simple Bingham model does not provide dequate explanation and a more sophistate model is required. The Newtonian part of the model has to be replaced by power liquid. For example, according to Ferraris at el¹ concrete behaves as shown in Figure 1.7. However, for most practical purposes, this kind of figures isn't used in regular engineering practice.

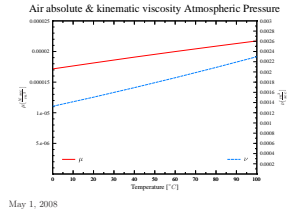
1.5.3 Kinematic Viscosity

The kinematic viscosity is another way to look at the viscosity. The reason for this new definition is that some experimental data are given in this form. These results also explained better using the new definition. The kinematic viscosity embraces both the viscosity and density properties of a fluid. The above equation shows that the dimensions of ν to be square meter per second, [m^2/sec], which are acceleration units (a combination of kinematic terms). This fact explains the name “kinematic” viscosity. The kinematic viscosity is defined as

$$\nu = \frac{\mu}{\rho} \tag{1.16}$$

¹C. Ferraris, F. de Larrard and N. Martys, Materials Science of Concrete VI, S. Mindess and J. Skalny, eds., 215-241 (2001)

The gas density increases with the temperature. The increase of the absolute viscosity with the temperature is enough to overcome the increase of density and thus, the kinematic viscosity also increase with the temperature for many materials.



May 1, 2008

1.5.4 Estimation of The Viscosity

The absolute viscosity of many fluids relatively doesn't change with the pressure but very sensitive to temperature. For isothermal flow, the viscosity can be considered constant in many cases. The variations of air and water as a function of the temperature at atmospheric pressure are plotted in Figures 1.8 and 1.9.

Some common materials (pure and mixture) have expressions that provide an estimate. For many gases, Sutherland's equation is used and according to the literature, provides reasonable results² for the range of $-40^{\circ}C$ to $1600^{\circ}C$

$$\mu = \mu_0 \frac{0.555 T_{i0} + Suth}{0.555 T_{in} + Suth} \left(\frac{T}{T_0} \right)^{\frac{3}{2}} \quad (1.17)$$

Where

μ	viscosity at input temperature T
μ_0	reference viscosity at reference temperature, T_{i0}
T_{in}	input temperature in degrees Kelvin
T_{i0}	reference temperature in degrees Kelvin
$Suth$	Suth is Sutherland's constant and it is presented in the Table 1.1

Example 1.3:

Calculate the viscosity of air at 800K based on Sutherland's equation. Use the data provide in Table 1.1.

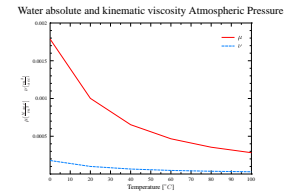
SOLUTION

Applying the constants from Suthelnd's table provides

$$\mu = 0.00001827 \times \frac{0.555 \times 524.07 + 120}{0.555 \times 800 + 120} \times \left(\frac{800}{524.07} \right)^{\frac{3}{2}} \sim 2.51 \cdot 10^{-5} \left[\frac{N \cdot sec}{m^2} \right]$$

²This author is ambivalent about statement.

Fig. -1.8: Air viscosity as a function of the temperature.



May 1, 2008

Fig. -1.9: Water viscosity as a function temperature.

Material	coefficients	chemical formula	Sutherland	$T_{i0}[K]$	$\mu_0(N\ sec/m^2)$
ammonia		NH_3	370	527.67	0.00000982
standard air			120	524.07	0.00001827
carbon dioxide		CO_2	240	527.67	0.00001480
carbon monoxide		CO	118	518.67	0.00001720
hydrogen		H_2	72	528.93	0.0000876
nitrogen		N_2	111	540.99	0.0001781
oxygen		O_2	127	526.05	0.0002018
sulfur dioxide		SO_2	416	528.57	0.0001254

Table -1.1: The list for Sutherland's equation coefficients for selected materials.

The viscosity increases almost by 40%. The observed viscosity is about $\sim 3.710^{-5}$.

Liquid Metals

Liquid metal can be considered as a Newtonian fluid for many applications. Furthermore, many aluminum alloys are behaving as a Newtonian liquid until the first solidification appears (assuming steady state thermodynamics properties). Even when there is a solidification (mushy zone), the metal behavior can be estimated as a Newtonian material (further reading can be done in this author's book "Fundamentals of Die Casting Design"). Figure 1.10 exhibits several liquid metals (from The Reactor Handbook, Vol. Atomic Energy Commission AECD-3646 U.S. Government Printing Office, Washington D.C. May 1995 p. 258.)

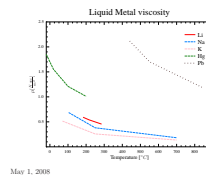


Fig. -1.10: Liquid metals viscosity as a function of the temperature.

The General Viscosity Graphs

In case "ordinary" fluids where information is limit, Hougen et al suggested to use graph similar to compressibility chart. In this graph, if one point is well documented, other points can be estimated. Furthermore, this graph also shows the trends. In Figure 1.11 the relative viscosity $\mu_r = \mu/\mu_c$ is plotted as a function

Substance	formula	Temperature $T [^{\circ}C]$	Viscosity [$\frac{N \cdot sec}{m^2}$]
	$i - C_4H_{10}$	23	0.0000076
	CH_4	20	0.0000109
	CO_2	20	0.0000146
oxygen	O_2	20	0.0000203
mercury vapor	Hg	380	0.0000654

Table -1.2: Viscosity of selected gases.

Substance	formula	Temperature $T [^{\circ}C]$	Viscosity [$\frac{N \cdot sec}{m^2}$]
	$(C_2H_5)O$	20	0.000245
	C_6H_6	20	0.000647
	Br_2	26	0.000946
	C_2H_5OH	20	0.001194
	Hg	25	0.001547
	H_2SO_4	25	0.01915
Olive Oil		25	0.084
Castor Oil		25	0.986
Glucose		25	5-20
Corn Oil		20	0.072
SAE 30		-	0.15-0.200
SAE 50		$\sim 25^{\circ}C$	0.54
SAE 70		$\sim 25^{\circ}C$	1.6
Ketchup		$\sim 20^{\circ}C$	0,05
Ketchup		$\sim 25^{\circ}C$	0,098
Benzene		$\sim 20^{\circ}C$	0.000652
Firm glass		-	$\sim 1 \times 10^7$
Glycerol		20	1.069

Table -1.3: Viscosity of selected liquids.

chemical component	Molecular Weight	T_c [K]	P_c [Bar]	μ_c [$\frac{N \cdot sec}{m^2}$]
H_2	2.016	33.3	12.9696	3.47
He	4.003	5.26	2.289945	2.54
Ne	20.183	44.5	27.256425	15.6
Ar	39.944	151	48.636	26.4
Xe	131.3	289.8	58.7685	49.
Air "mix"	28.97	132	36.8823	19.3
CO_2	44.01	304.2	73.865925	19.0
O_2	32.00	154.4	50.358525	18.0
C_2H_6	30.07	305.4	48.83865	21.0
CH_4	16.04	190.7	46.40685	15.9
Water		647.096 K	22.064 [MPa]	

Table -1.4: The properties at the critical stage and their values of selected materials.

of relative temperature, T_r . μ_c is the viscosity at critical condition and μ is the viscosity at any given condition. The lines of constant relative pressure, $P_r = P/P_c$ are drawn. The lower pressure is, for practical purpose, $\sim 1[bar]$.

The critical pressure can be evaluated in the following three ways. The simplest way is by obtaining the data from Table 1.4 or similar information. The second way, if the information is available and is close enough to the critical point, then the critical viscosity is obtained as

$$\mu_c = \frac{\overbrace{\mu}^{\text{given}}}{\underbrace{\mu_r}_{\text{figure 1.11}}} \quad (1.18)$$

The third way, when none is available, is by utilizing the following approximation

$$\mu_c = \sqrt{M T_c \tilde{v}_c^{2/3}} \quad (1.19)$$

Where \tilde{v}_c is the critical molecular volume and M is molecular weight. Or

$$\mu_c = \sqrt{M P_c^{2/3} T_c^{-1/6}} \quad (1.20)$$

Calculate the reduced pressure and the reduced temperature and from the Figure 1.11 obtain the reduced viscosity.

Example 1.4:

Estimate the viscosity of oxygen, O_2 at $100^\circ C$ and $20[Bar]$.

SOLUTION

The critical condition of oxygen are $P_c = 50.35[\text{Bar}]$ $T_c = 154.4$ $\mu_c = 18 \left[\frac{\text{N sec}}{\text{m}^2} \right]$ The value of the reduced temperature is

$$T_r \sim \frac{373.15}{154.4} \sim 2.41$$

The value of the reduced pressure is

$$P_r \sim \frac{20}{50.35} \sim 0.4$$

From Figure 1.11 it can be obtained $\mu_r \sim 1.2$ and the predicted viscosity is

$$\mu = \mu_c \overbrace{\left(\frac{\mu}{\mu_c} \right)}^{\text{Table}} = 18 \times 1.2 = 21.6[\text{N sec}/\text{m}^2]$$

The observed value is $24[\text{N sec}/\text{m}^2]^3$.

Viscosity of Mixtures

In general the viscosity of liquid mixture has to be evaluated experimentally. Even for homogeneous mixture, there isn't silver bullet to estimate the viscosity. In this book, only the mixture of low density gases is discussed for analytical expression. For most cases, the following Wilke's correlation for gas at low density provides a result in a reasonable range.

$$\mu_{mix} = \sum_{i=1}^n \frac{x_i \mu_i}{\sum_{j=1}^n x_j \Phi_{ij}} \quad (1.21)$$

where Φ_{ij} is defined as

$$\Phi_{ij} = \frac{1}{\sqrt{8}} \sqrt{1 + \frac{M_i}{M_j}} \left(1 + \sqrt{\frac{\mu_i}{\mu_j}} \sqrt[4]{\frac{M_j}{M_i}} \right)^2 \quad (1.22)$$

Here, n is the number of the chemical components in the mixture. x_i is the mole fraction of component i , and μ_i is the viscosity of component i . The subscript i should be used for the j index. The dimensionless parameter Φ_{ij} is equal to one when $i = j$. The mixture viscosity is highly nonlinear function of the fractions of the components.

Example 1.5:

Calculate the viscosity of a mixture (air) made of 20% oxygen, O_2 and 80% nitrogen N_2 for the temperature of 20°C .

³Kyama, Makita, Rev. Physical Chemistry Japan Vol. 26 No. 2 1956.

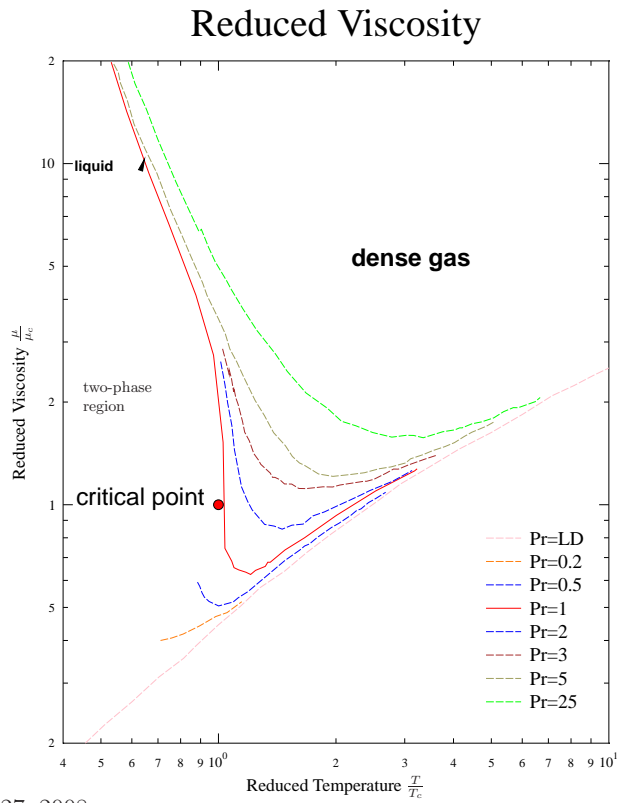
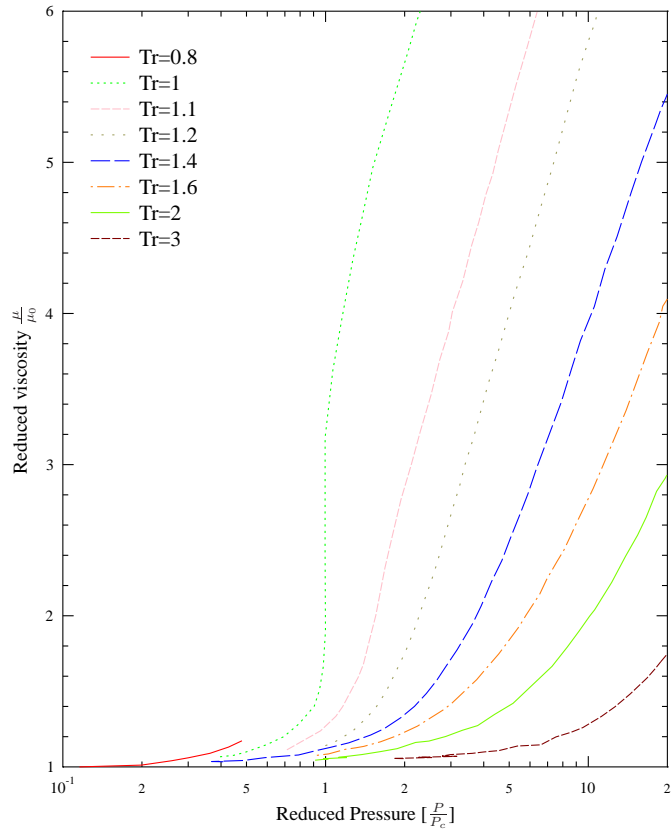


Fig. -1.11: Reduced viscosity as function of the reduced temperature.

SOLUTION

The following table summarize the known details

i	Component	Molecular Weight, M	Mole Fraction, x	Viscosity, μ
1	O_2	32.	0.2	0.0000203
2	N_2	28.	0.8	0.00001754



June 2, 2008

Fig. -1.12: Reduced viscosity as function of the reduced temperature.

i	j	M_i/M_j	μ_i/μ_j	Φ_{ij}
1	1	1.0	1.0	1.0
	2	1.143	1.157	1.0024
2	1	0.875	.86	0.996
	2	1.0	1.0	1.0

$$\mu_{mix} \sim \frac{0.2 \times 0.0000203}{0.2 \times 1.0 + 0.8 \times 1.0024} + \frac{0.8 \times 0.00001754}{0.2 \times 0.996 + 0.8 \times 1.0} \sim 0.0000181 \left[\frac{N \text{ sec}}{m^2} \right]$$

The observed value is $\sim 0.0000182 \left[\frac{N \text{ sec}}{m^2} \right]$.

In very low pressure, in theory, the viscosity is only a function of the tem-

perature with a “simple” molecular structure. For gases with very long molecular structure or complexity structure these formulas cannot be applied. For some mixtures of two liquids it was observed that at a low shear stress, the viscosity is dominated by a liquid with high viscosity and at high shear stress to be dominated by a liquid with the low viscosity liquid. The higher viscosity is more dominate at low shear stress. Reiner and Phillippoff suggested the following formula

$$\frac{dU_x}{dy} = \left(\frac{1}{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \left(\frac{\tau_{xy}}{\tau_s}\right)^2}} \right) \tau_{xy} \quad (1.23)$$

Where the term μ_∞ is the experimental value at high shear stress. The term μ_0 is the experimental viscosity at shear stress approaching zero. The term τ_s is the characteristic shear stress of the mixture. An example for values for this formula, for Molten Sulfur at temperature $120^\circ C$ are $\mu_\infty = 0.0215 \left(\frac{N \text{ sec}}{m^2}\right)$, $\mu_0 = 0.00105 \left(\frac{N \text{ sec}}{m^2}\right)$, and $\tau_s = 0.0000073 \left(\frac{kN}{m^2}\right)$. This equation (1.23) provides reasonable value only up to $\tau = 0.001 \left(\frac{kN}{m^2}\right)$.

Figure 1.12 can be used for a crude estimate of dense gases mixture. To estimate the viscosity of the mixture with n component Hougen and Watson’s method for pseudocritical properties is adapted. In this method the following is defined as

$$P_{c \text{ mix}} = \sum_{i=1}^n x_i P_{c_i} \quad (1.24)$$

$$T_{c \text{ mix}} = \sum_{i=1}^n x_i T_{c_i} \quad (1.25)$$

and

$$\mu_{c \text{ mix}} = \sum_{i=1}^n x_i \mu_{c_i} \quad (1.26)$$

1.5.5 Bulk Modulus

Similar to solids (hook’s law), liquids have a property that describes the volume change as results of pressure change for constant temperature. It can be noted that this property is not the result of the equation of state but related to it. The bulk modulus is defined as

$$B_T = -v \left(\frac{\partial P}{\partial v} \right)_T \quad (1.27)$$

Using the identity of $v = 1/\rho$ transfers equation (1.27) into

$$B_T = \rho \left(\frac{\partial P}{\partial \rho} \right)_T \quad (1.28)$$

The bulk modulus for several liquids is presented in Table 1.5.

Table -1.5: The bulk modulus for selected material with the critical temperature and pressure $na \rightarrow$ not available and $nf \rightarrow$ not found (exist but was not found in the literature).

chemical component	Bulk Modulus $10^9 \frac{N}{m}$	T_c	P_c
Acetic Acid	2.49	593K	57.8 [Bar]
Acetone	0.80	508 K	48 [Bar]
Benzene	1.10	562 K	4.74 [MPa]
Carbon Tetrachloride	1.32	556.4 K	4.49 [MPa]
Ethyl Alcohol	1.06	514 K	6.3 [Mpa]
Gasoline	1.3	nf	nf
Glycerol	4.03-4.52	850 K	7.5 [Bar]
Mercury	26.2-28.5	1750 K	172.00 [MPa]
Methyl Alcohol	0.97	Est 513	Est 78.5 [Bar]
Nitrobenzene	2.20	nf	nf
Olive Oil	1.60	nf	nf
Paraffin Oil	1.62	nf	nf
SAE 30 Oil	1.5	na	na
Seawater	2.34	na	na
Toluene	1.09	591.79 K	4.109 [MPa]
Turpentine	1.28	na	na
Water	2.15-2.174	647.096 K	22.064 [MPa]

In the literature, additional expansions for similar parameters are defined. The thermal expansion is defined as

$$\beta_P = \frac{1}{v} \left(\frac{\partial v}{\partial T} \right)_P \quad (1.29)$$

This parameter indicates the change of volume due to temperature change when the pressure is constant. Another definition is referred as coefficient of tension and it is defined as

$$\beta_v = \frac{1}{P} \left(\frac{\partial P}{\partial T} \right)_v \quad (1.30)$$

This parameter indicates the change of the pressure due to the change of temperature (where $v = constant$). These definitions are related to each other. This relationship is obtained by the observation that the pressure as a function of the temperature and specific volume as

$$P = f(T, v) \tag{1.31}$$

The full pressure derivative is

$$dP = \left(\frac{\partial P}{\partial T}\right)_v dT + \left(\frac{\partial P}{\partial v}\right)_T dv \tag{1.32}$$

On constant pressure lines, $dP = 0$, and therefore equation (1.32) is

$$0 = \left(\frac{\partial P}{\partial T}\right)_v dT + \left(\frac{\partial P}{\partial v}\right)_T dv \tag{1.33}$$

From equation (1.33) follows that

$$\left.\frac{dv}{dT}\right|_{P=const} = -\frac{\left(\frac{\partial P}{\partial T}\right)_v}{\left(\frac{\partial P}{\partial v}\right)_T} \tag{1.34}$$

Equation (1.34) indicates that relationship for these three coefficients is

$$\beta_T = -\frac{\beta_v}{\beta_P} \tag{1.35}$$

The last equation (1.35) sometimes is used in measurement of the bulk modulus.

The increase of the pressure increases the bulk modulus due to the molecules increase of the rejecting forces between each other when they are closer. In contrast, the temperature increase results in reduction of the bulk of modulus because the molecular are further away.

Example 1.6:

Calculate the modulus of liquid elasticity that reduced 0.035 per cent of its volume by applying a pressure of 5[Bar].

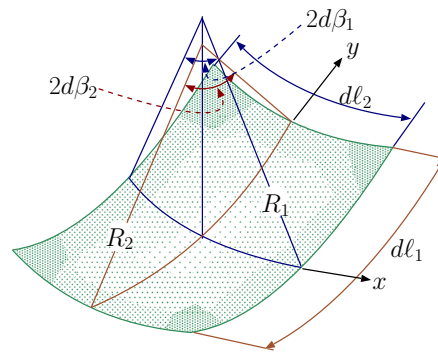


Fig. -1.13: Surface Tension control volume analysis.

SOLUTION

Using the definition for the bulk modulus

$$\beta_T = -v \frac{\partial P}{\partial v} \sim \frac{v}{\Delta v} \Delta P = \frac{5}{0.00035} \sim 14285.714 [\text{Bar}]$$

Example 1.7:

Calculate the pressure needed to apply on water to reduce its volume by 1 per cent. Assume the temperature to be 20°C.

SOLUTION

Using the definition for the bulk modulus

$$\Delta P \sim \beta_T \frac{\Delta v}{v} \sim 2.15 \cdot 10^9 \cdot 0.01 = 2.15 \cdot 10^7 [\text{N/m}^2] = 215 [\text{Bar}]$$

1.6 Surface Tension

The surface tension manifested itself by a rise or depression of the liquid at the free surface edge. Surface tension is also responsible for the creation of the drops and bubbles. It also responsible for the breakage of a liquid jet into other medium/phase to many drops (atomization). The surface tension is force per length and is measured by [N/m] and is acting to stretch the surface.

Surface tension results from a sharp change in the density between two adjacent phases or materials. There is a common misconception for the source of the surface tension. In many (physics, surface tension, and fluid mechanics) books explained that the surface tension is a result from unbalance molecular cohesive forces. This explanation is wrong since it is in conflict with Newton's second law (see example ?). This erroneous explanation can be traced to Adam's book but earlier source may be found.

The relationship between the surface tension and the pressure on the two sides of the surface is based on geometry. Consider a small element of surface. The pressure on one side is P_i and the pressure on the other side is P_o . When the surface tension is constant, the horizontal forces cancel each other because symmetry. In the vertical direction, the surface tension forces are pulling the surface upward. Thus, the pressure difference has to balance the surface tension. The forces in the vertical direction reads

$$(P_i - P_o) dl_1 dl_2 = \Delta P dl_1 dl_2 = 2\sigma dl_1 \sin \beta_1 + 2\sigma dl_2 \sin \beta_2 \quad (1.36)$$

For a very small area, the angles are very small and thus ($\sin \beta \sim \beta$). Furthermore, it can be noticed that $d\ell_i \sim 2 R_i d\beta_i$. Thus, the equation (1.36) can be simplified as

$$\Delta P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (1.37)$$

Equation (1.37) predicts that pressure difference increase with inverse of the radius. There are two extreme cases: one) radius of infinite and radius of finite size. The second with two equal radii. The first case is for an infinite long cylinder for which the equation (1.37) is reduced to

$$\Delta P = \sigma \left(\frac{1}{R} \right) \quad (1.38)$$

Other extreme is for a sphere for which the main radii are the same and equation (1.37) is reduced to

$$\Delta P = \frac{2\sigma}{R} \quad (1.39)$$

Where R is the radius of the sphere. A soap bubble is made of two layers, inner and outer, thus the pressure inside the bubble is

$$\Delta P = \frac{4\sigma}{R} \quad (1.40)$$

Example 1.8:

A Tank filled with liquid, which contains n bubbles with equal radii, r . Calculate the minimum work required to increase the pressure in tank by ΔP . Assume that the liquid bulk modulus is infinity.

SOLUTION

The work is due to the change of the bubbles volume. The work is

$$w = \int_{r_0}^{r_f} \Delta P(v) dv \quad (1.41)$$

The minimum work will be for a reversible process. The reversible process requires very slow compression. It is worth noting that for very slow process, the temperature must remain constant due to heat transfer. The relationship between pressure difference and the radius is described by equation (1.39) for reversible process. Hence the work is

$$w = \int_{r_0}^{r_f} \overbrace{\frac{2\sigma}{r}}^{\Delta P} \overbrace{4\pi r^2 dr}^{dv} = 8\pi\sigma \int_{r_0}^{r_f} r dr = 4\pi\sigma (r_f^2 - r_0^2) \quad (1.42)$$

Where, r_0 is the radius at the initial stage and r_f is the radius at the final stage.

The work for n bubbles is then $4\pi\sigma n(r_f^2 - r_0^2)$. It can be noticed that the work is negative, that is the work is done on the system.

1.6.1 Wetting of Surfaces

To explain the source of the contact angle, consider the point where three phases became in contact. This contact point occurs due to free surface reaching a solid boundary. The surface tension occurs between gas phase (G) to liquid phase (L) and also occurs between the solid (S) and the liquid phases as well as between the gas phase and the solid phase. In Figure 1.14, forces diagram is shown when control volume is chosen so that the masses of the solid, liquid, and gas can be ignored. Regardless to the magnitude of the surface tensions (except to zero) the forces cannot be balanced for the description of straight lines. For example, forces balanced along the line of solid boundary is

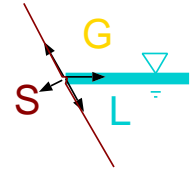


Fig. -1.14: Forces in Contact angle.

$$\sigma_{gs} - \sigma_{ls} - \sigma_{lg} \cos \beta = 0 \quad (1.43)$$

and in the tangent direction to the solid line the forces balance is

$$F_{solid} = \sigma_{lg} \sin \beta \quad (1.44)$$

substituting equation (1.44) into equation (1.43) yields

$$\sigma_{gs} - \sigma_{ls} = \frac{F_{solid}}{\tan \beta} \quad (1.45)$$

For $\beta = \pi/2 \implies \tan \beta = \infty$. Thus, the solid reaction force must be zero. The gas solid surface tension is different from the liquid solid surface tension and hence violating equation (1.43).

The surface tension forces must be balanced, thus, a contact angle is created to balance it. The contact angle is determined by whether the surface tension between the gas solid (gs) is larger or smaller than the surface tension of liquid solid (ls) and the local geometry. It must be noted that the solid boundary isn't straight. The surface tension is a molecular phenomenon, thus depend on the locale structure of the surface and it provides the balance for these local structures.

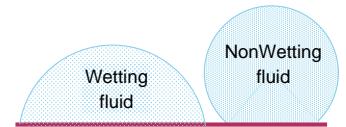


Fig. -1.15: Description of wetting and non-wetting fluids.

The connection of the three phases—materials—mediums creates two situations which are categorized as wetting or non-wetting. There is a common definition of wetting the surface. If the angle of the contact between three materials is larger than 90° then it is non-wetting. On the other hand, if the angle is below than 90° the material is wetting the surface (see Figure 1.15). The angle is determined by properties of the liquid, gas medium and the solid surface. And a small change on the solid surface can change the wetting condition to non-wetting. In fact there are commercial sprays that are intent to change the surface from wetting to non wetting. This fact is the reason that no reliable data can be provided with the exception to pure substances and perfect geometries. For example, water is described in many books as a wetting fluid. This statement is correct in most cases, however, when solid surface is made or coated with certain materials, the water is changed to be wetting (for example 3M selling product to “change” water to non-wetting). So, the wetness of fluids is a function of the solid as well.

Table -1.6: The contact angle for air, distilled water with selected materials to demonstrate the inconsistency.

chemical component	Contact Angle	Source
Steel	$\pi/3.7$	[1]
Steel,Nickel	$\pi/4.74$	[2]
Nickel	$\pi/4.74$ to $\pi/3.83$	[1]
Nickel	$\pi/4.76$ to $\pi/3.83$	[3]
Chrome-Nickel Steel	$\pi/3.7$	[4]
Silver	$\pi/6$ to $\pi/4.5$	[5]
Zink	$\pi/3.4$	[4]
Bronze	$\pi/3.2$	[4]
Copper	$\pi/4$	[4]
Copper	$\pi/3$	[7]
Copper	$\pi/2$	[8]

- 1 R. Siegel, E. G. Keshock (1975) “Effects of reduced gravity on nucleate boiling bubble dynamics in saturated water,” AIChE Journal Volume 10 Issue 4, Pages 509 - 517. 1975
- 2 Bergles A. E. and Rohsenow W. M. ”The determination of forced convection surface-boiling heat transfer, ASME, J. Heat Transfer, vol 1 pp 365 - 372.
- 3 Tolubinsky, V.I. and Ostrovsky, Y.N. (1966) “On the mechanism of boiling heat transfer”,. International Journal of Heat and Mass Transfer, Vol. 9, No 12, pages 1465-1470.
- 4 Arefeva E.I., Aladev O, I.T., (1958) “wlijanii smatchivaemosti na teploobmen pri

- kipenii," Injenerno Fizitcheskij Jurnal, 11-17 1(7) In Russian.
- 5 Labuntsov D. A. (1939) "Approximate theory of heat transfer by developed nucleate boiling" In Sussian Izvestiya An SSSR , Energetika I transport, No 1.
 - 6 Basu, N., Warriar, G. R., and Dhir, V. K., (2002) "Onset of Nucleate Boiling and Active Nucleation Site Density during Subcooled Flow Boiling," ASME Journal of Heat Transfer, Vol. 124, pages 717 -728.
 - 7 Gaetner, R. F., and Westwater, J. W., (1960) "Population of Active Sites in Nucleate Boiling Heat Transfer," Chem. Eng. Prog. Symp., Ser. 56.
 - 8 Wang, C. H., and Dhir, V. K., (1993), "Effect of Surface Wettability on Active Nucleation Site Density During Pool Boiling of Water on a Vertical Surface," J. Heat Transfer 115, pp. 659-669

To explain the contour of the surface, and the contact angle consider simple "wetting" liquid contacting a solid material in two-dimensional shape as depicted in Figure 1.16. To solve the shape of the liquid surface, the pressure difference between the two sides of free surface has to be balanced by the surface tension. In Figure 1.16 describes the raising of the liquid as results of the surface tension. The surface tension reduces the pressure in the liquid above the liquid line (the dotted line in the Figure 1.16). The pressure just below the surface is $-g h(x) \rho$ (this pressure difference will be explained in more details in Chapter 4). The pressure, on the gas side, is the atmospheric pressure. This problem is a two dimensional problem and equation (1.38) is applicable to it. Appalling equation (1.38) and using the pressure difference yields

$$g h(x), \rho = \frac{\sigma}{R(x)} \quad (1.46)$$

The radius of any continuous function, $h = h(x)$, is

$$R(x) = \frac{\left(1 + [\dot{h}(x)]^2\right)^{3/2}}{\ddot{h}(x)} \quad (1.47)$$

Where \dot{h} is the derivative of h with respect to x .

Equation (1.47) can be derived either by forcing a circle at three points at $(x, x+dx, \text{ and } x+2dx)$ and thus finding the the diameter or by geometrical analysis of triangles

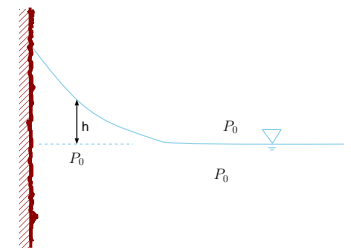


Fig. -1.16: Description of liquid surface.

build on points x and $x+dx$ (perpendicular to the tangent at these points). Substituting equation (1.47) into equation (1.46) yields

$$g h(x) \rho = \frac{\sigma}{\frac{\left(1 + \left[\dot{h}(x)\right]^2\right)^{3/2}}{\ddot{h}(x)}} \quad (1.48)$$

Equation (1.50) is non-linear differential equation for height and can be written as

$$\frac{g h \rho}{\sigma} \left(1 + \left[\frac{dh}{dx}\right]^2\right)^{3/2} - \frac{d^2h}{dx^2} = 0 \quad (1.49)$$

With the boundary conditions that specify either the derivative $\dot{h}(x = r) = 0$ (symmetry) and the derivative at $\dot{h}x = \beta$ or heights in two points or other combinations. An alternative presentation of equation (1.50) is

$$g h \rho = \frac{\sigma \ddot{h}}{\left(1 + \dot{h}^2\right)^{3/2}} \quad (1.50)$$

Integrating equation (1.50) transforms into

$$\int \frac{g \rho}{\sigma} h dh = \int \frac{\ddot{h}}{\left(1 + \dot{h}^2\right)^{3/2}} dh \quad (1.51)$$

The constant $Lp \sigma / \rho g$ is referred to as Laplace's capillarity constant. The units of this constant are meter squared. The differential dh is \dot{h} . Using dummy variable and the identity $\dot{h} = \xi$ and hence, $\ddot{h} = \dot{\xi} = d\xi$ transforms equation (1.51) into

$$\int \frac{1}{Lp} h dh = \int \frac{\xi d\xi}{\left(1 + \xi^2\right)^{3/2}} \quad (1.52)$$

After the integration equation (1.52) becomes

$$\frac{h^2}{2 Lp} + constant = -\frac{1}{\left(1 + \dot{h}^2\right)^{1/2}} \quad (1.53)$$

At infinity, the height and the derivative of the height must be zero so $constant + 0 = -1/1$ and hence, $constant = -1$.

$$1 - \frac{h^2}{2 Lp} = \frac{1}{\left(1 + \dot{h}^2\right)^{1/2}} \quad (1.54)$$

Equation (1.54) is a first order differential equation that can be solved by variables separation⁴. Equation (1.54) can be rearranged to be

$$\left(1 + \dot{h}^2\right)^{1/2} = \frac{1}{1 - \frac{h^2}{2Lp}} \quad (1.55)$$

Squaring both sides and moving the one to the right side yields

$$\dot{h}^2 = \left(\frac{1}{1 - \frac{h^2}{2Lp}}\right)^2 - 1 \quad (1.56)$$

The last stage of the separation is taking the square root of both sides to be

$$\dot{h} = \frac{dh}{dx} = \sqrt{\left(\frac{1}{1 - \frac{h^2}{2Lp}}\right)^2 - 1} \quad (1.57)$$

or

$$\frac{dh}{\sqrt{\left(\frac{1}{1 - \frac{h^2}{2Lp}}\right)^2 - 1}} = dx \quad (1.58)$$

Equation (1.59) can be integrated to yield

$$\int \frac{dh}{\sqrt{\left(\frac{1}{1 - \frac{h^2}{2Lp}}\right)^2 - 1}} = x + constant \quad (1.59)$$

The constant is determined by the boundary condition at $x = 0$. For example if $h(x=0) = h_0$ then $constant = h_0$. This equation is studied extensively in classes on surface tension. Furthermore, this equation describes the dimensionless parameter that affects this phenomenon and this parameter will be studied in Chapter ?. This book is introductory, therefore this discussion on surface tension equation will be limited.

1.6.1.1 Capillarity

⁴This equation has an analytical solution which is $x = Lp\sqrt{4 - (h/Lp)^2} - Lp \operatorname{acosh}(2Lp/h) + constant$ where Lp is the Laplace constant. Shamefully, this author doesn't know how to show it in a two lines derivations.

1.6. SURFACE TENSION

The capillary forces referred to the fact that surface tension causes liquid to rise or penetrate into area (volume), otherwise it will not be there. It can be shown that the height that the liquid raised in a tube due to the surface tension is

$$h = \frac{2 \sigma \cos \beta}{g \Delta \rho r} \quad (1.60)$$

Where $\Delta \rho$ is the difference of liquid density to the gas density and r is the radius of tube.

But this simplistic equation is unusable and useless unless the contact angle (assuming that the contact angle is constant or a repressive average can be found or provided or can be measured) is given. However, in reality there is no readily information for contact angle⁵ and therefore this equation is useful to show the trends. The maximum that the contact angle can be obtained in equation (1.60) when $\beta = 0$ and thus $\cos \beta = 1$. This angle is obtained when a perfect half a sphere shape exist of the liquid surface. In that case equation (1.60) becomes

$$h_{max} = \frac{2 \sigma}{g \Delta \rho r} \quad (1.61)$$

Figure 1.18 exhibits the height as a function of the radius of the tube. The height based on equation (1.61) is shown in Figure 1.17 as blue line. The actual height is shown in the red line. Equation (1.61) provides reasonable results only in a certain range. For a small tube radius, equation (1.49) proved better results because the curve approaches hemispherical shaper (small gravity effect). For large radii equation (1.49) approaches the strait line (the liquid line) strong gravity effect. On the other hand, for extremely small radii equation (1.61) indicates that the high height which indicates a negative pressure. The liquid at a certain pressure will be vaporized and will breakdown the model upon this equation was constructed. Furthermore, the small scale indicates that the simplistic and continuous approach is not appropriate and a different model is needed. The conclusion of this discussion are shown in Figure 1.17. The acutal dimension for many liquids (even water) is about 1-5 [mm].

The discussion above was referred to “wetting” contact angle. The depression of the liquid occurs in a “negative” contact angle similarly to “wetting.” The depression height, h is similar to equation (1.61) with a minus sign. However, the

⁵Actually, there are information about the contact angle. However, that information conflict each other and no real information is available see Table 1.6.

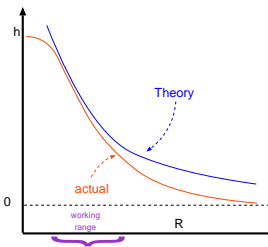


Fig. -1.17: The raising height as a function of the radii.

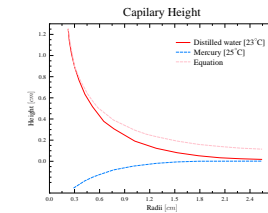


Fig. -1.18: The raising height as a function of the radius.

gravity is working against the surface tension and reducing the range and quality of the predictions of equation (1.61). The measurements of the height of distilled water and mercury are presented in Figure 1.18. The experimental results of these materials are with agreement with the discussion above.

The surface tension of a selected material is given in Table 1.7.

In conclusion, the surface tension issue is important only in case where the radius is very small and gravity is negligible. The surface tension depends on the two materials or mediums that it separates.

Example 1.9:

Calculate the diameter of a water droplet to attain pressure difference of $1000[N/m^2]$. You can assume that temperature is $20^\circ C$.

SOLUTION

The pressure inside the droplet is given by equation (1.39).

$$D = 2R = \frac{2\sigma}{\Delta P} = \frac{4 \times 0.0728}{1000} \sim 2.912 \cdot 10^{-4} [m]$$

Example 1.10:

Calculate the pressure difference between a droplet of water at $20^\circ C$ when the droplet has a diameter of 0.02 cm .

SOLUTION

using equation

$$\Delta P = \frac{2\sigma}{r} \sim \frac{2 \times 0.0728}{0.0002} \sim 728.0 [N/m^2]$$

Example 1.11:

Calculate the maximum force necessary to lift a thin wire ring of $0.04[m]$ diameter from a water surface at $20^\circ C$. Neglect the weight of the ring.

SOLUTION

$$F = 2(2\pi r \sigma) \cos \beta$$

The actual force is unknown since the contact angle is unknown. However, the maximum Force is obtained when $\beta = 0$ and thus $\cos \beta = 1$. Therefore,

$$F = 4\pi r \sigma = 4 \times \pi \times 0.04 \times 0.0728 \sim .0366 [N]$$

In this value the gravity is not accounted for.

Table -1.7: The surface tension for selected materials at temperature 20°C when not mentioned.

chemical component	Surface Tension $\frac{mN}{m}$	T	correction $\frac{mN}{m K}$
Acetic Acid	27.6	20°C	n/a
Acetone	25.20	-	-0.1120
Aniline	43.4	22°C	-0.1085
Benzene	28.88	-	-0.1291
Benzylalcohol	39.00	-	-0.0920
Benzylbenzoate	45.95	-	-0.1066
Bromobenzene	36.50	-	-0.1160
Bromobenzene	36.50	-	-0.1160
Bromoform	41.50	-	-0.1308
Butyronitrile	28.10	-	-0.1037
Carbon disulfid	32.30	-	-0.1484
Quinoline	43.12	-	-0.1063
Chloro benzene	33.60	-	-0.1191
Chloroform	27.50	-	-0.1295
Cyclohexane	24.95	-	-0.1211
Cyclohexanol	34.40	25°C	-0.0966
Cyclopentanol	32.70	-	-0.1011
Carbon Tetrachloride	26.8	-	n/a
Carbon disulfid	32.30	-	-0.1484
Chlorobutane	23.10	-	-0.1117
Ethyl Alcohol	22.3	-	n/a
Ethanol	22.10	-	-0.0832
Ethylbenzene	29.20	-	-0.1094
Ethylbromide	24.20	-	-0.1159
Ethylene glycol	47.70	-	-0.0890
Formamide	58.20	-	-0.0842
Gasoline	~ 21	-	n/a
Glycerol	64.0	-	-0.0598
Helium	0.12	-269°C	n/a
Mercury	425-465.0	-	-0.2049
Methanol	22.70	-	-0.0773
Methyl naphthalene	38.60	-	-0.1118
Methyl Alcohol	22.6	-	n/a
Neon	5.15	-247°C	n/a
Nitrobenzene	43.90	-	-0.1177
Olive Oil	43.0-48.0	-	-0.067
Perfluoroheptane	12.85	-	-0.0972
Perfluorohexane	11.91	-	-0.0935

Table -1.7: The surface tension for selected materials (continue)

chemical component	Surface Tension $\frac{mN}{m}$	T	correction $\frac{mN}{m \cdot K}$
Perfluorooctane	14.00	-	-0.0902
Phenylisothiocyanate	41.50	-	-0.1172
Propanol	23.70	25°C	-0.0777
Pyridine	38.00	-	-0.1372
Pyrrol	36.60	-	-0.1100
SAE 30 Oil	n/a	-	n/a
Seawater	54-69	-	n/a
Toluene	28.4	-	-0.1189
Turpentine	27	-	n/a
Water	72.80	-	-0.1514
o-Xylene	30.10	-	-0.1101
m-Xylene	28.90	-	-0.1104

CHAPTER 2

Review of Thermodynamics

In this chapter, a review of several definitions of common thermodynamics terms is presented. This introduction is provided to bring the student back to current place with the material.

2.1 *Basic Definitions*

The following basic definitions are common to thermodynamics and will be used in this book.

Work

In mechanics, the work was defined as

$$\text{mechanical work} = \int \mathbf{F} \cdot d\ell = \int PdV \quad (2.1)$$

This definition can be expanded to include two issues. The first issue that must be addressed, that work done on the surroundings by the system boundaries similarly is positive. Two, there is a transfer of energy so that its effect can cause work. It must be noted that electrical current is a work while heat transfer isn't.

System

This term will be used in this book and it is defined as a continuous (at least partially) fixed quantity of matter. The dimensions of this material can be changed. In this definition, it is assumed that the system speed is significantly lower than that of the speed of light. So, the mass can be assumed constant even though the true conservation law applied to the combination of mass energy (see Einstein's law). In fact for almost all engineering purpose this law is reduced to two separate laws of mass conservation and energy conservation.

Our system can receive energy, work, etc as long the mass remain constant the definition is not broken.

Thermodynamics First Law

This law refers to conservation of energy in a non accelerating system. Since all the systems can be calculated in a non accelerating systems, the conservation is applied to all systems. The statement describing the law is the following.

$$Q_{12} - W_{12} = E_2 - E_1 \quad (2.2)$$

The system energy is a state property. From the first law it directly implies that for process without heat transfer (adiabatic process) the following is true

$$W_{12} = E_1 - E_2 \quad (2.3)$$

Interesting results of equation (2.3) is that the way the work is done and/or intermediate states are irrelevant to final results. There are several definitions/separations of the kind of works and they include kinetic energy, potential energy (gravity), chemical potential, and electrical energy, etc. The internal energy is the energy that depends on the other properties of the system. For example for pure/homogeneous and simple gases it depends on two properties like temperature and pressure. The internal energy is denoted in this book as E_U and it will be treated as a state property.

The potential energy of the system is depended on the body force. A common body force is the gravity. For such body force, the potential energy is mgz where g is the gravity force (acceleration), m is the mass and the z is the vertical height from a datum. The kinetic energy is

$$K.E. = \frac{mU^2}{2} \quad (2.4)$$

Thus the energy equation can be written as

$$\frac{mU_1^2}{2} + mgz_1 + E_{U1} + Q = \frac{mU_2^2}{2} + mgz_2 + E_{U2} + W \quad (2.5)$$

For the unit mass of the system equation (2.5) is transformed into

$$\frac{U_1^2}{2} + gz_1 + E_{u1} + q = \frac{U_2^2}{2} + gz_2 + E_{u2} + w \quad (2.6)$$

where q is the energy per unit mass and w is the work per unit mass. The “new” internal energy, E_u , is the internal energy per unit mass.

Since the above equations are true between arbitrary points, choosing any point in time will make it correct. Thus differentiating the energy equation with respect to time yields the rate of change energy equation. The rate of change of the energy transfer is

$$\frac{DQ}{Dt} = \dot{Q} \quad (2.7)$$

In the same manner, the work change rate transferred through the boundaries of the system is

$$\frac{DW}{Dt} = \dot{W} \quad (2.8)$$

Since the system is with a fixed mass, the rate energy equation is

$$\dot{Q} - \dot{W} = \frac{D E_U}{Dt} + mU \frac{DU}{Dt} + m \frac{D B_f z}{Dt} \quad (2.9)$$

For the case where the body force, B_f , is constant with time like in the case of gravity equation (2.9) reduced to

$$\dot{Q} - \dot{W} = \frac{D E_U}{Dt} + mU \frac{DU}{Dt} + m g \frac{D z}{Dt} \quad (2.10)$$

The time derivative operator, D/Dt is used instead of the common notation because it referred to system property derivative.

Thermodynamics Second Law

There are several definitions of the second law. No matter which definition is used to describe the second law it will end in a mathematical form. The most common mathematical form is Clausius inequality which state that

$$\oint \frac{\delta Q}{T} \geq 0 \quad (2.11)$$

The integration symbol with the circle represent integral of cycle (therefor circle) in with system return to the same condition. If there is no lost, it is referred as a reversible process and the inequality change to equality.

$$\oint \frac{\delta Q}{T} = 0 \quad (2.12)$$

The last integral can go though several states. These states are independent of the path the system goes through. Hence, the integral is independent of the path. This observation leads to the definition of entropy and designated as S and the derivative of entropy is

$$ds \equiv \left(\frac{\delta Q}{T} \right)_{\text{rev}} \quad (2.13)$$

Performing integration between two states results in

$$S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T} \right)_{\text{rev}} = \int_1^2 dS \quad (2.14)$$

One of the conclusions that can be drawn from this analysis is for reversible and adiabatic process $dS = 0$. Thus, the process in which it is reversible and adiabatic, the entropy remains constant and referred to as isentropic process. It can be noted that there is a possibility that a process can be irreversible and the right amount of heat transfer to have zero change entropy change. Thus, the reverse conclusion that zero change of entropy leads to reversible process, isn't correct.

For reversible process equation (2.12) can be written as

$$\delta Q = T dS \quad (2.15)$$

and the work that the system is doing on the surroundings is

$$\delta W = P dV \quad (2.16)$$

Substituting equations (2.15) (2.16) into (2.10) results in

$$T dS = dE_U + P dV \quad (2.17)$$

Even though the derivation of the above equations were done assuming that there is no change of kinetic or potential energy, it still remain valid for all situations. Furthermore, it can be shown that it is valid for reversible and irreversible processes.

Enthalpy

It is a common practice to define a new property, which is the combination of already defined properties, the enthalpy of the system.

$$H = E_U + PV \quad (2.18)$$

The specific enthalpy is enthalpy per unit mass and denoted as, h .

Or in a differential form as

$$dH = dE_U + dPV + P dV \quad (2.19)$$

Combining equations (2.18) the (2.17) yields

$$TdS = dH - VdP \quad (2.20)$$

For isentropic process, equation (2.17) is reduced to $dH = VdP$. The equation (2.17) in mass unit is

$$Tds = du + Pdv = dh - \frac{dP}{\rho} \quad (2.21)$$

when the density enters through the relationship of $\rho = 1/v$.

Specific Heats

The change of internal energy and enthalpy requires new definitions. The first change of the internal energy and it is defined as the following

$$C_v \equiv \left(\frac{\partial E_u}{\partial T} \right) \quad (2.22)$$

And since the change of the enthalpy involve some kind of work is defined as

$$C_p \equiv \left(\frac{\partial h}{\partial T} \right) \quad (2.23)$$

The ratio between the specific pressure heat and the specific volume heat is called the ratio of the specific heat and it is denoted as, k .

$$k \equiv \frac{C_p}{C_v} \quad (2.24)$$

For solid, the ratio of the specific heats is almost 1 and therefore the difference between them is almost zero. Commonly the difference for solid is ignored and both are assumed to be the same and therefore referred as C . This approximation less strong for liquid but not by that much and in most cases it applied to the calculations. The ratio the specific heat of gases is larger than one.

Equation of state

Equation of state is a relation between state variables. Normally the relationship of temperature, pressure, and specific volume define the equation of state for gases. The simplest equation of state referred to as ideal gas. and it is defined as

$$P = \rho RT \quad (2.25)$$

Application of Avogadro's law, that "all gases at the same pressures and temperatures have the same number of molecules per unit of volume," allows the calculation of a "universal gas constant." This constant to match the standard units results in

$$\bar{R} = 8.3145 \frac{kJ}{kmol K} \quad (2.26)$$

Thus, the specific gas can be calculate as

$$R = \frac{\bar{R}}{M} \quad (2.27)$$

The specific constants for select gas at 300K is provided in table 2.1.

Table -2.1: Properties of Various Ideal Gases [300K]

Gas	Chemical Formula	Molecular Weight	$R \left[\frac{kJ}{KgK} \right]$	$C_v \left[\frac{kJ}{KgK} \right]$	$C_P \left[\frac{kJ}{KgK} \right]$	k
Air	-	28.970	0.28700	1.0035	0.7165	1.400
Argon	Ar	39.948	0.20813	0.5203	0.3122	1.400
Butane	C_4H_{10}	58.124	0.14304	1.7164	1.5734	1.091
Carbon Dioxide	CO_2	44.01	0.18892	0.8418	0.6529	1.289
Carbon Monoxide	CO	28.01	0.29683	1.0413	0.7445	1.400
Ethane	C_2H_6	30.07	0.27650	1.7662	1.4897	1.186
Ethylene	C_2H_4	28.054	0.29637	1.5482	1.2518	1.237
Helium	He	4.003	2.07703	5.1926	3.1156	1.667
Hydrogen	H_2	2.016	4.12418	14.2091	10.0849	1.409
Methane	CH_4	16.04	0.51835	2.2537	1.7354	1.299
Neon	Ne	20.183	0.41195	1.0299	0.6179	1.667
Nitrogen	N_2	28.013	0.29680	1.0416	0.7448	1.400
Octane	C_8H_{18}	114.230	0.07279	1.7113	1.6385	1.044
Oxygen	O_2	31.999	0.25983	0.9216	0.6618	1.393
Propane	C_3H_8	44.097	0.18855	1.6794	1.4909	1.327
Steam	H_2O	18.015	0.48152	1.8723	1.4108	1.327

From equation (2.25) of state for perfect gas it follows

$$d(Pv) = RdT \quad (2.28)$$

For perfect gas

$$dh = dE_u + d(Pv) = dE_u + d(RT) = f(T) \text{ (only)} \quad (2.29)$$

From the definition of enthalpy it follows that

$$d(Pv) = dh - dE_u \quad (2.30)$$

Utilizing equation (2.28) and substituting into equation (2.30) and dividing by dT yields

$$C_p - C_v = R \quad (2.31)$$

This relationship is valid only for ideal/perfect gases.

The ratio of the specific heats can be expressed in several forms as

$$C_v = \frac{R}{k-1} \quad (2.32)$$

$$C_p = \frac{kR}{k-1} \quad (2.33)$$

The specific heat ratio, k value ranges from unity to about 1.667. These values depend on the molecular degrees of freedom (more explanation can be obtained in Van Wylen "F. of Classical thermodynamics." The values of several gases can be approximated as ideal gas and are provided in Table (2.1).

The entropy for ideal gas can be simplified as the following

$$s_2 - s_1 = \int_1^2 \left(\frac{dh}{T} - \frac{dP}{\rho T} \right) \quad (2.34)$$

Using the identities developed so far one can find that

$$s_2 - s_1 = \int_1^2 C_p \frac{dT}{T} - \int_1^2 \frac{R dP}{P} = C_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \quad (2.35)$$

Or using specific heat ratio equation (2.35) transformed into

$$\frac{s_2 - s_1}{R} = \frac{k}{k-1} \ln \frac{T_2}{T_1} - \ln \frac{P_2}{P_1} \quad (2.36)$$

For isentropic process, $\Delta s = 0$, the following is obtained

$$\ln \frac{T_2}{T_1} = \ln \left(\frac{P_2}{P_1} \right)^{\frac{k-1}{k}} \quad (2.37)$$

There are several famous identities that results from equation (2.37) as

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1} \right)^{\frac{k-1}{k}} = \left(\frac{P_2}{P_1} \right)^{k-1} \quad (2.38)$$

The ideal gas model is a simplified version of the real behavior of real gas. The real gas has a correction factor to account for the deviations from the ideal gas model. This correction factor referred as the compressibility factor and defined as

$$Z = \frac{PV}{RT} \quad (2.39)$$

CHAPTER 3

Review of Mechanics

This author would like to express his gratitude to Dan Olsen (former Minneapolis city Engineer) and his friend Richard Hackbarth.

This chapter provides a review of important definitions and concepts from Mechanics (statics and dynamics). These concepts and definitions will be used in this book and a review is needed.

3.1 Center of Mass

The center of mass is divided into two sections, first, center of the mass and two, center of area (two-dimensional body with equal distribution mass).

3.1.1 Center of the Mass

In many engineering problems, the center of mass is required to make the calculations. This concept is derived from the fact that a body has a center of mass/gravity which interacts with other bodies and that this force acts on the center (equivalent force). It turns out that this concept is very useful in calculating rotations, moment of inertia, etc. The center of mass doesn't depend on the coordinate system and on the way it is calculated. The physical meaning of the center of mass is that if a straight line force acts on the body in away through the center of gravity, the body will not rotate. In other words, if a body will be held by one point it will be enough to hold the body in the direction of the center of mass. Note, if the body isn't be held through the center of mass, then a moment in additional to force is required (to prevent the body for rotating). It is convenient to use the Cartesian system to explain this concept. Suppose that the body has a distribution of the mass (density, ρ) as a function of the location. The density "normally" defined as mass per volume. Here, the the line density is referred to density mass per unit length in the x direction.

In x coordinate, the center will be defined as

$$\bar{x} = \frac{1}{m} \int_V x \overbrace{\rho(x) dV}^{dm} \quad (3.1)$$

Here, the dV element has finite dimensions in y - z plane and infinitesimal dimension in x direction see Figure 3.1. Also, the mass, m is the total mass of the object. It can be noticed that center of mass in the x -direction isn't affected by the distribution in the y nor by z directions. In same fashion the center of mass can be defined in the other directions as following

$$\bar{x}_i = \frac{1}{m} \int_V x_i \rho(x_i) dV \quad (3.2)$$

where x_i is the direction of either, x , y or z . The density, $\rho(x_i)$ is the line density as function of x_i . Thus, even for solid and uniform density the line density is a function of the geometry.

3.1.2 Center of Area

In the previous case, the body was a three dimensional shape. There are cases where the body can be approximated as a two-dimensional shape because the body is with a thin with uniform density. Consider a uniform thin body with constant thickness shown in Figure 3.2 which has density, ρ . Thus, equation (3.1) can be transferred into

$$\bar{x} = \frac{1}{\underbrace{tA}_V} \rho \int_V x \overbrace{\rho t dA}^{dm} \quad (3.3)$$

The density, ρ and the thickness, t , are constant and can be canceled. Thus equation (3.3) can be transferred into

$$\bar{x}_i = \frac{1}{A} \int_A x_i dA \quad (3.4)$$

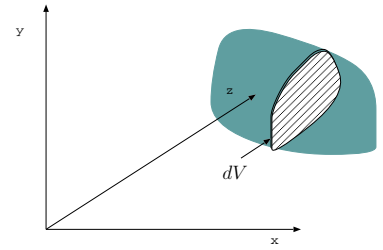


Fig. -3.1: Description of how the center of mass is calculated.

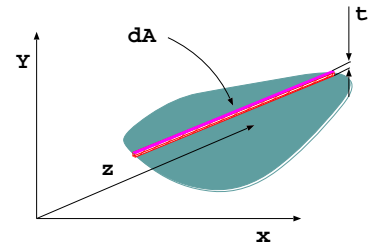


Fig. -3.2: Thin body center of mass/area schematic.

when the integral now over only the area as oppose over the volume.

Finding the centeroid location should be done in the most convenient coordinate system since the location is coordinate independent.

3.2 Moment of Inertia

As it was divided for the body center of mass, the moment of inertia is divided into moment of inertia of mass and area.

3.2.1 Moment of Inertia for Mass

The moment of inertia turns out to be an essential part for the calculations of rotating bodies. Furthermore, it turns out that the moment of inertia has much wider applicability. Moment of inertia of mass is defined as

$$I_{rrm} = \int_m \rho r^2 dm \quad (3.5)$$

If the density is constant then equation (3.5) can be transformed into

$$I_{rrm} = \rho \int_V r^2 dV \quad (3.6)$$

The moment of inertia is independent of the coordinate system used for the calculation, but dependent on the location of axis of rotation relative to the body. Some people define the radius of gyration as an equivalent concepts for the center of mass concept and which means if all the mass were to locate in the one point/distance and to obtain the same of moment of inertia.

$$r_k = \sqrt{\frac{I_m}{m}} \quad (3.7)$$

The body has a different moment of inertia for every coordinate/axis and they are

$$\begin{aligned} I_{xx} &= \int_V r_x^2 dm = \int_V (y^2 + z^2) dm \\ I_{yy} &= \int_V r_y^2 dm = \int_V (x^2 + z^2) dm \\ I_{zz} &= \int_V r_z^2 dm = \int_V (x^2 + y^2) dm \end{aligned} \quad (3.8)$$

3.2.2 Moment of Inertia for Area

3.2.2.1 General Discussion

For body with thickness, t and uniform density the following can be written

$$I_{xxm} = \int_m r^2 dm = \rho t \int_A r^2 dA \quad (3.9)$$

moment of
inertia for
area

The moment of inertia about axis is x can be defined as

$$I_{xx} = \int_A r^2 dA = \frac{I_{xxm}}{\rho t} \quad (3.10)$$

where r is distance of dA from the axis x and t is the thickness. Any point distance can be calculated from axis x as

$$x = \sqrt{y^2 + z^2} \quad (3.11)$$

Thus, equation (3.10) can be written as

$$I_{xx} = \int_A (y^2 + z^2) dA \quad (3.12)$$

In the same fashion for other two coordinates as

$$I_{yy} = \int_A (x^2 + z^2) dA \quad (3.13)$$

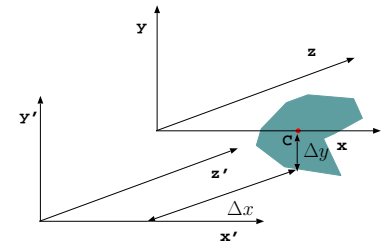


Fig. -3.3: The schematic that explains the summation of moment of inertia.

$$I_{zz} = \int_A (x^2 + y^2) dA \quad (3.14)$$

3.2.2.2 The Parallel Axis Theorem

The moment of inertial can be calculated for any axis. The knowledge about one axis can help calculating the moment of inertia for a parallel axis. Let I_{xx} the moment of inertia about axis xx which is at the center of mass/area.

The moment of inertia for axis x' is

$$I_{x'x'} = \int_A r'^2 dA = \int_A (y'^2 + z'^2) dA = \int_A [(y + \Delta y)^2 + (z + \Delta z)^2] dA \quad (3.15)$$

equation (3.15) can be expanded as

$$I_{x'x'} = \overbrace{\int_A (y^2 + z^2) dA}^{I_{xx}} + \overbrace{2 \int_A (y \Delta y + z \Delta z) dA}^{=0} + \int_A ((\Delta y)^2 + (\Delta z)^2) dA \quad (3.16)$$

The first term in equation (3.16) on the right hand side is the moment of inertia about axis x and the second term is zero. The second term is zero because it is the integral of center about center thus is zero. The third term is a new term and can be written as

$$\int_A \overbrace{((\Delta y)^2 + (\Delta z)^2)}^{\text{constant}} dA = \overbrace{((\Delta y)^2 + (\Delta z)^2)}^{r^2} \overbrace{\int_A dA}^A = r^2 A \quad (3.17)$$

Hence, the relationship between the moment of inertia at xx and parallel axis $x'x'$ is

$$I_{x'x'} = I_{xx} + r^2 A \quad (3.18)$$

The moment of inertia of several areas is the sum of moment inertia of each area see Figure 3.4 and therefore,

$$I_{xx} = \sum_{i=1}^n I_{xxi} \quad (3.19)$$

If the same areas are similar thus

$$I_{xx} = \sum_{i=1}^n I_{xxi} = n I_{xxi} \quad (3.20)$$

Equation (3.20) is very useful in the calculation of the moment of inertia utilizing the moment of inertia of known bodies. For example, the moment of inertia of half a circle is half of whole circle for axis at the center of circle. The moment of inertia can then move the center of area. of the

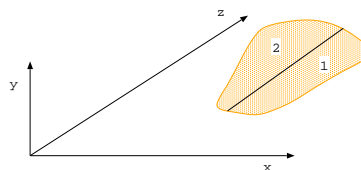


Fig. -3.4: The schematic to explain the summation of moment of inertia.

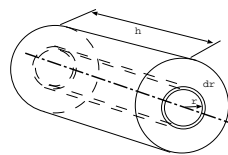


Fig. -3.5: Cylinder with the element for calculation moment of inertia.

3.2.3 Examples of Moment of Inertia

Example 3.1:

Calculate the moment of inertia for the mass of the cylinder about center axis which height of h and radius, r_0 , as shown in Figure 3.5. The material is with a uniform density and homogeneous.

SOLUTION

The element can be calculated using cylindrical coordinate. Here the convenient element is a shell of thickness dr which shown in Figure 3.5 as

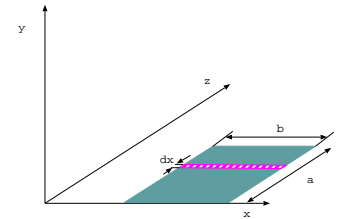
$$I_{rr} = \rho \int_V r^2 dm = \rho \int_0^{r_0} r^2 \overbrace{h 2\pi r dr}^{dV} = \rho h 2\pi \frac{r_0^4}{4} = \frac{1}{2} \rho h \pi r_0^4 = \frac{1}{2} m r_0^2$$

The radius of gyration is

$$r_k = \sqrt{\frac{\frac{1}{2} m r_0^2}{m}} = \frac{r_0}{\sqrt{2}}$$

Example 3.2:

Calculate the moment of inertia of the rectangular shape shown in Figure 3.6.



SOLUTION

The moment of inertia is calculated utilizing equation (3.12) as following

$$I_{xx} = \int_A \left(\overbrace{y^2}^0 + z^2 \right) dA = \int_0^a z^2 \overbrace{bdz}^{dA} = \frac{a^3 b}{3}$$

Fig. -3.6: Description of rectangular in x-y plane for calculation of moment of inertia.

Example 3.3:

As measure to study the assumption of zero thickness consider a simple shape to see the effects of this assumption. Calculate the moment of inertia about the center of mass of a square shape with a thickness, t compare the results to a square shape with zero thickness.

SOLUTION

The moment of inertia of transverse slice about y' (see Figure mech:fig:squareEll)

is

$$dI_{xxm} = \rho \overbrace{dy}^t \frac{\overbrace{ba^3}^{I_{xx}}}{12} \tag{3.21}$$

The transformation into from local axis x to center axis, x' can be done as following

$$dI_{x'x'm} = \rho dy \left(\frac{\overbrace{ba^3}^{I_{xx}}}{12} + \underbrace{\frac{z^2}{r^2}}_{r^2} \underbrace{ba}_A \right) \tag{3.22}$$

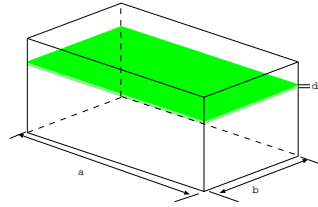


Fig. -3.7: A square element for the calculations of inertia of two-dimensional to three-dimensional deviations.

The total moment of inertia can be obtained by integration of equation (3.22) to write as

$$I_{xxm} = \rho \int_{-t/2}^{t/2} \left(\frac{ba^3}{12} + z^2 ba \right) dz = \rho t \frac{abt^2 + a^3b}{12} \tag{3.23}$$

Comparison with the thin body results in

$$\frac{I_{xx} \rho t}{I_{xxm}} = \frac{ba^3}{t^2 ba + ba^3} = \frac{1}{1 + \frac{t^2}{a^2}} \tag{3.24}$$

It can be noticed right away that equation (3.24) indicates that ratio approaches one when thickness ratio approaches zero, $I_{xxm}(t \rightarrow 0) \rightarrow 1$. Additionally it can be noticed that the ratio a^2/t^2 is the only contributor to the error¹. The results are present in Figure 3.8. I can be noticed that the error is significant very fast even for small values of t/a while the with of the box, b has no effect on the error.

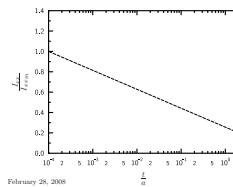


Fig. -3.8: The ratio of the moment of inertia of two-dimensional to three-dimensional.

Example 3.4:

Calculate the center of area and moment of inertia of the parabola, $y = \alpha x^2$, shown in Figure 3.9.

¹This ratio is a dimensionless number that commonly has no special name. This author suggests to call this ratio as the B number.

SOLUTION

For $y = b$ the value of $x = \sqrt{b/\alpha}$. First the area inside the parabola calculated as

$$A = 2 \int_0^{\sqrt{b/\alpha}} \overbrace{(b - \alpha \xi^2)}^{dA/2} d\xi = \frac{2(3\alpha - 1)}{3} \left(\frac{b}{\alpha}\right)^{\frac{3}{2}}$$

The center of area can be calculated utilizing equation (3.4). The center of every element is at, $\left(\alpha \xi^2 + \frac{b - \alpha \xi^2}{2}\right)$ the element area is used before and therefore

$$x_c = \frac{1}{A} \int_0^{\sqrt{b/\alpha}} \overbrace{\left(\alpha \xi^2 + \frac{b - \alpha \xi^2}{2}\right)}^{x_c} \overbrace{(b - \alpha \xi^2) d\xi}^{dA} = \frac{3\alpha b}{15\alpha - 5} \quad (3.25)$$

The moment of inertia of the area about the center can be found using in equation (3.25) can be done in two steps first calculate the moment of inertia in this coordinate system and then move the coordinate system to center. Utilizing equation (3.12) and doing the integration from 0 to maximum y provides

$$I_{x'x'} = 4 \int_0^b \xi^2 \overbrace{\sqrt{\frac{\xi}{\alpha}}}^{dA} d\xi = \frac{2b^{7/2}}{7\sqrt{\alpha}}$$

Utilizing equation (3.18)

$$I_{xx} = I_{x'x'} - A \Delta x^2 = \frac{4b^{7/2}}{7\sqrt{\alpha}} - \frac{3\alpha - 1}{3} \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \left(\frac{3\alpha b}{15\alpha - 5}\right)^2$$

or after working the details results in

$$I_{xx} = \frac{\sqrt{b} (20b^3 - 14b^2)}{35\sqrt{\alpha}}$$

3.2.4 Product of Inertia

In addition to the moment of inertia, the product of inertia is commonly used. Here only the product of the area is defined and discussed. The product of inertia defined as

$$I_{x_i x_j} = \int_A x_i x_j dA \quad (3.26)$$

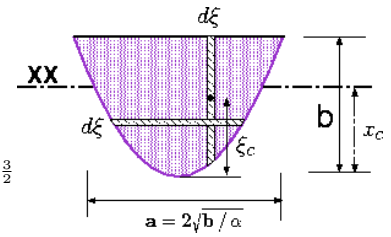


Fig. -3.9: Description of parabola for calculation of moment of inertia and center of area.

For example, the product of inertia for x and y axes is

$$I_{xy} = \int_A xy dA \tag{3.27}$$

Product of inertia can be positive or negative value as oppose the moment of inertia. The calculation of the product of inertia isn't different much for the calculation of the moment of inertia. The units of the product of inertia are the same as for moment of inertia.

Transfer of Axis Theorem

Same as for moment of inertia there is also similar theorem.

$$I_{x'y'} = \int_A x' y' dA = \int_A (x + \Delta x)(y + \Delta y) dA \tag{3.28}$$

expanding equation (3.28) results in

$$I_{x'y'} = \underbrace{\int_A xy dA}_{I_{xy}} + \underbrace{\Delta y \int_A x dA}_0 + \underbrace{\Delta x \int_A y dA}_0 + \underbrace{\int_A \Delta x \Delta y dA}_{\Delta x \Delta y A} \tag{3.29}$$

The final form is

$$I_{x'y'} = I_{xy} + \Delta x \Delta y A \tag{3.30}$$

There are several relationships should be mentioned

$$I_{xy} = I_{yx} \tag{3.31}$$

Symmetrical area has zero product of inertia.

Example 3.5:
Calculate the product of inertia of straight edge triangle.

SOLUTION

The equation of the line is

$$y = \frac{a}{b}x + a$$

The product of inertia at the center is zero. The total product of inertia is

$$I_{x'y'} = 0 + \underbrace{\frac{\Delta x}{3}}_a \underbrace{\frac{\Delta y}{3}}_b \underbrace{\left(\frac{ab}{2}\right)}_A = \frac{a^2 b^2}{18}$$

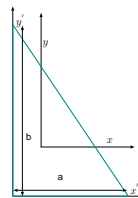


Fig. -3.10: Product of inertia for triangle.

3.2.5 Principal Axes of Inertia

The inertia matrix or inertia tensor is

$$\begin{vmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{vmatrix} \quad (3.32)$$

In linear algebra it was shown that for some angle equation (3.32) can be transform into

$$\begin{vmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{vmatrix} \quad (3.33)$$

System which creates equation (3.33) referred as principle system.

3.3 Newton's Laws of Motion

These laws can be summarized in two statements one, for every action by body **A** on Body **B** there is opposite reaction by body **B** on body **A**. Two, which can expressed in mathematical form as

$$\sum \mathbf{F} = \frac{D(mU)}{Dt} \quad (3.34)$$

It can be noted that D replaces the traditional d since the additional meaning which be added. Yet, it can be treated as the regular derivative. This law apply to any body and any body can "broken" into many small bodies which connected to each other. These small "bodies" when became small enough equation (3.34) can be transformed to a continuous form as

$$\sum \mathbf{F} = \int_V \frac{D(\rho U)}{Dt} dV \quad (3.35)$$

The external forces are equal to internal forces the forces between the "small" bodies are cancel each other. Yet this examination provides a tool to study what happened in the fluid during operation of the forces.

Since the derivative with respect to time is independent of the volume, the derivative can be taken out of the integral and the alternative form can be written as

$$\sum \mathbf{F} = \frac{D}{Dt} \int_V \rho U dV \quad (3.36)$$

The velocity, U is a derivative of the location with respect to time, thus,

$$\sum \mathbf{F} = \frac{D^2}{Dt^2} \int_V \rho r dV \quad (3.37)$$

where r is the location of the particles from the origin.

The external forces are typically divided into two categories: body forces and surface forces. The body forces are forces that act from a distance like magnetic field or gravity. The surface forces are forces that act on the surface of the body (pressure, stresses). The same as in the dynamic class, the system acceleration called the internal forces. The acceleration is divided into three categories: Centrifugal, $\omega \times (\mathbf{r} \times \omega)$, Angular, $\mathbf{r} \times \dot{\omega}$, Coriolis, $2(\mathbf{U}_r \times \omega)$. The radial velocity is denoted as U_r .

3.4 Angular Momentum and Torque

The angular momentum of body, dm , is defined as

$$L = \mathbf{r} \times \mathbf{U} dm \quad (3.38)$$

The angular momentum of the entire system is calculated by integration (summation) of all the particles in the system as

$$L_s = \int_m \mathbf{r} \times \mathbf{U} dm \quad (3.39)$$

The change with time of angular momentum is called torque, in analogous to the momentum change of time which is the force.

$$T_\tau = \frac{DL}{Dt} = \frac{D}{Dt} (\mathbf{r} \times \mathbf{U} dm) \quad (3.40)$$

where T_τ is the torque. The torque of entire system is

$$T_{\tau s} = \int_m \frac{DL}{Dt} = \frac{D}{Dt} \int_m (\mathbf{r} \times \mathbf{U} dm) \quad (3.41)$$

It can be noticed (well, it can be proved utilizing vector mechanics) that

$$T_\tau = \frac{D}{Dt} (\mathbf{r} \times \mathbf{U}) = \frac{D}{Dt} (\mathbf{r} \times \frac{D\mathbf{r}}{Dt}) = \frac{D^2 \mathbf{r}}{Dt^2} \quad (3.42)$$

To understand these equations a bit better, consider a particle moving in x–y plane. A force is acting on the particle in the same plane (x–y) plane. The velocity can be written as $U = u\hat{i} + v\hat{j}$ and the location from the origin can be written as $\mathbf{r} = x\hat{i} + y\hat{j}$. The force can be written, in the same fashion, as $\mathbf{F} = F_x\hat{i} + F_y\hat{j}$. Utilizing equation (3.38) provides

$$\mathbf{L} = \mathbf{r} \times \mathbf{U} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ u & v & 0 \end{pmatrix} = (xv - yu)\hat{k} \quad (3.43)$$

Utilizing equation (3.40) to calculate the torque as

$$T_\tau = \mathbf{r} \times \mathbf{F} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{pmatrix} = (x F_y - y F_x) \hat{k} \quad (3.44)$$

Since the torque is a derivative with respect to the time of the angular momentum it is also can be written as

$$x F_y - y F_x = \frac{D}{Dt} [(xv - yu) dm] \quad (3.45)$$

The torque is a vector and the various components can be represented as

$$T_{\tau_x} = \hat{i} \cdot \frac{D}{Dt} \int_m \mathbf{r} \times \mathbf{U} dm \quad (3.46)$$

In the same way the component in y and z can be obtained.

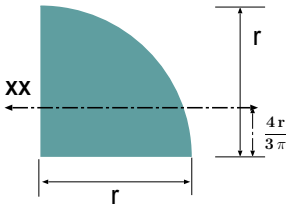
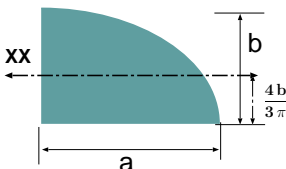
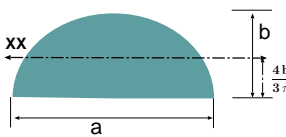
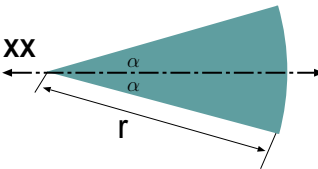
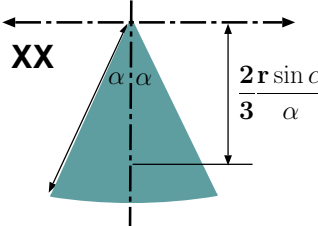
3.4.1 Tables of geometries

The following tables present several moment of inertias of commonly used geometries.

Table -3.1: Moments of Inertia for various plane surfaces about their center of gravity (full shapes)

Shape Name	Picture description	x_c, y_c	A	I_x
Rectangle		$\frac{b}{2}; \frac{a}{2}$	ab	$\frac{ab^3}{12}$
Triangle		$\frac{a}{3}$	$\frac{ab}{3}$	$\frac{ab^3}{36}$
Circle		$\frac{b}{2}$	$\frac{\pi b^2}{4}$	$\frac{\pi b^4}{64}$
Ellipse		$\frac{b}{2} \frac{b}{2}$	$\frac{\pi ab}{4}$	$\frac{Ab^2}{64}$
$y = \alpha x^2$ Parabola		$\frac{3\alpha b}{15\alpha - 5}$	$\frac{6\alpha - 2}{3} \times \left(\frac{b}{\alpha}\right)^{\frac{3}{2}}$	$\frac{\sqrt{b}(20b^3 - 14b^2)}{35\sqrt{\alpha}}$

Table -3.2: Moment of inertia for various plane surfaces about their center of gravity

Shape Name	Picture description	x_c, y_c	A	I_{xx}
Quadrant of Circle		$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$	$r^4 \left(\frac{\pi}{16} - \frac{4}{9\pi} \right)$
Ellipsoidal Quadrant		$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$	$ab^3 \left(\frac{\pi}{16} - \frac{4}{9\pi} \right)$
Half of Elliptic		$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$	$ab^3 \left(\frac{\pi}{16} - \frac{4}{9\pi} \right)$
Circular Sector		0	$2\alpha r^2$	$\frac{r^4}{4} \left(\alpha - \frac{1}{2} \sin 2\alpha \right)$
Circular Sector		$\frac{2}{3} \frac{r \sin \alpha}{\alpha}$	$2\alpha r^2$	$I_{x'x'} = \frac{r^4}{4} \left(\alpha + \frac{1}{2} \sin 2\alpha \right)$

CHAPTER 4

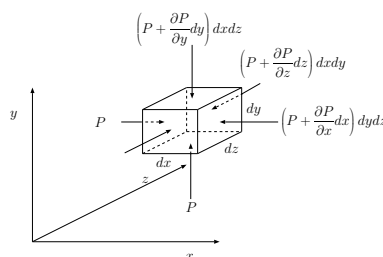
Fluids Statics

4.1 Introduction

The simplest situation that can occur in the study of fluid is when the fluid is at rest or queasy rest. This topic was introduced to most students in previous study of rigid body. However, here this topic will be more vigorously examined. Furthermore, the student will be exposed to stability analysis probably for the first time. Later, the methods discussed here will be expanded to more complicated dynamics situations.

4.2 The Hydrostatic Equation

A fluid element with dimensions of dx , dy , and dz is motionless in the accelerated system, with acceleration, a as shown in Figure 4.1. The system is in a body force field, $g_G(x, y, z)$. The combination of an acceleration and the body force results in effective body force which is



$$\mathbf{g}_G - a = g_{\text{eff}} \quad (4.1) \quad \text{Fig. -4.1: Description of a fluid element in accelerated system under body forces.}$$

Equation (4.1) can be reduced and simplified for the case of no acceleration, $a = 0$.

In these derivations, several assumptions must be made. The first assumption is that the change in the pressure is a continuous function. There is no requirement that the pressure has to be a monotonous function e.g. that pressure can

increase and later decrease. The changes of the second derivative pressure are not significant compared to the first derivative ($\partial P/\partial n \times dl \gg \partial^2 P/\partial n^2$). where n is the steepest direction of the pressure derivative and dl is the infinitesimal length. This mathematical statement simply requires that the pressure can deviate in such a way that the average on infinitesimal area can be found and expressed as only one direction. The net pressure force on the faces in the x direction results in

$$d\mathbf{F} = - \left(\frac{\partial P}{\partial x} \right) dy dx \hat{i} \quad (4.2)$$

In the same fashion, the calculations of the three directions result in the total net pressure force as

$$\sum_{\text{surface}} F = - \left(\frac{\partial P}{\partial x} \hat{i} + \frac{\partial P}{\partial y} \hat{j} + \frac{\partial P}{\partial z} \hat{k} \right) \quad (4.3)$$

The term in the parentheses in equation (4.3) referred to in the literature as the pressure gradient. This mathematical operation has a geometrical interpretation. If the pressure, P , was a two-dimensional height (that is only a function of x and y) then the gradient is the steepest ascent of the height (to the valley). The second point is that the gradient is a vector (that is, it has a direction). Even though, the pressure is treated, now, as a scalar function (there no reference to the shear stress in part of the pressure) the gradient is a vector. For example, the dot product of the following is

$$\hat{i} \bullet \mathbf{grad}P = \frac{\partial P}{\partial x} \quad (4.4)$$

In general, if the coordinates were to “rotate/transform” to a new system which has a different orientation the dot product results in

$$\hat{i}_n \bullet \mathbf{grad}P = \frac{\partial P}{\partial n} \quad (4.5)$$

where \hat{i}_n is the unit vector in the n direction and $\partial/\partial n$ is a derivative in that direction.

As before, the effective gravity force is utilized in case where the gravity is the only body force and in an accelerated system. The body (element) is in rest and therefore the net force is zero

$$\sum_{\text{total}} \mathbf{F} = \sum_{\text{surface}} \mathbf{F} + \sum_{\text{body}} \mathbf{F} \quad (4.6)$$

Hence, the utilizing the above derivations one can obtain

$$-\mathbf{grad}P dx dy dz + \rho g_{\text{eff}} dx dy dz = 0 \quad (4.7)$$

or

$$\text{grad}P = \rho g_{\text{eff}} \quad (4.8)$$

Some refer to equation (4.8) as the Fluid Static Equation. This equation can be integrated and therefore solved. However, there are several physical implications to this equation which should be discussed and are presented here. First, a discussion on a simple condition and will continue in more challenging situations.

4.3 Pressure and Density in a Gravitational Field

In this section, a discussion on the pressure and the density in various conditions is presented.

4.3.1 Constant Density in Gravitational Field

The simplest case is when the density, ρ , pressure, P , and temperature, T (in a way no function of the location) are constant. Traditionally, the z coordinate is used as the (negative) direction of the gravity¹. The effective body force is

$$g_{\text{eff}} = -g \hat{k} \quad (4.9)$$

Utilizing equation (4.9) and substituting it into equation (4.8) results into three simple partial differential equations. These equations are

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0 \quad (4.10)$$

and

$$\frac{\partial P}{\partial z} = -\rho g \quad (4.11)$$

Equations (4.10) can be integrated to yield

$$P(x, y) = \text{constant} \quad (4.12)$$

and constant in equation (4.12) can be absorbed by the integration of equation (4.11) and therefore

$$P(x, y, z) = -\rho g z + \text{constant} \quad (4.13)$$

The integration constant is determined from the initial conditions or another

¹This situation were the tradition is appropriated, it will be used. There are fields where x or y are designed to the direction of the gravity and opposite direction. For this reason sometime there will be a deviation from the above statement.

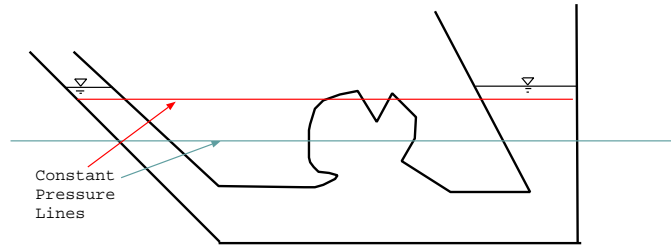


Fig. -4.2: Pressure lines a static fluid with a constant density.

point. For example, if at point z_0 the pressure is P_0 then the equation (4.13) becomes

$$P(z) - P_0 = -\rho g(z - z_0) \quad (4.14)$$

It is evident from equation (4.13) that the pressure depends only on z and/or the constant pressure lines are in the plane of x and y . Figure 4.2 describes the constant pressure lines in the container under the gravity body force. The pressure lines are continuous even in area where there is a discontinuous fluid. The reason that a solid boundary doesn't break the continuity of the pressure lines is because there is always a path to some of the planes.

It is convenient to reverse the direction of z to get rid of the negative sign and to define h as the dependent of the fluid that is $h \equiv -(z - z_0)$ so equation (4.14) becomes

$$P(h) - P_0 = \rho g h \quad (4.15)$$

In the literature, the right hand side of the equation (4.15) is defined as piezometric pressure.

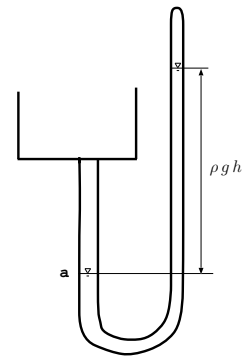


Fig. -4.3: A schematic to explain the measure of the atmospheric pressure.

4.3.2 Pressure Measurement

4.3.2.1 Measuring the Atmospheric Pressure

One of the application of this concept is the idea of measuring the atmospheric pressure. Consider a situation described in Figure 4.3. The liquid is filling the tube and is brought into a steady state. The pressure above the liquid on the right side is the vapor pressure. Using liquid with a very low vapor pressure like mercury, will result in a device that can measure the pressure without additional information (the temperature).

Example 4.1:

Calculate the atmospheric pressure at 20°C . The high of the Mercury is 0.76 [m] and the gravity acceleration is 9.82 [m/sec] . Assume that the mercury vapor pressure is 0.000179264 [kPa] . The description of the height is given in Figure 4.3. The mercury density is $13545.85\text{ [kg/m}^3\text{]}$.

SOLUTION

The pressure is uniform or constant plane perpendicular to the gravity. Hence, knowing any point on this plane provides the pressure anywhere on the plane. The atmospheric pressure at point **a** is the same as the pressure on the right hand side of the tube. Equation (4.15) can be utilized and it can be noticed that pressure at point **a** is

$$P_a = \rho g h + P_{\text{vapor}} \quad (4.16)$$

The density of the mercury is given along with the gravity and therefore,

$$P_a = 13545.85 \times 9.82 \times 0.76 \sim 101095.39\text{ [Pa]} \sim 1.01\text{ [Bar]}$$

The vapor pressure is about 1×10^{-4} percent of the total results.

The main reason the mercury is used because its of large density and the fact that it is in a liquid phase in most of the measurement range. The third reason is the low vapor (partial) pressure of the mercury. The partial pressure of mercury is in the range of the 0.000001793 [Bar] which is insignificant compared to the total measurement as can be observed from the above example.

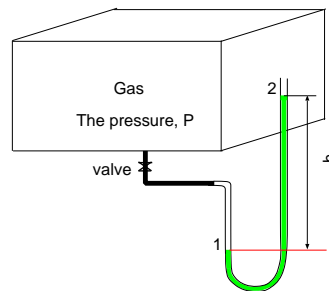


Fig. -4.4: Schematic of gas measurement utilizing the “U” tube.

4.3.2.2 Pressure Measurement

The idea describes the atmospheric measurement can be extended to measure the

pressure gas chambers. Consider a chamber filled with gas needed to be measured (see Figure 4.4). One technique is to attach “U” tube to the chamber and measure the pressure. This way, the gas is prevented from escaping and its pressure can be measured with a minimal interference to the gas (some gas enters to the tube).

The gas density is significantly lower than the liquid density and therefore can be neglected. The pressure at point “1” is

$$P_1 = P_{atmos} + \rho g h \quad (4.17)$$

Since the atmospheric pressure was measured previously (the technique was shown in the previous section) the pressure of the chamber can be measured.

4.3.2.3 Magnified Pressure Measurement

For situations where the pressure difference is very small, engineers invented more sensitive measuring device. This device is built around the fact that the height is a function of the densities difference. In the previous technique, the density of one side was neglected (the gas side) compared to other side (liquid). This technique utilizes the opposite range. The two sides of the densities are very close to each other so the height became large. Figure 4.5 shows a typical and simple schematic of such an instrument. If the pressure differences between P_1 and P_2 is small this instrument can “magnified” height, h_1 and provide “better” accuracy reading. This device is based on the following mathematical explanation.

In steady state, the pressure balance (only differences) is

$$P_1 + g \rho_1 (h_1 + h_2) = P_2 + g h_2 \rho_2 \quad (4.18)$$

It can be noticed that the “missing height” is canceled between the two sides. It can be noticed that h_1 can be positive or negative or zero and it depends on the ratio that two containers filled with the light density liquid. Additionally, it can be observed that h_1 is relatively small because $A_1 \gg A_2$. The densities of the liquid are chosen so that they are close to each other but not equal. The densities of the liquids are chosen to be much heavier than the measured gas density. Thus, in writing equation (4.18) the gas density was neglected. The pressure difference can be expressed as

$$P_1 - P_2 = g [\rho_2 h_2 - \rho_1 (h_1 + h_2)] \quad (4.19)$$

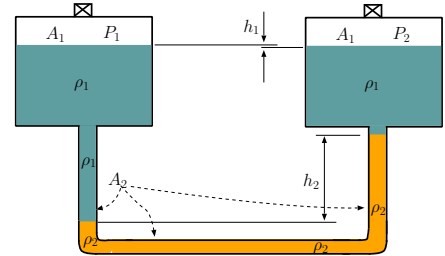


Fig. -4.5: Schematic of sensitive measurement device.

If the liquid volume in the two containers is known, it provides the relationship between h_1 and h_2 . For example, if the volumes in two containers is equal then

$$-h_1 A_1 = h_2 A_2 \longrightarrow h_1 = -\frac{h_2 A_2}{A_1} \quad (4.20)$$

Liquid volumes do not necessarily have to be equal. Additional parameter, the volume ratio, will be introduced when the volumes ratio isn't equal. The calculations as results of this additional parameter does not cause significant complications. Here, this ratio equals to one and it simplify the equation (4.20). But this ratio easily can be inserted into the derivations. With the equation for height (4.20) equation (4.18) becomes

$$P_1 - P_2 = g h_2 \left(\rho_2 - \rho_1 \left(1 - \frac{A_2}{A_1} \right) \right) \quad (4.21)$$

or the height is

$$h_2 = \frac{P_1 - P_2}{g \left[(\rho_2 - \rho_1) + \rho_1 \frac{A_2}{A_1} \right]} \quad (4.22)$$

For the small value of the area ratio, $A_2/A_1 \ll 1$, then equation (4.22) becomes

$$h_2 = \frac{P_1 - P_2}{g (\rho_2 - \rho_1)} \quad (4.23)$$

Some refer to the density difference shown in equation (4.23) as "magnification factor" since it replace the regular density, ρ_2 .

4.3.3 Varying Density in a Gravity Field

There are several cases that will be discussed here which are categorized as gases, liquids and other. In the gas phase, the equation of state is simply the ideal gas model or the ideal gas with the compressibility factor (sometime referred to as real gas). The equation of state for liquid can be approximated or replaced by utilizing the bulk modulus. These relationships will be used to find the functionality between pressure, density and location.

4.3.3.1 Gas Phase under Hydrostatic Pressure

Ideal Gas under Hydrostatic Pressure

The gas density vary gradually with the pressure. As first approximation, the ideal gas model can be employed to describe the density. Thus equation (4.11) becomes

$$\frac{\partial P}{\partial z} = -\frac{g P}{RT} \quad (4.24)$$

Separating the variables and changing the partial derivatives to full derivative (just a notation for this case) results in

$$\frac{dP}{P} = -\frac{g dz}{RT} \quad (4.25)$$

Equation (4.25) can be integrated from point "0" to any point to yield

$$\ln \frac{P}{P_0} = -\frac{g}{RT} (z - z_0) \quad (4.26)$$

It is convenient to rearrange equation (4.26) to the following

$$\frac{P}{P_0} = e^{-\left(\frac{g(z-z_0)}{RT}\right)} \quad (4.27)$$

Here the pressure ratio is related to the height exponentially. Equation (4.27) can be expanded to show the difference to standard assumption of constant pressure as

$$\frac{P}{P_0} = 1 - \frac{\overbrace{\frac{h \rho_0 g}{P_0}}^{(z-z_0)g}}{RT} + \frac{(z-z_0)^2 g}{6 RT} + \dots \quad (4.28)$$

Or in a simplified form where the transformation of $h = (z - z_0)$ to be

$$\frac{P}{P_0} = 1 + \frac{\rho_0 g}{P_0} \left(h - \overbrace{\frac{h^2}{6} + \dots}^{\text{correction factor}} \right) \quad (4.29)$$

Equation (4.29) is useful in mathematical derivations but should be ignored for practical use².

Real Gas under Hydrostatic Pressure

The mathematical derivations for ideal gas can be reused as a foundation for the real gas model ($P = Z\rho RT$). For a large range of P/P_c and T/T_c , the value of the compressibility factor, Z , can be assumed constant and therefore can be swallowed into equations (4.27) and (4.28). The compressibility is defined in equation (2.39). The modified equation is

$$\frac{P}{P_0} = e^{-\left(\frac{g(z-z_0)}{ZRT}\right)} \quad (4.30)$$

²These derivations are left for a mathematical mind person. These deviations have a limited practical purpose. However, they are presented here for students who need to answer questions on this issue.

Or in a series form which is

$$\frac{P}{P_0} = 1 - \frac{(z - z_0)g}{ZRT} + \frac{(z - z_0)^2 g}{6ZRT} + \dots \quad (4.31)$$

Without going through the mathematics, the first approximation should be noticed that the compressibility factor, Z enter the equation as h/Z and not just h . Another point that is worth discussing is the relationship of Z to other gas properties. In general, the relationship is very complicated and in some ranges Z cannot be assumed constant. In these cases, a numerical integration must be carried out.

4.3.3.2 Liquid Phase Under Hydrostatic Pressure

The bulk modulus was defined in equation (1.28). The simplest approach is to assume that the bulk modulus is constant (or has some representative average). For these cases, there are two differential equations that needed to be solved. Fortunately, here, only one hydrostatic equation depends on density equation. So, the differential equation for density should be solved first. The governing differential density equation (see equation (1.28)) is

$$\rho = B_T \frac{\partial \rho}{\partial P} \quad (4.32)$$

The variables for equation (4.32) should be separated and then the integration can be carried out as

$$\int_{P_0}^P dP = \int_{\rho_0}^{\rho} B_T \frac{d\rho}{\rho} \quad (4.33)$$

The integration of equation (4.33) yields

$$P - P_0 = B_T \ln \frac{\rho}{\rho_0} \quad (4.34)$$

Equation (4.34) can be represented in a more convenient form as

$$\rho = \rho_0 e^{\frac{P - P_0}{B_T}} \quad (4.35)$$

Equation (4.35) is the counterpart for the equation of state of ideal gas for the liquid phase. Utilizing equation (4.35) in equation (4.11) transformed into

$$\frac{\partial P}{\partial z} = -g\rho_0 e^{\frac{P - P_0}{B_T}} \quad (4.36)$$

Equation (4.36) can be integrated to yield

$$\frac{B_T}{g\rho_0} e^{\frac{P - P_0}{B_T}} = z + Constant \quad (4.37)$$

It can be noted that B_T has units of pressure and therefore the ratio in front of the exponent in equation (4.37) has units of length. The integration constant, with units of length, can be evaluated at any specific point. If at $z = 0$ the pressure is P_0 and the density is ρ_0 then the constant is

$$\text{Constant} = \frac{B_T}{g \rho_0} \quad (4.38)$$

This constant, $B_T/g \rho_0$, is a typical length of the problem. Additional discussion will be presented in the dimensionless issues chapter (currently under construction). The solution becomes

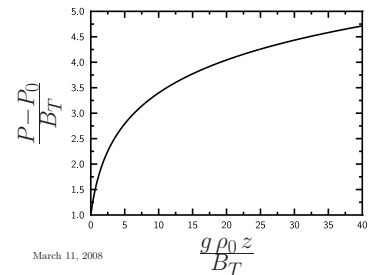
$$\frac{B_T}{g \rho_0} \left(e^{\frac{P-P_0}{B_T}} - 1 \right) = z \quad (4.39)$$

Or in a dimensionless form

$$\left(e^{\frac{P-P_0}{B_T}} - 1 \right) = \frac{z g \rho_0}{B_T} \quad (4.40)$$

The solution is presented in equation (4.39) and is plotted in Figure 4.6. The solution is a reverse function (that is not $P = f(z)$ but $z = f(P)$) it is a monotonous function which is easy to solve for any numerical value (that is only one z corresponds to any Pressure). Sometimes, the solution is presented as

$$\frac{P}{P_0} = \frac{B_T}{P_0} \ln \left(\frac{g \rho_0 z}{B_T} + 1 \right) + 1 \quad (4.41)$$



An approximation of equation (4.40) is presented for historical reasons and in order to compare the constant density assumption. The exponent can be expanded as

$$\left(\underbrace{\text{piezometric pressure}}_{(P - P_0)} + \overbrace{\frac{B_T}{2} \left(\frac{P - P_0}{B_T} \right)^2 + \frac{B_T}{6} \left(\frac{P - P_0}{B_T} \right)^3 + \dots}^{\text{corrections}} \right) = z g \rho_0 \quad (4.42)$$

It can be noticed that equation (4.42) is reduced to the standard equation when the normalized pressure ratio, P/B_T is small ($\ll 1$). Additionally, it can be observed that the correction is on the left hand side and not as the “traditional” correction on the piezometric pressure side.

4.3.4 The Pressure Effects Because Temperature Variations

4.3.4.1 The Basic Analysis

There are situations when the main change of the density results from other effects. For example, when the temperature field is not uniform, the density is affected and thus the pressure is a location function (for example, in the atmosphere the temperature is assumed to be a linear with the height under certain conditions.). A bit more complicate case is when the gas is a function of the pressure and another parameter. Air can be a function of the temperature field and the pressure. For the atmosphere, it is commonly assumed that the temperature is a linear function of the height.

Here, a simple case is examined for which the temperature is a linear function of the height as

$$\frac{dT}{dh} = -C_x \quad (4.43)$$

where h here referred to height or distance. Hence, the temperature–distance function can be written as

$$T = Constant - C_x h \quad (4.44)$$

where the *Constant* is the integration constant which can be obtained by utilizing the initial condition. For $h = 0$, the temperature is T_0 and using it leads to

$$T = T_0 - C_x h \quad (4.45)$$

Combining equation (4.45) with (4.11) results in

$$\frac{\partial P}{\partial h} = -\frac{g P}{R(T_0 - C_x h)} \quad (4.46)$$

Separating the variables in equation (4.46) and changing the formal ∂ to the informal d to obtain

$$\frac{dP}{P} = -\frac{g dh}{R(T_0 - C_x h)} \quad (4.47)$$

Defining a new variable³ as $\xi = (T_0 - C_x h)$ for which $\xi_0 = T_0 - C_x h_0$ and $d/d\xi = -C_x d/dh$. Using these definitions results in

$$\frac{dP}{P} = \frac{g}{RC_x} \frac{d\xi}{\xi} \quad (4.48)$$

After integration of equation (4.47) and reusing (the reverse definitions) the variables transformed the result into

$$\ln \frac{P}{P_0} = \frac{g}{RC_x} \ln \frac{T_0 - C_x h}{T_0} \quad (4.49)$$

Or in a more convenient form as

$$\frac{P}{P_0} = \left(\frac{T_0 - C_x h}{T_0} \right)^{\left(\frac{g}{RC_x} \right)} \quad (4.50)$$

It can be noticed that equation (4.50) is a monotonous function which decreases with height because the term in the brackets is less than one. This situation is roughly representing the pressure in the atmosphere and results in a temperature decrease. It can be observed that C_x has a “double role” which can change the pressure ratio. Equation (4.50) can be approximated by two approaches/ideas. The first approximation for a small distance, h , and the second approximation for a small temperature gradient. It can be recalled that the following expansions are

$$\frac{P}{P_0} = \lim_{h \rightarrow 0} \left(1 - \frac{C_x}{T_0} h \right)^{\frac{g}{RC_x}} = 1 - \underbrace{\frac{g h}{T_0 R}}_{\frac{g h \rho_0}{P_0}} - \underbrace{\frac{(R g C_x - g^2) h^2}{2 T_0^2 R^2}}_{\text{correction factor}} - \dots \quad (4.51)$$

Equation (4.51) shows that the first two terms are the standard terms (negative sign is as expected i.e. negative direction). The correction factor occurs only at the third term which is important for larger heights. It is worth to point out that the above statement has a qualitative meaning when additional parameter is added. However, this kind of analysis will be presented in the dimensional analysis chapter⁴.

The second approximation for small C_x is

$$\frac{P}{P_0} = \lim_{C_x \rightarrow 0} \left(1 - \frac{C_x}{T_0} h \right)^{\frac{g}{RC_x}} = e^{-\frac{g h}{R T_0}} - \frac{g h^2 C_x}{2 T_0^2 R} e^{-\frac{g h}{R T_0}} - \dots \quad (4.52)$$

³A colleague asked this author to insert this explanation for his students. If you feel that it is too simple, please, just ignore it.

⁴These concepts are very essential in all the thermo-fluid science. I am grateful to my adviser E.R.G. Eckert who was the pioneer of the dimensional analysis in heat transfer and was kind to show me some of his ideas.

Equation (4.52) shows that the correction factor (lapse coefficient), C_x , influences at only large values of height. It has to be noted that these equations (4.51) and (4.52) are not properly represented without the characteristic height. It has to be inserted to make the physical significance clearer.

Equation (4.50) represents only the pressure ratio. For engineering purposes, it is sometimes important to obtain the density ratio. This relationship can be obtained from combining equations (4.50) and (4.45). The simplest assumption to combine these equations is by assuming the ideal gas model, equation (2.25), to yield

$$\frac{\rho}{\rho_0} = \frac{PT_0}{P_0T} = \underbrace{\left(1 - \frac{C_x h}{T_0}\right)^{\left(\frac{\gamma}{R C_x}\right)}}_{\frac{P}{P_0}} \underbrace{\left(1 + \frac{C_x h}{T}\right)}_{\frac{T_0}{T}} \quad (4.53)$$

Advance material can be skipped

4.3.4.2 The Stability Analysis

It is interesting to study whether this solution (4.50) is stable and if so under what conditions. Suppose that for some reason, a small slab of material moves from a layer at height, h , to layer at height $h + dh$ (see Figure 4.7) What could happen? There are two main possibilities

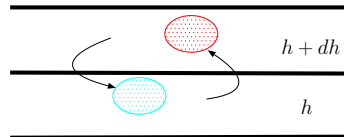


Fig. -4.7: Two adjoin layers for stability analysis.

one: the slab could return to the original layer or two: stay at the new layer (or even move further, higher heights). The first case is referred to as the stable condition and the second case referred to as the unstable condition. The whole system falls apart and does not stay if the analysis predicts unstable conditions. A weak wind or other disturbances can make the unstable system to move to a new condition.

This question is determined by the net forces acting on the slab. Whether these forces are toward the original layer or not. The two forces that act on the slab are the gravity force and the surroundings pressure (buoyant forces). Clearly, the slab is in equilibrium with its surroundings before the movement (not necessarily stable). Under equilibrium, the body forces that acting on the slab are equal to zero. That is, the surroundings “pressure” forces (buoyancy forces) are equal to gravity forces. The buoyancy forces are proportional to the ratio of the density of the slab to surrounding layer density. Thus, the stability question is whether the slab density from layer h , $\rho'(h)$ undergoing a free expansion is higher or lower than the density of the layer $h + dh$. If $\rho'(h) > \rho(h + dh)$ then the situation is stable. The term $\rho'(h)$ is slab from layer h that had undergone the free expansion.

The reason that the free expansion is chosen to explain the process that the slab undergoes when it moves from layer h to layer $h + dh$ is because it is the

simplest. In reality, the free expansion is not far way from the actual process. The two processes that occurred here are thermal and the change of pressure (at the speed of sound). The thermal process is in the range of [cm/sec] while the speed of sound is about 300 [m/sec]. That is, the pressure process is about thousands times faster than the thermal process. The second issue that occurs during the “expansion” is the shock (in the reverse case $[h + dh] \rightarrow h$). However, this shock is insignificant (check book on Fundamentals of Compressible Flow Mechanics by this author on the French problem).

The slab density at layer $h + dh$ can be obtained using equation (4.53) as following

$$\frac{\rho(h + dh)}{\rho(h)} = \frac{P T_0}{P_0 T} = \left(1 - \frac{C_x dh}{T_0}\right)^{\left(\frac{g}{R C_x}\right)} \left(1 + \frac{C_x dh}{T}\right) \quad (4.54)$$

The pressure and temperature change when the slab moves from layer at h to layer $h + dh$. The process, under the above discussion and simplifications, can be assumed to be adiabatic (that is, no significant heat transfer occurs in the short period of time). The little slab undergoes isentropic expansion as following for which (see equation (2.25))

$$\frac{\rho'(h + dh)}{\rho(h)} = \left(\frac{P'(h + dh)}{P(h)}\right)^{1/k} \quad (4.55)$$

When the symbol ' denotes the slab that moves from layer h to layer $h + dh$. The pressure ratio is given by equation (4.50) but can be approximated by equation (4.51) and thus

$$\frac{\rho'(h + dh)}{\rho(h)} = \left(1 - \frac{g dh}{T(h) R}\right)^{1/k} \quad (4.56)$$

Again using the ideal gas model for equation (4.57) transformed into

$$\frac{\rho'(h + dh)}{\rho(h)} = \left(1 - \frac{\rho g dh}{P}\right)^{1/k} \quad (4.57)$$

Expanding equation (4.57) in Taylor series results in

$$\left(1 - \frac{\rho g dh}{P}\right)^{1/k} = 1 - \frac{g \rho dh}{P k} - \frac{(g^2 \rho^2 k - g^2 \rho^2) dh^2}{2 P^2 k^2} - \dots \quad (4.58)$$

The density at layer $h + dh$ can be obtained from (4.54) and then it is expanded in Taylor series as

$$\frac{\rho(h + dh)}{\rho(h)} = \left(1 - \frac{C_x dh}{T_0}\right)^{\left(\frac{g}{R C_x}\right)} \left(1 + \frac{C_x dh}{T}\right) \sim 1 - \left(\frac{g \rho}{P} - \frac{C_x}{T}\right) dh + \dots \quad (4.59)$$

The comparison of the right hand terms of equations (4.59) and (4.58) provides the conditions to determine the stability.

From a mathematical point of view, to keep the inequality for a small dh only the first term need to be compared as

$$\frac{g \rho}{P k} > \frac{g \rho}{P} - \frac{C_x}{T} \tag{4.60}$$

After rearrangement of the inequality (4.60) and using the ideal gas identity, it transformed to

$$\begin{aligned} \frac{C_x}{T} &> \frac{(k-1) g \rho}{k P} \\ C_x &< \frac{k-1}{k} \frac{g}{R} \end{aligned} \tag{4.61}$$

The analysis shows that the maximum amount depends on the gravity and gas properties. It should be noted that this value should be changed a bit since the k should be replaced by polytropic expansion n . When lapse rate C_x is equal to the right hand side of the inequality, it is said that situation is neutral. However, one has to bear in mind that this analysis only provides a range and isn't exact. Thus, around this value additional analysis is needed ⁵.

One of the common question this author has been asked is about the forces of continuation. What is the source of the force(s) that make this situation when unstable continue to be unstable? Supposed that the situation became unstable and the layers have been exchanged, would the situation become stable now? One has to remember that temperature gradient forces continuous heat transfer which the source temperature change after the movement to the new layer. Thus, the unstable situation is continuously unstable.

4.3.5 Gravity Variations Effects on Pressure and Density

Until now the study focus on the change of density and pressure of the fluid. Equation (4.11) has two terms on the right hand side, the density, ρ and the body force, g . The body force was assumed until now to be constant. This assumption must be deviated when the distance from the body source is significantly change. At first glance, the body force is independent of the fluid. The source of the gravity force in gas is another body, while the gravity force source in liquid can be the liquid itself. Thus, the discussion

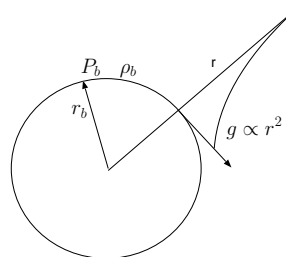


Fig. -4.8: The varying gravity effects on density and pressure.

⁵The same issue of the floating ice. See example for the floating ice in cup.

is separated into two different issues. The issues of magnetohydrodynamics are too advance for undergraduate student and therefore, will not be introduced here.

4.3.5.1 Ideal Gas in Varying Gravity

In physics, it was explained that the gravity is a function of the distance from the center of the planet/body. Assuming that the pressure is affected by this gravity/body force. The gravity force is reversely proportional to r^2 . The gravity force can be assumed that for infinity, $r \rightarrow \infty$ the pressure is about zero. Again, equation (4.11) can be used (semi one directional situation) when r is used as direction and thus

$$\frac{\partial P}{\partial r} = -\rho \frac{G}{r^2} \quad (4.62)$$

where here G denotes the general gravity constant. The regular method of separation is employed to obtain

$$\int_{P_b}^P \frac{dP}{P} = -\frac{G}{RT} \int_{r_b}^r \frac{dr}{r^2} \quad (4.63)$$

where the subscript b denotes the conditions at the body surface. The integration of equation (4.63) results in

$$\ln \frac{P}{P_b} = -\frac{G}{RT} \left(\frac{1}{r_b} - \frac{1}{r} \right) \quad (4.64)$$

Or in a simplified form as

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = e^{-\frac{G}{RT} \frac{r-r_b}{r r_b}} \quad (4.65)$$

Equation (4.65) demonstrates that the pressure is reduced with the distance. It can be noticed that for $r \rightarrow r_b$ the pressure is approaching $P \rightarrow P_b$. This equation confirms that the density in outer space is zero $\rho(\infty) = 0$. As before, equation (4.65) can be expanded in taylor series as

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = 1 - \overbrace{\frac{G(r-r_b)}{RT}}^{\text{standard}} - \overbrace{\frac{(2GRT + G^2 r_b)(r-r_b)^2}{2r_b(RT)^2}}^{\text{correction factor}} + \dots \quad (4.66)$$

Notice that G isn't our beloved and familiar g and also that $G r_b / RT$ is a dimensionless number (later in dimensionless chapter about it and its meaning).

4.3.5.2 Real Gas in Varying Gravity

The regular assumption of constant compressibility, Z , is employed. It has to remember when this assumption isn't accurate enough, numerical integration is a

possible solution. Thus, equation (4.63) is transformed into

$$\int_{P_b}^P \frac{dP}{P} = -\frac{G}{ZRT} \int_{r_b}^r \frac{dr}{r^2} \quad (4.67)$$

With the same process as before for ideal gas case, one can obtain

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = e^{-\frac{G}{ZRT} \frac{r-r_b}{r r_b}} \quad (4.68)$$

Equation (4.65) demonstrates that the pressure is reduced with the distance. It can be observed that for $r \rightarrow r_b$ the pressure is approaching $P \rightarrow P_b$. This equation confirms that the density in outer space is zero $\rho(\infty) = 0$. As before Taylor series for equation (4.65) is

$$\frac{\rho}{\rho_b} = \frac{P}{P_b} = \underbrace{1 - \frac{G(r-r_b)}{ZRT}}_{\text{standard}} - \underbrace{\frac{(2GZRT + G^2 r_b)(r-r_b)^2}{2r_b(ZRT)^2}}_{\text{correction factor}} + \dots \quad (4.69)$$

It can be noted that compressibility factor can act as increase or decrease of the ideal gas model depending on whether it is above one or below one.

4.3.5.3 Liquid in Under Varying Gravity

For comparison reason consider the deepest location in the ocean which is about 11,000 [m]. If the liquid "equation of state" (4.35) is used with the hydrostatic fluid equation results in

$$\frac{\partial P}{\partial r} = -\rho_0 e^{\frac{P-P_0}{B_T}} \frac{G}{r^2} \quad (4.70)$$

which the solution of equation (4.70) is

$$e^{\frac{P_0-P}{B_T}} = \text{Constant} - \frac{B_T g \rho_0}{r} \quad (4.71)$$

Since this author is not aware to which practical situation this solution should be applied, it is left for the reader to apply according to problem, if applicable.

4.3.6 Liquid Phase

While for most practical purposes, the Cartesian coordinates provides sufficient treatment to the problem, there are situations where the spherical coordinates must be considered and used.

Derivations of the fluid static in spherical coordinates are

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) + 4\pi G\rho = 0 \quad (4.72)$$

Or in a vector form as

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P \right) + 4\pi G\rho = 0 \quad (4.73)$$

4.4 Fluid in a Accelerated System

Up to this stage, body forces were considered as one-dimensional. The general, the linear acceleration have three components as opposed to the previous case of only one. However, the previous derivations can be easily extended. Equation (4.8) can be transformed into a different coordinate system where the main coordinate is in the direction of the effective gravity. Thus, the previous method can be used and there is no need to solve new three (or two) different equations. As before, the constant pressure plane is perpendicular to the direction of the effective gravity. Generally the acceleration is divided into two categories: linear and angular and they will be discussed in this order.

4.4.1 Fluid in a Linearly Accelerated System

For example, in a two dimensional system, for the effective gravity

$$g_{eff} = a \hat{i} + g \hat{k} \quad (4.74)$$

where the magnitude of the effective gravity is

$$|g_{eff}| = \sqrt{g^2 + a^2} \quad (4.75)$$

and the angle/direction can be obtained from

$$\tan\beta = \frac{a}{g} \quad (4.76)$$

Perhaps the best way to explain the linear acceleration is by examples. Consider the following example to illustrate the situation.

Example 4.2:

What is the angle of the liquid surface for a container in an accelerated system of $a = 5[m/sec]$?

SOLUTION

This question is one of the traditional question the fluid static and is straight for-

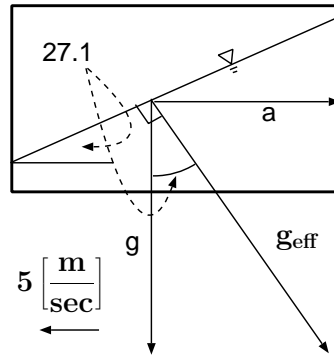


Fig. -4.9: The effective gravity is for accelerated cart.

ward. This requires to find the angle of the effective body force. This angle/direction can be found using the following

$$\tan^{-1} \beta = \tan^{-1} \frac{a}{g} = \frac{5}{9.81} \sim 27.01^\circ$$

The magnitude of the effective acceleration is

$$|g_{eff}| = \sqrt{5^2 + 9.81^2} = 11.015[m/sec^2]$$

Example 4.3:

Cart partially is filled with liquid and is sliding on an inclined plane as shown in Figure 4.10. Calculate the shape of the surface. If there is a resistance what will be the angle? What happen when the slope angle is straight (the cart is dropping straight down)?

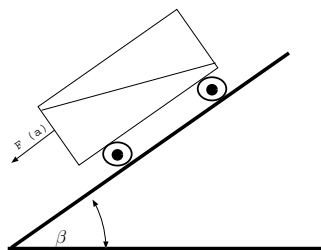


Fig. -4.10: A cart slide on inclined plane

SOLUTION

(a)

The angle can be found when the acceleration of the cart is found. If there is no resistance, the acceleration in the cart direction is determined from

$$a = g \sin \beta \tag{4.77}$$

The effective body force is acting perpendicular to the slope. Thus, the liquid surface is parallel to the surface of the inclination surface.

(b)

In case of resistance force (either of friction due to the air or resistance in the wheels) reduces the acceleration of the cart. In that case the effective body force moves closer to the gravity forces. The net body force depends on the mass of the liquid and the net acceleration is

$$a = g - \frac{F_{net}}{m} \quad (4.78)$$

The angle of the surface, $\alpha < \beta$, is now

$$\tan \alpha = \frac{g - \frac{F_{net}}{m}}{g \cos \beta} \quad (4.79)$$

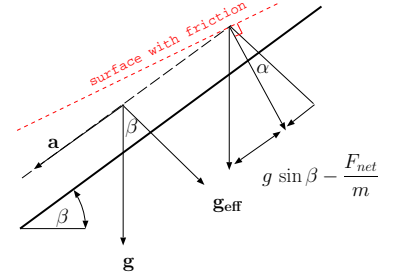


Fig. -4.11: Forces diagram of cart sliding on inclined plane

(c)

In the case when the angle of the inclination turned to be straight (direct falling) the effective body force is zero. The pressure is uniform in the tank and no pressure difference can be found. So, the pressure at any point in the liquid is the same and equal to the atmospheric pressure.

4.4.2 Angular Acceleration Systems: Constant Density

For simplification reasons, the first case that deal here is a rotation in a perpendicular to the gravity. That is the effective body force can be written as

$$\mathbf{g}_{eff} = -g \hat{k} + \omega^2 r \hat{r} \quad (4.80)$$

The lines of constant pressure are not straight lines but lines in a parabolic shape. The angle of the line depends on the radius as

$$\frac{dx}{dr} = -\frac{g}{\omega^2 r} \quad (4.81)$$

Equation (4.81) can be integrated as

$$x - x_0 = \frac{\omega^2 r^2}{2g} \quad (4.82)$$

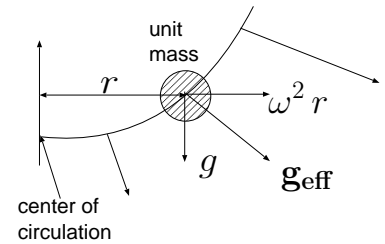


Fig. -4.12: Schematic to explain the angular angle.

Notice that the integration constant was substituted by x_0 . The constant pressure will be along

$$P - P_0 = \rho g \left[(x_0 - x) + \frac{\omega^2 r^2}{2g} \right] \tag{4.83}$$

4.5 Fluid Forces on Surfaces

The forces that fluids (at static conditions) exerts on surfaces are very important for engineering purposes. This section deals with these calculations. These calculations are divided into two categories, straight surfaces and curved surfaces.

4.5.1 Fluid Forces on Straight Surfaces

A motivation is needed before going through the routine of derivations. Initially, a simple case will be examined. Later, how the calculations can be simplified will be shown.

Example 4.4:

Consider a rectangular shape gate as shown in Figure 4.13. Calculate the minimum forces, F_1 and F_2 to maintain the gate in position. Assuming that the atmospheric pressure can be ignored.

SOLUTION

The forces can be calculated by looking at the moment around point "O." The element of moment is $a d\xi$ for the width of the gate and is

$$dM = \overbrace{P}^{dF} \underbrace{a d\xi}_{dA} (\ell + \xi)$$

The pressure, P can be expressed as a function ξ as the following

$$P = g \rho (\ell + \xi) \sin \beta$$

The liquid total moment on the gate is

$$M = \int_0^b g \rho (\ell + \xi) \sin \beta a d\xi (\ell + \xi)$$

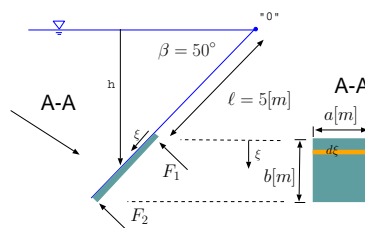


Fig. -4.13: Rectangular area under pressure.

The integral can be simplified as

$$M = g a \rho \sin \beta \int_0^b (\ell + \xi)^2 d\xi \quad (4.84)$$

The solution of the above integral is

$$M = g \rho a \sin \beta \left(\frac{3 b \ell^2 + 3 b^2 \ell + b^3}{3} \right)$$

This value provides the moment that F_1 and F_2 should extract. Additional equation is needed. It is the total force, which is

$$F_{total} = \int_0^b g \rho (\ell + \xi) \sin \beta a d\xi$$

The total force integration provides

$$F_{total} = g \rho a \sin \beta \int_0^b (\ell + \xi) d\xi = g \rho a \sin \beta \left(\frac{2 b \ell + b^2}{2} \right)$$

The forces on the gate have to provide

$$F_1 + F_2 = g \rho a \sin \beta \left(\frac{2 b \ell + b^2}{2} \right)$$

Additionally, the moment of forces around point "O" is

$$F_1 \ell + F_2(\ell + b) = g \rho a \sin \beta \left(\frac{3 b \ell^2 + 3 b^2 \ell + b^3}{3} \right)$$

The solution of these equations is

$$F_1 = \frac{(3 \ell + b) a b g \rho \sin \beta}{6}$$

$$F_2 = \frac{(3 \ell + 2 b) a b g \rho \sin \beta}{6}$$

The above calculations are time consuming and engineers always try to make life simpler. Looking at the above calculations, it can be observed that there is a moment of area in equation (4.84) and also a center of area. These concepts have been introduced in Chapter 3. Several represented areas for which moment of inertia and center of area have been tabulated in Chapter 3. These tabulated values can be used to solve this kind of problems.

Symmetrical Shapes

Consider the two-dimensional symmetrical area that are under pressure as shown in Figure 4.14. The symmetry is around any axes parallel to axis x . The total force and moment that the liquid extracting on the area need to be calculated. First, the force is

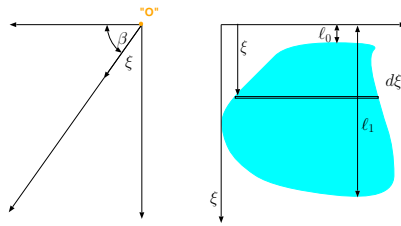


Fig. -4.14: Schematic of submerged area to explain the center forces and moments

$$F = \int_A P dA = \int (P_{atmos} + \rho g h) dA = A P_{atmos} + \rho g \int_{\ell_0}^{\ell_1} \overbrace{(\xi + \ell_0) \sin \beta}^{h(\xi)} dA \tag{4.85}$$

In this case, the atmospheric pressure can include any additional liquid layer above layer “touching” area. The “atmospheric” pressure can be set to zero.

The boundaries of the integral of equation (4.85) refer to starting point and ending points not to the start area and end area. The integral in equation (4.85) can be further developed as

$$F_{total} = A P_{atmos} + \rho g \sin \beta \left(\ell_0 A + \overbrace{\int_{\ell_0}^{\ell_1} \xi dA}^{x_c A} \right) \tag{4.86}$$

In a final form as

$$F_{total} = A [P_{atmos} + \rho g \sin \beta (\ell_0 + x_c)] \tag{4.87}$$

The moment of the liquid on the area around point “O” is

$$M_y = \int_{\xi_0}^{\xi_1} P(\xi) \xi dA \tag{4.88}$$

$$M_y = \int_{\xi_0}^{\xi_1} (P_{atmos} + g \rho \overbrace{\xi \sin \beta}^{h(\xi)}) \xi dA \tag{4.89}$$

Or separating the parts as

$$M_y = P_{atmos} \overbrace{\int_{\xi_0}^{\xi_1} \xi dA}^{x_c A} + g \rho \sin \beta \overbrace{\int_{\xi_0}^{\xi_1} \xi^2 dA}^{I_{x'x'}} \quad (4.90)$$

The moment of inertia, $I_{x'x'}$, is about the axis through point "O" into the page. Equation (4.90) can be written in more compact form as

$$M_y = P_{atmos} x_c A + g \rho \sin \beta I_{x'x'} \quad (4.91)$$

Example 4.4 can be generalized to solve any two forces needed to balance the area/gate. Consider the general symmetrical body shown in figure 4.15 which has two forces that balance the body. Equations (4.87) and (4.91) can be combined the moment and force acting on the general area. If the "atmospheric pressure" can be zero or include additional layer of liquid. The forces balance reads

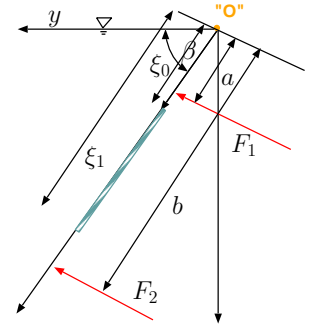


Fig. -4.15: The general forces acting on submerged area.

$$F_1 + F_2 = A [P_{atmos} + \rho g \sin \beta (\ell_0 + x_c)] \quad (4.92)$$

and moments balance reads

$$F_1 a + F_2 b = P_{atmos} x_c A + g \rho \sin \beta I_{x'x'} \quad (4.93)$$

The solution of these equations is

$$F_1 = \frac{\left[\left(\rho \sin \beta - \frac{P_{atmos}}{g b} \right) x_c + \ell_0 \rho \sin \beta + \frac{P_{atmos}}{g} \right] b A - I_{x'x'} \rho \sin \beta}{g (b - a)} \quad (4.94)$$

and

$$F_2 = \frac{I_{x'x'} \rho \sin \beta - \left[\left(\rho \sin \beta - \frac{P_{atmos}}{g a} \right) x_c + \ell_0 \rho \sin \beta + \frac{P_{atmos}}{g} \right] a A}{g (b - a)} \quad (4.95)$$

In the solution, the forces can be negative or positive, and the distance a or b can be positive or negative. Additionally, the atmospheric pressure can contain either an additional liquid layer above the "touching" area or even atmospheric pressure simply can be set up to zero. In symmetrical area only two forces are required since the moment is one dimensional. However, in non-symmetrical area

there two different moments and therefor three forces are required. Thus, additional equation is required. This equation is for the additional moment around the x axis (see for explanation in Figure 4.16). The moment around the y axis is given by equation (4.91) and the total force is given by (4.87). The moment around the x axis (which was arbitrary chosen) should be

$$M_x = \int_A y P dA \tag{4.96}$$

Substituting the components for the pressure transforms equation (4.96) into

$$M_x = \int_A y (P_{atmos} + \rho g \xi \sin \beta) dA \tag{4.97}$$

The integral in equation (4.96) can be written as

$$M_x = P_{atmos} \overbrace{\int_A y dA}^{A y_c} + \rho g \sin \beta \overbrace{\int_A \xi y dA}^{I_{x' y'}} \tag{4.98}$$

The compact form can be written as

$$M_x = P_{atmos} A y_c + \rho g \sin \beta I_{x' y'} \tag{4.99}$$

The product of inertia was presented in Chapter 3. These equations (4.87), (4.91) and (4.99) provide the base for solving any problem for straight area under pressure with uniform density. There are many combinations of problems (e.g. two forces and moment) but no general solution is provided. Example to illustrate the use of these equations is provided.

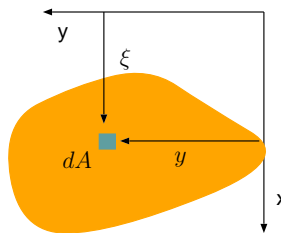


Fig. -4.16: The general forces acting on non symmetrical straight area.

Example 4.5:
Calculate the forces that are required to balance the triangular shape shown in the Figure 4.17.

SOLUTION

The three equations that needs to be solved are

$$F_1 + F_2 + F_3 = F_{total} \tag{4.100}$$

The moment around x axis is

$$F_1 b = M_y \tag{4.101}$$

The moment around y axis is

$$F_1 \ell_1 + F_2 (a + \ell_0) + F_3 \ell_0 = M_x \quad (4.102)$$

The right hand side of these equations are given before in equations (4.87), (4.91) and (4.99).

The moment of inertia of the triangle around x is made of two triangles (as shown in the Figure (4.17) for triangle 1 and 2). Triangle 1 can be calculated as the moment of inertia around its center which is $\ell_0 + 2 * (\ell_1 - \ell_0)/3$. The height of triangle 1 is $(\ell_1 - \ell_0)$ and its width b and thus, moment of inertia about its center is $I_{xx1} = b(\ell_1 - \ell_0)^3/36$. The moment of inertia for triangle 1 about y is

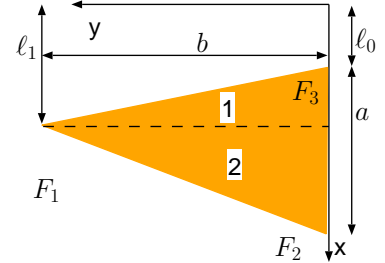


Fig. -4.17: The general forces acting on non symmetrical straight area.

$$I_{xx1} = \frac{b(\ell_1 - \ell_0)^3}{36} + \frac{A_1}{3} \left(\ell_0 + \frac{2(\ell_1 - \ell_0)}{3} \right)^2$$

The height of the triangle 2 is $a - (\ell_1 - \ell_0)$ and its width b and thus, the moment of inertia about its center is

$$I_{xx2} = \frac{b[a - (\ell_1 - \ell_0)]^3}{36} + \frac{A_2}{3} \left(\ell_1 + \frac{[a - (\ell_1 - \ell_0)]}{3} \right)^2$$

and the total moment of inertia

$$I_{xx} = I_{xx1} + I_{xx2}$$

The product of inertia of the triangle can be obtain by integration. It can be noticed that upper line of the triangle is $y = \frac{(\ell_1 - \ell_0)x}{b} + \ell_0$. The lower line of the triangle is $y = \frac{(\ell_1 - \ell_0 - a)x}{b} + \ell_0 + a$.

$$I_{xy} = \int_0^b \left[\int_{\frac{(\ell_1 - \ell_0)x}{b} + \ell_0}^{\frac{(\ell_1 - \ell_0 - a)x}{b} + \ell_0 + a} x y dx \right] dy = \frac{2 a b^2 \ell_1 + 2 a b^2 \ell_0 + a^2 b^2}{24}$$

The solution of this set equations is

$$F_1 = \left[\frac{a b}{3} \right] \frac{(g (6 \ell_1 + 3 a) + 6 g \ell_0) \rho \sin \beta + 8 P_{atmos}}{24},$$

$$\frac{F_2}{\left[\frac{ab}{3}\right]} = - \frac{\left((3\ell_1 - 14a) - \ell_0 \left(\frac{12\ell_1}{a} - 27 \right) + \frac{12\ell_0^2}{a} \right) g \rho \sin \beta}{\left(\left(\frac{24\ell_1}{a} - 24 \right) + \frac{48\ell_0}{a} \right) P_{atmos}},$$

$$\frac{F_3}{\left[\frac{ab}{3}\right]} = \frac{\left(\left(a - \frac{15\ell_1}{a} \right) + \ell_0 \left(27 - \frac{12\ell_1}{a} \right) + \frac{12\ell_0^2}{a} \right) g \rho \sin \beta}{\left(\left(\frac{24\ell_1}{a} + 24 \right) + \frac{48\ell_0}{a} \right) P_{atmos}}$$

4.5.1.1 Pressure Center

In the literature, pressure centers are commonly defined. These definitions are mathematical in nature and has physical meaning of equivalent force that will act through this center. The definition is derived or obtained from equation (4.91) and equation (4.99). The pressure center is the distance that will create the moment with the hydrostatic force on point "O." Thus, the pressure center in the x direction is

$$x_p = \frac{1}{F} \int_A x P dA \quad (4.103)$$

In the same way, the pressure center in the y direction is defined as

$$y_p = \frac{1}{F} \int_A y P dA \quad (4.104)$$

To show relationship between the pressure center and the other properties, it can be found by setting the atmospheric pressure and ℓ_0 to zero as following

$$x_p = \frac{g \rho \sin \beta I_{x'x'}}{A \rho g \sin \beta x_c} \quad (4.105)$$

Expanding $I_{x'x'}$ according to equation (3.15) results in

$$x_p = \frac{I_{xx}}{x_c A} + x_c \quad (4.106)$$

and in the same fashion in y direction

$$y_p = \frac{I_{yy}}{y_c A} + y_c \quad (4.107)$$

It has to emphasis that these definitions are useful only for case where the atmospheric pressure can be neglected or canceled and where ℓ_0 is zero. Thus, these limitations diminish the usefulness of pressure center definitions. In fact, the reader can find that direct calculations can sometimes simplify the problem.

4.5.1.2 Multiply Layers

In the previous sections, the density was assumed to be constant. For non constant density the derivations aren't "clean" but are similar. Consider straight/flat body that is under liquid with a varying density⁶. If density can be represented by average density, the force that is acting on the body is

$$F_{total} = \int_A g \rho h dA \sim \bar{\rho} \int_A g h dA \quad (4.108)$$

In cases where average density cannot be represented reasonably⁷, the integral has to be carried out. In cases where density is non-continuous, but constant in segments, the following can be said

$$F_{total} = \int_A g \rho h dA = \int_{A_1} g \rho_1 h dA + \int_{A_2} g \rho_2 h dA + \dots + \int_{A_n} g \rho_n h dA \quad (4.109)$$

As before for single density, the following can be written

$$F_{total} = g \sin \beta \left[\rho_1 \int_{A_1}^{\overbrace{x_{c1} A_1}} \xi dA + \rho_2 \int_{A_2}^{\overbrace{x_{c2} A_2}} \xi dA + \dots + \rho_n \int_{A_n}^{\overbrace{x_{cn} A_n}} \xi dA \right] \quad (4.110)$$

Or in a compact form and in addition considering the "atmospheric" pressure can be written as

$$F_{total} = P_{atmos} A_{total} + g \sin \beta \sum_{i=1}^n \rho_i x_{ci} A_i \quad (4.111)$$

where the density, ρ_i is the density of the layer i and A_i and x_{ci} are geometrical properties of the area which is in contact with that layer. The atmospheric pressure can be entered into the calculation in the same way as before. Moreover, the atmospheric pressure can include all the layer(s) that do(es) not with the "contact" area.

The moment around axis y , M_y under the same considerations as before is

$$M_y = \int_A g \rho \xi^2 \sin \beta dA \quad (4.112)$$

⁶This statement also means that density is a monotonous function. Why? Because of the buoyancy issue. It also means that the density can be a non-continuous function.

⁷A qualitative discussion on what is reasonably is not presented here. However, if the variation of the density is within 10% and/or the accuracy of the calculation is minimal, the reasonable average can be used.

After similar separation of the total integral, one can find that

$$M_y = g \sin \beta \sum_{i=1}^n \rho_i I_{x'x'_i} \tag{4.113}$$

If the atmospheric pressure also enters into the calculations one can find that

$$M_y = P_{atmos} x_c A_{total} + g \sin \beta \sum_{i=1}^n \rho_i I_{x'x'_i} \tag{4.114}$$

In the same fashion one can obtain the moment for x axis as

$$M_x = P_{atmos} y_c A_{total} + g \sin \beta \sum_{i=1}^n \rho_i I_{x'y'_i} \tag{4.115}$$

To illustrate how to work with these equations the following example is provided.

Example 4.6:

Consider the hypothetical Figure 4.18. The last layer is made of water with density of $1000[kg/m^3]$. The densities are $\rho_1 = 500[kg/m^3]$, $\rho_2 = 800[kg/m^3]$, $\rho_3 = 850[kg/m^3]$, and $\rho_4 = 1000[kg/m^3]$. Calculate the forces at points a_1 and b_1 . Assume that the layers are stables without any movement between the liquids. Also neglect all mass transfer phenomena that may occur. The heights are: $h_1 = 1[m]$, $h_2 = 2[m]$, $h_3 = 3[m]$, and $h_4 = 4[m]$. The forces distances are $a_1 = 1.5[m]$, $a_2 = 1.75[m]$, and $b_1 = 4.5[m]$. The angle of inclination is $\beta = 45^\circ$.

SOLUTION

Since there are only two unknowns, only two equations are needed, which are (4.114) and (4.111). The solution method of this example is applied for cases with less layers (for example by setting the specific height difference to be zero). Equation (4.114) can be used by modifying it, as can be noticed that instead of using the regular atmospheric pressure the new "atmospheric" pressure can be used as

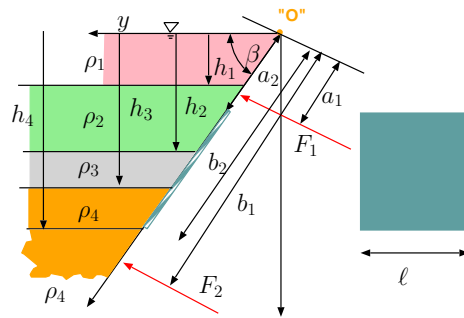


Fig. -4.18: The effects of multi layers density on static forces.

$$P_{atmos}' = P_{atmos} + \rho_1 g h_1$$

The distance for the center for each area is at the middle of each of the “small” rectangular. The geometries of each areas are

$$\begin{aligned} x_{c1} &= \frac{a_2 + \frac{h_2}{\sin \beta}}{2} & A_1 &= \ell \left(\frac{h_2}{\sin \beta} - a_2 \right) & I_{x'x'_1} &= \frac{\ell \left(\frac{h_2}{\sin \beta} - a_2 \right)^3}{36} + (x_{c1})^2 A_1 \\ x_{c2} &= \frac{h_2 + h_3}{2 \sin \beta} & A_2 &= \frac{\ell}{\sin \beta} (h_3 - h_2) & I_{x'x'_2} &= \frac{\ell (h_3 - h_2)^3}{36 \sin \beta} + (x_{c2})^2 A_2 \\ x_{c3} &= \frac{h_3 + h_4}{2 \sin \beta} & A_3 &= \frac{\ell}{\sin \beta} (h_4 - h_3) & I_{x'x'_3} &= \frac{\ell (h_4 - h_3)^3}{36 \sin \beta} + (x_{c3})^2 A_3 \end{aligned}$$

After inserting the values, the following equations are obtained

Thus, the first equation is

$$F_1 + F_2 = P_{atmos} \overbrace{\ell (b_2 - a_2)}^{A_{total}} + g \sin \beta \sum_{i=1}^3 \rho_{i+1} x_{c_i} A_i$$

The second equation is (4.114) to be written for the moment around the point “O” as

$$F_1 a_1 + F_2 b_1 = P_{atmos} \overbrace{\frac{(b_2 + a_2)}{2} \ell (b_2 - a_2)}^{x_c A_{total}} + g \sin \beta \sum_{i=1}^3 \rho_{i+1} I_{x'x'_i}$$

The solution for the above equation is

$$\begin{aligned} F_1 &= \frac{2 b_1 g \sin \beta \sum_{i=1}^3 \rho_{i+1} x_{c_i} A_i - 2 g \sin \beta \sum_{i=1}^3 \rho_{i+1} I_{x'x'_i}}{2 b_1 - 2 a_1} \\ &= \frac{(b_2^2 - 2 b_1 b_2 + 2 a_2 b_1 - a_2^2) \ell P_{atmos}}{2 b_1 - 2 a_1} \\ F_2 &= \frac{2 g \sin \beta \sum_{i=1}^3 \rho_{i+1} I_{x'x'_i} - 2 a_1 g \sin \beta \sum_{i=1}^3 \rho_{i+1} x_{c_i} A_i}{2 b_1 - 2 a_1} + \\ &= \frac{(b_2^2 + 2 a_1 b_2 + a_2^2 - 2 a_1 a_2) \ell P_{atmos}}{2 b_1 - 2 a_1} \end{aligned}$$

The solution isn't provided in the complete long form since it makes things messy. It is simpler to compute the terms separately. A mini source code for the calculations is provided in the the text source. The intermediate results in SI units ([m], [m²], [m⁴]) are:

$$\begin{aligned} x_{c1} &= 2.2892 & x_{c2} &= 3.5355 & x_{c3} &= 4.9497 \\ A_1 &= 2.696 & A_2 &= 3.535 & A_3 &= 3.535 \\ I_{x'x'_1} &= 14.215 & I_{x'x'_2} &= 44.292 & I_{x'x'_3} &= 86.718 \end{aligned}$$

The final answer is

$$F_1 = 304809.79[N]$$

and

$$F_2 = 958923.92[N]$$

4.5.2 Force on Curved Surfaces

The pressure is acting on surfaces perpendicular to the direction of the surface (no shear forces assumption). At this stage, the pressure is treated as a scalar function. The element force is

$$d\mathbf{F} = -P \hat{n} d\mathbf{A} \quad (4.116)$$

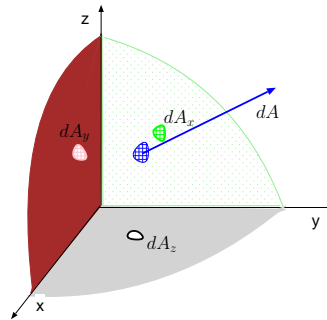


Fig. -4.19: The forces on curved area.

Here, the conventional notation is used which is to denote the area, dA , outward as positive. The total force on the area will be the integral of the unit force

$$\mathbf{F} = - \int_A P \hat{n} d\mathbf{A} \quad (4.117)$$

The result of the integral is a vector. So, if the y component of the force is needed, only a dot product is needed as

$$dF_y = d\mathbf{F} \cdot \hat{j} \quad (4.118)$$

From this analysis (equation (4.118)) it can be observed that the force in the direction of y , for example, is simply the integral of the area perpendicular to y as

$$F_y = \int_A P dA_y \quad (4.119)$$

The same can be said for the x direction. The force in the z direction is

$$F_z = \int_A h g \rho dA_z \quad (4.120)$$

The force which acting on the z direction is the weight of the liquid above the projected area plus the atmospheric pressure. This force component can be combined with the other components in the other directions to be

$$F_{total} = \sqrt{F_z^2 + F_x^2 + F_y^2} \quad (4.121)$$

And the angle in “ xz ” plane is

$$\tan \theta_{xz} = \frac{F_z}{F_x} \quad (4.122)$$

and the angle in the other plane, “ $y z$ ” is

$$\tan \theta_{zy} = \frac{F_z}{F_y} \quad (4.123)$$

The moment due to the curved surface require integration to obtain the value. There are no readily made expressions for these 3–dimensional geometries. However, for some geometries there are readily calculated center of mass and when combined with two other components provide the moment (force with direction line).

Cut–Out Shapes Effects

There are bodies with a shape that the vertical direction (z direction) is “cut–out” aren’t continuous. Equation (4.120) implicitly means that the net force on the body in z direction is only the actual liquid above it. For example, Figure 4.20 shows a floating body with cut–out slot into it. The atmospheric pressure acts on the area with continuous lines. Inside the slot, the atmospheric pressure with its piezometric pressure is canceled by the upper part of the slot. Thus, only the net force is the actual liquid in the slot which is acting on the body. Additional point that is worth mentioning is that the depth where the cut–out occur is insignificant (neglecting the change in the density).

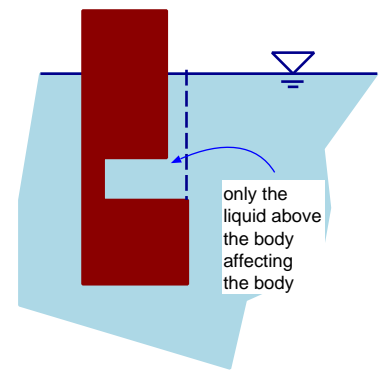


Fig. -4.20: Schematic of Net Force on floating body.

Example 4.7:

Calculate the force and the moment around point “O” that is acting on the dam (see Figure (4.21)). The dam is made of an arc with the angle of $\theta_0 = 45^\circ$ and radius of $r = 2[m]$. You can assume that the liquid density is constant and equal to $1000 [kg/m^3]$. The gravity is $9.8[m/sec^2]$ and width of the dam is $b = 4[m]$. Compare the different methods of computations, direct and indirect.

SOLUTION

The force in the x direction is

$$F_x = \int_A P \overbrace{r \cos \theta}^{dA_x} \quad (4.124)$$

Note that the direction of the area is taken into account (sign). The differential area that will be used is, $b r d\theta$ where b is the width of the dam (into the page). The

pressure is only a function of θ and it is

$$P = P_{atmos} + \rho g r \sin \theta$$

The force that is acting on the x direction of the dam is $A_x \times P$. When the area A_x is $b r d\theta \cos \theta$. The atmospheric pressure does cancel itself (at least if the atmospheric pressure on both sides of the dam is the same.). The net force will be

$$F_x = \int_0^{\theta_0} \overbrace{\rho g r \sin \theta}^P \overbrace{b r \cos \theta d\theta}^{dA_x}$$

The integration results in

$$F_x = \frac{\rho g b r^2}{2} (1 - \cos^2(\theta_0))$$

Another way to do this calculation is by calculating the pressure at mid point and then multiply it by the projected area, A_x (see Figure 4.22) as

$$F_x = \rho g \overbrace{b r \sin \theta_0}^{A_x} \overbrace{\frac{r \sin \theta_0}{2}}^{x_c} = \frac{\rho g b r}{2} \sin^2 \theta$$

Notice that $dA_x(\cos \theta)$ and $A_x(\sin \theta)$ are different, why?

The values to evaluate the last equation are provided in the question and simplify substitute into it as

$$F_x = \frac{1000 \times 9.8 \times 4 \times 2}{2} \sin(45^\circ) = 19600.0[N]$$

Since the last two equations are identical (use the sinuous theorem to prove it $\sin^2 \theta + \cos^2 = 1$), clearly the discussion earlier was right (not a good proof LOL⁸). The force in the y direction is the area times width.

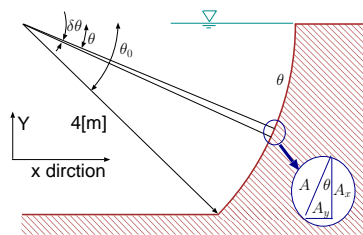


Fig. -4.21: Dam is a part of a circular shape.

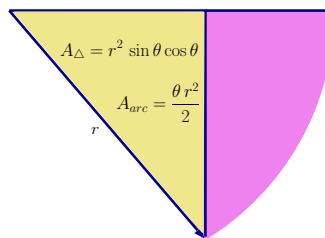


Fig. -4.22: Area above the dam arc subtract triangle.

$$F_y = - \left(\overbrace{\frac{\theta_0 r^2}{2} - \frac{r^2 \sin \theta_0 \cos \theta_0}{2}}^A \right) b g \rho \sim 22375.216[N]$$

⁸Well, just a demonstration!

The center area (purple area in Figure 4.22) should be calculated as

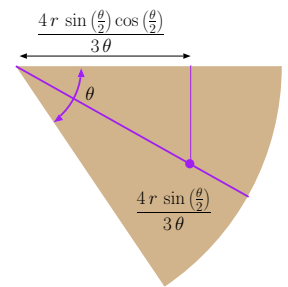
$$y_c = \frac{y_c A_{arc} - y_c A_{triangle}}{A}$$

The center area above the dam requires to know the center area of the arc and triangle shapes. Some mathematics are required because the shift in the arc orientation. The arc center (see Figure 4.23) is at

$$y_{c_{arc}} = \frac{4 r \sin^2 \left(\frac{\theta}{2} \right)}{3 \theta}$$

All the other geometrical values are obtained from Tables 3.1 and 3.2. and substituting the proper values results in

$$y_{c_r} = \frac{\frac{A_{arc}}{\theta r^2} \overbrace{4 r \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)}^{y_c} - \frac{y_c}{3} \overbrace{\frac{A_{triangle}}{\sin \theta r^2}}^{r^2 \sin \theta \cos \theta}}{\frac{2}{\theta r^2} - \frac{2}{r^2 \sin \theta \cos \theta}}$$



This value is the reverse value and it is

$$y_{c_r} = 1.65174[m]$$

Fig. -4.23: Area above the dam arc calculation for the center.

The result of the arc center from point "O" (above calculation area) is

$$y_c = r - y_{c_r} = 2 - 1.65174 \sim 0.348[m]$$

The moment is

$$M_v = y_c F_y \sim 0.348 \times 22375.2 \sim 7792.31759[N \times m]$$

The center pressure for x area is

$$x_p = x_c + \frac{I_{xx}}{x_c A} = \frac{r \cos \theta_0}{2} + \frac{\frac{I_{xx}}{36} (r \cos \theta_0)^3}{\frac{2}{x_c} (r \cos \theta_0)} = \frac{5 r \cos \theta_0}{9}$$

The moment due to hydrostatic pressure is

$$M_h = x_p F_x = \frac{5 r \cos \theta_0}{9} F_x \sim 15399.21[N \times m]$$

The total moment is the combination of the two and it is

$$M_{total} = 23191.5[N \times m]$$

For direct integration of the moment is done as following

$$dF = P dA = \int_0^{\theta_0} \rho g \sin \theta b r d\theta$$

and element moment is

$$dM = dF \times \ell = dF \overbrace{2r \sin\left(\frac{\theta}{2}\right)}^{\ell} \cos\left(\frac{\theta}{2}\right)$$

and the total moment is

$$M = \int_0^{\theta_0} dM$$

or

$$M = \int_0^{\theta_0} \rho g \sin \theta b r 2r \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta$$

The solution of the last equation is

$$M = \frac{g r \rho (2\theta_0 - \sin(2\theta_0))}{4}$$

The vertical force can be obtained by

$$F_v = \int_0^{\theta_0} P dA_v$$

or

$$F_v = \int_0^{\theta_0} \overbrace{\rho g r \sin \theta}^P \overbrace{r d\theta \cos \theta}^{dA_v} = \frac{g r^2 \rho}{2} (1 - \cos(\theta_0)^2)$$

Here, the traditional approach was presented first, and the direct approach was presented second. It is much simpler now to use the second method. In fact, there are many programs or hand held devices that can carry numerical integration by inserting the function and the boundaries.

To demonstrate this point further, consider a more general case of a polynomial function. The reason that a polynomial function was chosen is that almost all the continuous functions can be represented by a Taylor series, and thus, this example provides for practical purposes of the general solution for curved surfaces.

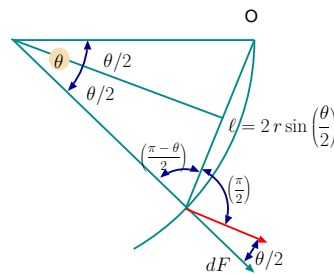


Fig. -4.24: Moment on arc element around Point "O."

Example 4.8:

For the liquid shown in Figure 4.25, calculate the moment around point “O” and the force created by the liquid per unit depth. The function of the dam shape is $y = \sum_{i=1}^n a_i x^i$ and it is a monotonous function (this restriction can be relaxed somewhat). Also calculate the horizontal and vertical forces.

SOLUTION

The calculations are done per unit depth (into the page) and do not require the actual depth of the dam.

The element force (see Figure 4.25) in this case is

$$dF = \underbrace{h}_P \underbrace{g \rho}_{dA} \sqrt{dx^2 + dy^2}$$

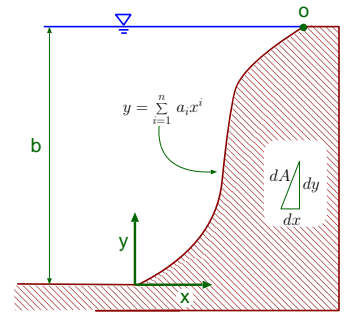


Fig. -4.25: Polynomial shape dam description for the moment around point “O” and force calculations.

The size of the differential area is the square root of the dx^2 and dy^2 (see Figure 4.25). It can be noticed that the differential area that is used here should be multiplied by the depth. From mathematics, it can be shown that

$$\sqrt{dx^2 + dy^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The right side can be evaluated for any given function. For example, in this case describing the dam function is

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\sum_{i=1}^n i a_i x^{i-1}\right)^2}$$

The value of x_b is where $y = b$ and can be obtained by finding the first and positive root of the equation of

$$0 = \sum_{i=1}^n a_i x^i - b$$

To evaluate the moment, expression of the distance and angle to point “O” are needed (see Figure 4.26).

The distance between the point on the dam at x to the point “O” is

$$\ell(x) = \sqrt{(b - y)^2 + (x_b - x)^2}$$

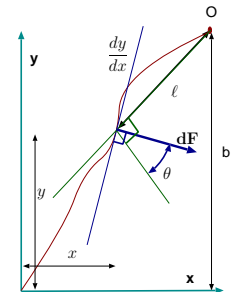


Fig. -4.26: The difference between the slop and the direction angle.

The angle between the force and the distance to point "O" is

$$\theta(x) = \tan^{-1} \left(\frac{dy}{dx} \right) - \tan^{-1} \left(\frac{b-y}{x_b-x} \right)$$

The element moment in this case is

$$dM = \overbrace{\ell(x) (b-y) g \rho \sqrt{1 + \left(\frac{dy}{dx} \right)^2}}^{dF} \cos \theta(x) dx$$

To make this example less abstract, consider the specific case of $y = 2x^6$. In this case, only one term is provided and x_b can be calculated as following

$$x_b = \sqrt[6]{\frac{b}{2}}$$

Notice that $\sqrt[6]{\frac{b}{2}}$ is measured in meters. The number "2" is a dimensional number with units of $[1/m^5]$. The derivative at x is

$$\frac{dy}{dx} = 12x^5$$

and the derivative is dimensionless (a dimensionless number). The distance is

$$\ell = \sqrt{(b - 2x^6)^2 + \left(\sqrt[6]{\frac{b}{2}} - x \right)^2}$$

The angle can be expressed as

$$\theta = \tan^{-1} (12x^5) - \tan^{-1} \left(\frac{b - 2x^6}{\sqrt[6]{\frac{b}{2}} - x} \right)$$

The total moment is

$$M = \int_0^{\sqrt[6]{b}} \ell(x) \cos \theta(x) (b - 2x^6) g \rho \sqrt{1 + 12x^5} dx$$

This integral doesn't have analytical solution. However, for a given value b this integral can be evaluate. The horizontal force is

$$F_h = b \rho g \frac{b}{2} = \frac{\rho g b^2}{2}$$

The vertical force per unit depth is the volume above the dam as

$$F_v = \int_0^{\sqrt[6]{b}} (b - 2x^6) \rho g dx = \rho g \frac{5b^{\frac{7}{6}}}{7}$$

In going over these calculations, the calculations of the center of the area were not carried out. This omission saves considerable time. In fact, trying to find the center of the area will double the work. This author find this method to be simpler for complicated geometries while the indirect method has advantage for very simple geometries.

4.6 Buoyancy and Stability

One of the oldest known scientific research on fluid mechanics relates to buoyancy due to question of money was carried by Archimedes. Archimedes principle is related to question of density and volume. While Archimedes did not know much about integrals, he was able to capture the essence. Here, because this material is presented in a different era, more advance mathematics will be used. While the question of the stability was not scientifically examined in the past, the floating vessels structure (more than 150 years ago) show some understanding⁹.

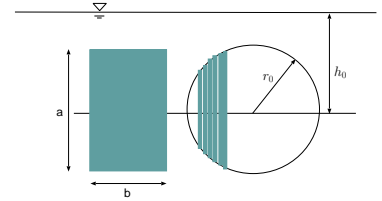


Fig. -4.27: Schematic of Immersed Cylinder.

The total forces the liquid exerts on a body are considered as a buoyancy issue. To understand this issue, consider a cubical and a cylindrical body that is immersed in liquid and center in a depth of, h_0 as shown in Figure 4.27. The force to hold the cylinder at the place must be made of integration of the pressure around the surface of the square and cylinder bodies. The forces on square geometry body are made only of vertical forces because the two sides cancel each other. However, on the vertical direction, the pressure on the two surfaces are different. On the upper surface the pressure is $\rho g(h_0 - a/2)$. On the lower surface the pressure is $\rho g(h_0 + a/2)$. The force due to the liquid pressure per unit depth (into the page) is

$$F = \rho g ((h_0 - a/2) - (h_0 + a/2)) \ell b = -\rho g a b \ell = -\rho g V \quad (4.125)$$

In this case the ℓ represents a depth (into the page). Rearranging equation (4.125) to be

$$\frac{F}{V} = \rho g \quad (4.126)$$

The force on the immersed body is equal to the weight of the displaced liquid. This analysis can be generalized by noticing two things. All the horizontal forces

⁹This topic was the author's class name in high school. It was taught by people like these, 150 years ago and more, ship builders who know how to calculate GM but weren't aware of scientific principles behind it. If the reader wonders why such a class is taught in a high school, perhaps the name can explain it: Sea Officers High School.

are canceled. Any body that has a projected area that has two sides, those will cancel each other. Another way to look at this point is by approximation. For any two rectangle bodies, the horizontal forces are canceling each other. Thus even these bodies are in contact with each other, the imaginary pressure make it so that they cancel each other.

On the other hand, any shape is made of many small rectangles. The force on every rectangular shape is made of its weight of the volume. Thus, the total force is made of the sum of all the small rectangles which is the weight of the sum of all volume.

In illustration of this concept, consider the cylindrical shape in Figure 4.27. The force per area (see Figure 4.28) is

$$dF = \overbrace{\rho g (h_0 - r \sin \theta)}^P \overbrace{\sin \theta r d\theta}^{dA_{vertical}} \quad (4.127)$$

The total force will be the integral of the equation (4.127)

$$F = \int_0^{2\pi} \rho g (h_0 - r \sin \theta) r d\theta \sin \theta \quad (4.128)$$

Rearranging equation (4.127) transforms it to

$$F = r g \rho \int_0^{2\pi} (h_0 - r \sin \theta) \sin \theta d\theta \quad (4.129)$$

The solution of equation (4.129) is

$$F = -\pi r^2 \rho g \quad (4.130)$$

The negative sign indicate that the force acting upwards. While the horizontal force is

$$F_v = \int_0^{2\pi} (h_0 - r \sin \theta) \cos \theta d\theta = 0 \quad (4.131)$$

Typical examples to explain the buoyancy are of the vessel with thin walls put upside down into liquid. The second example of the speed of the floating bodies. Since there are no better examples, these examples are a must.

Example 4.9:

A cylindrical body, shown in Figure 4.29, is floating in liquid with density, ρ_l . The body was inserted into liquid in a such a way that the air had remained in it. Express the maximum wall thickness, t , as a function of the density of the wall, ρ_s liquid density, ρ_l and the surroundings air temperature, T_1 for the body to float. In the case where thickness is half the maximum, calculate the pressure inside the container. The container diameter is w . Assume that the wall thickness is small compared with the other dimensions ($t \ll w$ and $t \ll h$).

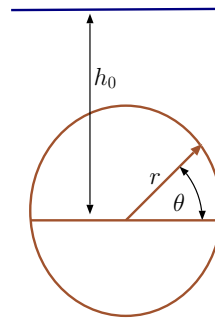


Fig. -4.28: The floating forces on Immersed Cylinder (4.130)

SOLUTION

The air mass in the container is

$$m_{air} = \underbrace{\pi w^2 h}_V \frac{\overbrace{P_{atmos}}^{\rho_{air}}}{RT}$$

The mass of the container is

$$m_{container} = \left(\underbrace{\pi w^2 + 2\pi w h}_A \right) t \rho_s$$

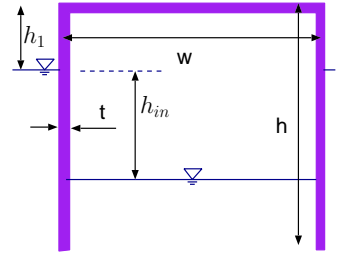


Fig. -4.29: Schematic of a thin wall floating body.

The liquid amount enters into the cavity is such that the air pressure in the cavity equals to the pressure at the interface (in the cavity). Note that for the maximum thickness, the height, h_1 has to be zero. Thus, the pressure at the interface can be written as

$$P_{in} = \rho_l g h_{in}$$

On the other hand, the pressure at the interface from the air point of view (ideal gas model) should be

$$P_{in} = \frac{m_{air} R T_1}{\underbrace{h_{in} \pi w^2}_V}$$

Since the air mass didn't change and it is known, it can be inserted into the above equation.

$$\rho_l g h_{in} + P_{atmos} = P_{in} = \frac{(\pi w^2 h) \overbrace{P_{atmos}}^{\rho} R T_1}{h_{in} \pi w^2}$$

The last equation can be simplified into

$$\rho_l g h_{in} + P_{atmos} = \frac{h P_{atmos}}{h_{in}}$$

And the solution for h_{in} is

$$h_{in} = -\frac{P_{atmos} + \sqrt{4g h P_{atmos} \rho_l + P_{atmos}^2}}{2g \rho_l}$$

and

$$h_{in} = \frac{\sqrt{4g h P_{atmos} \rho_l + P_{atmos}^2} - P_{atmos}}{2g \rho_l}$$

The solution must be positive, so that the last solution is the only physical solution.

Example 4.10:

Calculate the minimum density an infinitely long equilateral triangle (three equal sides) has to be so that the sharp end is in the water.

Extreme Cases

Advance material can be skipped

The solution demonstrates that when $h \rightarrow 0$ then $h_{in} \rightarrow 0$. When the gravity approaches zero (macro gravity) then

$$h_{in} = \frac{P_{atmos}}{\rho_l g} + h - \frac{h^2 \rho_l g}{P_{atmos}} + \frac{2 h^3 \rho_l^2 g^2}{P_{atmos}^2} - \frac{5 h^4 \rho_l^3 g^3}{P_{atmos}^3} + \dots$$

This “strange” result shows that bodies don’t float in the normal sense. When the floating is under vacuum condition, the following height can be expanded into

$$h_{in} = \sqrt{\frac{h P_{atmos}}{g \rho_l}} + \frac{P_{atmos}}{2 g \rho_l} + \dots$$

which shows that the large quantity of liquid enters into the container as it is expected. _____ End advance material _____

Archimedes theorem states that the force balance is at displaced weight liquid (of the same volume) should be the same as the container, the air. Thus,

$$\underbrace{\pi w^2 (h - h_{in}) g}_{\text{net displayed water}} = \underbrace{(\pi w^2 + 2 \pi w h) t \rho_s g}_{\text{container}} + \underbrace{\pi w^2 h \left(\frac{P_{atmos}}{R T_1} \right) g}_{\text{air}}$$

If air mass is neglected the maximum thickness is

$$t_{max} = \frac{2 g h w \rho_l + P_{atmos} w - w \sqrt{4 g h P_{atmos} \rho_l + P_{atmos}^2}}{(2 g w + 4 g h) \rho_l \rho_s}$$

The condition to have physical value for the maximum thickness is

$$2 g h \rho_l + P_{atmos} \geq \sqrt{4 g h P_{atmos} \rho_l + P_{atmos}^2}$$

The full solution is

$$t_{max} = - \frac{(w R \sqrt{4 g h P_{atmos} \rho_l + P_{atmos}^2} - 2 g h w R \rho_l - P_{atmos} w R) T_1 + 2 g h P_{atmos} w \rho_l}{(2 g w + 4 g h) R \rho_l \rho_s T_1}$$

In this analysis the air temperature in the container immediately after insertion in the liquid has different value from the final temperature. It is reasonable as the first approximation to assume that the process is adiabatic and isentropic. Thus, the temperature in the cavity immediately after the insertion is

$$\frac{T_i}{T_f} = \left(\frac{P_i}{P_f} \right)$$

The final temperature and pressure were calculated previously. The equation of state is

$$P_i = \frac{m_{air} R T_i}{V_i}$$

The new unknown must provide additional equation which is

$$V_i = \pi w^2 h_i$$

Thickness Below The Maximum

For the half thickness $t = \frac{t_{max}}{2}$ the general solution for any given thickness below maximum is presented. The thickness is known, but the liquid displacement is still unknown. The pressure at the interface (after long time) is

$$\rho_l g h_{in} + P_{atmos} = \frac{\pi w^2 h \frac{P_{atmos}}{R T_1} R T_1}{(h_{in} + h_1) \pi w^2}$$

which can be simplified to

$$\rho_l g h_{in} + P_{atmos} = \frac{h P_{atmos}}{h_{in} + h_1}$$

The second equation is Archimedes' equation, which is

$$\pi w^2 (h - h_{in} - h_1) = (\pi w^2 + 2 \pi w h) t \rho_s g + \pi w^2 h \left(\frac{P_{atmos}}{R T_1} \right) g$$

Example 4.11:

A body is pushed into the liquid to a distance, h_0 and left at rest. Calculate acceleration and time for a body to reach the surface. The body's density is $\alpha \rho_l$, where α is ratio between the body density to the liquid density and ($0 < \alpha < 1$). Is the body volume important?

SOLUTION

The net force is

$$F = \underbrace{V g \rho_l}_{\text{liquid weight}} - \underbrace{V g \alpha \rho_l}_{\text{body weight}} = V g \rho_l (1 - \alpha)$$

But on the other side the internal force is

$$F = m a = \overbrace{V \alpha \rho_l}^m a$$

Thus, the acceleration is

$$a = g \left(\frac{1 - \alpha}{\alpha} \right)$$

If the object is left at rest (no movement) thus time will be ($h = 1/2 a t^2$)

$$t = \sqrt{\frac{2h\alpha}{g(1 - \alpha)}}$$

If the object is very light ($\alpha \rightarrow 0$) then

$$t_{min} = \sqrt{\frac{2h\alpha}{g}} + \frac{\sqrt{2gh}\alpha^{\frac{3}{2}}}{2g} + \frac{3\sqrt{2gh}\alpha^{\frac{5}{2}}}{8g} + \frac{5\sqrt{2gh}\alpha^{\frac{7}{2}}}{16g} + \dots$$

From the above equation, it can be observed that only the density ratio is important. This idea can lead to experiment in “large gravity” because the acceleration can be magnified and it is much more than the reverse of free falling.

Example 4.12:

In some situations, it is desired to find equivalent of force of a certain shape to be replaced by another force of a “standard” shape. Consider the force that acts on a half sphere. Find equivalent cylinder that has the same diameter that

SOLUTION

The force act on the half sphere can be found by integrating the forces around the sphere. The element force is

$$dF = (\rho_L - \rho_S) g \underbrace{r \cos \phi}_{h} \underbrace{\cos \theta \cos \phi}_{\frac{dA_x}{dA}} \underbrace{r^2 d\theta d\phi}_{dA}$$

The total force is then

$$F_x = \int_0^\pi \int_0^\pi (\rho_L - \rho_S) g \cos^2 \phi \cos^2 \theta r^3 d\theta d\phi$$

The result of the integration the force on sphere is

$$F_s = \frac{\pi^2 (\rho_L - \rho_S) r^3}{4}$$

The force on equivalent cylinder is

$$F_c = \pi r^2 (\rho_L - \rho_S) h$$

These forces have to be equivalent and thus

$$\frac{\pi r^2 (\rho_L - \rho_S) r^3}{4} = \pi r^2 (\rho_L - \rho_S) h$$

Thus the height is

$$\frac{h}{r} = \frac{\pi}{4}$$

4.6.1 Stability

Figure 4.30 shows a body made of hollow balloon and a heavy sphere connected by a thin and light rod. This arrangement has mass centroid close to the middle of the sphere. The buoyant center is below the middle of the balloon. If this arrangement is inserted into liquid and it will be floating, the balloon will be on the top and sphere on the bottom. Tilting the body with a small angle from its resting position creates a shift in the forces direction (examine Figure 4.30b).

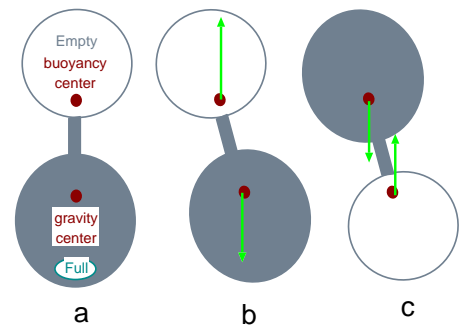


Fig. -4.30: Schematic of floating bodies.

These forces create a moment which wants to return the body to the resting (original) position. When the body is at the position shown in Figure 4.30c, the body is unstable and any tilt from the original position creates moment that will further continue to move the body from its original position. This analysis doesn't violate the second law of thermodynamics. Moving bodies from an unstable position is in essence like a potential.

A wooden cubic (made of pine, for example) is inserted into water. Part of the block floats above water line. The cubic mass (gravity) centroid is in the middle of the cubic. However the buoyant center is the middle of the volume under the water (see Figure 4.31). This situation is similar to Figure 4.30c. However, any experiment of this cubic wood shows that it is stable locally. Small amount of tilting of the cubic results in returning to the original position.

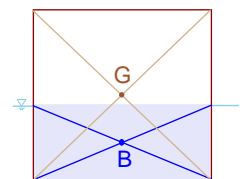


Fig. -4.31: Schematic of floating cubic.

When tilting a larger amount than $\pi/4$, it results in a flipping into the next stable

position. The cubic is stable in six positions (every cubic has six faces). In fact, in any of these six positions, the body is in situation like in 4.30c. The reason for this local stability of the cubic is that other positions are less stable. If one draws the stability (later about this criterion) as a function of the rotation angle will show a sinusoidal function with four picks in a whole rotation.

So, the body stability must be based on the difference between the body's local positions rather than the "absolute" stability. That is, the body is "stable" in some points more than others in their vicinity. These points are raised from the buoyant force analysis. When the body is tilted at a small angle, β , the immersed part of the body center changes to a new location, \mathbf{B}' as shown in Figure 4.32. The center of the mass (gravity) is still in the old location since the body did not change. The stability of the body is divided into three categories.

If the new immerse volume creates a new center in such way that the couple forces (gravity and buoyancy) try to return the body, the original state is referred as the stable body and vice versa. The third state is when the couple forces do have zero moment, it is referred to as the neutral stable.

The body, shown in Figure 4.32, when given a tilted position, move to a new buoyant center, \mathbf{B}' . This deviation of the buoyant center from the old buoyant center location, \mathbf{B} , should to be calculated. This analysis is based on the difference of the displaced liquid. The right green area (volume) in Figure 4.32 is displaced by the same area (really the volume) on left since the weight of the body didn't change¹⁰ so the total immersed section is constant. For small angle, β , the moment is calculated as the integration of the small force shown in the Figure 4.32 as ΔF . The displacement of the buoyant center can be calculated by examining the moment these forces are creating. The body weight creates opposite moment to balance the moment of the displaced liquid volume.

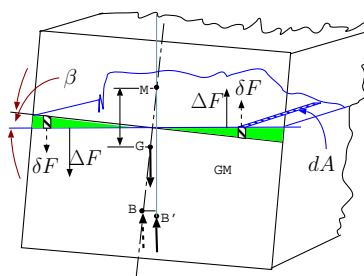


Fig. -4.32: Stability analysis of floating body.

$$\overline{BB'} W = M \tag{4.132}$$

Where M is the moment created by the displaced areas (volumes), $\overline{BB'}$ is the distance between points \mathbf{B} and point \mathbf{B}' , and, W referred to the weight of the body. It can be noticed that the distance $\overline{BB'}$ is an approximation for small angles (neglecting the vertical component.). So the perpendicular distance, $\overline{BB'}$, should be

$$\overline{BB'} = \frac{M}{W} \tag{4.133}$$

¹⁰It is correct to state: area only when the body is symmetrical. However, when the body is not symmetrical, the analysis is still correct because the volume and not the area is used.

The moment M can be calculated as

$$M = \int_A \underbrace{g \rho_l x \beta dA}_{\delta F} = g \rho_l \beta \int_A x^2 dA \quad (4.134)$$

The integral in the right side of equation (4.134) is referred to as the area moment of inertia and was discussed in Chapter 3. The distance, $\overline{BB'}$ can be written from equation (4.134) as

$$\overline{BB'} = \frac{g \rho_l I_{xx}}{\rho_s V_{body}} \quad (4.135)$$

The point where the gravity force direction is intersecting with the center line of the cross section is referred as metacentric point, \mathbf{M} . The location of the metacentric point can be obtained from the geometry as

$$\overline{BM} = \frac{\overline{BB'}}{\sin \beta} \quad (4.136)$$

And combining equations (4.135) with (4.136) yields

$$\overline{BM} = \frac{g \rho_l \beta I_{xx}}{g \rho_s \sin \beta V_{body}} = \frac{\rho_l I_{xx}}{\rho_s V_{body}} \quad (4.137)$$

For small angle ($\beta \sim 0$)

$$\lim_{\beta \rightarrow 0} \frac{\sin \beta}{\beta} \sim 1 \quad (4.138)$$

It is remarkable that the results is independent of the angle. Looking at Figure 4.32 ,the geometrical quantities can be related as

$$\overline{GM} = \frac{\overline{BM}}{\rho_s V_{body}} - \overline{BG} \quad (4.139)$$

To understand these principles consider the following examples.

Example 4.13:

A solid block of wood of uniform density, $\rho_s = \alpha \rho_l$ where ($0 \leq \alpha \leq 1$) is floating in a liquid. Construct a graph that shows the relationship of the \overline{GM} as a function of ratio height to width. Show that the block's length, L , is insignificant for this analysis.

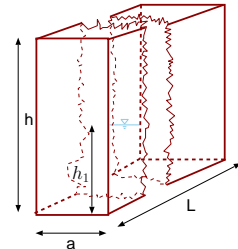


Fig. -4.33: Cubic body dimensions for stability analysis.

SOLUTION

Equation (4.139) requires that several quantities should be expressed. The moment of inertia for a block is given in Table 3.1 and is $I_{xx} = \frac{La^3}{12}$. Where L is the length into the page. The distance \overline{BG} obtained from Archimedes' theorem and can be expressed as

$$W = \rho_s \overbrace{a h L}^V = \rho_l \overbrace{a h_1 L}^{\text{immersed volume}} \implies h_1 = \frac{\rho_s}{\rho_l} h$$

Thus, the distance \overline{BG} is (see Figure 4.31)

$$\overline{BG} = \frac{h}{2} - \overbrace{\frac{\rho_s}{\rho_l} h}^{h_1} \frac{1}{2} = \frac{h}{2} \left(1 - \frac{\rho_s}{\rho_l} \right) \tag{4.140}$$

$$GM = \frac{\cancel{g} \rho_l \overbrace{\frac{L a^3}{12}}^{I_{xx}}}{\cancel{g} \rho_s \underbrace{a h L}_V} - \frac{h}{2} \left(1 - \frac{\rho_s}{\rho_l} \right)$$

Simplifying the above equation provides

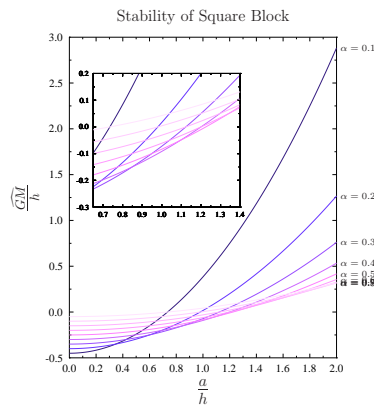
$$\frac{\overline{GM}}{h} = \frac{1}{12\alpha} \left(\frac{a}{h} \right)^2 - \frac{1}{2} (1 - \alpha) \tag{4.141}$$

where α is the density ratio. Notice that \overline{GM}/h isn't a function of the depth, L . This equation leads to the condition where the maximum height above which the body is not stable any more as

$$\frac{a}{h} \geq \sqrt{6(1 - \alpha)\alpha} \tag{4.142}$$

One of the interesting point that the above analysis was that there is a point above which the ratio of the height to the body width is not stable any more. In cylindrical shape equivalent equation to equation (4.142) can be expressed. For cylinder (circle) the moment of inertia is $I_{xx} = \pi b^4/64$. The distance \overline{BG} is the same as for the square shape (cubic) (see above (4.140)). Thus, the equation is

$$\frac{\overline{GM}}{h} = \frac{g}{64\alpha} \left(\frac{b}{h} \right)^2 - \frac{1}{2} (1 - \alpha)$$



April 16, 2008

Fig. -4.34: Stability of cubic body infinity long.

And the condition for maximum height for stability is

$$\frac{b}{h} \geq \sqrt{32(1-\alpha)\alpha}$$

This kind of analysis can be carried for different shapes and the results are shown for these two shapes in Figure 4.35.

It can be noticed that the square body is more stable than the circular shape body.

Principle Main Axes

Any body has infinite number of different axes around which moment of inertia can be calculated. For each of these axes there is a different moment of inertia. With the exception of the circular shape, every geometrical shape has an axis in which the moment of inertia is without the product of inertia. This axis is where the main rotation of the body will occur. Some analysis of floating bodies are done by breaking the rotation of arbitrary axis to rotate around the two main axes. For stability analysis, it is enough to find if the body is stable around the smallest moment of inertia. For example, a square shape body has larger moment of inertia around diagonal. The difference between the previous calculation and the moment of inertia around the diagonal is

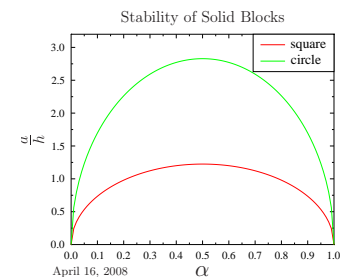


Fig. -4.35: The maximum height reverse as a function of density ratio.

$$\Delta I_{xx} = \frac{\overbrace{\sqrt{2} a \left(\frac{\sqrt{3} a}{2}\right)^3}^{I \text{ diagonal axis}}}{6} - \frac{\underbrace{a^4}_{\text{"normal" axis}}}{12} \sim 0.07 a^4$$

Which show that if the body is stable at main axes, it must be stable at the "diagonal" axis. Thus, this problem is reduced to find the stability for principle axis.

Unstable Bodies

What happen when one increases the height ratio above the maximum height ratio? The body will flip into the side and turn to the next stable point (angle). This is not a hypothetical question, but rather practical. This happens when a ship is overloaded with containers above the maximum height. In commercial ships, the fuel is stored at the bottom of the ship and thus the mass center (point G) is changing during the voyage. So, the ship that was stable (positive GM) leaving the initial port might became unstable (negative GM) before reaching the destination port.

Example 4.14:

In order to make the ship construction to be a hydrodynamic body is to make the

body as narrow as possible. What is the minimum width of the two opposite sides triangular attached to each other see Figure ???. What is the minimum ratio width of the triangle to have a stable body up to $a/h = 1$.

SOLUTION

under construction

4.6.1.1 Stability of Body with Shifting Mass Centroid

Ships and other floating bodies carry liquid or have a load which changes the mass location during tilting of the floating body. For example, a ship that carries wheat grains where the cargo is not properly secured to the ship. The movement of the load (grains, furniture, and/or liquid) does not occur in the same speed as the body itself or the displaced outside liquid. Sometimes, the slow reaction of the load, for stability analysis, is enough to be ignored. Exact analysis requires taking into account these shifting mass speeds. However, here, the extreme case where the load reacts in the same speed as the tilting of the ship/floating body is examined. For practical purposes, it is used as a limit for the stability analysis. There are situations where the real case approaches to this extreme. These situations involve liquid with a low viscosity (like water, alcohol) and ship with low natural frequency (later on the frequency of the ships). Moreover, in this analysis, the dynamics are ignored and only the statics is examined (see Figure 4.36).

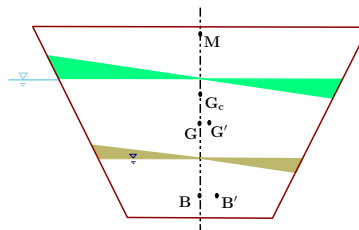


Fig. -4.36: The effects of liquid movement on the \overline{GM} .

A body is loaded with liquid “B” and is floating in a liquid “A” as shown in Figure 4.36. When the body is given a tilting position the body displaces the liquid on the outside. At the same time, the liquid inside is changing its mass centroid. The moment created by the inside displaced liquid is

$$M_{in} = g \rho_{lB} \beta I_{xxB} \tag{4.143}$$

Note that I_{xxB} isn't the same as the moment of inertia of the outside liquid interface.

The change in the mass centroid of the liquid "A" then is

$$\overline{G_1 G'_1} = \frac{\int \rho_B \beta I_{xxB}}{\underbrace{\int V_B \rho_B}_{\substack{\text{Inside} \\ \text{liquid} \\ \text{weight}}}} = \frac{I_{xxB}}{V_B} \quad (4.144)$$

Equation (4.144) shows that $\overline{GG'}$ is only a function of the geometry. This quantity, $\overline{G_1 G'_1}$, is similar for all liquid tanks on the floating body.

The total change of the vessel is then calculated similarly to center area calculations.

$$\int m_{total} \overline{GG'} = \int m_{body} \overline{GG'} + \int m_f \overline{G_1 G'_1} \quad (4.145)$$

For more than one tank, it can be written as

$$\overline{GG'} = \frac{g}{W_{total}} \sum_{i=1}^n \overline{G_i G'_i} \rho_i V_i = \frac{g}{W_{total}} \sum_{i=1}^n \frac{I_{xxb_i}}{V_{b_i}} \quad (4.146)$$

A new point can be defined as G_c . This point is the intersection of the center line with the vertical line form G' .

$$\overline{G G_c} = \frac{\overline{GG'}}{\sin \beta} \quad (4.147)$$

The distance that was used before \overline{GM} is replaced by the criterion for stability by $\overline{G_c M}$ and is expressed as

$$\overline{G_c M} = \frac{g \rho_A I_{xxA}}{\rho_s V_{body}} - \overline{BG} - \frac{1}{m_{total}} \frac{I_{xxb}}{V_b} \quad (4.148)$$

If there are more than one tank partially filled with liquid, the general formula is

$$\overline{G_c M} = \frac{g \rho_A I_{xxA}}{\rho_s V_{body}} - \overline{BG} - \frac{1}{m_{total}} \sum_{i=1}^n \frac{I_{xxb_i}}{V_{b_i}} \quad (4.149)$$

One way to reduce the effect of the moving mass center due to liquid is done by substituting a single tank with several tanks. The moment of inertial of the combine two tanks is smaller than the moment of inertial of a single tank. Increasing the number of tanks reduces the moment of inertia. The engineer could design the tanks in such a way that the moment of inertia is operationally changed. This control of the stability, \overline{GM} , can be achieved by having some tanks spanning across the entire body with tanks spanning on parts of the body. Movement of the liquid (mostly the fuel and water) provides way to control the stability, GM , of the ship.

4.6.1.2 Metacentric Height, \overline{GM} , Measurement

The metacentric height can be measured by finding the change in the angle when a weight is moved on the floating body.

Moving the weight, T a distance, d then the moment created is

$$M_{weight} = T d \quad (4.150)$$

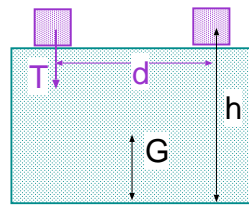


Fig. -4.37: Measurement of GM of floating body.

This moment is balanced by

$$M_{righting} = W_{total} \overline{GM}_{new} \theta \quad (4.151)$$

Where, W_{total} , is the total weight of the floating body including measuring weight. The angle, θ , is measured as the difference in the orientation of the floating body. The metacentric height is

$$\overline{GM}_{new} = \frac{T d}{W_{total} \theta} \quad (4.152)$$

If the change in the \overline{GM} can be neglected, equation (4.152) provides the solution. The calculation of \overline{GM} can be improved by taking into account the effect of the measuring weight. The change in height of G is

$$g m_{total} G_{new} = g m_{ship} G_{actual} + g T h \quad (4.153)$$

Combining equation (4.153) with equation (4.152) results in

$$\overline{GM}_{actual} = \overline{GM}_{new} \frac{m_{total}}{m_{ship}} - h \frac{T}{m_{ship}} \quad (4.154)$$

The weight of the ship is obtained from looking at the ship depth.

4.6.1.3 Stability of Submerged Bodies

The analysis of submerged bodied is different from the stability when the body lay between two fluid layers with different density. When the body is submerged in a single fluid layer, then none of the changes of buoyant centroid occurs. Thus, the mass centroid must be below the buoyant centroid in order to have stable condition.

However, all fluids have density varied in some degree. In cases where the density changes significantly, it must be taken into account. For an example of such a case is an object floating in a solar pond where the upper layer is made of water with lower salinity than the bottom layer(change up to 20% of the density). When the floating object is immersed in the two layers, the stability analysis must

take into account the changes of the displaced liquids of the two liquid layers. The calculations for such cases are a bit more complicated but based on the similar principles. Generally, this density change helps to increase the stability of the floating bodies. This analysis is out of the sscape of this book (for now).

4.6.1.4 Stability of None Systematical or “Strange” Bodies

While most floating bodies are symmetrical or semi-symmetrical, there are situations where the body has a “strange” and/or un-symmetrical body. Consider the first strange body that has an abrupt step change as shown in Figure 4.38. The body weight doesn’t change during the rotation that the green area on the left and the green area on right are the same (see Figure 4.38). There are two situations that can occur. After the tilting, the upper part of the body is above the liquid or part of the body is submerged under the water. The mathematical condition for the border is when $b = 3a$. For the case of $b < 3a$ the calculation of moment of inertia are similar to the previous case. The moment created by change in the displaced liquid (area) act in the same fashion as the before. The center of the moment is needed be found. This point is the intersection of the liquid line with the brown middle line. The moment of inertia should be calculated around this axis.

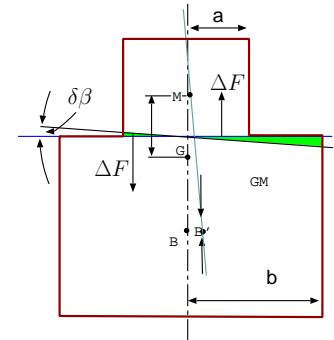


Fig. -4.38: Calculations of \overline{GM} for abrupt shape body.

For the case where $b < 3a$ some part is under the liquid. The amount of area under the liquid section depends on the tilting angle. These calculations are done as if none of the body under the liquid. This point is intersection point liquid with lower body and it is needed to be calculated. The moment of inertia is calculated around this point (note the body is “ended” at end of the upper body). However, the moment to return the body is larger than actually was calculated and the bodies tend to be more stable (also for other reasons).

4.6.1.5 Neutral frequency of Floating Bodies

This case is similar to pendulum (or mass attached to spring). The governing equation for the pendulum is

$$\ell \ddot{\beta} - g \beta = 0 \tag{4.155}$$

Where here ℓ is length of the rode (or the line/wire) connecting the mass with the rotation point. Thus, the frequency of pendulum is $\frac{1}{2\pi} \sqrt{\frac{g}{\ell}}$ which measured in Hz .

The period of the cycle is $2\pi\sqrt{\ell/g}$. Similar situation exists in the case of floating bodies. The basic differential equation is used to balance and is

$$\overbrace{I\ddot{\beta}}^{\text{rotation}} - \overbrace{V\rho_s\overline{GM}\beta}^{\text{rotating moment}} = 0 \quad (4.156)$$

In the same fashion the frequency of the floating body is

$$\frac{1}{2\pi}\sqrt{\frac{V\rho_s\overline{GM}}{I_{body}}} \quad (4.157)$$

and the period time is

$$2\pi\sqrt{\frac{I_{body}}{V\rho_s\overline{GM}}} \quad (4.158)$$

In general, the larger \overline{GM} the more stable the floating body is. Increase in \overline{GM} increases the frequency of the floating body. If the floating body is used for transport humans and/or other creatures or sensitive cargo it requires to reduce the \overline{GM} so that the traveling will be smother.

4.6.2 Surface Tension

The surface tension is one of the mathematically complex topic and related to many phenomena like boiling, coating, etc. In this section, only simplified topics like constant value will be discussed.

Example 4.15:

In interaction of the molecules shown in Figure ? describe the existence of surface tension. Explain why this description is erroneous?

SOLUTION

The upper layer of the molecules have unbalanced force towards the liquid phase. Newton's law states when there is unbalanced force, the body should be accelerate. However, in this case, the liquid is not in motion. Thus, the common explanation is wrong.

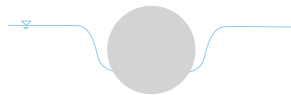


Fig. -4.39: A heavy needle is floating on a liquid.

Example 4.16:

Needle is made of steel and is heavier than water and many other liquids. However, the surface tension between the needle and the liquid hold the needle above the liquid. After certain diameter, the needle cannot be held by the liquid. Calculate the maximum diameter needle that can be inserted into liquid without drowning.

SOLUTION

Under Construction

4.7 Rayleigh–Taylor Instability

Rayleigh–Taylor instability (or RT instability) is named after Lord Rayleigh and G. I. Taylor. There are situations where a heavy liquid layer is placed over a lighter fluid layer. This situation has engineering implications in several industries. For example in die casting, liquid metal is injected in a cavity filled with air. In poor designs or other situations, some air is not evacuated and stay in small cavity on the edges of the shape to be casted. Thus, it can create a situation where the liquid metal is above the air but cannot penetrate into the cavity because of instability.

This instability deals with a dense, heavy fluid that is being placed above a lighter fluid in a gravity field perpendicular to interface. Example for such systems are dense water over oil (liquid–liquid), or water over air (gas–liquid). The original Rayleigh’s paper deals with the dynamics and density variations. For example, density variations according to the bulk modulus (see section 4.3.3.2) are always stable but unstable of the density is in the reversed order.

Supposed that a liquid density is arbitrary function of the height. This distortion can be as a result of heavy fluid above the lighter liquid. This analysis asks the question what happen when a small amount of liquid from above layer enter into the lower layer? Whether this liquid continue and will grow or will it return to its original conditions? The surface tension is the opposite mechanism that will returns the liquid to its original place. This analysis is referred to the case of infinite or very large surface. The simplified case is the two different uniform densities. For example a heavy fluid density, ρ_L , above lower fluid with lower density, ρ_G .

For perfectly straight interface, the heavy fluid will stay above the lighter fluid. If the surface will disturbed, some of heavy liquid moves down. This disturbance can grow or returned to its original situation. This condition is determined by competing forces, the surface density, and the buoyancy forces. The fluid above the depression is in equilibrium with the sounding pressure since the material is extending to infinity. Thus, the force that acting to get the above fluid down is the buoyancy force of the fluid in the depression.

The depression is returned to its original position if the surface forces are large enough. In that case, this situation is considered to be stable. On the other hand, if the surface forces

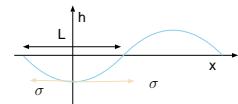


Fig. -4.40: Description of depression to explain the Rayleigh–Taylor instability.

(surface tension) are not sufficient, the situation is unstable and the heavy liquid enters into the liquid fluid zone and vice versa. As usual there is the neutral stable when the forces are equal. Any continuous function can be expanded in series of cosines. Thus, example of a cosine function will be examined. The conditions that required from this function will be required from all the other functions. The disturbance is of the following

$$h = -h_{max} \cos \frac{2\pi x}{L} \quad (4.159)$$

where h_{max} is the maximum depression and L is the characteristic length of the depression. The depression has different radius as a function of distance from the center of the depression, x . The weakest point is at $x = 0$ because symmetrical reasons the surface tension does not act against the gravity as shown in Figure (4.40). Thus, if the center point of the depression can “hold” the intrusive fluid then the whole system is stable.

The radius of any equation is expressed by equation (1.47). The first derivative of \cos around zero is \sin which is approaching zero or equal to zero. Thus, equation (1.47) can be approximated as

$$\frac{1}{R} = \frac{d^2h}{dx^2} \quad (4.160)$$

For equation (??) the radius is

$$\frac{1}{R} = -\frac{4\pi^2 h_{max}}{L^2} \quad (4.161)$$

According to equation (1.38) the pressure difference or the pressure jump is due to the surface tension at this point must be

$$P_H - P_L = \frac{4 h_{max} \sigma \pi^2}{L^2} \quad (4.162)$$

The pressure difference due to the gravity at the edge of the disturbance is then

$$P_H - P_L = g (\rho_H - \rho_L) h_{max} \quad (4.163)$$

Comparing equations (4.162) and (4.163) show that if the relationship is

$$\frac{4\sigma\pi^2}{L^2} > g (\rho_H - \rho_L) \quad (4.164)$$

It should be noted that h_{max} is irrelevant for this analysis as it is canceled. The point where the situation is neutral stable

$$L_c = \sqrt{\frac{4\pi^2\sigma}{g(\rho_H - \rho_L)}} \quad (4.165)$$

An alternative approach to analyze this instability is suggested here. Consider the situation described in Figure 4.41. If all the heavy liquid “attempts” to move straight down, the lighter liquid will “prevent” it. The lighter liquid needs to move up at the same time but in a different place. The heavier liquid needs to move in one side and the lighter liquid in another location. In this process the heavier liquid “enter” the lighter liquid in one point and creates a depression as shown in Figure 4.41.

To analyze it, considered two control volumes bounded by the blue lines in Figure 4.41. The first control volume is made of a cylinder with a radius r and the second is the depression below it. The “extra” lines of the depression should be ignored, they are not part of the control volume. The horizontal forces around the control volume are canceling each other. At the top, the force is atmospheric pressure times the area. At the cylinder bottom, the force is $\rho g h \times A$. This acts against the gravity force which make the cylinder to be in equilibrium with its surroundings if the pressure at bottom is indeed $\rho g h$.

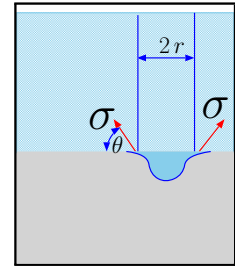


Fig. -4.41: Description of depression to explain the instability.

For the depression, the force at the top is the same force at the bottom of the cylinder. At the bottom, the force is the integral around the depression. It can be approximated as a flat cylinder that has depth of $r \pi/4$ (read the explanation in the example 4.12) This value is exact if the shape is a perfect half sphere. In reality, the error is not significant. Additionally when the depression occurs, the liquid level is reduced a bit and the lighter liquid is filling the missing portion. Thus, the force at the bottom is

$$F_{bottom} \sim \pi r^2 \left[\left(\frac{\pi r}{4} + h \right) (\rho_L - \rho_G) g + P_{atmos} \right] \quad (4.166)$$

The net force is then

$$F_{bottom} \sim \pi r^2 \left(\frac{\pi r}{4} \right) (\rho_L - \rho_G) g \quad (4.167)$$

The force that hold this column is the surface tension. As shown in Figure 4.41, the total force is then

$$F_{\sigma} = 2 \pi r \sigma \cos \theta \quad (4.168)$$

The forces balance on the depression is then

$$2 \pi r \sigma \cos \theta \sim \pi r^2 \left(\frac{\pi r}{4} \right) (\rho_L - \rho_G) g \quad (4.169)$$

The radius is obtained by

$$r \sim \sqrt{\frac{2 \pi \sigma \cos \theta}{(\rho_L - \rho_G) g}} \quad (4.170)$$

The maximum surface tension is when the angle, $\theta = \pi/2$. At that case, the radius is

$$r \sim \sqrt{\frac{2 \pi \sigma}{(\rho_L - \rho_G) g}} \quad (4.171)$$

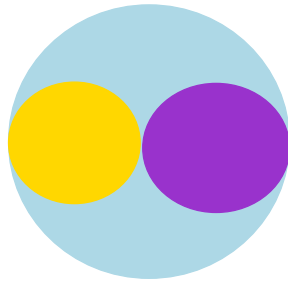


Fig. -4.42: The cross section of the interface. The purple color represents the maximum heavy liquid raising area. The yellow color represents the maximum lighter liquid that are “going down.”

The maximum possible radius of the depression depends on the geometry of the container. For the cylindrical geometry, the maximum depression radius is about half for the container radius (see Figure 4.42). This radius is limited because the lighter liquid has to enter at the same time into the heavier liquid zone. Since the “exchange” volumes of these two process are the same, the specific radius is limited. Thus, it can be written that the minimum radius is

$$r_{mintube} = 2 \sqrt{\frac{2 \pi \sigma}{g (\rho_L - \rho_G)}} \quad (4.172)$$

The actual radius will be much larger. The heavier liquid can stay on top of the lighter liquid without being turned upside down when the radius is smaller than the equation 4.172. This analysis introduces a new dimensional number that will be discussed in a greater length in the Dimensionless chapter. In equation (4.172) the angle was assumed to be 90 degrees. However, this angle is never obtained. The actual value of this angle is about $\pi/4$ to $\pi/3$ and in only extreme cases the angle exceed this value (considering dynamics). In Figure 4.42, it was shown that the depression and the raised area are the same. The actual area of the depression

is only a fraction of the interfacial cross section and is a function. For example, the depression is larger for square area. These two scenarios should be inserted into equation 4.168 by introducing an experimental coefficient.

Example 4.17:

Estimate the minimum radius to insert liquid aluminum into a representative tube at a temperature of 600[K]. Assume that the surface tension is 400[mN/m]. The density of the aluminum is 2400kg/m³.

SOLUTION

The depression radius is assumed to be significantly smaller and thus equation (4.171) can be used. The density of air is negligible as can be seen from the temperature compared to the aluminum density.

$$r \sim \sqrt{\frac{8\pi \overbrace{0.4}^{\sigma}}{2400 \times 9.81}}$$

The minimum radius is $r \sim 0.02[m]$ which demonstrates the assumption of $h \gg r$ was appropriate.

CHAPTER 5

Multi-Phase Flow

5.1 Introduction

Traditionally, the topic of multi-phase flow is ignored in an introductory class on fluid mechanics. For many engineers, this class will be the only opportunity to be exposed to this topic. The knowledge in this topic without any doubts, is required for many engineering problems. Calculations of many kinds of flow deals with more than one phase or material flow¹. The author believes that the trends and effects of multiphase flow could and should be introduced and considered by engineers. In the past, books on multiphase flow were written more as a literature review or heavy on the mathematics. It is recognized that multiphase flow is still evolving. In fact, there is not a consensus to the exact map of many flow regimes. This book attempts to describe these issues as a fundamentals of physical aspects and less as a literature review. This chapter provides information that is more or less in consensus². Additionally, the nature of multiphase flow requires solving many equations. Thus, in many books the representations is by writing the whole set governing equations. Here, it is believed that the interactions/calculations requires a full year class and hence, only the trends and simple calculations are described.

5.2 History

The study of multi-phase flow started for practical purposes after World War II. Initially the models were using simple assumptions. For simple models, there are

¹An example, there was a Ph.D. working for the government who analyzed filling cavity with liquid metal (aluminum), who did not consider the flow as two-phase flow and ignoring the air. As result, his analysis is in the twilight zone not in the real world.

²Or when the scientific principles simply dictate.

two possibilities (1) the fluids/materials are flowing in well homogeneous mixed (where the main problem to find the viscosity), (2) the fluids/materials are flowing separately where the actual total loss pressure can be correlated based on the separate pressure loss of each of the material. If the pressure loss was linear then the total loss will be the summation of the two pressure losses (of the lighter liquid (gas) and the heavy liquid). Under this assumption the total is not linear and experimental correlation was made. The flow patterns or regimes were not considered. This was suggested by Lockhart and Martinelli who use a model where the flow of the two fluids are independent of each other. They postulate that there is a relationship between the pressure loss of a single phase and combine phases pressure loss as a function of the pressure loss of the other phase. It turned out this idea provides a good crude results in some cases.

Researchers that followed Lockhart and Martinelli looked for a different map for different combination of phases. When it became apparent that specific models were needed for different situations, researchers started to look for different flow regimes and provided different models. Also the researchers looked at the situation when the different regimes are applicable. Which leads to the concept of flow regime maps. Taitle and Duckler suggested a map based on five non-dimensional groups which are considered as the most useful today. However, Taitle and Duckler's map is not universal and it is only applied to certain liquid-gas conditions. For example, Taitle-Duckler's map is not applicable for microgravity.

5.3 *What to Expect From This Chapter*

As oppose to the tradition of the other chapters in this book and all other Potto project books, a description of what to expect in this chapter is provided. It is an attempt to explain and convince all the readers that the multi-phase flow must be included in introductory class on fluid mechanics³. Hence, this chapter will explain the core concepts of the multiphase flow and their relationship, and importance to real world.

This chapter will provide: a category of combination of phases, the concept of flow regimes, multi-phase flow parameters definitions, flow parameters effects on the flow regimes, partial discussion on speed of sound of different regimes, double choking phenomenon (hopefully), and calculation of pressure drop of simple homogeneous model. This chapter will introduce these concepts so that the engineer not only be able to understand a conversation on multi-phase but also, and more importantly, will know and understand the trends. However, this chapter will not provide a discussion of transient problems, phase change or transfer processes during flow, and actual calculation of pressure of the different regimes.

³This author feels that he is in a unique position to influence many in the field of fluid mechanics. This fact is due to the sheer number of the downloaded Potto books. The number of the downloads of the book on Fundamental of compressible flow has exceed more than 100,000 in about two and half years. It also provides an opportunity to bring the latest advances in the fields since this author does not need to "sell" the book to a publisher or convince a "committee."

5.4 Kind of Multi-Phase Flow

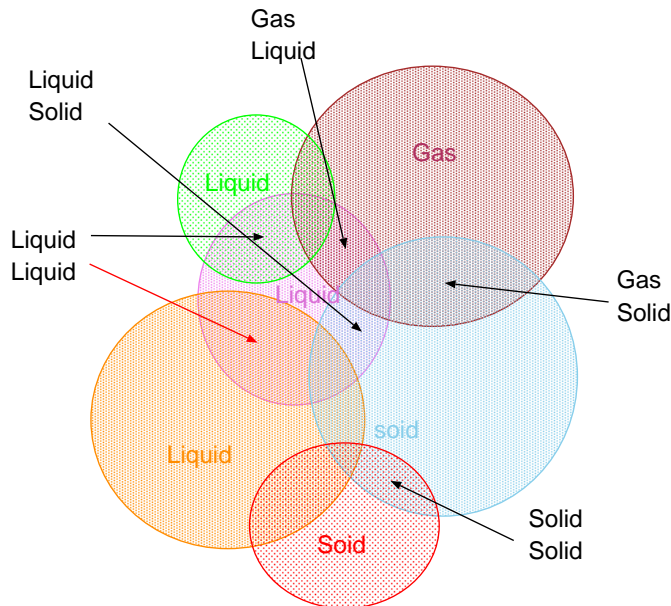


Fig. -5.1: lines a static fluid with a constant density.

All the flows are a form of multiphase flow. The discussion in the previous chapters is only as approximation when multiphase can be “reduced” into a single phase flow. For example, consider air flow that was discussed and presented earlier as a single phase flow. Air is not a pure material but a mixture of many gases. In fact, many properties of air are calculated as if the air is made of well mixed gases of Nitrogen and Oxygen. The results of the calculations of a mixture do not change much if it is assumed that the air flow as stratified flow⁴ of many concentration layers (thus, many layers (infinite) of different materials). Practically for many cases, the homogeneous assumption is enough and suitable. However, this assumption will not be appropriate when the air is stratified because of large body forces, or a large acceleration. Adopting this assumption might lead to a larger error. Hence, there are situations when air flow has to be considered as multiphase flow and this effect has to be taken into account.

In our calculation, it is assumed that air is made of only gases. The creation of clean room is a proof that air contains small particles. In almost all situations, the cleanness of the air or the fact that air is a mixture is ignored. The engineering accuracy is enough to totally ignore it. Yet, there are situations where cleanness of

⁴Different concentration of oxygen as a function of the height. While the difference of the concentration between the top to bottom is insignificant, nonetheless it exists.

the air can affect the flow. For example, the cleanness of air can reduce the speed of sound. In the past, the breaks in long trains were activated by reduction of the compressed line (a patent no. 360070 issued to George Westinghouse, Jr., March 29, 1887). In a four (4) miles long train, the breaks would started to work after about 20 seconds in the last wagon. Thus, a 10% change of the speed of sound due to dust particles in air could reduce the stopping time by 2 seconds (50 meter difference in stopping) and can cause an accident.

One way to categorize the multiphase is by the materials flows, For example, the flow of oil and water in one pipe is a multiphase flow. This flow is used by engineers to reduce the cost of moving crude oil through a long pipes system. The “average” viscosity is meaningless since in many cases the water follows around the oil. The water flow is the source of the friction. However, it is more common to categorize the flow by the distinct phases that flow in the tube. Since there are three phases, they can be solid–liquid, solid–gas, liquid–gas and solid–liquid–gas flow. This notion eliminates many other flow categories that can and should be included in multiphase flow. This category should include any distinction of phase/material. There are many more categories, for example, sand and grain (which are “solids”) flow with rocks and is referred to solid–solid flow. The category of liquid–gas should be really viewed as the extreme case of liquid–liquid where the density ratio is extremely large. The same can be said for gas–gas flow. For the gas, the density is a strong function of the temperature and pressure. Open Channel flow is, although important, is only an extreme case of liquid–gas flow and is a sub category of the multiphase flow.

The multiphase is an important part of many processes. The multiphase can be found in nature, living bodies (bio–fluids), and industries. Gas–solid can be found in sand storms, and avalanches. The body inhales solid particle with breathing air. Many industries are involved with this flow category such as dust collection, fluidized bed, solid propellant rocket, paint spray, spray casting, plasma and river flow with live creatures (small organisms to large fish) flow of ice berg, mud flow etc. The liquid–solid, in nature can be blood flow, and river flow. This flow also appears in any industrial process that are involved in solidification (for example die casting) and in moving solid particles. Liquid–liquid flow is probably the most common flow in the nature. Flow of air is actually the flow of several light liquids (gases). Many natural phenomenon are multiphase flow, for an example, rain. Many industrial process also include liquid–liquid such as painting, hydraulic with two or more kind of liquids.

5.5 *Classification of Liquid-Liquid Flow Regimes*

The general discussion on liquid–liquid will be provided and the gas–liquid flow will be discussed as a special case. Generally, there are two possibilities for two different materials to flow (it is also correct for solid–liquid and any other combination). The materials can flow in the same direction and it is referred as co–current flow. When the materials flow in the opposite direction, it is referred as counter–current.

In general, the co-current is the more common. Additionally, the counter-current flow must have special configurations of long length of flow. Generally, the counter-current flow has a limited length window of possibility in a vertical flow in conduits with the exception of magnetohydrodynamics. The flow regimes are referred to the arrangement of the fluids.

The main difference between the liquid-liquid flow to gas-liquid flow is that gas density is extremely lighter than the liquid density. For example, water and air flow as oppose to water and oil flow. The other characteristic that is different between the gas flow and the liquid flow is the variation of the density. For example, a reduction of the pressure by half will double the gas volumetric flow rate while the change in the liquid is negligible. Thus, the flow of gas-liquid can have several flow regimes in one situation while the flow of liquid-liquid will (probably) have only one flow regime.

5.5.1 Co-Current Flow

In Co-Current flow, two liquids can have three main categories: vertical, horizontal, and what ever between them. The vertical configuration has two cases, up or down. It is common to differentiate between the vertical (and near vertical) and horizontal (and near horizontal). There is no exact meaning to the word “near vertical” or “near horizontal” and there is no consensus on the limiting angles (not to mention to have limits as a function with any parameter that determine the limiting angle). The flow in inclined angle (that not covered by the word “near”) exhibits flow regimes not much different from the other two. Yet, the limits between the flow regimes are considerably different. This issue of incline flow will not be covered in this chapter.

5.5.1.1 Horizontal Flow

The typical regimes for horizontal flow are stratified flow (open channel flow, and non open channel flow), dispersed bubble flow, plug flow, and annular flow. For low velocity (low flow rate) of the two liquids, the heavy liquid flows on the bottom and lighter liquid flows on the top⁵ as depicted in

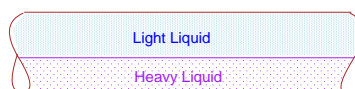


Fig. -5.2: Stratified flow in horizontal tubes when the liquids flow is very slow.

Figure 5.2. This kind of flow regime is referred to as horizontal flow. When the flow rate of the lighter liquid is almost zero, the flow is referred to as open channel flow. This definition (open channel flow) continues for small amount of lighter liquid as long as the heavier flow can be calculated as open channel flow (ignoring the lighter liquid). The geometries (even the boundaries) of open channel flow are very diverse. Open channel flow appears in many nature (river) as well in industrial

⁵With the exception of the extremely smaller diameter where Rayleigh-Taylor instability is an important issue.

process such as the die casting process where liquid metal is injected into a cylinder (tube) shape. The channel flow will be discussed in a greater detail in Open Channel Flow chapter.

As the lighter liquid (or the gas phase) flow rate increases (superficial velocity), the friction between the phases increase. The superficial velocity is referred to as the velocity that any phase will have if the other phase was not exist. This friction is one of the cause for the instability which manifested itself as waves and changing the surface from straight line to a different configuration (see Figure 5.3). The wave shape is created to keep the gas and the liquid velocity equal and at the same time to have shear stress to be balance by surface tension. The configuration of the cross section not only depend on the surface tension, and other physical properties of the fluids but also on the material of the conduit.

As the lighter liquid velocity increases two things can happen (1) wave size increase and (2) the shape of cross section continue to deform. Some referred to this regime as wavy stratified flow but this definition is not accepted by all as a category by itself. In fact, all the two phase flow are categorized by wavy flow which will proven later. There are two paths that can occur on the heavier liquid flow rate. If the heavier

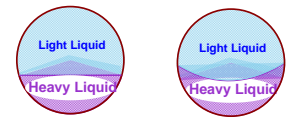


Fig. -5.3: Kind of Stratified flow in horizontal tubes.

flow rate is small, then the wave cannot reach to the crown and the shape is deformed to the point that all the heavier liquid is around the periphery. This kind of flow regime is referred to as annular flow. If the heavier liquid flow rate is larger⁶ than the distance, for the wave to reach the conduit crown is smaller. At some point, when the lighter liquid flow increases, the heavier liquid wave reaches to the crown of the pipe. At this stage, the flow pattern is referred to as slug flow or plug flow. Plug flow is characterized by regions of lighter liquid filled with drops of the heavier liquid with Plug (or Slug) of the heavier liquid (with bubble of the lighter liquid). These plugs are separated by large “chunks” that almost fill the entire tube. The plugs are flowing in a succession (see Figure 5.4). The pressure drop of this kind of regime is significantly larger than the stratified flow. The slug flow cannot be assumed to be as homogeneous flow nor it can exhibit some average viscosity. The “average” viscosity depends on the flow and thus making it as insignificant way to do the calculations. Further increase of the lighter liquid flow rate move the flow regime into annular flow. Thus, the possibility to go through slug flow regime depends on if there is enough liquid flow rate.

Choking occurs in compressible flow when the flow rate is above a certain point. All liquids are compressible to some degree. For liquid which the density is a strong and primary function of the pressure, choking oc-



Fig. -5.4: Plug flow in horizontal tubes when the liquids flow is faster slow.

⁶The liquid level is higher.

curs relatively closer/sooner. Thus, the flow that starts as a stratified flow will turn into a slug flow or stratified wavy⁷ flow after a certain distance depends on the heavy flow rate (if this category is accepted). After a certain distance, the flow becomes annular or the flow will choke. The choking can occur before the annular flow regime is obtained depending on the velocity and compressibility of the lighter liquid. Hence, as in compressible flow, liquid-liquid flow has a maximum combined of the flow rate (both phases). This maximum is known as double choking phenomenon.

The reverse way is referred to the process where the starting point is high flow rate and the flow rate is decreasing. As in many fluid mechanics and magnetic fields, the return path is not the exact same way. There is even a possibility to return on different flow regime. For example, flow that had slug flow in its path can be returned as stratified wavy flow. This phenomenon is referred to as hysteresis.

Flow that is under small angle from the horizontal will be similar to the horizontal flow. However, there is no consensus how far the “near” means. Qualitatively, the “near” angle depends on the length of the pipe. The angle decreases with the length of the pipe. Besides the length, other parameters can affect the “near.”

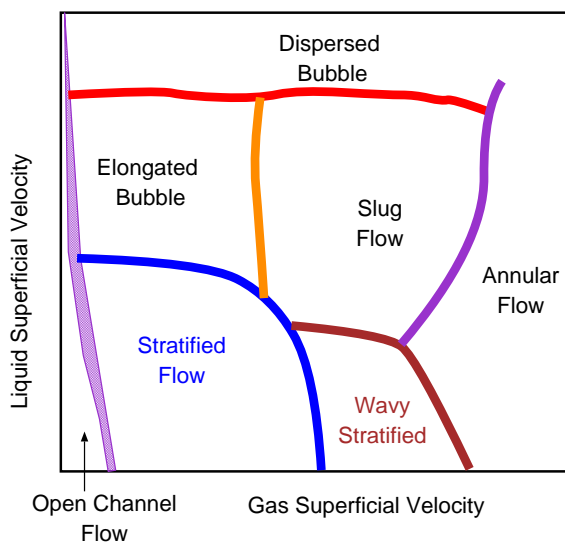


Fig. -5.5: Modified Mandhane map for flow regime in horizontal tubes.

The results of the above discussion are depicted in Figure 5.5. As many things in multiphase, this map is only characteristic of the “normal” conditions, e.g. in normal gravitation, weak to strong surface tension effects (air/water in “normal”

⁷Well, all the flow is wavy, thus it is arbitrary definition.

gravity), etc.

5.5.1.2 Vertical Flow

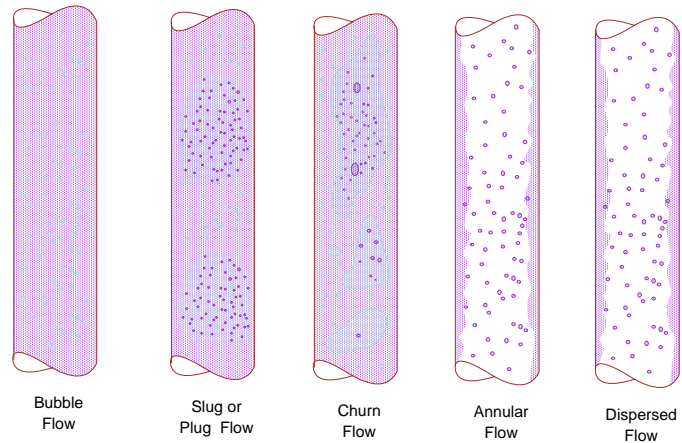


Fig. -5.6: lines a static fluid with a constant density.

The vertical flow has two possibilities, with the gravity or against it. In engineering application, the vertical flow against the gravity is more common used. There is a difference between flowing with the gravity and flowing against the gravity. The buoyancy is acting in two different directions for these two flow regimes. For the flow against gravity, the lighter liquid has a buoyancy that acts as an “extra force” to move it faster and this effect is opposite for the heavier liquid. The opposite is for the flow with gravity. Thus, there are different flow regimes for these two situations. The main reason that causes the difference is that the heavier liquid is more dominated by gravity (body forces) while the lighter liquid is dominated by the pressure driving forces.

Flow Against Gravity

For vertical flow against gravity, the flow cannot start as a stratified flow. The heavier liquid has to occupy almost the entire cross section before it can flow because of the gravity forces. Thus, the flow starts as a bubble flow. The increase of the lighter liquid flow rate will increase the number of bubbles until some bubbles start to collide. When many bubbles collide, they create a large bubble and the flow is referred to as slug flow or plug flow (see Figure 5.6). Notice, the different mechanism in creating the plug flow in horizontal flow compared to the vertical flow.

Further increase of lighter liquid flow rate will increase the slug size as more bubbles collide to create “super slug”; the flow regime is referred as elongated bubble flow. The flow is less stable as more turbulent flow and several “super

slug” or churn flow appears in more chaotic way, see Figure 5.6. After additional increase of “super slug” , all these “elongated slug” unite to become an annular flow. Again, it can be noted the difference in the mechanism that create annular flow for vertical and horizontal flow. Any further increase transforms the outer liquid layer into bubbles in the inner liquid. Flow of near vertical against the gravity in two–phase does not deviate from vertical. The choking can occur at any point depends on the fluids and temperature and pressure.

5.5.1.3 Vertical Flow Under Micro Gravity

The above discussion mostly explained the flow in a vertical configuration when the surface tension can be neglected. In cases where the surface tension is very important. For example, out in space between gas and liquid (large density difference) the situation is different. The flow starts as dispersed bubble (some call it as “gas continuous”) because the gas phase occupies most of column. The liquid flows through a trickle or channeled flow that only partially wets part of the tube. The interaction between the phases is minimal and can be considered as the “open channel flow” of the vertical configuration. As the gas flow increases, the liquid becomes more turbulent and some parts enter into the gas phase as drops. When the flow rate of the gas increases further, all the gas phase change into tiny drops of liquid and this kind of regime referred to as mist flow. At a higher rate of liquid flow and a low flow rate of gas, the regime liquid fills the entire void and the gas is in small bubble and this flow referred to as bubbly flow. In the medium range of the flow rate of gas and liquid, there is pulse flow in which liquid is moving in frequent pulses. The common map is based on dimensionless parameters. Here, it is presented in a dimension form to explain the trends (see Figure 5.7). In the literature, Figure 5.7 presented in dimensionless coordinates. The abscissa is a function of combination of Froude ,Reynolds, and Weber numbers. The ordinate is a combination of flow rate ratio and density ratio.

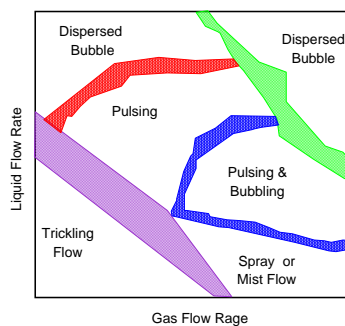


Fig. -5.7: A dimensional vertical flow map under very low gravity against the gravity.

Flow With The Gravity

As opposed to the flow against gravity, this flow can starts with stratified flow. A good example for this flow regime is a water fall. The initial part for this flow is more significant. Since the heavy liquid can be supplied from the “wrong” point/side, the initial part has a larger section compared to the flow against the gravity flow. After the flow has settled, the flow continues in a stratified configuration. The transitions between the flow regimes is similar to stratified flow. However, the

points where these transitions occur are different from the horizontal flow. While this author is not aware of an actual model, it must be possible to construct a model that connects this configuration with the stratified flow where the transitions will be dependent on the angle of inclinations.

5.6 Multi-Phase Flow Variables Definitions

Since the gas-liquid system is a specific case of the liquid-liquid system, both will be united in this discussion. However, for the convenience of the terms “gas and liquid” will be used to signify the lighter and heavier liquid, respectively. The liquid-liquid (also gas-liquid) flow is an extremely complex three-dimensional transient problem since the flow conditions in a pipe may vary along its length, over its cross section, and with time. To simplify the descriptions of the problem and yet to retain the important features of the flow, some variables are defined so that the flow can be described as a one-dimensional flow. This method is the most common and important to analyze two-phase flow pressure drop and other parameters. Perhaps, the only serious missing point in this discussion is the change of the flow along the distance of the tube.

5.6.1 Multi-Phase Averaged Variables Definitions

The total mass flow rate through the tube is the sum of the mass flow rates of the two phases

$$\dot{m} = \dot{m}_G + \dot{m}_L \quad (5.1)$$

It is common to define the mass velocity instead of the regular velocity because the “regular” velocity changes along the length of the pipe. The gas mass velocity is

$$G_G = \frac{\dot{m}_G}{A} \quad (5.2)$$

Where A is the entire area of the tube. It has to be noted that this mass velocity does not exist in reality. The liquid mass velocity is

$$G_L = \frac{\dot{m}_L}{A} \quad (5.3)$$

The mass flow of the tube is then

$$G = \frac{\dot{m}}{A} \quad (5.4)$$

It has to be emphasized that this mass velocity is the actual velocity.

The volumetric flow rate is not constant (since the density is not constant) along the flow rate and it is defined as

$$Q_G = \frac{G_G}{\rho_G} = U_{sG} \quad (5.5)$$

and for the liquid

$$Q_L = \frac{G_L}{\rho_L} \quad (5.6)$$

For liquid with very high bulk modulus (almost constant density), the volumetric flow rate can be considered as constant. The total volumetric volume vary along the tube length and is

$$Q = Q_L + Q_G \quad (5.7)$$

Ratio of the gas flow rate to the total flow rate is called the 'quality' or the "dryness fraction" and is given by

$$X = \frac{\dot{m}_G}{\dot{m}} = \frac{G_G}{G} \quad (5.8)$$

In a similar fashion, the value of $(1 - X)$ is referred to as the "wetness fraction." The last two fractions remain constant along the tube length as long the gas and liquid masses remain constant. The ratio of the gas flow cross sectional area to the total cross sectional area is referred as the void fraction and defined as

$$\alpha = \frac{A_G}{A} \quad (5.9)$$

This fraction is vary along tube length since the gas density is not constant along the tube length. The liquid fraction or liquid holdup is

$$L_H = 1 - \alpha = \frac{A_L}{A} \quad (5.10)$$

It must be noted that Liquid holdup, L_H is not constant for the same reasons the void fraction is not constant.

The actual velocities depend on the other phase since the actual cross section the phase flows is dependent on the other phase. Thus, a superficial velocity is commonly defined in which if only one phase is using the entire tube. The gas superficial velocity is therefore defined as

$$U_{sG} = \frac{G_G}{\rho_G} = \frac{X \dot{m}}{\rho_G A} = Q_G \quad (5.11)$$

The liquid superficial velocity is

$$U_{sL} = \frac{G_L}{\rho_L} = \frac{(1 - X) \dot{m}}{\rho_L A} = Q_L \quad (5.12)$$

Since $U_{sL} = Q_L$ and similarly for the gas then

$$U_m = U_{sG} + U_{sL} \quad (5.13)$$

Where U_m is the averaged velocity. It can be noticed that U_m is not constant along the tube.

The average superficial velocity of the gas and liquid are different. Thus, the ratio of these velocities is referred to as the slip velocity and is defined as the following

$$SLP = \frac{U_G}{U_L} \quad (5.14)$$

Slip ratio is usually greater than unity. Also, it can be noted that the slip velocity is not constant along the tube.

For the same velocity of phases ($SLP = 1$), the mixture density is defined as

$$\rho_m = \alpha \rho_G + (1 - \alpha) \rho_L \quad (5.15)$$

This density represents the density taken at the “frozen” cross section (assume the volume is the cross section times infinitesimal thickness of dx).

The average density of the material flowing in the tube can be evaluated by looking at the definition of density. The density of any material is defined as $\rho = m/V$ and thus, for the flowing material it is

$$\rho = \frac{\dot{m}}{Q} \quad (5.16)$$

Where Q is the volumetric flow rate. Substituting equations (5.1) and (5.7) into equation (5.16) results in

$$\rho_{average} = \frac{\overbrace{X \dot{m}_G}^{\dot{m}_G} + \overbrace{(1-X) \dot{m}_L}^{\dot{m}_L}}{Q_G + Q_L} = \frac{X \dot{m} + (1-X) \dot{m}}{\underbrace{X \dot{m}}_{Q_G} + \underbrace{(1-X) \dot{m}}_{Q_L}} \quad (5.17)$$

Equation (5.17) can be simplified by canceling the \dot{m} and noticing the $(1-X) + X = 1$ to become

$$\rho_{average} = \frac{1}{\frac{X}{\rho_G} + \frac{(1-X)}{\rho_L}} \quad (5.18)$$

The average specific volume of the flow is then

$$v_{average} = \frac{1}{\rho_{average}} = \frac{X}{\rho_G} + \frac{(1-X)}{\rho_L} = X v_G + (1-X) v_L \quad (5.19)$$

The relationship between X and α is

$$X = \frac{\dot{m}_G}{\dot{m}_G + \dot{m}_L} = \frac{\rho_G U_G \overbrace{A \alpha}^{A_G}}{\underbrace{\rho_L U_L A(1-\alpha)}_{A_L} + \rho_G U_G A \alpha} = \frac{\rho_G U_G \alpha}{\rho_L U_L(1-\alpha) + \rho_G U_G \alpha} \quad (5.20)$$

If the slip is one $SLP = 1$, thus equation (5.20) becomes

$$X = \frac{\rho_G \alpha}{\rho_L(1-\alpha) + \rho_G \alpha} \quad (5.21)$$

5.7 Homogeneous Models

Before discussing the homogeneous models, it is worthwhile to appreciate the complexity of the flow. For the construction of fluid basic equations, it was assumed that the flow is continuous. Now, this assumption has to be broken, and the flow is continuous only in many chunks (small segments). Furthermore, these segments are not defined but results of the conditions imposed on the flow. In fact, the different flow regimes are examples of typical configuration of segments of continuous flow. Initially, it was assumed that the different flow regimes can be neglected at least for the pressure loss (not correct for the heat transfer). The single phase was studied earlier in this book and there is a considerable amount of information about it. Thus, the simplest is to use it for approximation.

The average velocity (see also equation (5.13)) is

$$U_m = \frac{Q_L + Q_G}{A} = U_{sL} + U_{sG} = U_m \quad (5.22)$$

It can be noted that the continuity equation is satisfied as

$$\dot{m} = \rho_m U_m A \quad (5.23)$$

Example 5.1:

Under what conditions equation (5.23) is correct?

SOLUTION

Under construction

The governing momentum equation can be approximated as

$$\dot{m} \frac{dU_m}{dx} = -A \frac{dP}{dx} - S \tau_w - A \rho_m g \sin \theta \quad (5.24)$$

or modifying equation (5.24) as

$$-\frac{dP}{dx} = -\frac{S}{A}\tau_w - \frac{\dot{m}}{A}\frac{dU_m}{dx} + \rho_m g \sin \theta \quad (5.25)$$

The energy equation can be approximated as

$$\frac{dq}{dx} - \frac{dw}{dx} = \dot{m}\frac{d}{dx}\left(h_m + \frac{U_m^2}{2} + g x \sin \theta\right) \quad (5.26)$$

5.7.1 Pressure Loss Components

In a tube flowing upward in incline angle θ , the pressure loss is affected by friction loss, acceleration, and body force (gravitation). These losses are non-linear and depend on each other. For example, the gravitation pressure loss reduce the pressure and thus the density must change and hence, acceleration must occur. However, for small distances (dx) and some situations, this dependency can be neglected. In that case, from equation (5.25), the total pressure loss can be written as

$$\frac{dP}{dx} = \overbrace{\frac{dP}{dx}}^{\text{friction}} \Big|_f + \overbrace{\frac{dP}{dx}}^{\text{acceleration}} \Big|_a + \overbrace{\frac{dP}{dx}}^{\text{gravity}} \Big|_g \quad (5.27)$$

Every part of the total pressure loss will be discussed in the following section.

5.7.1.1 Friction Pressure Loss

The frictional pressure loss for a conduit can be calculated as

$$-\frac{dP}{dx} \Big|_f = \frac{S}{A}\tau_w \quad (5.28)$$

Where S is the perimeter of the fluid. For calculating the frictional pressure loss in the pipe is

$$-\frac{dP}{dx} \Big|_f = \frac{4\tau_w}{D} \quad (5.29)$$

The wall shear stress can be estimated by

$$\tau_w = f\frac{\rho_m U_m^2}{2} \quad (5.30)$$

The friction factor is measured for a single phase flow where the average velocity is directly related to the wall shear stress. There is not available experimental data for the relationship of the averaged velocity of the two (or more) phases and wall shear stress. In fact, this friction factor was not measured for the “averaged” viscosity of the two phase flow. Yet, since there isn’t anything better, the experimental data that was developed and measured for single flow is used.

The friction factor is obtained by using the correlation

$$f = C \left(\frac{\rho_m U_m D}{\mu_m} \right)^{-n} \quad (5.31)$$

Where C and n are constants which depend on the flow regimes (turbulent or laminar flow). For laminar flow $C = 16$ and $n = 1$. For turbulent flow $C = 0.079$ and $n = 0.25$. There are several suggestions for the average viscosity. For example, Duckler suggest the following

$$\mu_m = \frac{\mu_G Q_G}{Q_G + Q_L} + \frac{\mu_L Q_L}{Q_G + Q_L} \quad (5.32)$$

Duckler linear formula does not provide always good approximation and Cichilli suggest similar to equation (5.33) average viscosity as

$$\mu_{average} = \frac{1}{\frac{X}{\mu_G} + \frac{(1-X)}{\mu_L}} \quad (5.33)$$

Or simply make the average viscosity depends on the mass fraction as

$$\mu_m = X \mu_G + (1 - X) \mu_L \quad (5.34)$$

Using this formula, the friction loss can be estimated.

5.7.1.2 Acceleration Pressure Loss

The acceleration pressure loss can be estimated by

$$-\frac{dP}{dx} \Big|_a = \dot{m} \frac{dU_m}{dx} \quad (5.35)$$

The acceleration pressure loss (can be positive or negative) results from change of density and the change of cross section. Equation (5.35) can be written as

$$-\frac{dP}{dx} \Big|_a = \dot{m} \frac{d}{dx} \left(\frac{\dot{m}}{A \rho_m} \right) \quad (5.36)$$

Or in an explicit way equation (5.36) becomes

$$-\frac{dP}{dx} \Big|_a = \dot{m}^2 \left[\begin{array}{l} \text{pressure loss due to} \\ \text{density change} \\ \underbrace{\frac{1}{A} \frac{d}{dx} \left(\frac{1}{\rho_m} \right)} \\ \text{pressure loss due to} \\ \text{area change} \\ \underbrace{\frac{1}{\rho_m A^2} \frac{dA}{dx}} \end{array} \right] \quad (5.37)$$

There are several special cases. The first case where the cross section is constant, $dA/dx = 0$. In second case is where the mass flow rates of gas and liquid is constant in which the derivative of X is zero, $dX/dx = 0$. The third special case is for constant density of one phase only, $d\rho_L/dx = 0$. For the last point, the private case is where densities are constant for both phases.

5.7.1.3 Gravity Pressure Loss

Gravity was discussed in Chapter 4 and is

$$\left. \frac{dP}{dx} \right|_g = g \rho_m \sin \theta \quad (5.38)$$

The density change during the flow can be represented as a function of density. The density in equation (5.38) is the density without the “movement” (the “static” density).

5.7.1.4 Total Pressure Loss

The total pressure between two points, (a and b) can be calculated with integration as

$$\Delta P_{ab} = \int_a^b \frac{dP}{dx} dx \quad (5.39)$$

and therefore

$$\Delta P_{ab} = \overbrace{\Delta P_{abf}}^{\text{friction}} + \overbrace{\Delta P_{aba}}^{\text{acceleration}} + \overbrace{\Delta P_{abg}}^{\text{gravity}} \quad (5.40)$$

5.7.2 Lockhart Martinelli Model

The second method is by assumption that every phase flow separately. One such popular model by Lockhart and Martinelli⁸. Lockhart and Martinelli built model based on the assumption that the separated pressure loss are independent from each other. Lockhart Martinelli parameters are defined as the ratio of the pressure loss of two phases and pressure of a single phase. Thus, there are two parameters as shown below.

$$\phi_G = \sqrt{\left. \frac{dP}{dx} \right|_{TP} / \left. \frac{dP}{dx} \right|_{SG}} \quad (5.41)$$

⁸This method was considered a military secret, private communication with Y., Taitle

Where the TP denotes the two phases and SG denotes the pressure loss for the single gas phase. Equivalent definition for the liquid side is

$$\phi_L = \sqrt{\left. \frac{dP}{dx} \right|_{TP} / \left. \frac{dP}{dx} \right|_{SL}} \Big|_f \quad (5.42)$$

Where the SL denotes the pressure loss for the single liquid phase.

The ratio of the pressure loss for a single liquid phase and the pressure loss for a single gas phase is

$$\Xi = \sqrt{\left. \frac{dP}{dx} \right|_{SL} / \left. \frac{dP}{dx} \right|_{SG}} \Big|_f \quad (5.43)$$

where Ξ is Martinelli parameter.

It is assumed that the pressure loss for both phases are equal.

$$\left. \frac{dP}{dx} \right|_{SG} = \left. \frac{dP}{dx} \right|_{SL} \quad (5.44)$$

The pressure loss for the liquid phase is

$$\left. \frac{dP}{dx} \right|_L = \frac{2 f_L U_L^2 \rho_l}{D_L} \quad (5.45)$$

For the gas phase, the pressure loss is

$$\left. \frac{dP}{dx} \right|_G = \frac{2 f_G U_G^2 \rho_l}{D_G} \quad (5.46)$$

Simplified model is when there is no interaction between the two phases.

To insert the Diagram.

5.8 Solid-Liquid Flow

Solid-liquid system is simpler to analyze than the liquid-liquid system. In solid-liquid, the effect of the surface tension are very minimal and can be ignored. Thus, in this discussion, it is assumed that the surface tension is insignificant compared to the gravity forces. The word "solid" is not really mean solid but a combination of many solid particles. Different combination of solid particle creates different "liquid." Therefore, there will be a discussion about different particle size and different geometry (round, cubic, etc). The uniformity is categorizing the particle sizes, distribution, and geometry. For example, analysis of small coal particles in water is different from large coal particles in water.

The density of the solid can be above or below the liquid. Consider the case where the solid is heavier than the liquid phase. It is also assumed that the "liquids" density does not change significantly and it is far from the choking point. In that case there are four possibilities for vertical flow:

1. The flow with the gravity and lighter density solid particles.
2. The flow with the gravity and heavier density solid particles.
3. The flow against the gravity and lighter density solid particles.
4. The flow against the gravity and heavier density solid particles.

All these possibilities are different. However, there are two sets of similar characteristics, possibility, 1 and 4 and the second set is 2 and 3. The first set is similar because the solid particles are moving faster than the liquid velocity and vice versa for the second set (slower than the liquid). The discussion here is about the last case (4) because very little is known about the other cases.

5.8.1 Solid Particles with Heavier Density $\rho_S > \rho_L$

Solid–liquid flow has several combination flow regimes.

When the liquid velocity is very small, the liquid cannot carry the solid particles because there is not enough resistance to lift up the solid particles. A particle in a middle of the vertical liquid flow experience several forces. The force balance of spherical particle in field viscous fluid (creeping flow) is

$$\underbrace{\frac{\pi D^3 g (\rho_S - \rho_L)}{6}}_{\text{gravity and buoyancy forces}} = \underbrace{\frac{C_{D\infty} \pi D^2 \rho_L U_L^2}{8}}_{\text{drag forces}} \quad (5.47)$$

Where $C_{D\infty}$ is the drag coefficient and is a function of Reynolds number, Re , and D is the equivalent radius of the particles. The Reynolds number defined as

$$Re = \frac{U_L D \rho_L}{\mu_L} \quad (5.48)$$

Inserting equating (5.48) into equation (5.47) become

$$\overbrace{f(Re)}^{C_{D\infty}(U_L)} U_L^2 = \frac{4Dg(\rho_S - \rho_L)}{3\rho_L} \quad (5.49)$$

Equation (5.49) relates the liquid velocity that needed to maintain the particle “floating” to the liquid and particles properties. The drag coefficient, $C_{D\infty}$ is complicated function of the Reynolds number. However, it can be approximated for several regimes. The first regime is for $Re < 1$ where Stokes’ Law can be approximated as

$$C_{D\infty} = \frac{24}{Re} \quad (5.50)$$

In transitional region $1 < Re < 1000$

$$C_{D\infty} = \frac{24}{Re} \left(1 + \frac{1}{6} Re^{2/3} \right) \tag{5.51}$$

For larger Reynolds numbers, the Newton's Law region, $C_{D\infty}$, is nearly constant as

$$C_{D\infty} = 0.44 \tag{5.52}$$

In most cases of solid-liquid system, the Reynolds number is in the second range⁹. For the first region, the velocity is small to lift the particle unless the density difference is very small (that very small force can lift the particles). In very large range (especially for gas) the choking might be approached. Thus, in many cases the middle region is applicable.

So far the discussion was about single particle. When there are more than one particle in the cross section, then the actual velocity that every particle experience depends on the void fraction. The simplest assumption that the change of the cross section of the fluid create a parameter that multiply the single particle as

$$C_{D\infty|\alpha} = C_{D\infty} f(\alpha) \tag{5.53}$$

When the subscript α is indicating the void, the function $f(\alpha)$ is not a linear function. In the literature there are many functions for various conditions.

Minimum velocity is the velocity when the particle is "floating". If the velocity is larger, the particle will drift with the liquid. When the velocity is lower, the particle will sink into the liquid. When the velocity of liquid is higher than the minimum velocity many particles will be floating. It has to remember that not all the particle are uniform in size or shape. Consequently, the minimum velocity is a range of velocity rather than a sharp transition point.

As the solid particles are not pushed by a pump but moved by the forces the fluid applies to them. Thus, the only velocity that can be applied is the fluid velocity. Yet, the solid particles can be supplied at different rate. Thus, the discussion will be focus on the fluid velocity. For small gas/liquid velocity, the particles are what some call fixed fluidized bed. Increasing the fluid velocity beyond a minimum will move the particles and it is referred to as mix fluidized bed. Additional increase of the fluid velocity will move

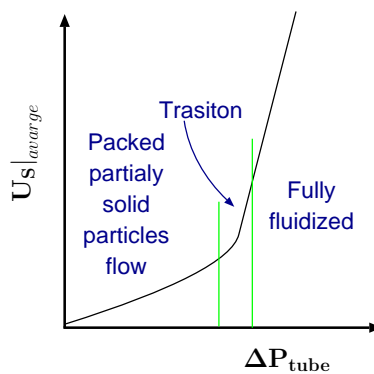


Fig. 5.8: The terminal velocity that left the solid particles.

⁹ It be wonderful if flow was in the last range? The critical velocity could be found immediately.

all the particles and this is referred to as fully fluidized bed. For the case of liquid, further increase will create a slug flow. This slug flow is when slug shape (domes) are almost empty of the solid particle. For the case of gas, additional increase create “tunnels” of empty almost from solid particles. Additional increase in the fluid velocity causes large turbulence and the ordinary domes are replaced by churn type flow or large bubbles that are almost empty of the solid particles. Further increase of the fluid flow increases the empty spots to the whole flow. In that case, the sparse solid particles are dispersed all over. This regimes is referred to as Pneumatic conveying (see Figure 5.9).

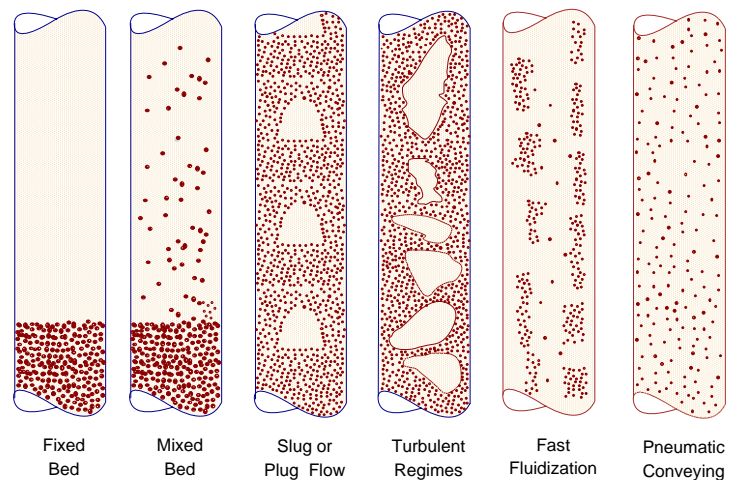


Fig. -5.9: The flow patterns in solid-liquid flow.

One of the main difference between the liquid and gas flow in this category is the speed of sound. In the gas phase, the speed of sound is reduced dramatically with increase of the solid particles concentration (further reading *Fundamentals of Compressible Flow*” chapter on Fanno Flow by this author is recommended). Thus, the velocity of gas is limited when reaching the Mach somewhere between $1/\sqrt{k}$ and 1 since the gas will be choked (neglecting the double choking phenomenon). Hence, the length of conduit is very limited. The speed of sound of the liquid does not change much. Hence, this limitation does not (effectively) exist for most cases of solid-liquid flow.

5.8.2 Solid With Lighter Density $\rho_S < \rho$ and With Gravity

This situation is minimal and very few cases exist. However, it must be pointed out that even in solid-gas, the fluid density can be higher than the solid (especially with

micro gravity). There was very little investigations and known about the solid–liquid flowing down (with the gravity). Furthermore, there is very little knowledge about the solid–liquid when the solid density is smaller than the liquid density. There is no known flow map for this kind of flow that this author is aware of.

Nevertheless, several conclusions and/or expectations can be drawn. The issue of minimum terminal velocity is not exist and therefor there is no fixed or mixed fluidized bed. The flow is fully fluidized for any liquid flow rate. The flow can have slug flow but more likely will be in fast Fluidization regime. The forces that act on the spherical particle are the buoyancy force and drag force. The buoyancy is accelerating the particle and drag force are reducing the speed as

$$\frac{\pi D^3 g(\rho_S - \rho_L)}{6} = \frac{C_{D\infty} \pi D^2 \rho_L (U_S - U_L)^2}{8} \quad (5.54)$$

From equation 5.54, it can observed that increase of the liquid velocity will increase the solid particle velocity at the same amount. Thus, for large velocity of the fluid it can be observed that $U_L/U_S \rightarrow 1$. However, for a small fluid velocity the velocity ratio is very large, $U_L/U_S \rightarrow 0$. The affective body force “seems” by the particles can be in some cases larger than the gravity. The flow regimes will be similar but the transition will be in different points.

The solid–liquid horizontal flow has some similarity to horizontal gas–liquid flow. Initially the solid particles will be carried by the liquid to the top. When the liquid velocity increase and became turbulent, some of the particles enter into the liquid core. Further increase of the liquid velocity appear as somewhat similar to slug flow. However, this author have not seen any evidence that show the annular flow does not appear in solid–liquid flow.

5.9 Counter–Current Flow

This discussion will be only on liquid–liquid systems (which also includes liquid–gas systems). This kind of flow is probably the most common to be realized by the masses. For example, opening a can of milk or juice. Typically if only one hole is opened on the top of the can, the liquid will flow in pulse regime. Most people know that two holes are needed to empty the can easily and continuously. Otherwise, the flow will be in a pulse regime.

In most cases, the possibility to have counter–current flow is limited to having short length of tubes. In only certain configurations of the infinite long pipes the counter–current flow can exist. In that case, the pressure difference and gravity (body forces) dominates the flow. The inertia components of the flow, for long tubes, cannot compensate for the pressure gradient. In short tube, the pressure difference in one phase can be positive while the pressure difference in the other phase can be negative. The pressure difference in the interface must be finite. Hence, the counter–current flow can have opposite pressure gradient for short conduit. But in most cases, the heavy phase (liquid) is pushed by the gravity and lighter phase (gas) is driven by the pressure difference.



Fig. -5.10: Counter-current flow in a can (the left figure) has only one hole thus pulse flow and a flow with two holes (right picture).

The counter-current flow occurs, for example, when cavity is filled or emptied with a liquid. The two phase regimes “occurs” mainly in entrance to the cavity. For example, Figure 5.10 depicts emptying of can filled with liquid. The air is “attempting” to enter the cavity to fill the vacuum created thus forcing pulse flow. If there are two holes, in some cases, liquid flows through one hole and the air through the second hole and the flow will be continuous. It also can be noticed that if there is one hole (orifice) and a long and narrow tube, the liquid will stay in the cavity (neglecting other phenomena such as dripping flow.).

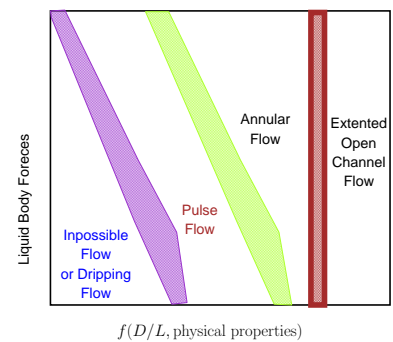


Fig. -5.11: Counter-flow in vertical tubes map.

There are three flow regimes¹⁰ that have been observed. The first flow pattern is pulse flow regime. In this flow regime, the phases flow turns into different direction (see Figure 5.12). The name pulse flow is used to signify that the flow is flowing in pulses that occurs in a certain frequency. This is opposed to counter-current solid-gas flow when almost no pulse was observed. Initially, due to the gravity, the heavy liquid is leaving the can. Then the pressure in the can is reduced compared to the outside and some lighter liquid (gas) entered into the can. Then, the pressure in the can increase, and some heavy liquid will starts to flow. This process continue until almost the liquid is evacuated (some liquid stay due the surface tension). In many situations, the volume flow rate of the two phase is almost equal. The duration the cycle depends on several factors. The cycle duration can be replaced by frequency. The analysis of the frequency is much more complex issue and will not be dealt here.

¹⁰Caution! this statement should be considered as “so far found”. There must be other flow regimes that were not observed or defined. For example, elongated pulse flow was observed but measured. This field hasn’t been well explored. There are more things to be examined and to be studied.

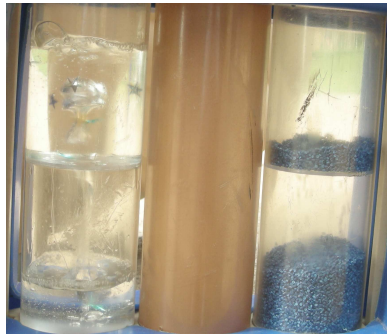


Fig. -5.12: Pictures of Counter-current flow in liquid-gas and solid-gas configurations. The container is made of two compartments. The upper compartment is filled with the heavy phase (liquid, water solution, or small wood particles) by rotating the container. Even though the solid-gas ratio is smaller, it can be noticed that the solid-gas is faster than the liquid-gas flow.

Annular Flow in Counter-current flow

The other flow regime is annular flow in which the heavier phase is on the periphery of the conduit (In the literature, there are someone who claims that heavy liquid will be inside). The analysis is provided, but somehow it contradicts with the experimental evidence. Probably, one or more of the assumptions that the analysis based is erroneous). In very small diameters of tubes the counter-current flow is not possible because of the surface tension (see section 4.7). The ratio of the diameter to the length with some combinations of the physical properties (surface tension etc) determines the point where the counter flow can start. At this point, the pulsing flow will start and larger diameter will increase the flow and turn the flow into annular flow. Additional increase of the diameter will change the flow regime into extended open channel flow. Extended open channel flow retains the characteristic of open channel that the lighter liquid (almost) does not effect the heavier liquid flow. Example of such flow in the nature is water falls in which water flows down and air (wind) flows up.

The driving force is the second parameter which effects the flow existence. When the driving (body) force is very small, no counter-current flow is possible. Consider the can in zero gravity field, no counter-current flow possible. However, if the can was on the sun (ignoring the heat transfer issue), the flow regime in the can moves from pulse to annular flow. Further increase of the body force will move

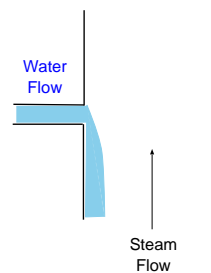


Fig. -5.13: Flood in vertical pipe.

the flow to be in the extended “open channel flow.”

In the vertical co-current flow there are two possibilities, flow with gravity or against it. As opposed to the co-current flow, the counter-current flow has no possibility for these two cases. The heavy liquid will flow with the body forces (gravity). Thus it should be considered as non-existent flow.

5.9.1 Horizontal Counter-Current Flow

Up to this point, the discussion was focused on the vertical tubes. In horizontal tubes, there is an additional flow regime which is stratified. Horizontal flow is different from vertical flow from the stability issues. A heavier liquid layer can flow above a lighter liquid. This situation is unstable for large diameter but as in static (see section (4.7) page 108) it can be considered stable for small diameters. A flow in a very narrow tube with heavy fluid above the lighter fluid should be considered as a separate issue.

When the flow rate of both fluids is very small, the flow will be stratified counter-current flow. The flow will change to pulse flow when the heavy liquid flow rate increases. Further increase of the flow will result in a single phase flow regime. Thus, closing the window of this kind of flow. Thus, this increase terminates the two phase flow possibility. The flow map of the horizontal flow is different from the vertical flow and is shown in Figure 5.14. A flow in an angle of inclination is closer to vertical flow unless the angle of inclination is very small. The stratified counter flow has a lower pressure loss (for the liquid side). The change to pulse flow increases the pressure loss dramatically.

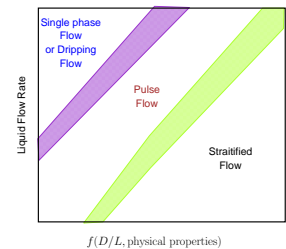


Fig. -5.14: A flow map to explain the horizontal counter-current flow.

5.9.2 Flooding and Reversal Flow

The limits of one kind the counter-current flow regimes, that is stratified flow are discussed here. This problem appears in nuclear engineering (or boiler engineering) where there is a need to make sure that liquid (water) inserted into the pipe reaching the heating zone. When there is no water (in liquid phase), the fire could melt or damage the boiler. In some situations, the fire can be too large or/and the water supply failed below a critical value the water turn into steam. The steam will flow in the opposite direction. To analyze this situation consider a two dimensional conduit with a liquid inserted in the left side as depicted in Figure 5.13. The liquid velocity at very low gas velocity is constant but not uniform. Further increase of the gas velocity will reduce the average liquid velocity. Additional increase of the gas velocity will bring it to a point where the liquid will flow in a reverse direction and/or disappear (dried out).

A simplified model for this situation is for a two dimensional configuration where the liquid is flowing down and the gas is flowing up as shown in Figure 5.15. It is assumed that both fluids are flowing in a laminar regime and steady state. Additionally, it is assumed that the entrance effects can be neglected. The liquid flow rate, Q_L , is unknown. However, the pressure difference in the (x direction) is known and equal to zero. The boundary conditions for the liquid is that velocity at the wall is zero and the velocity at the interface is the same for both phases $U_G = U_L$ or $\tau_i|_G = \tau_i|_L$. As it will be shown later, both conditions cannot coexist. The model can be improved by considering turbulence, mass transfer, wavy interface, etc¹¹. This model is presented to exhibit the trends and the special features of counter-current flow. Assuming the pressure difference in the flow direction for the gas is constant and uniform. It is assumed that the last assumption does not contribute or change significantly the results. The underline rational for this assumption is that gas density does not change significantly for short pipes (for more information look for the book "Fundamentals of Compressible Flow" in Potta book series in the Fanno flow chapter.)

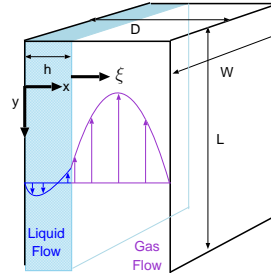


Fig. -5.15: A diagram to explain the flood in a two dimension geometry.

The liquid film thickness is unknown and can be expressed as a function of the above boundary conditions. Thus, the liquid flow rate is a function of the boundary conditions. On the liquid side, the gravitational force has to be balanced by the shear forces as

$$\frac{d\tau_{xy}}{dx} = \rho_L g \tag{5.55}$$

The integration of equation (5.55) results in

$$\tau_{xy} = \rho_L g x + C_1 \tag{5.56}$$

The integration constant, C_1 , can be found from the boundary condition where $\tau_{xy}(x = h) = \tau_i$. Hence,

$$\tau_i = \rho_L g h + C_1 \tag{5.57}$$

The integration constant is then $C_i = \tau_i - \rho_L g h$ which leads to

$$\tau_{xy} = \rho_L g (x - h) + \tau_i \tag{5.58}$$

Substituting the newtonian fluid relationship into equation (5.58) to obtained

$$\mu_L \frac{dU_y}{dx} = \rho_L g (x - h) + \tau_i \tag{5.59}$$

¹¹The circular configuration is under construction and will be appeared as a separated article momentarily.

or in a simplified form as

$$\frac{dU_y}{dx} = \frac{\rho_L g (x - h)}{\mu_L} + \frac{\tau_i}{\mu_L} \quad (5.60)$$

Equation (5.60) can be integrate to yield

$$U_y = \frac{\rho_L g}{\mu_L} \left(\frac{x^2}{2} - h x \right) + \frac{\tau_i x}{\mu_L} + C_2 \quad (5.61)$$

The liquid velocity at the wall, [$U(x = 0) = 0$], is zero and the integration coefficient can be found to be

$$C_2 = 0 \quad (5.62)$$

The liquid velocity profile is then

$$U_y = \frac{\rho_L g}{\mu_L} \left(\frac{x^2}{2} - h x \right) + \frac{\tau_i x}{\mu_L} \quad (5.63)$$

The velocity at the liquid–gas interface is

$$U_y(x = h) = \frac{\tau_i h}{\mu_L} - \frac{\rho_L g h^2}{2 \mu_L} \quad (5.64)$$

The velocity can vanish (zero) inside the film in another point which can be obtained from

$$0 = \frac{\rho_L g}{\mu_L} \left(\frac{x^2}{2} - h x \right) + \frac{\tau_i x}{\mu_L} \quad (5.65)$$

The solution for equation (5.65) is

$$x|_{@U_L=0} = 2h - \frac{2\tau_i}{\mu_L g \rho_L} \quad (5.66)$$

The maximum x value is limited by the liquid film thickness, h . The minimum shear stress that start to create reversible velocity is obtained when $x = h$ which is

$$0 = \frac{\rho_L g}{\mu_L} \left(\frac{h^2}{2} - h h \right) + \frac{\tau_i h}{\mu_L} \quad (5.67)$$

$$\hookrightarrow \tau_{i0} = \frac{h g \rho_L}{2}$$

If the shear stress is below this critical shear stress τ_{i0} then no part of the liquid will have a reversed velocity. The notation of τ_{i0} denotes the special value at which a starting shear stress value is obtained to have reversed flow. The point where the liquid flow rate is zero is important and it is referred to as initial flashing point.

The flow rate can be calculated by integrating the velocity across the entire liquid thickness of the film.

$$\frac{Q}{w} = \int_0^h U_y dx = \int_0^h \left[\frac{\rho_L g}{\mu_L} \left(\frac{x^2}{2} - h x \right) + \frac{\tau_i x}{\mu_L} \right] dx \quad (5.68)$$

Where w is the thickness of the conduit (see Figure 5.15). Integration equation (5.68) results in

$$\frac{Q}{w} = \frac{h^2 (3\tau_i - 2gh\rho_L)}{6\mu_L} \quad (5.69)$$

It is interesting to find the point where the liquid mass flow rate is zero. This point can be obtained when equation (5.69) is equated to zero. There are three solutions for equation (5.69). The first two solutions are identical in which the film height is $h = 0$ and the liquid flow rate is zero. But, also, the flow rate is zero when $3\tau_i = 2gh\rho_L$. This request is identical to the demand in which

$$\tau_{i \text{ critical}} = \frac{2gh\rho_L}{3} \quad (5.70)$$

This critical shear stress, for a given film thickness, reduces the flow rate to zero or effectively “drying” the liquid (which is different then equation (5.67)).

For this shear stress, the critical upward interface velocity is

$$U_{critical}|_{interface} = \frac{\left(\frac{2}{3} - \frac{1}{2}\right)}{6} \left(\frac{\rho_L g h^2}{\mu_L} \right) \quad (5.71)$$

The wall shear stress is the last thing that will be done on the liquid side. The wall shear stress is

$$\tau_L|_{@wall} = \mu_L \frac{dU}{dx} \Big|_{x=0} = \mu_L \left(\frac{\rho_L g}{\mu_L} \left(2x \overset{0}{-} h \right) + \frac{\tau_i}{3} \frac{1}{\mu_L} \right) \Big|_{x=0} \quad (5.72)$$

Simplifying equation (5.72)¹² becomes (notice the change of the sign accounting for the direction)

$$\tau_L|_{@wall} = \frac{gh\rho_L}{3} \quad (5.73)$$

¹²Also noticing that equation (5.70) has to be equal $gh\rho_L$ to support the weight of the liquid.

Again, the gas is assumed to be in a laminar flow as well. The shear stress on gas side is balanced by the pressure gradient in the y direction. The momentum balance on element in the gas side is

$$\frac{d\tau_{xyG}}{dx} = \frac{dP}{dy} \quad (5.74)$$

The pressure gradient is a function of the gas compressibility. For simplicity, it is assumed that pressure gradient is linear. This assumption means or implies that the gas is incompressible flow. If the gas was compressible with an ideal gas equation of state then the pressure gradient is logarithmic. Here, for simplicity reasons, the linear equation is used. In reality the logarithmic equation should be used (a discussion can be found in "Fundamentals of Compressible Flow" a Potto project book). Thus, equation (5.74) can be rewritten as

$$\frac{d\tau_{xyG}}{dx} = \frac{\Delta P}{\Delta y} = \frac{\Delta P}{L} \quad (5.75)$$

Where $\Delta y = L$ is the entire length of the flow and ΔP is the pressure difference of the entire length. Utilizing the Newtonian relationship, the differential equation is

$$\frac{d^2 U_G}{dx^2} = \frac{\Delta P}{\mu_G L} \quad (5.76)$$

Equation (5.76) can be integrated twice to yield

$$U_G = \frac{\Delta P}{\mu_G L} x^2 + C_1 x + C_2 \quad (5.77)$$

This velocity profile must satisfy zero velocity at the right wall. The velocity at the interface is the same as the liquid phase velocity or the shear stress are equal. Mathematically these boundary conditions are

$$U_G(x = D) = 0 \quad (5.78)$$

and

$$\begin{aligned} U_G(x = h) &= U_L(x = h) & (a) & \quad \text{or} & (5.79) \\ \tau_G(x = h) &= \tau_L(x = h) & (b) & \end{aligned}$$

Applying B.C. (5.78) into equation (5.77) results in

$$\begin{aligned} U_G = 0 &= \frac{\Delta P}{\mu_G L} D^2 + C_1 D + C_2 & (5.80) \\ \hookrightarrow C_2 &= -\frac{\Delta P}{\mu_G L} D^2 + C_1 D \end{aligned}$$

Which leads to

$$U_G = \frac{\Delta P}{\mu_G L} (x^2 - D^2) + C_1 (x - D) \quad (5.81)$$

At the other boundary condition, equation (5.79)(a), becomes

$$\frac{\rho_L g h^2}{6 \mu_L} = \frac{\Delta P}{\mu_G L} (h^2 - D^2) + C_1 (h - D) \quad (5.82)$$

The last integration constant, C_1 can be evaluated as

$$C_1 = \frac{\rho_L g h^2}{6 \mu_L (h - D)} - \frac{\Delta P (h + D)}{\mu_G L} \quad (5.83)$$

With the integration constants evaluated, the gas velocity profile is

$$U_G = \frac{\Delta P}{\mu_G L} (x^2 - D^2) + \frac{\rho_L g h^2 (x - D)}{6 \mu_L (h - D)} - \frac{\Delta P (h + D) (x - D)}{\mu_G L} \quad (5.84)$$

The velocity in Equation (5.84) is equal to the velocity equation (5.63) when ($x = h$). However, in that case, it is easy to show that the gas shear stress is not equal to the liquid shear stress at the interface (when the velocities are assumed to be the equal). The difference in shear stresses at the interface due to this assumption, of the equal velocities, cause this assumption to be not physical.

The second choice is to use the equal shear stresses at the interface, condition (5.79)(b). This condition requires that

$$\mu_G \frac{dU_G}{dx} = \mu_L \frac{dU_L}{dx} \quad (5.85)$$

The expressions for the derivatives are

$$\overbrace{\frac{2h \Delta P}{L}}^{\text{gas side}} + \mu_G C_1 = \overbrace{\frac{2gh \rho_L}{3}}^{\text{liquid side}} \quad (5.86)$$

As result, the integration constant is

$$C_1 = \frac{2gh \rho_L}{3 \mu_G} - \frac{2h \Delta P}{\mu_G L} \quad (5.87)$$

The gas velocity profile is then

$$U_G = \frac{\Delta P}{\mu_G L} (x^2 - D^2) + \left(\frac{2gh \rho_L}{3 \mu_G} - \frac{2h \Delta P}{\mu_G L} \right) (x - D) \quad (5.88)$$

The gas velocity at the interface is then

$$U_G|_{@x=h} = \frac{\Delta P}{\mu_G L} (h^2 - D^2) + \left(\frac{2 g h \rho_L}{3 \mu_G} - \frac{2 h \Delta P}{\mu_G L} \right) (h - D) \quad (5.89)$$

This gas interface velocity is different than the velocity of the liquid side. The velocity at interface can have a “slip” in very low density and for short distances. The shear stress at the interface must be equal, if no special effects occurs. Since there no possibility to have both the shear stress and velocity on both sides of the interface, different thing(s) must happen. It was assumed that the interface is straight but is impossible. Then if the interface becomes wavy, the two conditions can co-exist.

The wall shear stress is

$$\tau_G|_{@wall} = \mu_G \left. \frac{dU_G}{dx} \right|_{x=D} = \mu_G \left(\frac{\Delta P 2x}{\mu_G L} + \left(\frac{2 g h \rho_L}{3 \mu_G} - \frac{2 h \Delta P}{\mu_G L} \right) \right)_{x=D} \quad (5.90)$$

or in a simplified form as

$$\tau_G|_{@wall} = \frac{2 \Delta P (D - h)}{L} + \frac{2 g h \rho_L}{3} \quad (5.91)$$

The Required Pressure Difference

The pressure difference to create the flooding (drying) has to take into account the fact that the surface is wavy. However, as first estimate the waviness of the surface can be neglected. The estimation of the pressure difference under the assumption of equal shear stress can be applied. In the same fashion the pressure difference under the assumption the equal velocity can be calculated. The actual pressure difference can be between these

two assumptions but not must be between them. This model and its assumptions are too simplistic and the actual pressure difference is larger. However, this explanation is to show magnitudes and trends and hence it provided here.

To calculate the required pressure that cause the liquid to dry, the total balance is needed. The control volume include the gas and liquid volumes. Figure 5.16 describes the general forces that acts on the control volume. There are two

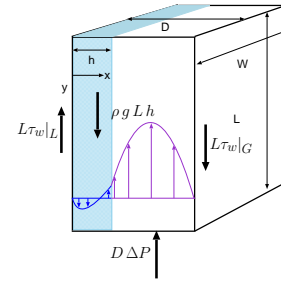


Fig. -5.16: General forces diagram to calculate the pressure difference in a two dimension geometry.

forces that act against the gravity and two forces with the gravity. The gravity force on the gas can be neglected in most cases. The gravity force on the liquid is the liquid volume times the liquid volume as

$$F_{gL} = \rho g \overbrace{hL}^{\text{Volme/w}} \quad (5.92)$$

The total momentum balance is (see Figure 5.16)

$$F_{gL} + \overbrace{L}^{A/w} \tau_{w_g} = \overbrace{L}^{A/w} \tau_{w_l} + \overbrace{D \Delta P}^{\text{force due to pressure}} \quad (5.93)$$

Substituting the different terms into (5.93) result in

$$\rho g L h + L \left(\frac{2 \Delta P (D - h)}{L} + \frac{2 g h \rho_L}{3} \right) = L \frac{g h \rho_L}{3} + D \Delta P \quad (5.94)$$

Simplifying equation (5.94) results in

$$\frac{4 \rho g L h}{3} = (2 h - D) \Delta P \quad (5.95)$$

or

$$\Delta P = \frac{4 \rho g L h}{3 (2 h - D)} \quad (5.96)$$

This analysis shows far more reaching conclusion that initial anticipation expected. The interface between the two liquid flowing together is wavy. Unless the derivations or assumptions are wrong, this analysis equation (5.96) indicates that when $D > 2 h$ is a special case (extend open channel flow).

5.10 Multi-Phase Conclusion

For the first time multi-phase is included in a standard introductory textbook on fluid mechanics. There are several points that should be noticed in this chapter. There are many flow regimes in multi-phase flow that "regular" fluid cannot be used to solve it such as flooding. In that case, the appropriate model for the flow regime should be employed. The homogeneous models or combined models like Lockhart-Martinelli can be employed in some cases. In other case where more accurate measurement are needed a specific model is required. Perhaps as a side conclusion but important, the assumption of straight line is not appropriate when two liquid with different viscosity are flowing.

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