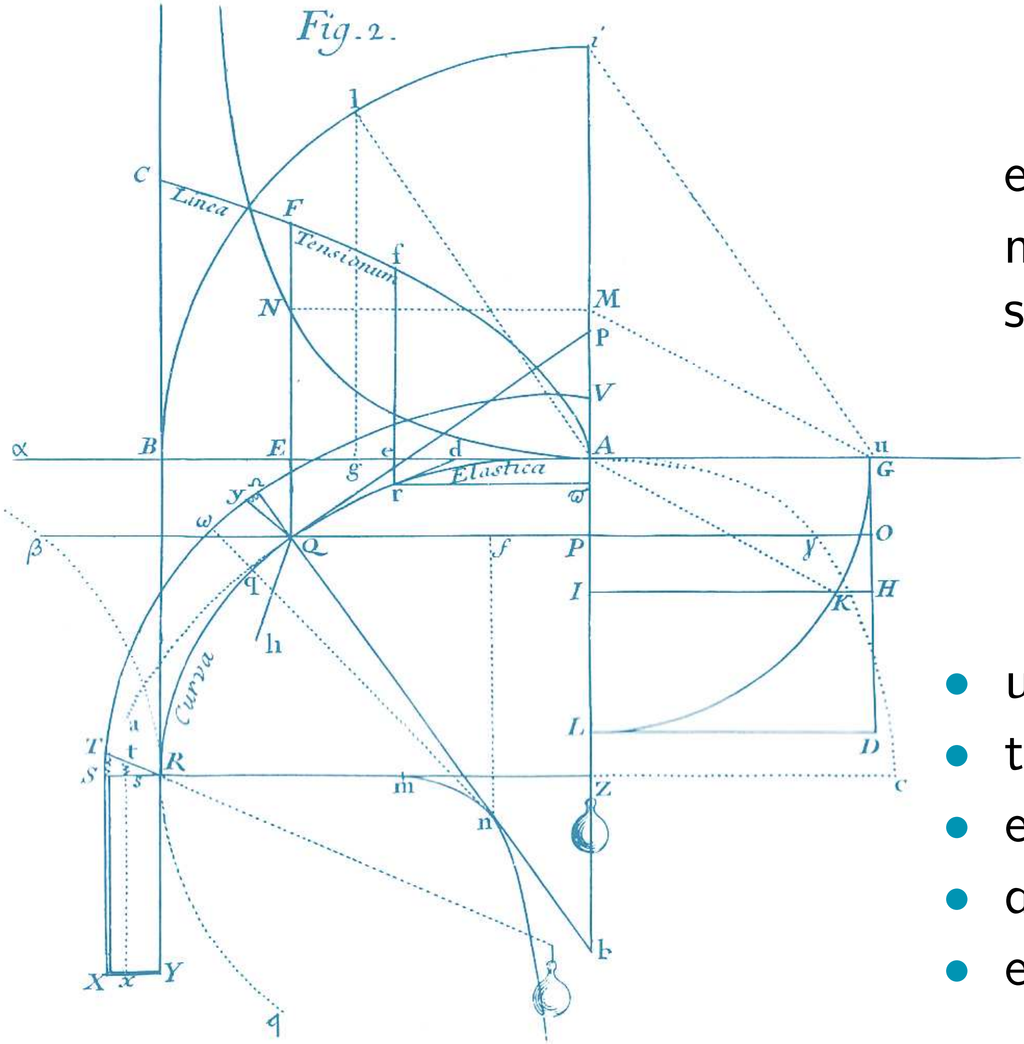
beam bending
– euler bernoulli vs timoshenko –



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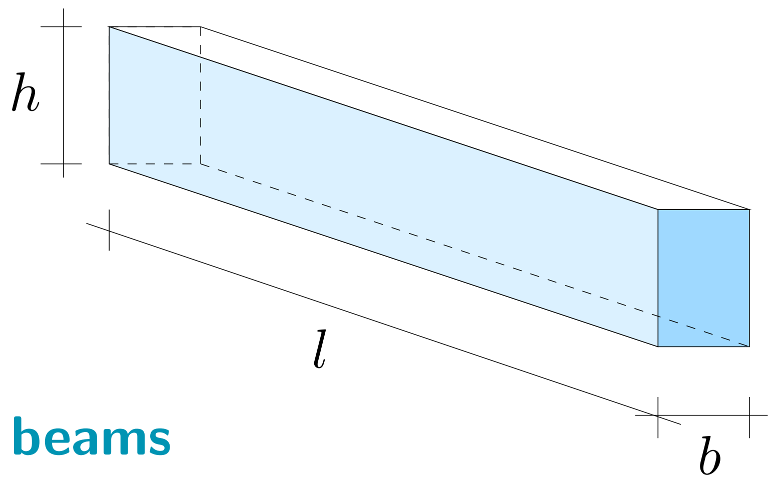
- uniaxial bending
- timoshenko beam theory
- euler bernoulli beam theory
- differential equation
- examples

kinematic assumptions

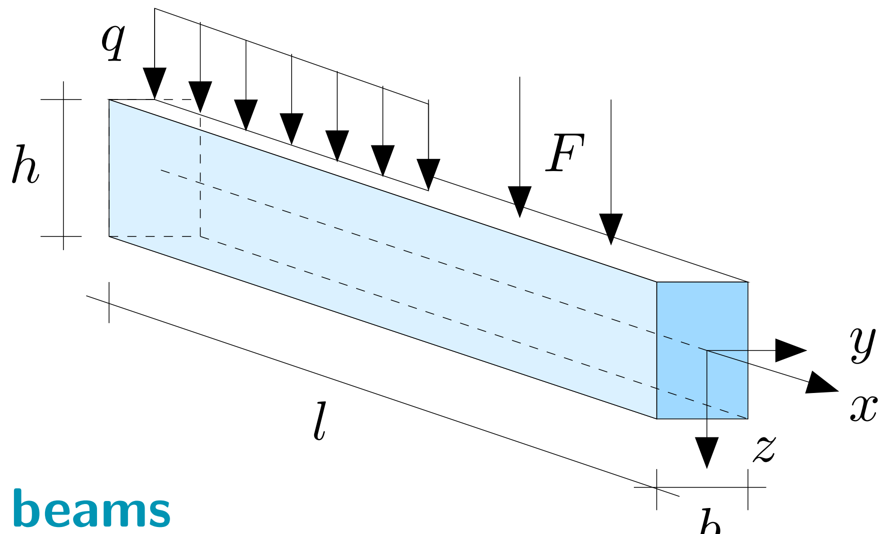


beams

[1] width and height $b, h \ll$ length l

[2] forces **orthogonal** to beam axes

kinematic assumptions



beams

[1] width and height $b, h \ll$ length l

[2] forces **orthogonal** to beam axes

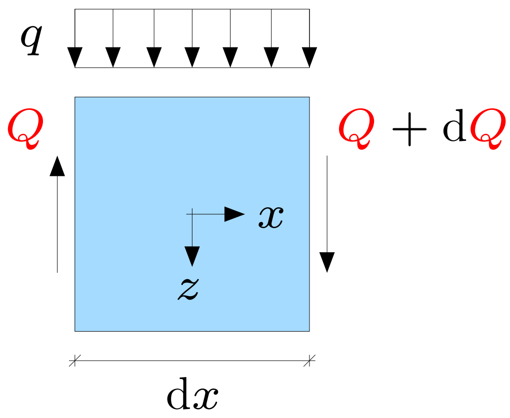
planar / uniaxial bending

[3] external **forces** in x-z-**plane**

[4] cross section **symmetric** to z-axis

[5] coordinate system in axis of gravity

governing equations of beam theory

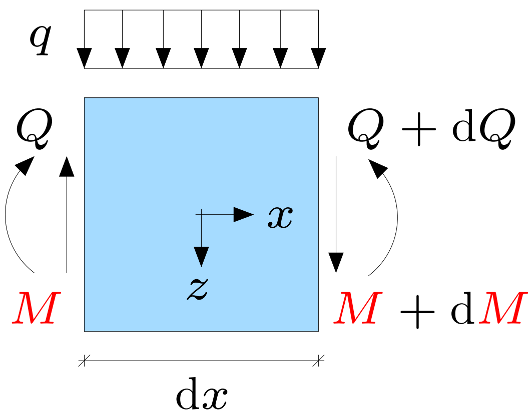


- equilibrium

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q$$

$$Q = \int \tau \,\mathrm{d}A$$

governing equations of beam theory



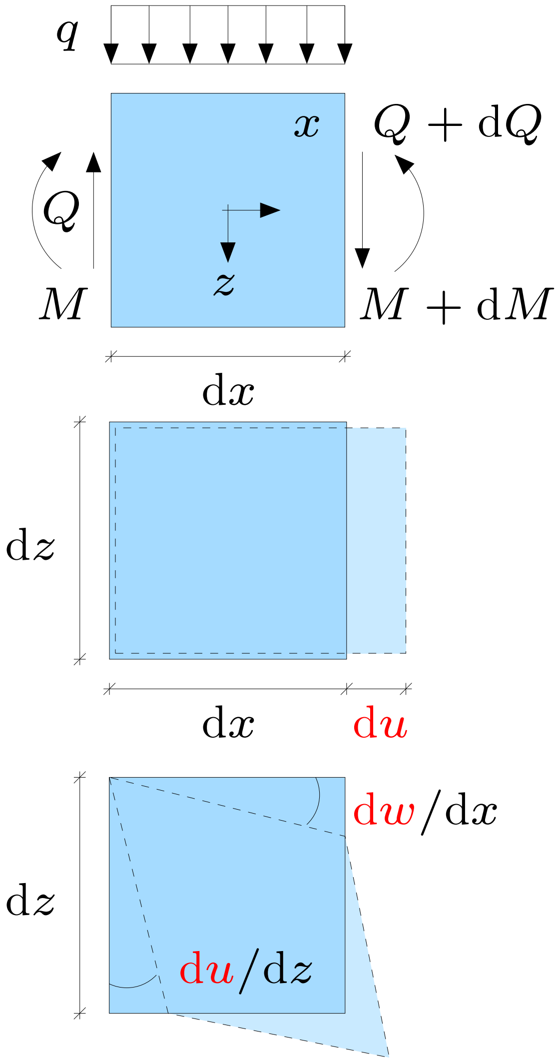
- equilibrium

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = Q$$

$$Q = \int \tau \, \mathrm{d}A \qquad M = \int z \, \sigma \, \mathrm{d}A$$

uniaxial bending

governing equations of beam theory



- equilibrium

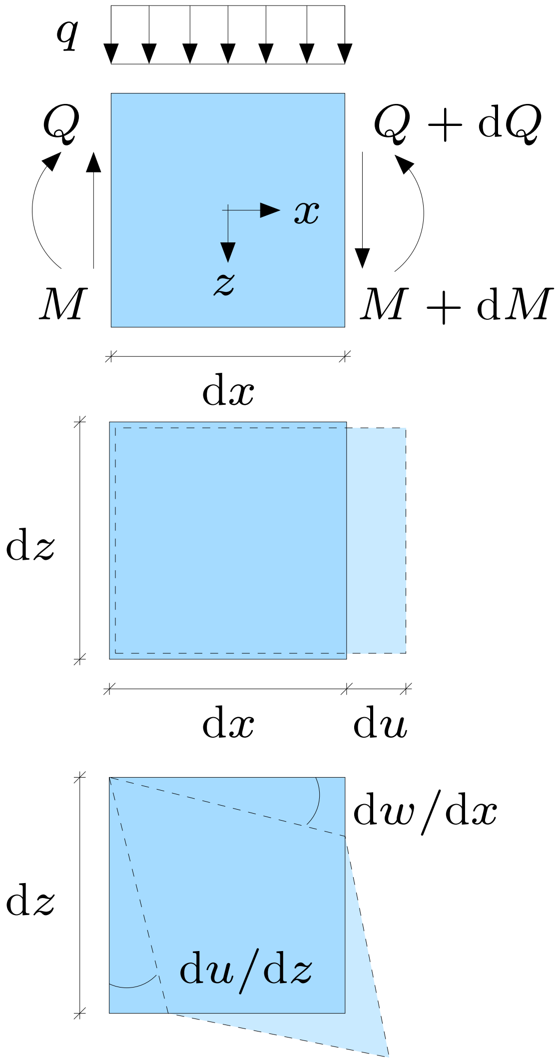
$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = Q$$

$$Q = \int \tau \, \mathrm{d}A \qquad M = \int z \, \sigma \, \mathrm{d}A$$

- kinematics

$$\epsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \gamma = \frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z}$$

governing equations of beam theory



- equilibrium

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = Q$$

$$Q = \int \tau \, \mathrm{d}A \qquad M = \int z \, \sigma \, \mathrm{d}A$$

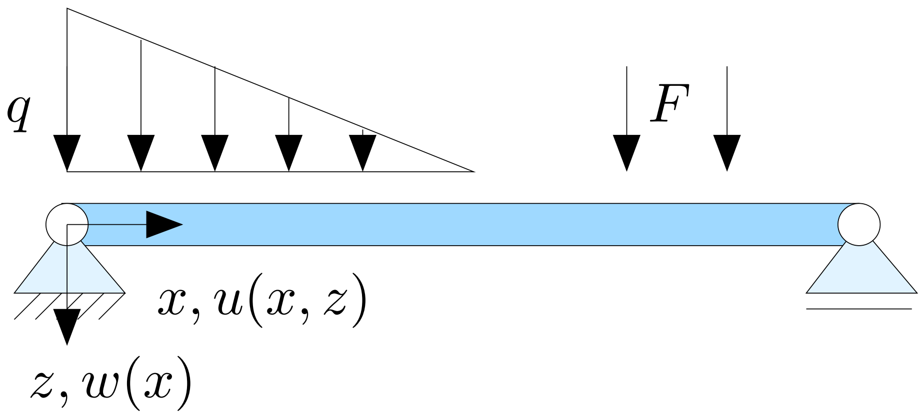
- kinematics

$$\epsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \gamma = \frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z}$$

- constitutive equations

$$\sigma = E\,\epsilon \qquad \tau = G\,\gamma$$

kinematic assumptions



[1] the **deflection** w is independent of z $\qquad w = w(x)$

all points of a cross section undergo the same displacement in z-dirction

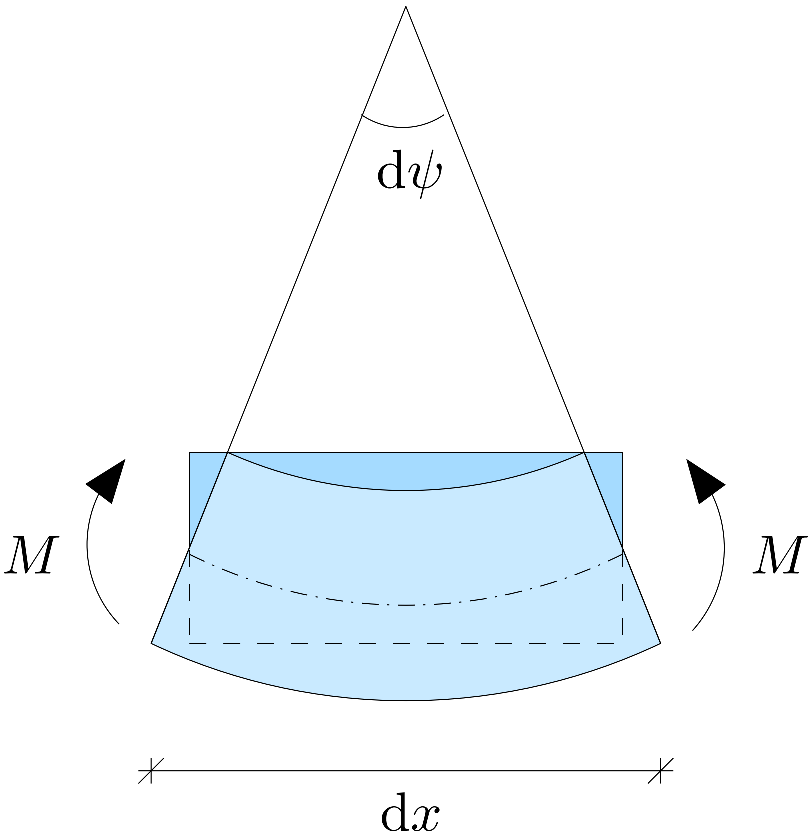
the beam hight remains unchanged

[2] **planar** cross sections remain **planar**

cross sections undergo a deflection w and a rotation ψ $\qquad u = \psi(x)\, z$

sufficiently accurate for **slender beams** at **small strains**

moment - angle



- normal stress

$$\sigma = E\,\epsilon = E\,\frac{\mathrm{d}u}{\mathrm{d}x} \qquad u = \psi(x)\,z$$
$$\sigma = E\,\psi' z$$

- bending moment

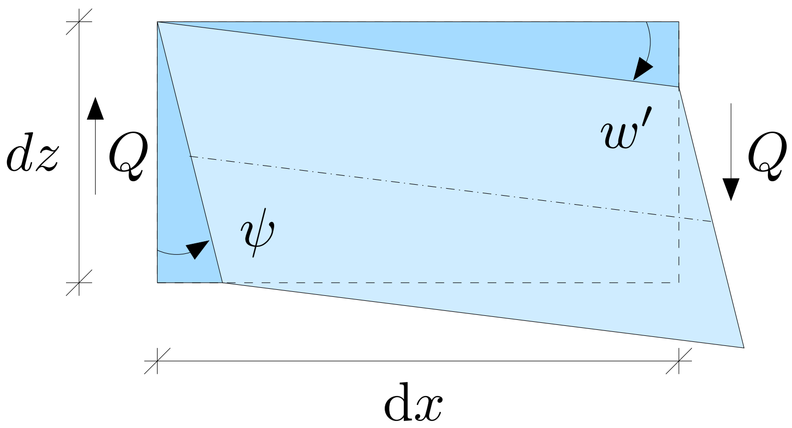
$$M = \int z\sigma\,\mathrm{d}A = \int zE\;\psi' z\,\mathrm{d}A$$
$$I = \int z^2\,\mathrm{d}A$$ ...area moment of inertia

- elasticity for bending moment

$$M = EI\,\psi'$$
EI ...bending stiffness

changes in the angle ψ are proportional to the bending moment M

shear force - shear strain



- shear stress

$$\tau = G\,\gamma = G\,[\,\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z}\,] \qquad w = w(x) \quad u = \psi(x)\,z$$

$$\tau = G\,[w' + \psi]$$

- shear force

$$Q = \int \tau\,\mathrm{d}A = \int G[w' + \psi]\,\mathrm{d}A$$

κ ...shear factor since $\tau \neq$ const.

- elasticity for shear force

$$Q = GA_\kappa\,[w' + \psi]$$

$GA_\kappa = \kappa\,GA$ ...shear stiffness

under shear force Q beam element experiences shear strain $[w' + \psi]$

kinematic assumptions

[1] the **deflection** w is independent of z
all points of a cross section undergo the same deflection in z-direction

$$w = w(x)$$

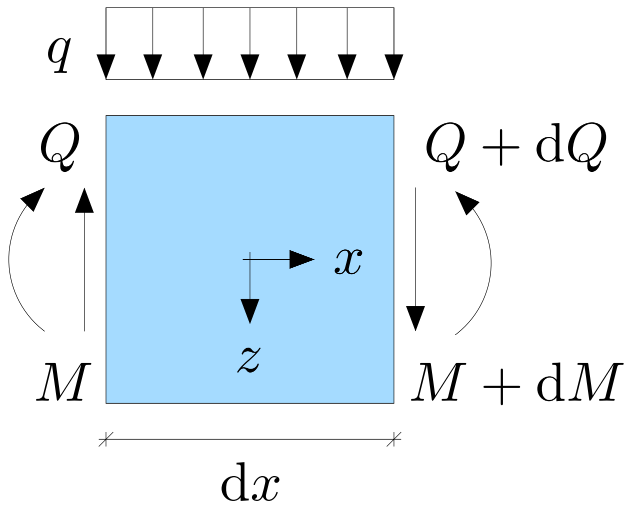
[2] **planar** cross sections remain **planar**
cross sections undergo a deflection w and a rotation ψ

$$u = \psi(x)\, z$$

Stephen Timoshenko [1878-1972]

timoshenko beam theory

governing equations for timoshenko beams



- equilibrium

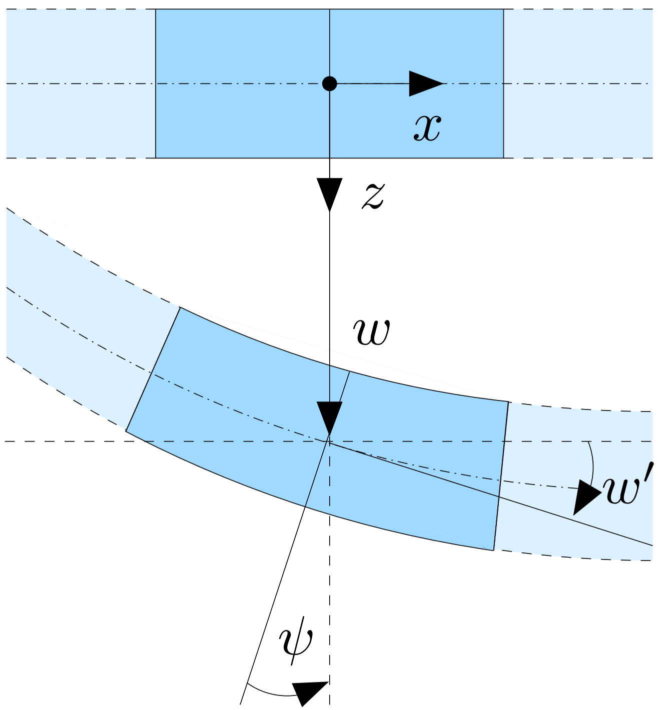
$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = Q$$

- constitutive equations

$$M = EI\,\psi' \qquad Q = GA_\kappa\,[w' + \psi]$$

four equations for shear force Q, moment M, angle ψ, and deflection w

bernoulli hypothesis



- constitutive equation for shear force
  $Q = GA_\kappa[w' + \psi]$

- **bernoulli beam**
  $GA_\kappa \to \infty$

- for finite shear force Q
  $w' + \psi = 0$
  no changes in angle
  kinematic assumption replaces const eqn

cross sections that are orthogonal to the beam axis remain orthogonal

kinematic assumptions

[1] the **deflection** w is independent of z
all points of a cross section undergo the same deflection in z-direction

$$w = w(x)$$

[2] **planar** cross sections remain **planar**
cross sections undergo a deflection w and a rotation ψ

$$u = \psi(x)\, z$$

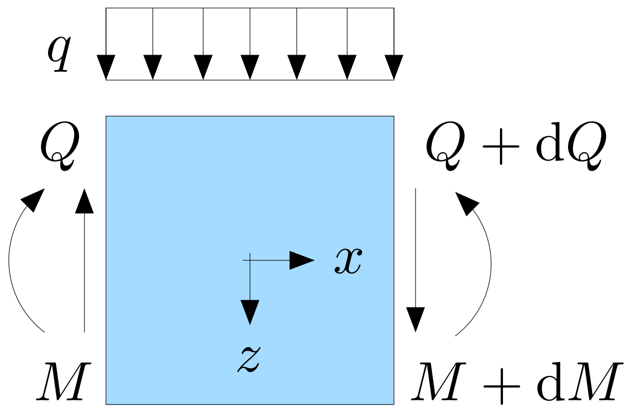
[3] cross sections that are **orthogonal** to the beam axis remain **orthogonal**
shear force does not induce changes in angle

$$w' + \psi = 0$$

Jacob Bernoulli [1654-1705]

bernoulli beam theory

governing equations for bernoulli beams



$$Q' = -q \qquad M' = Q$$

$$\psi' = \frac{M}{EI} \qquad w' = -\psi$$

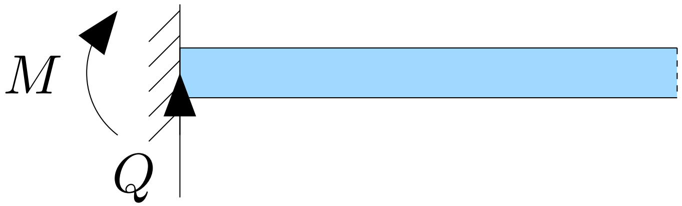
differential eqn for bending – statically determined systems $w'' = -\dfrac{M}{EI}$

integrate twice to obtain deflection w for given M and EI

differential eqn for bending – statically indetermined systems $EI w^{IV} = q$

integrate four times to obtain deflection w for given q and $EI = \text{const.}$

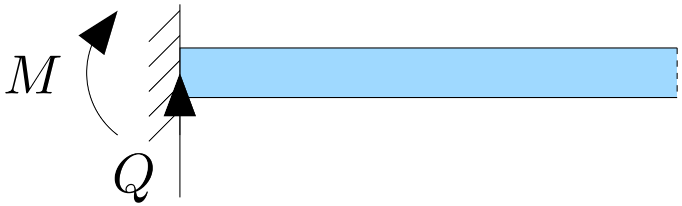
boundary conditions



- clamped

$$w = 0 \qquad w' = 0 \qquad Q \neq 0 \qquad M \neq 0$$

boundary conditions



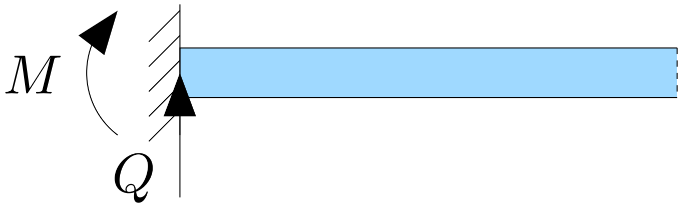
- clamped

  $w = 0 \qquad w' = 0 \qquad Q \neq 0 \qquad M \neq 0$

- simply supported

  $w = 0 \qquad w' \neq 0 \qquad Q \neq 0 \qquad M = 0$

boundary conditions

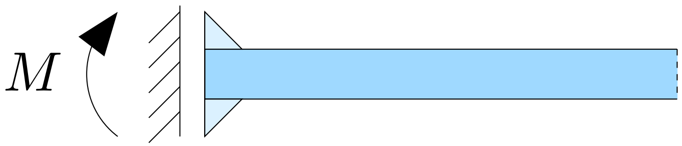


- clamped

  $w = 0 \qquad w' = 0 \qquad Q \neq 0 \qquad M \neq 0$

- simply supported

  $w = 0 \qquad w' \neq 0 \qquad Q \neq 0 \qquad M = 0$



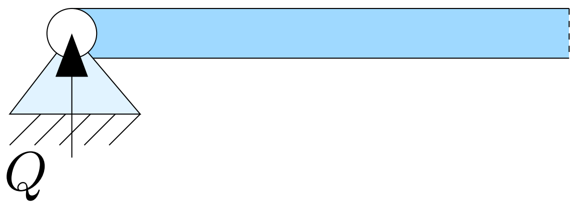
- vertically supported

  $w \neq 0 \qquad w' = 0 \qquad Q = 0 \qquad M \neq 0$

boundary conditions

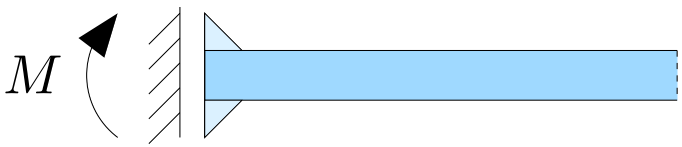
- clamped

  $w = 0 \qquad w' = 0 \qquad Q \neq 0 \qquad M \neq 0$



- simply supported

  $w = 0 \qquad w' \neq 0 \qquad Q \neq 0 \qquad M = 0$



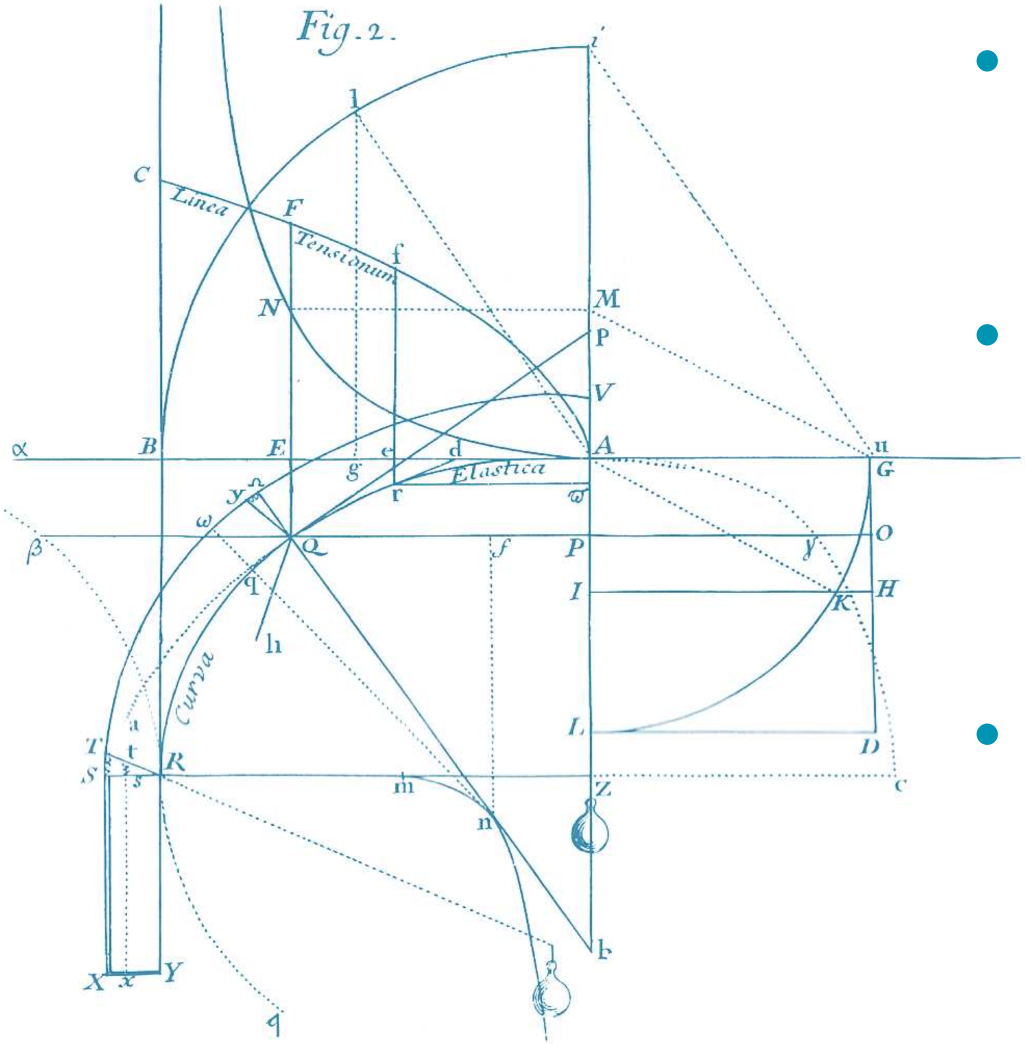
- vertically supported

  $w \neq 0 \qquad w' = 0 \qquad Q = 0 \qquad M \neq 0$

- free end

  $w \neq 0 \qquad w' \neq 0 \qquad Q = 0 \qquad M = 0$

balkenbiegung
– governing equations for beam bending –



- **bernoulli beam**

  $Q' = -q \qquad M' = Q$

  $\psi' = M/EI \qquad w' = -\psi$

- **2nd order differential eqn**

  $$w'' = -\frac{M}{EI}$$

  for statically determined systems integrate twice to obtain w

- **4th order differential eqn**

  $EI\, w^{IV} = q$

  for statically indetermined stystems integrate four times to obtain w