Basin of attraction of triangular maps with applications *

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Abstract

We consider planar triangular maps $x_{n+1} = f_0(u_n) + f_1(u_n)x_n, u_{n+1} = \phi(u_n)$. These maps preserve the fibration of the plane given by $\mathcal{F} = \{\phi(u) = c, c \in \text{Image}(\phi)\}$. We assume that there exists an invariant attracting fiber $\{u = u_*\}$ for the dynamical system generated by ϕ and we study the limit dynamics of those points in the basin of attraction of this invariant fiber, assuming that either it contains a global attractor, or it is filled by fixed or 2-periodic points. We apply our results to several examples.

1 Introduction

In this paper we consider triangular systems of the form

$$
\begin{cases}
x_{n+1} & = f_0(u_n) + f_1(u_n)\, x_n, \\
u_{n+1} & = \phi(u_n),
\end{cases}
\tag{1}
$$

where $\{x_n\}$ and $\{u_n\}$ are real sequences, and f_0, f_1 and ϕ are continuous functions.

Observe that system (1) preserves the fibration of the plane given by $\mathcal{F} = \{\phi(u) = c, c \in \text{Image}(\phi)\}$, that is, it sends fibers of $\mathcal{F}$ to other ones. We will assume that there exists a point $u = u_*$ which is a local stable attractor of the subsystem $u_{n+1} = \phi(u_n)$. In this case, we say that system (1) has a *local attractive fiber* $\{u = u^*\}$. Our objective is to know if the asymptotic dynamics of the orbits corresponding to points in the *basin of attraction* of the *limit fiber* $\{u = u_*\}$ is characterized by the dynamics on this fiber, the *limit dynamics*. In all the cases, we will assume that the limit dynamics is very simple, that is: either (A) the fiber $\{u = u_*\}$ contains a global attractor, see Proposition 2; (B) the fiber is filled by fixed points, Theorem 6(a); or (C) it is filled by 2-periodic orbits, Theorem 6(b). Observe that there is no need to consider the case in which there is a global repellor on the fiber $\{u = u_*\}$, since in this situation it is clear that any orbit with initial conditions in the basin of attraction of this fiber is unbounded (otherwise there should be accumulation points in the fiber).

The paper is structured as follows. In Section 2 we present the main results (Proposition 2, Theorem 6 and Corollary 7) together with some motivating examples. The proofs are given in Section 3. In Section 4, and as an application of the main results, we study the limit dynamics of some linear quasi-homogeneous maps. The example considered there shows that, for this class of maps, the shape of the basins of attraction of the origin can have a certain level of complexity.

Finally we notice that the results obtained in this paper can be applied to study the dynamics of several types of difference equations.

2 Motivating examples and main results

We were motivated by the following illustrative example, which was inspired by an analogue for continuous systems in [6]. A particular case was also considered in [10], see also [2] and [13]. Consider the systems of type

$$\begin{cases} x_{n+1} & = \sum_{\ell \geq 0} f_\ell(u_n) x_n^\ell, \\ u_{n+1} & = \lambda u_n + o(u_n), \end{cases} \tag{2}$$

with $|\lambda| < 1$. These systems are such that $\{u = 0\}$ is a global limit fiber, having a global attractor for the restricted dynamics on $\{u = 0\}$. However, there are cases having unbounded orbits (x_n, u_n), with $u_n \to 0$, in their the global dynamics.

Example A: Hyperbolic globally attracting fiber with a restricted global attractor but having unbounded solutions. Consider the system

$$\begin{cases} x_{n+1} & = \mu\, x_n + a\, u_n^j x_n^\ell, \\ u_{n+1} & = \lambda\, u_n, \end{cases} \tag{3}$$

with $|\lambda| < 1$, $|\mu| \leq 1$ and $a \in \mathbb{R}$. Of course, this system preserves the fibration $\{u = c; c \in \mathbb{R}\}$, and it has a *global* attracting limit fiber $\{u = 0\}$. On this fiber the dynamics is as follows: the

origin is attractive if $|\mu| < 1$; the fiber is a continuum of fixed points if $\mu = 1$; and the fiber is a continuum of 2–periodic orbits if $\mu = -1$. However, as shown in Proposition 1, if $j(\ell - 1) > 0$, then there are initial conditions giving rise to unbounded solutions, and therefore the global dynamics is not characterized by the dynamics on the global limit fiber.

Proposition 1. *The curve*

$$\Gamma = \left\{ u^j x^{\ell-1} = \frac{\lambda^\alpha - \mu}{a}, \ \ where \ \alpha = -\frac{j}{\ell - 1} \right\}$$

is invariant for system (3). Moreover, for any initial condition $(x_0, u_0) \in \Gamma$ the associated orbit is given by $x_n = (\lambda^\alpha)^n x_0$, $u_n = \lambda^n u_0$. If $j(\ell - 1) > 0$, then any initial condition $(x_0, u_0) \in \Gamma$ gives rise to unbounded solutions, with $u_n \to 0$.

Proof. Imposing that system (3) has solutions of the form $x_n = (\lambda^\alpha)^n x_0$, $u_n = \lambda^n u_0$, we easily get

$$\lambda^\alpha x_0 = \mu x_0 + a \lambda^{nj+\alpha n(\ell-1)} u_0^j x_0^\ell.$$

Observe that if $nj + \alpha n(\ell - 1) = 0$ (that is when $\alpha = -j/(\ell - 1)$) we obtain that Γ is an invariant curve. Moreover, if $j(\ell - 1) > 0$, then $|\lambda^\alpha| > 1$, and the orbits on Γ are unbounded solutions. ∎

Example A shows that, in general, for systems of type (2) with terms of degree greater or equal than 2 in x_n, we cannot expect that the global dynamics in the basin of attraction of the limit fiber $\{u = 0\}$ is characterized by the dynamics on the fiber. However, it remains to explore the case of systems of type (1) with affine terms in x_n. For these systems, the possible limit dynamics that we consider are the cases (A), (B) and (C) mentioned in the introduction.

The first result concerns the case (A), when the the limit fiber contains a global attractor. This case is characterized by the fact that $|f_1(u_*)| < 1$, and the attractor is $x_* = f_0(u_*)/(1 - f_1(u_*))$;

Proposition 2. *Consider the system (1) with f_0 and f_1 continuous and $|f_1(u_*)| < 1$. Suppose that $u = u_*$ is an attractive point of $u_{n+1} = \phi(u_n)$. Then, for all initial condition (x_0, u_0) with u_0 in the basin of attraction of $u = u_*$, we have $\lim_{n \to \infty} (x_n, u_n) = (f_0(u_*)/(1 - f_1(u_*)), u_*)$.*

We want to point out that the above proposition is not a local result. In fact, the convergence is guaranteed for all (x_0, u_0) such that u_0 is in the basin of attraction of $u = u_*$.

In case (B), when the limit fiber $\{u = u_*\}$ consists of fixed points, there are systems such that the orbits starting in the basin of attraction of the limit fiber converge to the fixed points. The following example illustrates this remark.

Example B: Hyperbolic attracting fiber and fast enough convergence to the limit dynamics.
Consider the system given by

$$\begin{cases} x_{n+1} & = (1 + a\, u_n)\, x_n, \\ u_{n+1} & = \lambda\, u_n, \end{cases}$$

with $|\lambda| < 1$, and $a \in \mathbb{R}$. Then

$$x_{n+1} = \left(\prod_{k=0}^{n} \left(1 + a\lambda^k u_0 \right) \right) x_0.$$

Observe that the infinite product $P(u_0) = \prod_{k=0}^{\infty} \left(1 + a\lambda^k u_0 \right)$ is convergent since, as $|\lambda| < 1$, we have

$$S(u_0) = \ln(P(u_0)) = \sum_{k=0}^{\infty} \ln(1 + a\lambda^k u_0) \sim \sum_{k=0}^{\infty} a\lambda^k u_0 = \frac{a u_0}{1 - \lambda} < \infty.$$

So, for each initial condition (x_0, u_0) the associated orbit converges to the fixed point $(P(u_0)x_0, 0)$.

However, the next example shows that the above situation does not occur when the convergence of $f_1(u_n)$ to 1 is too slow.

Example C: Hyperbolic attracting fiber but slow convergence to the limit dynamics. Consider the system

$$\begin{cases} x_{n+1} & = f_1(u_n)\, x_n, \\ u_{n+1} & = \lambda\, u_n, \end{cases}$$

with $|\lambda| < 1$, and

$$f_1(u) = \begin{cases} 1 - \dfrac{1}{\ln |u|} & \text{for } u \neq 0, \\ 1 & \text{for } u = 0. \end{cases}$$

Again, an straightforward computation gives

$$x_{n+1} = \left(\prod_{k=0}^{n} \left(1 - \frac{1}{k \ln |\lambda| + \ln |u_0|} \right) \right) x_0.$$

Using that $\displaystyle\sum_{k=0}^{\infty} \ln \left(1 - \frac{1}{k \ln |\lambda| + \ln |u_0|} \right) \sim \sum_{k=0}^{\infty} \frac{1}{k \ln |\lambda| + \ln |u_0|}$, which is a divergent series, we get that the infinite product $\displaystyle\prod_{k=0}^{\infty} \left(1 - \frac{1}{k \ln |\lambda| + \ln |u_0|} \right)$ is also divergent. Hence each initial condition (x_0, u_0) gives rise to an unbounded sequence $\{x_n\}$ and, therefore, the dynamics on the basin of attraction of the limit fiber $\{u = 0\}$ is not characterized by the dynamics on the fiber.

The above example leads us to introduce the following definition. As we will see in Example E below, this definition gives a good characterization of the speed of convergence of the terms $f_0(u_n)$ and $f_1(u_n)$ to guarantee the convergence of the orbits in the basin of attraction of the limit fiber.

Definition 3 (Fast enough convergence to the limit dynamics property)**.** *If u_* is a stable attractor of $u_{n+1} = \phi(u_n)$, we say that the attracting fiber $\{u = u_*\}$ of system (1) with $|f_1(u_*)| = 1$, has the fast enough convergence to the limit dynamics property if there exists $\varepsilon > 0$ such that if $\{u_n\}$ is a solution of $u_{n+1} = \phi(u_n)$ satisfying $|u_{n_0} - u_*| \leq \varepsilon$ for some n_0, then there exists p_n such that $|u_n - u_*| \leq p_n \leq \varepsilon$ for all $n \geq n_0$, $\lim_{n \to \infty} p_n = 0$, and also there exist two functions $V, W : [0, \varepsilon) \to \mathbb{R}^+$ such that for all $|u - u_*| < \varepsilon$ it is satisfied:*

H_1: $|f_0(u) - f_0(u_*)| \leq W(|u - u_*|) \leq \overline{W} \in \mathbb{R}$.

H_2: $|f_1(u) - f_1(u_*)| \leq V(|u - u_*|) \leq \overline{V} \in \mathbb{R}$.

H_3: $V(\nu)$ and $W(\nu)$ are non decreasing for $0 \leq \nu < \varepsilon$;

H_4: $S_W = \sum_{j=0}^{\infty} W(p_j) < \infty$ and $S_V = \sum_{j=0}^{\infty} V(p_j) < \infty$.

Definition 4. *Let $u = u_*$ be a stable attractive fixed point of $\phi(u)$. We say that $\phi(u)$ is locally contractive at $u = u_*$ if there exists an open neighborhood V of u_* such that any $u \in V \setminus \{u_*\}$*

$$|\phi(u) - u_*| < |u - u_*|. \tag{4}$$

Prior to stating the next results, we state the following result about local contractivity.

Proposition 5. *Let $\phi : \mathcal{U} \to \mathcal{U}$ be a continuous function and let u_* be a stable attractive fixed point of $\phi(u)$.*

(a) If ϕ is an orientation preserving function, then $\phi(u)$ is locally contractive at $u = u_*$.

(b) There exist orientation reversing functions $\phi(u)$ which are not locally contractive at $u = u_*$.

(c) If $\phi \in \mathcal{C}^1(\mathcal{U})$ and $u = u_*$ is an hyperbolic fixed point of $\phi(u)$ then $\phi(u)$ is locally contractive at $u = u_*$.

Now we are ready to present our result about the cases (B) and (C). Observe that if $f_1(u_*) = 1$ and $f_0(u_*) = 0$, then the limit fiber is filled by fixed points (if $f_0(u_*) \neq 0$ then there are unbounded solutions). When $f_1(u_*) = -1$, the fiber is filled by 2-periodic orbits and there is a fixed point given by $x_* = f_0(u_*)/2$.

Theorem 6. *Consider system (1) where f_0 and f_1 are continuous functions, and $|f_1(u_*)| = 1$. Suppose that $\{u = u_*\}$ is an attracting fiber satisfying the fast enough convergence to the limit dynamics property, and consider initial conditions (x_0, u_0) with u_0 in the basin of attraction of $u = u_*$ for the recurrence $u_{n+1} = \phi(u_n)$. Then*

(a) If $f_1(u_*) = 1$ and $f_0(u_*) = 0$, then there exists $\ell(x_0, u_0) \in \mathbb{R}$ such that $\lim_{n \to \infty} (x_n, u_n) = (\ell(x_0, u_0), u_*)$.

(b) If $f_1(u_*) = -1$ and additionally ϕ is locally contractive at $u = u_*$, then $\lim_{n \to \infty} (x_{2n}, u_{2n}) = (\ell(x_0, u_0), u_*)$ and $\lim_{n \to \infty} (x_{2n+1}, u_{2n+1}) = (f_0(u_*) - \ell(x_0, u_0), u_*)$.

Notice that the above result must not be interpreted in the sense that the limit $\ell(x_0, u_0)$ is different for each initial condition (x_0, u_0).

Example D: Hyperbolic attracting fiber and fast enough convergence to the limit dynamics via Theorem 6. Consider the systems given by

$$\begin{cases} x_{n+1} &= (a + b\,|u_n|^\alpha)\,x_n, \\ u_{n+1} &= \lambda\,u_n, \end{cases}$$

with $|\lambda| < 1$, $\alpha > 0$, $a \in \{-1, 1\}$ and $b \in \mathbb{R}$.

Fixing ε and taking $|u_0| < \varepsilon$; setting $p_n = |\lambda|^n |u_0|$, $W(u) \equiv 0$, $\overline{W} = 0$, $V(u) = B|u|^\alpha$ with $B = |b|$, $\overline{V} = B\varepsilon^\alpha$ we have that hypotheses H_1 and H_3 are trivially fulfilled. With respect hypotheses H_2 we have

$$|f_1(u) - f_1(0)| = B|u|^\alpha = V(|u|) \le B\varepsilon^\alpha = \overline{V}.$$

On the other hand $S_W = 0$ and

$$S_V = \sum_{j=0}^{\infty} V(p_j) = \sum_{j=0}^{\infty} V(|\lambda|^j |u_0|) = \sum_{j=0}^{\infty} B|\lambda|^{j\alpha} |u_0|^\alpha = \frac{B|u_0|^\alpha}{1 - |\lambda|^\alpha},$$

so H_4 is also fulfilled. From Theorem 6, for each initial condition (x_0, u_0) there exists n_0 such that $|u_{n_0}| < \varepsilon$, and also there exists $\ell(x_0, u_0) \in \mathbb{R}$ such that: if $a = 1$ then $\lim_{n \to \infty} (x_n, u_n) = (\ell(x_0, u_0), 0)$, and if $a = -1$, then $\lim_{n \to \infty} (x_{2n}, u_{2n}) = (\ell(x_0, u_0), 0)$ and $\lim_{n \to \infty} (x_{2n+1}, u_{2n+1}) = (-\ell(x_0, u_0), 0)$.

The next example shows that Theorem 6 is optimal when $u = u_*$ is not a hyperbolic attractor of $u_{n+1} = \phi(u_n)$.

Example E: Non-hyperbolic attracting fiber. Optimal characterization of fast enough and slow convergence to the limit dynamics. Consider the systems given by

$$\begin{cases} x_{n+1} &= (1 + b\,|u_n|^\alpha)\,x_n, \\ u_{n+1} &= |u_n| - a|u_n|^k, \end{cases} \tag{5}$$

with $a > 0$, $b \neq 0$, and $k > 1$.

Take $0 < |u_0| < \varepsilon$ for small enough ε, and such that u_0 is in the basin of attraction of 0. Now, we claim that if $\alpha > k - 1$, then there exists $\ell(x_0, u_0) \in \mathbb{R}$ such that $\lim_{n \to \infty} (x_n, u_n) = (\ell(x_0, u_0), 0)$; and if $\alpha \le k - 1$ then $\{x_n\}$ is a divergent sequence if $b > 0$, and $\lim_{n \to \infty} x_n = 0$ if $b < 0$. To prove this claim we use Theorem 1.1 of [11] (see also [12, Problem 174]), which gives the

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asymptotic behavior of certain sequences converging to a non-hyperbolic fixed point, obtaining that $u_n \sim 1/((k-1)\,a\,n)^{1/k-1}$. Thus, taking logarithms in Equation (5) we have,

$$\ln|x_{n+1}| = \sum_{j=1}^{\infty} \ln|1 + b\,|u_j|^{\alpha}| + \ln|x_0| \sim b \sum_{j=1}^{\infty} |u_j|^{\alpha} \sim \frac{b}{((k-1)a)^{\frac{\alpha}{k-1}}} \sum_{j=1}^{\infty} \frac{1}{j^{\frac{\alpha}{k-1}}}.$$

Hence if $\alpha > k-1$, then $\{\ln|x_n|\}$ is a convergent sequence, and if $\alpha \le k-1$ then $\lim_{n\to\infty} \ln|x_n| = \text{sign}(b)\infty$, and the claim is proved.

At this point we will see that the criterium given by Theorem 6 is sharp. By fixing $\varepsilon > 0$ small enough, taking $|u_0| < \varepsilon$, and using Theorem 1.1 of [11], we have that for the sequence $\{u_n\}$ defined by (5) and for all $\delta > 0$, there exist n_0 and $A > 0$ such that if $n > n_0$ then

$$|u_n| \le \frac{A}{n^{\frac{1}{k-1}}} \le \frac{1}{n^{\frac{1}{k-1}-\delta}}.$$

Setting

$$p_n = \frac{1}{n^{\frac{1}{k-1}-\delta}},$$

$W(u) \equiv 0$, $\overline{W} = 0$, $V(u) = |u|^{\alpha}$, and $\overline{V} = \varepsilon^{\alpha}$ we have that for any $\delta > 0$, the hypotheses H_1–H_3 are trivially fulfilled, and with respect H_4 we have that $S_W = 0$ and

$$S_V = \sum_{j=1}^{\infty} V(p_j) = \sum_{j=1}^{\infty} \frac{1}{j^{\frac{\alpha}{k-1}-\delta}}.$$

Observe that S_V is convergent if and only if $\frac{\alpha}{k-1} - \delta > 1$ for all $\delta > 0$ or, in other words, if and only if

$$\alpha > (k-1)(1+\delta) \text{ for all } \delta > 0.$$

Hence, Theorem 6 guarantees convergence of the sequence $\{(x_n, u_n)\}$ if $\alpha > k-1$ for all $b \ne 0$, which is the optimal value.

The next result shows that if the map (1) is differentiable, and $u = u_*$ is a hyperbolic attractor of $u_{n+1} = \phi(u_n)$, then the hypotheses of Theorem 6 are fulfilled. Furthermore, each point in the attracting fiber is the limit of an orbit of the basin of attraction of the fiber.

Corollary 7. *Consider system (1) where $f_0, f_1, \phi \in \mathcal{C}^1(\mathcal{U})$ where $\mathcal{U}$ is a neighborhood of $u = u_*$, and $|f_1(u_*)| = 1$. Suppose that $u = u_*$ is a hyperbolic attractor of ϕ. Then, for any u_0 in the basin of attraction of $u = u_*$ there exists $\ell(x_0, u_0) \in \mathbb{R}$ such that:*

(a) If $f_0(u_*) = 0$ and $f_1(u_*) = 1$, then $\lim_{n\to\infty} (x_n, u_n) = (\ell(x_0, u_0), u_*)$.

(b) If $f_1(u_*) = -1$, then $\lim_{n\to\infty} (x_{2n}, u_{2n}) = (\ell(x_0, u_0), u_*)$ and $\lim_{n\to\infty} (x_{2n+1}, u_{2n+1}) = (f_0(0) - \ell(x_0, u_0), u_*)$.

Furthermore, for any point $(x_*, u_*) \in \{u = u_*\}$ there exists an initial condition (x_0, u_0) in the basin of attraction of the limit fiber such that $\lim_{n\to\infty} (x_n, u_n) = (x_*, u_*)$.

3 Proofs of the main results

In this section we prove the main results of the paper: Proposition 2, Theorem 6 and Corollary 7.

Proof of Proposition 2. First observe that, from the continuity of f_1, there exists ε such that for all u such that $|u - u_*| < \varepsilon$, we have $|f_1(u)| < N$ with $N < 1$. Consider (x_0, u_0) such that u_0 is in the basin of attraction of $u = u_*$. From the hypotheses, we can assume that there exists n_0 such that for all $n \geq n_0$, we have $|u_n - u_*| < \varepsilon$ hence, since f_0 is continuous, $|f_0(u_n)| < M =: \sup_{|u-u_*|<\varepsilon} |f_0(u)|$. Thus we have $|x_{n+n_0+1}| \leq M + N|x_{n+n_0}|$. Applying this last inequality we obtain

$$
\begin{aligned}
|x_{n+n_0+1}| &\leq M + N|x_{n+n_0}| \leq M + N\left(M + N|x_{n+n_0-1}|\right) \leq \ldots \leq M\left(\sum_{i=0}^{n} N^i\right) + N^n|x_{n_0}| \\
&\leq \tfrac{M}{N-1} + |x_{n_0}|,
\end{aligned}
$$

and therefore the sequence is bounded. As a consequence $i := \liminf_{n\to\infty} \in \mathbb{R}$ and $s := \limsup_{n\to\infty} \in \mathbb{R}$. Hence, there are subsequences $\{(x_{n_k}, u_{n_k})\} \to (i, u_*)$ and $\{(x_{n_j}, u_{n_j})\} \to (s, u_*)$. For these subsequences, from equation (1), and using the continuity of f_0 and f_1, we have

$$
s = f_0(u_*) + f_1(u_*)s \text{ and } i = f_0(u_*) + f_1(u_*)i,
$$

hence we have that $i = s = f_0(u_*)/(1 - f_1(u_*))$. ∎

Proof of Proposition 5. To prove (a), since $u = u_*$ is a stable attractor, we can take $\varepsilon > 0$ such that for all u with $|u - u_*| < \varepsilon$, $\lim_{n\to\infty} \phi^n(u) = u_*$. In particular, u_* is the only fixed point in the neighborhood $(u_* - \varepsilon, u_* + \varepsilon)$. Observe that since ϕ preserves orientation, if $u > u_*$ then $\phi(u) > u_*$.

Assume that statement (a) is not true. Given a sequence $\{\varepsilon_n\} \to 0$, then for each n with $\varepsilon_n < \varepsilon$ we can find a point u_n such that

$$
|u_n - u_*| < \varepsilon_n \quad \text{and} \quad |\phi(u_n) - u_*| \geq |u_n - u_*|. \tag{6}
$$

Taking, if necessary, a subsequence, we can assume that $u_n > u_*$ for all n (the opposite assumption can be treated in a similar manner), and also that the subsequence is monotone decreasing. Notice that, since $u_n > u_*$ and ϕ preserves orientation, also $\phi(u_n) > u_*$ and the second inequality in (6) reads as

$$
\phi(u_n) \geq u_n \text{ for all } n \in \mathbb{N}.
$$

On the other hand, we can consider the sequence of iterates by ϕ of one point $w_0 > u_*$ such that $w_0 - u_* < \varepsilon$, obtaining $w_n := \phi^n(w_0) > u_*$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} w_n = u_*$. That is, we

can choose a subsequence w_{k_n} with the property that $w_{k_n} < u_n$ and, as before, being monotone decreasing. Considering the continuous function $\phi - Id$ we have that for all $n \in \mathbb{N}$,

$$(\phi - Id)(u_n) = \phi(u_n) - u_n \geq 0 \text{ and } (\phi - Id)(w_{k_n}) = \phi(w_{k_n}) - w_{k_n} < 0.$$

Hence, for all $n \in \mathbb{N}$ it exists t_n such that

$$w_{k_n} < t_n < u_n \ , \ \phi(t_n) = t_n.$$

It implies that $\lim_{n \to \infty} t_n = u_*$ and u_* is not isolated as a fixed point of ϕ. A contradiction with our assumptions.

In order to see (b) consider the function $\phi(u) = -u - u^2$. This map has two fixed points $u = 0$ and $u = -2$. Since $\phi'(-2) = 3 > 1$, the point $u = -2$ is a repellor of the function $\phi(u)$. The point $u = 0$ is not an hyperbolic fixed point because $\phi'(0) = -1$, but it is easy to see that the interval $\left[-2, \frac{1}{4}\right]$ is invariant under the action of ϕ and that for all $u \in \left(-2, \frac{1}{4}\right)$ and $\lim_{n \to \infty} \phi^n(u) = 0$. Then, $u_* = 0$ is an attracting fixed point of $\phi(u)$.

On the other hand, if $u > 0$ then $\phi(u)$ satisfies $|\phi(u)| = |u + u^2| > |u|$, hence condition (4) is not satisfied.

Statement (c) is a simple consequence of the mean value theorem.

$\blacksquare$

Proof of Theorem 6. In order to prove statement (a) consider (x_0, u_0) such that u_0 is in the basin of attraction of $u = u_*$. First we prove that the sequence x_n is bounded. From the hypotheses, if $\varepsilon > 0$ is small enough, there exists n_0 such that for all $n \geq n_0$, $|u_n - u_*| < \varepsilon$. Hence, to simplify the notation, in the following we will assume that the point (x_0, u_0) is such that $|u_0 - u_*| < \varepsilon$. Furthermore,

(1) From the fact that $W(\nu)$ is non decreasing for $0 \leq \nu < \varepsilon$, if n is large enough $|f_0(u_n) - f_0(u_*)| = |f_0(u_n)| \leq W(|u_n - u_*|) \leq W(p_n) \leq \overline{W}$.

(2) Analogously $|f_1(u_n)| \leq |f_1(u_*)| + V(|u_n - u_*|) \leq 1 + V(p_n)$.

In summary, we have $|x_{n+1} - f_0(u_*)| \leq W(p_n) + (1 + V(p_n)) |x_n|$. Applying this last inequality we obtain

$$
\begin{aligned}
|x_{n+1} - f_0(u_*)| \ &\leq W(p_n) + (1 + V(p_n)) |x_n| \\
&\leq W(p_n) + (1 + V(p_n)) W(p_{n-1}) + (1 + V(p_n)) (1 + V(p_{n-1})) |x_{n-1}| \\
&\leq \ldots \leq W(p_n) + \sum_{i=0}^{n-1} \left[W(p_i) \prod_{j=i+1}^{n} (1 + V(p_j)) \right] + \prod_{j=0}^{n} (1 + V(p_j)) |x_0|.
\end{aligned}
$$

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Observe that from hypotheses H_2 we have $V(0) = 0$, and by hypotheses H_3, for all $j = 0, \ldots n$ we have that $1 + V(p_j) \geq 1$. So

$$\prod_{j=i+1}^{n} 1 + V(p_j) \leq \prod_{j=0}^{n} 1 + V(p_j) \leq \prod_{j=0}^{\infty} 1 + V(p_j) =: P.$$

Observe that $P < +\infty$, because

$$S = \ln(P) = \sum_{j=0}^{\infty} \ln\left(1 + V(p_j)\right) \sim \sum_{j=0}^{\infty} V(p_j) = S_V < +\infty$$

Hence, regarding that if n is large enough then $W(p_n) \leq \overline{W}$, we have

$$|x_{n+1} - f_0(u_*)| \leq \overline{W} + P\left(\sum_{i=0}^{n-1} W(p_i)\right) + P|x_0| \leq \overline{W} + P\,S_W + P\,|x_0|,$$

so the sequence $\{x_n\}$ is bounded, and therefore also the sequence $\{(x_n, u_n)\}$ is bounded.

Now we are going to see that $\{x_n\}$ is a Cauchy sequence and therefore it has a limit $\ell(x_0, u_0)$. First observe that if R is a bound of $\{|x_n|\}$ then

$$|x_{n+1} - x_n| \leq |f_0(u_n)| + |f_1(u_n) - 1|\,|x_n| \leq W(p_n) + V(p_n)R.$$

Therefore, since the series S_W and S_V are convergent, then for all $\varepsilon > 0$ there exists $N > 0$ such that for all $n, m > N$

$$\begin{aligned} |x_{n+m} - x_n| &\leq |x_{n+m} - x_{n+m-1}| + \ldots + |x_{n+1} - x_n| \\ &\leq \sum_{j=n}^{n+m-1} W(p_j) + R\sum_{j=n}^{n+m-1} V(p_j) \leq \varepsilon, \end{aligned}$$

and therefore $\{x_n\}$ is a Cauchy sequence, which completes the proof of statement (a).

In order to see (b), observe that

$$\begin{aligned} x_{n+2} &= f_0(u_{n+1}) + f_1(u_{n+1})\left(f_0(u_n) + f_1(u_n)x_n\right) \\ &=: F_0(u_n) + F_1(u_n)x_n, \end{aligned}$$

where $F_0(u) = f_0(\phi(u)) + f_1(\phi(u))f_0(u)$ and $F_1(u) = f_1(\phi(u))f_1(u)$. After renaming $y_n = x_{2n}$, $v_n = u_{2n}$, $v_* = u_*$ and $\varphi = \phi \circ \phi$ we get the system

$$\begin{cases} y_{n+1} &= F_0(v_n) + F_1(v_n)y_n, \\ v_{n+1} &= \varphi(v_n), \end{cases} \tag{7}$$

which is a system of type (1). Notice that when we take (x_0, u_0) (resp. (x_1, u_1)) as initial condition and we apply (7) repeatedly we get (x_{2n}, u_{2n}) (resp. (x_{2n+1}, u_{2n+1})).

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We are going to show that system (7) satisfies the hypotheses H_1–H_4, and since $F_0(v_*) = 0$ and $F_1(v_*) = 1$, the result will follow from the convergence of the sequence $\{(y_n, v_n)\}$ guaranteed by statement (a).

Adding and subtracting both $f_0(v_*)$ and $f_1(\phi(v))f_0(v_*)$ to $F_0(v)$ we get,

$$\begin{aligned} |F_0(v)| &\leq |f_0(\phi(v)) - f_0(v_*)| + |f_1(\phi(v))| \, |f_0(v) - f_0(v_*)| + |f_0(v_*) \, (f_1(\phi(v)) + 1)| \\ &\leq W(|\phi(v) - v_*|) + |f_1(\phi(v))| \, W(|v - v_*|) + |f_0(v_*)| \, V(|\phi(v) - v_*|). \end{aligned}$$

Since u_* is a stable attractor of $u_{n+1} = \phi(u_n)$ and $\phi(u)$ is locally contractive at $u = u_*$, if $|v - v_*| < \varepsilon$ then $|\phi(v) - v_*| < |v - v_*|$. Using that W and V satisfy H_3, we get

$$\begin{aligned} |F_0(v)| &\leq W(|v - v_*|) + |f_1(\phi(v))| \, W(|v - v_*|) + |f_0(v_*)| \, V(|v - v_*|) \\ &\leq A\,W(|v - v_*|) + B\,V(|v - v_*|), \end{aligned}$$

where $A := 1 + \sup_{|v - v_*| \leq \varepsilon} |f_1(\phi(v))|$ and $B := |f_0(v_*)|$. Now we obtain a new function $\widetilde{W}(\nu) := AW(\nu) + BV(\nu)$, satisfying H_1 for system (7).

On the other hand, using the inequality $|ab - 1| = |a(b+1) - (a+1)| \leq |a| \, |b+1| + |a+1|$, we have

$$\begin{aligned} |F_1(v) - 1| &= |f_1(\phi(v))f_1(v) - 1| \leq |f_1(v)| \, |f_1(\phi(v)) + 1| + |f_1(v) + 1| \\ &\leq |f_1(v)| \, V(|\phi(v) - v_*|) + V(|v - v_*|) \\ &\leq |f_1(v)| \, V(|v - v_*|) + V(|v - v_*|) \\ &\leq C\,V(|\phi(v) - v_*|), \end{aligned}$$

where $C := \sup_{|v - v_*| \leq \varepsilon} |f_1(v)|$, obtaining the new function $\widetilde{V}(\nu) := CV(\nu)$, satisfying H_2 for system (7).

Now, it is straightforward to prove that $\widetilde{V}(v)$ and $\widetilde{W}(v)$ satisfy hypotheses H_3 and H_4. Hence each sequence $\{y_n\}$ is convergent, and therefore the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are convergent.

Finally, observe that setting $\ell_e(x_0, u_0) := \lim_{n \to \infty} x_{2n}$ and $\ell_o(x_0, u_0) := \lim_{n \to \infty} x_{2n+1}$, and using the continuity of f_0 and f_1, equation (1) yields $\ell_o = f_0(u_*) - \ell_e$. $\blacksquare$

Proof of Corollary 7. If $u = u_*$ is a hyperbolic attractor of ϕ, then for ε small enough, we have that $\mu := \sup_{|u - u_*| < \varepsilon} |\phi'(u)| < 1$. Setting $p_n = \mu^n |u_0 - u_*|$, $W(\nu) := A|\nu|$, and $V(\nu) := B|\nu|$, where A and B are the suprema in $|u - u_*| < \varepsilon$ of $|f_0'|$ and $|f_1'|$ respectively, and using the mean value theorem, one gets straightforwardly, that the hypotheses H_1–H_4 are satisfied. So statements (a) and (b) follow from Theorem 6.

Observe that for each point $(x_*, u_*) \in \{u = u_*\}$, the differential matrix of the map associated to system (1), $F(x, u) = (f_0(u) + f_1(u)x, \phi(u))$, is given by

$$DF(x_*, u_*) = \begin{pmatrix} 1 & f_0'(u_*) + f_1'(u_*)x_* \\ 0 & \phi'(u_*) \end{pmatrix},$$

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where $|\phi'(u_*)| < 1$. Hence from the stable manifold theorem (see [4] or [8] for instance) there is an invariant $\mathcal{C}^1$ curve, transversal to $\{u = u_*\}$ at (x_*, u_*), such that any initial condition on this curve gives rise to a solution with limit (x_*, u_*). ∎

4 An application: linear quasi-homogeneous maps

We say that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a *quasi-homogeneous function* with weights (α, β) and quasi-degree d if

$$f(\lambda^\alpha x, \lambda^\beta y) = \lambda^d f(x, y), \text{ for all } \lambda > 0.$$

Notice that if f is a quasi-homogeneous function with weights (α, β) then, for all constant $c \neq 0$, f is also a quasi-homogeneous function with weights $(c\alpha, c\beta)$. Hence we will consider $(\alpha, \beta) \in \mathbb{Z}^2$ with $\gcd(\alpha, \beta) = 1$.

We say that $F = (f, g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a *quasi-homogeneous map* with weights (α, β) and quasi-degree d if $f(x, y)$ and $g(x, y)$ are quasi-homogeneous functions with weights (α, β) and quasi-degrees $d\alpha$ and $d\beta$, respectively. This section deals with quasi-homogeneous maps of quasi-degree 1 (which are called *linear quasi-homogeneous maps*) with weights of different sign.

Assuming that $\alpha > 0$ and $\beta < 0$, from the definition it follows that in the class of analytic maps, any linear quasi-homogeneous map takes the form

$$F(x, y) = \left(x\, p(x^{-\beta} y^\alpha), y\, q(x^{-\beta} y^\alpha) \right),$$

where $p(z)$ and $q(z)$ are analytic functions. These maps preserve the fibration given by $\mathcal{F} = \{x^{-\beta} y^\alpha = h, h \in \mathbb{R}\}$, since they send the curves $x^{-\beta} y^\alpha = h$ to the curves $x^{-\beta} y^\alpha = h\, p(h)^{-\beta} q(h)^\alpha$. The dynamical system associated to F is:

$$\begin{cases} x_{n+1} &= x_n\, p(x_n^{-\beta} y_n^\alpha), \\ y_{n+1} &= y_n\, q(x_n^{-\beta} y_n^\alpha). \end{cases}$$

Applying the transformation $(x, y) \to (x, x^{-\beta} y^\alpha)$ and calling $u = x^{-\beta} y^\alpha$, the map $F(x, y)$ is transformed into

$$\tilde{F}(x, u) = \left(x\, p(u), u\, p(u)^{-\beta} q(u)^\alpha \right),$$

which is a triangular map, whose corresponding system is:

$$\begin{cases} x_{n+1} &= x_n\, p(u_n), \\ u_{n+1} &= u_n\, p(u_n)^{-\beta} q(u_n)^\alpha. \end{cases}$$

Notice that such a system is of the form (1) and the fiber $u = 0$ is invariant. In order to apply Proposition 2 to study the basin of attraction of the origin, we need that $|p_0| < 1$ and that $u = 0$ is an attractive point of the subsystem $u_{n+1} = u_n\, p(u_n)^{-\beta} q(u_n)^\alpha$, which is guaranteed by the

condition $|p_0^{-\beta} q_0^{\alpha}| < 1$, where $p_0 = p(0)$ and $q_0 = q(0)$. So, in this case we have $\lim\limits_{n \to \infty} (x_n, u_n) = (0, 0)$ for all $u_0 \in \mathcal{B}$, where $\mathcal{B}$ is the basin of attraction of $u = 0$, and any arbitrary value of x_0.

On the other hand, if we perform the similar change $(x, y) \to (x^{-\beta} y^{\alpha}, y)$ and we call $u = x^{-\beta} y^{\alpha}$, the map $F(x, y)$ is transformed into another triangular map

$$\bar{F}(u, y) = \left(u \, p(u)^{-\beta} q(u)^{\alpha}, y \, q(u) \right),$$

whose associated system is:

$$\begin{cases} u_{n+1} & = u_n \, p(u_n)^{-\beta} q(u_n)^{\alpha}, \\ y_{n+1} & = y_n \, q(u_n). \end{cases}$$

As before we can apply Proposition 2. Thus, if $|q_0| < 1$ and $|p_0^{-\beta} q_0^{\alpha}| < 1$, then $\lim\limits_{n \to \infty} (u_n, y_n) = (0, 0)$ for all $u_0 \in \mathcal{B}$, where $\mathcal{B}$ is the basin of attraction of $u = 0$, for the subsystem $u_{n+1} = u_n \, p(u_n)^{-\beta} q(u_n)^{\alpha}$ and y_0 is arbitrary. Hence, we get:

Proposition 8. *Assume that $|p_0| < 1$ and $|q_0| < 1$ and let $\mathcal{B}$ be the basin of attraction of $u = 0$ for the one-dimensional system $u_{n+1} = u_n \, p(u_n)^{-\beta} q(u_n)^{\alpha}$. Then for all (x_0, y_0) such that $x_0^{-\beta} y_0^{\alpha} \in \mathcal{B}$, the sequence $\{(x_n, y_n)\}$ tends to $(0, 0)$ as $n \to \infty$.*

Example F. As a particular easy case we can take $(\alpha, \beta) = (1, -1)$ and

$$\begin{cases} x_{n+1} & = x_n \, (a + b x_n y_n), \\ y_{n+1} & = y_n \, (c + d x_n y_n), \end{cases} \tag{8}$$

which is transformed in:

$$\begin{cases} x_{n+1} & = x_n \, (a + b u_n), \\ u_{n+1} & = u_n \, (a + b u_n) \, (c + d u_n), \end{cases}$$

with $|a| < 1$ and $|c| < 1$. To determine the basin of attraction of $u = 0$ for arbitrary values of a, b, c, d is not an easy task. In fact, the dynamics of the one dimensional system $u_{n+1} = \phi(u_n)$ with $\phi(u) := u(a + bu)(c + du)$ can be very complicated. For instance, if such map has a 3-periodic point, then it has periodic points for all the periods. It means that the *intrafibration* dynamics of the hyperbolas $xy = h$ for system (8) can be complicated.

One case for which the dynamics of $u_{n+1} = \phi(u_n)$ is simple is just when we have three fixed points with alternating stability. For instance, if we consider $a = 2/(3d), b = -1/(6d)$ and $c = d$, then we get three fixed points $u = 0, u = 1$ and $u = 2$. Moreover $\phi'(0) = \phi'(2) = 2/3$ so 0 and 2 are attractive points and $\phi'(1) = 7/6$ and 1 is a repelling point.

There are three preimages of the fixed point 1, that are 1, $p_1 = 1 - \sqrt{7}$ and $p_2 = 1 + \sqrt{7}$. It easily follows that $I_1 := (p_1, 1)$ is contained in $\mathcal{B}_0$, the basin of attraction of $u = 0$, while $J_1 := (1, p_2)$ is contained in $\mathcal{B}_2$, the basin of attraction of $u = 2$. Setting $I_2 = \phi^{-1}(I_1) \setminus I_1$ and

$I_{n+1} = \phi^{-1}(I_n)$ for $n > 2$; and $J_2 = \phi^{-1}(J_1) \setminus J_1$, $J_{n+1} = \phi^{-1}(J_n)$ we have that these intervals are interlaced as

$$\ldots I_7,\ J_6,\ I_5,\ J_4,\ I_3,\ J_2,\ I_1,\ J_1,\ I_2,\ J_3,\ I_4,\ J_5,\ I_6,\ \ldots$$

where the right extreme of each interval J_{2k} equals the left one of I_{2k-1} and the right extreme of each interval J_{2k-1} equals the left one of I_{2k}. These boundary points are exactly the preimages of the fixed point $u = 1$. Furthermore, these intervals have decreasing length, tending to the two-periodic points $q_\pm = 1 \pm \sqrt{13}$.

It is very easy to prove that ϕ sends $\mathbb{R} \setminus [q_-, q_+]$ to itself. Collecting all the above observations we get:

Lemma 9. *Consider the one dimensional discrete system generated by*

$$\phi(u) = \frac{1}{6} u \, (4 - u) \, (1 + u).$$

Let $\mathcal{B}_0$ (resp. $\mathcal{B}_2$ or $\mathcal{B}_\infty$) be the basin of attraction of $u = 0$ (resp. $u = 2$ or infinity) and let $\mathcal{O}_1 := \cup_{n \in \mathbb{N}} \phi^{-n}(1)$ be, the set of preimages of the repulsive fixed point $u = 1$. Then

$$\mathcal{B}_0 = \cup_{n \in \mathbb{N}} \phi^{-n}(I_1), \quad \mathcal{B}_2 = \cup_{n \in \mathbb{N}} \phi^{-n}(J_1) \quad \text{and} \quad \mathcal{B}_0 \cup \mathcal{B}_2 \cup \mathcal{O}_1 = (q_-, q_+),$$

where $q_\pm = 1 \pm \sqrt{13}$ is the unique orbit of $\phi(u)$ with minimal period 2. Moreover

$$\mathbb{R} = \mathcal{B}_0 \cup \mathcal{B}_2 \cup \mathcal{O}_1 \cup \mathcal{B}_\infty \cup \{p_-, p_+\}.$$

Once the basin of attraction of $u = 0$ is determined, we can come back to system (8).

Proposition 10. *Let $\mathcal{B}_0$ be the basin of attraction of $u = 0$ for $\phi(u) = u \, (4 - u) \, (1 + u) \, /6$. For $d \neq 0$, consider the system*

$$\begin{cases} x_{n+1} & = \dfrac{1}{6d} \, x_n \, (4 - x_n y_n), \\ y_{n+1} & = d \, y_n \, (1 + x_n y_n). \end{cases} \tag{9}$$

(i) For $2/3 < |d| < 1$,

    (a) If $x_0 y_0 \in \mathcal{B}_0$ then $\lim\limits_{n \to \infty} (x_n, y_n) = (0, 0)$; see Figure 1.

    (b) If $x_0 y_0 \notin \mathcal{B}_0$ then $\lim\limits_{n \to \infty} |x_n| + |y_n| = \infty$.

(ii) For $|d| \in \mathbb{R}^+ \setminus [2/3, 1]$, if $x_0 y_0 \neq 0$ then $\lim\limits_{n \to \infty} |x_n| + |y_n| = \infty$.

Proof. (i) By using Lemma 9, assertion (a) is a consequence of Proposition 8.

To prove (b), assume that $x_0 y_0 \notin \mathcal{B}_0$. Then, again by Lemma (9), the sequence $\{u_n\}$, where $u_n = x_n y_n$, has four possibilities when n goes to infinity: either it tends to 2, or after some iterates it is constant equal to 1, or it takes the values q_-, q_+, or it tends to infinity. Using

$y_{n+1} = dy_n(1 + u_n)$, we see that in the first case, when n is big enough $y_{n+1} \simeq 3d\, y_n$. Since $2/3 < |d| < 1$, it follows that $|y_n|$ tends to ∞ when n goes to infinity. The other cases follow similarly.

The proof of (ii) is a consequence of Lemma 9 and the dynamics of (9) on the invariant sets $xy = 0, xy = 1, xy = 2$ and $(xy - q_-)(xy - q_+) = 0$. For instance, on the third one,

$$\begin{cases} x_{n+1} &= \dfrac{1}{3d}\, x_n, \\ y_{n+1} &= 3d\, y_n, \end{cases}$$

and clearly the orbits go towards infinity on it. ∎

Notice that system (9) presents interesting bifurcation when $|d| \in \{2/3, 1\}$. In particular, as a consequence of the shape of $\mathcal{B}_0$, explained above, when $2/3 < |d| < 1$, the basin of attraction of $(0,0)$ of this system is formed by the union of infinitely many disjoint hyperbolic-shaped bands which accumulate to the hyperbolas $xy = q_\pm$; see Figure 1.
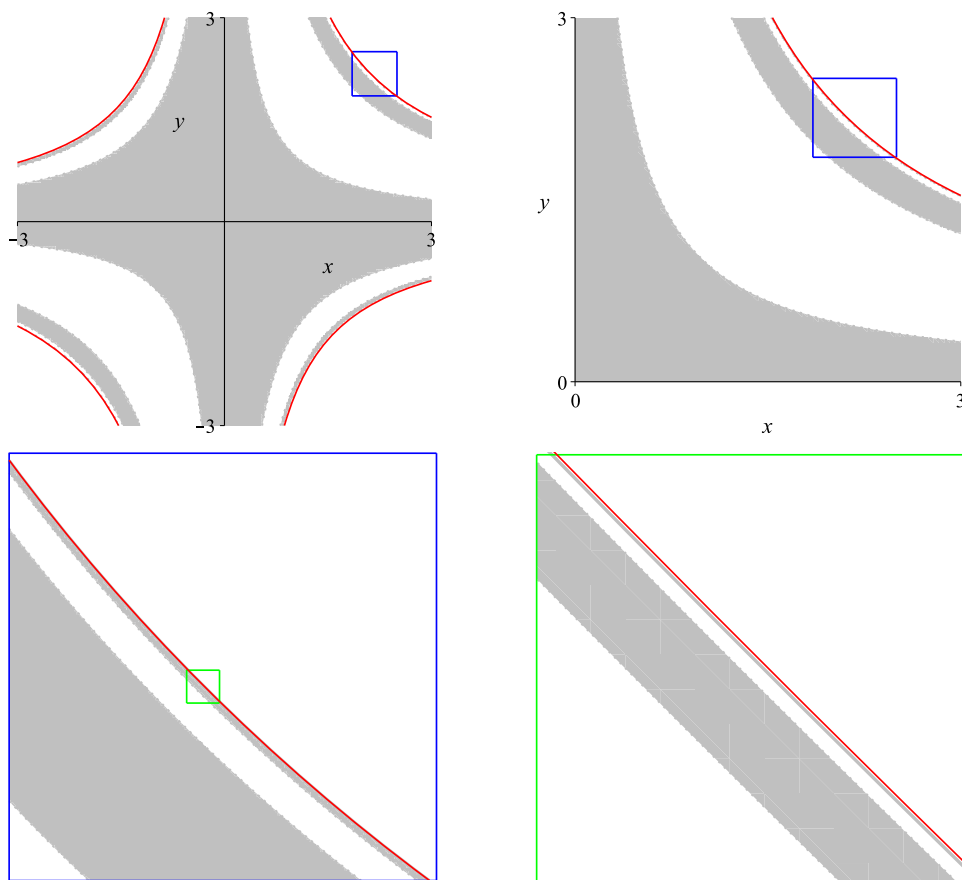

Figure 1: Details of the basin of attraction of $(0,0)$ of system (9) for $2/3 < |d| < 1$, shaded in grey.

5 Applicability to difference equations

As a final remark, we notice that the range of applications for the theory developed in this paper is wide, and the main results can be applied to study the global dynamics of different types of difference equations. For instance, equations of multiplicative type $x_{n+k} = x_n g(x_n \cdots x_{n+k-1})$ for $k \geq 2$, additive equations of the type $x_{n+2} = -bx_{n+1} + g(x_{n+1} + bx_n)$, or equations of the form $x_{n+3} = g(x_n x_{n+2})/x_{n+1}$, where in all the cases g is a $\mathcal{C}^1$ function. We will not develop the analysis here. The reader is referred to [5, Sections 4.2–4.4] for further details. These type of equations have been also considered in [1, 3, 7, 9].

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