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TITLE- Review of Modal Synthesis Techniques and a New Approach

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ABSTRACT

Modal synthesis is a technique of determining mode shapes and frequencies of a large discrete mass dynamic system by breaking the system into parts, analyzing the parts, and then re-assembling the total system using selected modal information from the parts. This paper provides an interpretive review of the state-of-art in both research and aerospace applications. Comments on each technique are presented. Based on these studies, a new approach is proposed.

The new approach emphasizes ease of interpretation, mathematical simplicity, accuracy of results, and computer capability. This method can yield a spectrum of system normal frequencies in the range of interest and their associated mode shapes. Actual boundaries are imposed between parts, using either rigid body and free-free elastic modes, or only constrained elastic modes of the parts. Both compatibility and equilibrium at boundaries are satisfied for modal coupling. A simple error control scheme based on convergence of eigenvalues of the total system is used to ensure adequate selection of modes from the parts.

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TECHNICAL MEMORANDUM

I. INTRODUCTION

Modal information is essential for performing vibration analysis of a structural system. When a structural system is very big, the following difficulties may be encountered:

A. The number of degrees-of-freedom (DOF) may exceed the available computer capability for eigensolutions.

B. The structural systems may be too large for modal vibration testing, especially when free-free boundary conditions must be simulated.

C. A large system, such as a space vehicle, is generally built in parts, and each part may be associated with different contractors in various distant locations. An assemblage of all parts for modal testing is costly, time consuming, and difficult to handle.

Thus, modal synthesis techniques provide the following advantages:

A. Computer capability is extended to large systems by reducing the size of matrices through partitioning and partial modal coupling.

B. Modal information may be obtained by analysis or testing of subsystems. The subsystems are easier to handle than the total system, and their analysis or testing may be performed by different sources.

C. Actual boundary conditions may be introduced in the synthesis scheme.

D. A design change in one part need only modify the modal data of the changed part. The changed modal data can then be coupled with the remaining unchanged parts.

II. REVIEW OF THE STATE-OF-ART

The concepts of modal synthesis were introduced by W. C. Hurty in a series of papers[6-9], the first appearing in 1960. However, no known application was made in the aerospace industry until the mid 60's. Bamford[2] completed a computer program using Hurty's method with modification, while Goldman[3,4] introduced a fresh approach. Since then, modal synthesis has become a popular topic in both research and industrial fields. The key point of modal synthesis is to impose a successful coupling scheme between subsystem modes, such that compatibility and equilibrium can be restored at the interfaces. The main contributors and their techniques are briefly summarized as follows:

A. Hurty[6-9] - This technique contains the following main steps:

1. Partition a system to several subsystems (parts).

2. For each part "i", set constraints at interfaces and compute mode shape matrices as follows:

$[\phi_e]_i$ = fixed constraint elastic normal modes,

$[\phi_c]_i$ = constraint modes,

$[\phi_r]_i$ = rigid body modes,

where $[\phi_c]_i$ are shapes of static displacements when each of the constraints is independently given a unit displacement, and $[\phi_e]_i$ may contain either all normal mode shapes or the shapes of only the few lowest modes.

Then by defining:

$[\phi]_i = [\phi_r | \phi_c | \phi_e]_i$ = mode shape matrix,

$[m]_i$ = mass matrix,

$[k]_i$ = stiffness matrix,

$[c]_i$ = damping matrix,

$\{f\}_i$ = load matrix,

the generalized subsystem matrices are computed as

$$[m_G]_i = [\phi]_i^T [m]_i [\phi]_i,$$

$$[k_G]_i = [\phi]_i^T [k]_i [\phi]_i,$$

$$[c_G]_i = [\phi]_i^T [c]_i [\phi]_i,$$

$$\{f_G\}_i = [\phi]_i^T \{f\}_i.$$

3. Set up system equations:

$$[m]\{\ddot{p}\} + [c]\{\dot{p}\} + [k]\{p\} = \{f(t)\},$$

where

$$[m] = \begin{bmatrix} [m_G]_1 & & \\ & [m_G]_2 & \\ & & \ddots \end{bmatrix}, \quad [k] = \begin{bmatrix} [k_G]_1 & & \\ & [k_G]_2 & \\ & & \ddots \end{bmatrix},$$

$$[c] = \begin{bmatrix} [c_G]_1 & & \\ & [c_G]_2 & \\ & & \ddots \end{bmatrix}, \quad \{f\} = \begin{Bmatrix} \{f_G\}_1 \\ \{f_G\}_2 \\ \vdots \end{Bmatrix},$$

and {p} is an assemblage of generalized coordinates of all subsystems.

4. Set up transformation matrix [β] for modal coupling by restoring geometric compatibility at the interfaces. Thus {P} can be expressed by its independent coordinates, {q}, as

$$\{p\} = [\beta]\ \{q\}.$$

Notice that [β] depends entirely upon the geometric configuration of the interfaces.

5. Hence the synthesized total system equations are developed using the transformation matrix [β]:

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F\},$$

where

$$[M] = [\beta]^T\ [m]\ [\beta],\quad [K] = [\beta]^T\ [k]\ [\beta],$$

$$[C] = [\beta]^T\ [c]\ [\beta],\quad \{F\} = [\beta]^T\ \{f\}.$$

B. Bamford[2] - A computer program based on Hurty's techniques was developed. In its subsystem processing, "attachment modes" are added for describing the shapes of motion caused by concentrated loads at unconstrained points. Thus for any subsystem "i", the mode shape matrix is defined as

$$[\phi]_i = [\phi_r\ |\phi_c|\ \phi_a|\ \phi_e]_i,$$

where $[\phi_a]_i$ contains all attachment mode shapes.

C. Bajan, Feng, and Jaszlics[1] - Technique involves modal coupling and modal substitution. The former couples the subsystem modes to obtain the total system modes, while the latter provides an iteration scheme for improving accuracy through error analysis. Key steps are as follows:

1. Perform modal analysis on each subsystem i, which has known mass matrix $[m]_i$ and stiffness matrix $[k]_i$, and obtain its mode shapes as

$$[\phi]_i = [\phi_c | \phi_e]_i ,$$

where $[\phi_c]_i$ contains all constraint modes and $[\phi_e]_i$ contains all elastic modes with fixed interfaces.

2. Subsystem matrices are assembled as

$$[m] = \begin{bmatrix} [m]_1 & & \\ & [m]_2 & \\ & & \ddots \end{bmatrix}, \quad [k] = \begin{bmatrix} [k]_1 & & \\ & [k]_2 & \\ & & \ddots \end{bmatrix},$$

$$[\phi] = \begin{bmatrix} [\phi]_1 & & \\ & [\phi]_2 & \\ & & \ddots \end{bmatrix}.$$

3. Define matrix [c], such that

$$\{u_s\} = [c]\{u\},$$

where $\{u_s\}$ is an assemblage of displacement vectors $\{u_s\}_i$ of all subsystems, and $\{u\}$ is the displacement vector of the system as a whole.

4. A matrix can be generated as

$$[T]_t = [\phi][c] = [T_c | T_r | T_d].$$

where $[T_c]$ contains all constraint modes, and $[T_r]$ contains subsystem elastic modes retained for modal coupling, while remainders $[T_d]$ are deleted. Thus the partial modal coupling is done by solving

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\},$$

where $[M] = [T]^T [m] [T]$,

$[K] = [T]^T [k] [T]$,

$[T] = [T_c \; T_r]$.

Solutions give

$$\{q\} = [Q]\{\eta\},$$

where $[Q]$ and $\{\eta\}$ are the participation matrix and the vector of generalized coordinates, respectively. Hence

$$\{u_s\} = ([T][Q])\{\eta\} = [\psi]\{\eta\},$$

where $[\psi]$ contains approximate mode shapes of the system through such partial coupling.

5. Set a new $[T]$ by retaining certain modes from $[\psi]$ and certain unused modes of $[T_d]$, which are defined as objective modes $[\psi_0]$ and replacement modes $[T_{rep}]$, respectively:

$$[T] = [\psi_0 \,|\, T_{rep}].$$

Then go through modal analysis as shown in step 4, and get new $[T]$. Such a process, which is called "modal substitution" will be repeated until satisfactory eigenvalues are obtained.

Replacement modes may also be obtained from deleted modes in $[\psi]$ of previous cycles when all modes in $[T_d]$ are used. Selection of replacement modes are based on their individual contribution of error to the eigensolution, which is automatized through a "modal selection algorithm".

D. Goldman[3,4] - Use rigid body modes and free-free elastic modes of parts for synthesis. Modal couplings are performed by eliminating the terms of internal forces at connection interfaces from equations of motion through the compatibility relations. Key steps are as follows:

1. Modal equations of motion of all subsystems are assembled as:

$$\{\ddot{p}_e\} + [\tilde{\omega}_e^2]\{p_e\} = [\phi_{eb}]^T\{F_c\},$$

$$\{\ddot{p}_r\} = [M_r]^{-1}[\phi_{rb}]^T\{F_c\},$$

where

$\{p_e\}$ = Generalized coordinates for elastic motions,

$\{p_r\}$ = Generalized coordinates for rigid body motions,

$[\tilde{\omega}_e^2]$ = Frequencies of elastic modes in subsystems,

$[\phi_{eb}]$ = Elastic mode shapes at interfaces of subsystems with unit generalized mass,

$[\phi_{rb}]$ = Rigid body mode shapes at interfaces of subsystems,

$\{F_c\}$ = All internal forces at connection interfaces.

2. The geometric compatibility regarding displacements at the connection interfaces provides relations as

$$[\phi_{eb}]\{p_e\} + [\phi_{rb}]\{p_r\} = \{0\}.$$

3. Multiplying the first equation of motion by $[\phi_{eb}]$ and the second one by $[\phi_{rb}]$, the sum of these two equations gives

$$\{F_c\} = [E]^{-1}[\phi_{eb}]\lceil\omega_e^2\rceil\{p_e\},$$

where $[E] = [\phi_{eb}][\phi_{eb}]^T + [\phi_{rb}][M_r]^{-1}[\phi_{rb}]^T.$

4. Thus, replacing $\{F_c\}$ by the above expression, the first equation of motion becomes

$$\{\ddot{p}_e\} + [L]\{p_e\} = \{0\},$$

where $[L] = \lceil\omega_e\rceil\left([I] - [\phi_{eb}]^T[E]^{-1}[\phi_{eb}]\right)\lceil\omega_e\rceil.$

This gives the form of eigensolution for the whole system.

E. Martin Marietta Co., Denver, Colorado[5] - Empirical coupling techniques are developed for computing modal data of a space vehicle which contains a main structure and several branch structures. Key steps are as follows:

1. Each branch i is treated as a cantilever by fixing it at its attaching point to the main structure, and then performing a modal analysis for each branch through the following equation of motions:

$$[m_b]_i\{\ddot{u}_b\}_i + [k_b]_i\{u_b\}_i = \{0\},$$

where $[m_b]$, $[k_b]$, and $\{u_b\}$ are the mass matrix, stiffness matrix, and displacements of branch i. The mode shapes so obtained are defined as $[\phi_b]_i$, and the generalized coordinates are $\{p_b\}_i$.

2. Then perform modal analysis for the main structure by treating all branches as rigid appendages:

$$[m_m]\{\ddot{u}_m\} + [k_m]\{u_m\} = \{0\},$$

where $[m_m]$, $[k_m]$ and $\{u_m\}$ are the mass matrix, stiffness matrix, and displacements for the main structure. The mode shapes so obtained are defined as $[\phi_m]$, and the generalized coordinates are $\{p_m\}$.

3. Assume that motion in any branch i is given by $\{u_b\}_i$ plus the rigid body motion $[T]_i \{u_m\}$ caused by the motion at the attaching point. Thus

$$\{u\} = \begin{bmatrix} I & & \\ T_1 & I & \\ T_2 & & I \\ \vdots & & \ddots \end{bmatrix} \begin{Bmatrix} \{u_m\} \\ \{u_b\}_1 \\ \{u_b\}_2 \\ \vdots \end{Bmatrix}$$

$$= \begin{bmatrix} [\phi_m] & & \\ [T_1][\phi_m] & [\phi_b]_1 & \\ [T_2][\phi_m] & & [\phi_b]_2 \\ \vdots & & \ddots \end{bmatrix} \begin{Bmatrix} \{p_m\} \\ \{p_b\}_1 \\ \{p_b\}_2 \\ \vdots \end{Bmatrix} = [\beta]\{p\} \quad .$$

4. Thus the system equation for eigensolution after modal coupling is

$$[M]\{\ddot{p}\} + [K]\{p\} = \{0\},$$

where

$$[M] = [\beta]^T \begin{bmatrix} [m_m] & & & \\ & [m_b]_1 & & \\ & & [m_b]_2 & \\ & & & \ddots \end{bmatrix} [\beta] \quad ,$$

$$[K] = [\beta]^T \begin{bmatrix} [K_m] & & & \\ & [K_b]_1 & & \\ & & [K_b]_2 & \\ & & & \ddots \end{bmatrix} [\beta] \quad .$$

III. COMMENTS ON EXISTING TECHNIQUES

A. Hurty and Bamford were concerned with determining response after obtaining modal information by synthesis. However, solving for modal information alone, the loading and damping terms introduced by Hurty and the attachment modes introduced by Bamford need not be included in the synthesis scheme. The reasons are:

(1) Only the homogeneous solutions are needed from the equations of motion.

(2) Damping effects to normal frequencies are negligible if the system is lightly damped. For heavily damped cases, the equations of motion will be coupled and solutions will be difficult to obtain.

However, if modal information of parts are obtained by testing, and such testing must introduce constraints which may not be at the interface between parts, Bamford's techniques will provide data for synthesis and a check of testing.

B. Bajan, Feng, and Jaszlics have laid out a clear scheme for obtaining constraint modes. Their ideas for automatic modal selection by error analysis and substitution

scheme are impressive. However, formulations are complicated and involve a lot of computations. The number of objective modes and replacement modes used for each cycle remains arbitrary. In addition, the matrix [c] is based on relations in cartesian coordinates between the whole system and subsystems and is not the coupling between subsystem modal vectors, as is applied.

C. The most attractive point of Goldman's technique is the use of free-free subsystem modes. Thus constraint modes as used by other techniques are not needed. In addition, the method always leads to an eigensolution of a single matrix [L] (see Section II E.4.). Thus the eigensolution for the synthesized total system equations are easy to handle. Since Goldman's system equation of elastic motions after coupling has a number of DOF equal to the sum of elastic DOF of the subsystems, the total DOF of the fully coupled system is not the same as the actual DOF of the system. This may introduce error.

D. Techniques used by Martin Marietta Company are empirical in nature and may yield close solutions when the branch systems are of far less effect to the main system in terms of mass and frequency range. In addition, since no relative motions are allowed among DOF's at the interface (constraint modes are not used), the number of mass points at the connection interface should be as few as possible; the best is one.

IV. THE NEW APPROACH

Based on previous studies, a new approach emphasizing ease of interpretation, mathematical simplicity, accuracy of results, and potential computer capability is presented. This new approach is guided by the following considerations:

Whenever eigensolution routines are capable of handling free-free systems, subsystems should be unconstrained, unless constraints physically exist at the boundary. Thus, computation for

constraint modes may be eliminated if free-free elastic modes are introduced from the parts.

The main goal of the approach is to yield modal information, and thus there is no need to consider forcing functions and damping. Concentrating on undamped free vibration will simplify the formulation and the computation scheme.

Connection interfaces between parts should be simple with as few DOF involved as possible.

Rigid body modes of subsystems are included since they have significant effect on the lowest elastic modes obtained through modal coupling. However, the effects on the higher modes are negligible.

The computation scheme should include a method for selecting proper modes, and in adequate number from subsystems for partial modal coupling. The method uses a simple error analysis technique, the main purpose of which is to insure convergence of results.

A. Subsystem Analysis

Partition a structural system in parts. Each part is considered as free-free unless it is physically constrained. Then perform modal analysis of each part (say part i) to obtain its normal frequencies $[\omega_i^2]$ and mode shapes $[\phi_i]$. The portion of $[\phi_i]$ at interfaces is designated as $[\phi_{ic}]$. Rigid body modes are treated as free-free elastic modes having zero frequencies. All mode shapes are normalized to unit generalized mass. For convenience of explanation, suppose a system is partitioned into two parts, A and B, which have n_A and n_B modes participating in the synthesis, respectively. Let n_r be the number of degrees-of-freedom at the connection interface.

For part A, we have

$$\{\ddot{p}_A\} + [\omega_A^2]\{p_A\} = [\phi_{AC}]^T\{F_A\} \qquad (1)$$

and

$$\{u_A\} = [\phi_A]\{p_A\} \quad , \tag{2}$$

where

$\{p_A\}$ = Vector of generalized coordinates

$\{u_A\}$ = Vector of displacements in physical coordinates

$\{F_A\}$ = Internal forces acting on the connection interface between parts.

Let $\{u_{AC}\}$ be the portion of $\{u_A\}$ on the connection interface, and the remaining portion be $\{u_{Ar}\}$, equation (2) can be partitioned as

$$\left\{\begin{matrix} u_{Ar} \\ \hline u_{AC} \end{matrix}\right\} = \left[\begin{matrix} \phi_{Ar} \\ \hline \phi_{AC} \end{matrix}\right]\{p_A\} \quad . \tag{3}$$

Using the same expressions for part B, we have

$$\{\ddot{p}_B\} + [\omega_B^2]\{p_B\} = [\phi_{BC}]^T\{F_B\} \quad , \tag{4}$$

$$\{u_B\} = [\phi_B]\{p_B\} \quad , \tag{5}$$

and

$$\left\{\begin{matrix} u_{Br} \\ \hline u_{BC} \end{matrix}\right\} = \left[\begin{matrix} \phi_{Br} \\ \hline \phi_{BC} \end{matrix}\right]\{p_B\} \quad . \tag{6}$$

B. Compatibility Conditions

The required conditions for geometric compatibility at the connection interface provide n_r equations as follows:

$$\{u_{AC}\} = \{u_{BC}\} \quad , \tag{7}$$

or

$$[\phi_{AC}]\{p_A\} = [\phi_{BC}]\{p_B\} \quad . \tag{8}$$

Letting $n_A > n_r$, $[\phi_{AC}]$ can be partitioned as

$$[\phi_{AC}] = [\phi^S_{AC} \,|\, \phi^r_{AC}] \quad , \tag{9}$$

where $[\phi^S_{AC}]$ is a square matrix, and $[\phi^r_{AC}]$ contains the remainders. Thus, from equation (8), we have

$$[\phi^S_{AC} \,|\, \phi^r_{AC}] \left\{ \begin{matrix} p^S_A \\ \hline p^r_A \end{matrix} \right\} = [\phi_{BC}]\{p_B\} \quad , \tag{10}$$

and

$$\{p^S_A\} = -[\phi^S_{AC}]^{-1}[\phi^r_{AC}]\{p^r_A\} + [\phi^S_{AC}]^{-1}[\phi_{BC}]\{p_B\} \quad . \tag{11}$$

This gives an expression for (n_A+n_B) generalized coordinates $\{p\}$ in terms of $(n_A+n_B-n_r)$ independent coordinates $\{q\}$:

$$\{p\} = [T]\{q\} \quad , \tag{12}$$

where

$$\{p\} = \left\{ \begin{matrix} p_A \\ p_B \end{matrix} \right\} = \left\{ \begin{matrix} p^S_A \\ p^r_A \\ p_B \end{matrix} \right\} \quad , \quad \{q\} = \left\{ \begin{matrix} p^r_A \\ p_B \end{matrix} \right\} \quad , \tag{13}$$

$$[T] = \begin{bmatrix} -[\phi_{AC}^{S}]^{-1}[\phi_{AC}^{r}] & [\phi_{AC}^{S}]^{-1}[\phi_{BC}] \\ [I] & [0] \\ [0] & [I] \end{bmatrix} . \qquad (14)$$

C. System Equations by Modal Coupling

The uncoupled modal information of the parts can be assembled as follows:

$$\begin{Bmatrix} \ddot{p}_A \\ \ddot{p}_B \end{Bmatrix} + \left[\begin{array}{c|c} \omega_A^2 & 0 \\ \hline 0 & \omega_B^2 \end{array}\right] \begin{Bmatrix} p_A \\ p_B \end{Bmatrix} = \left[\begin{array}{c|c} \phi_{AC}^T & 0 \\ \hline 0 & \phi_{BC}^T \end{array}\right] \begin{Bmatrix} F_A \\ F_B \end{Bmatrix} . \qquad (15)$$

By substituting compatibility equation (12) and then restoring symmetry, we have

$$[T]^T[T]\{\ddot{q}\} + [T]^T \left[\begin{array}{c|c} \omega_A^2 & 0 \\ \hline 0 & \omega_B^2 \end{array}\right] [T]\{q\} = [T]^T \left[\begin{array}{c|c} \phi_{AC}^T & 0 \\ \hline 0 & \phi_{BC}^T \end{array}\right] \begin{Bmatrix} F_A \\ F_B \end{Bmatrix} . \qquad (16)$$

Since the term on the right hand side of equation (16) is equal to zero (see next section), we get $(n_A+n_B-n_r)$ coupled system equations as:

$$[m]\{\ddot{q}\} + [k]\{q\} = \{0\} , \qquad (17)$$

where

$[m]$ = Pseudo mass matrix = $[T]^T[T]$,

$[k]$ = Pseudo stiffness matrix = $[T]^T \left[\begin{array}{c|c} \omega_A^2 & 0 \\ \hline 0 & \omega_B^2 \end{array}\right] [T]$.

Performing a modal analysis of equation (17), we obtain normal frequencies of the system and shape vectors $[\psi]$. Hence, the mode shapes of the total system are

$$[\phi] = \left[\begin{array}{c|c} \phi_A & 0 \\ \hline 0 & \phi_B \end{array}\right] [T][\psi] \quad . \tag{18}$$

D. Equilibrium Check

Since $\{F_A\}$ and $\{F_B\}$ are equal and opposite internal forces acting at the connection interface, and since there are no external forces imposed on the system, the system must be in equilibrium at the interface if the right hand side of equation (17) is truly a zero vector. From the right hand side of equation (16), we have

$$[T]^T \left[\begin{array}{c|c} \phi_{AC}^T & 0 \\ \hline 0 & \phi_{BC}^T \end{array}\right] \left\{\begin{array}{c} F_A \\ \hline F_B \end{array}\right\}$$

$$= \left[\begin{array}{c|c|c} -[\phi_{AC}^r]^T [\phi_{AC}^S]^{-1,T} & [I] & [0] \\ \hline [\phi_{BC}]^T [\phi_{AC}^S]^{-1,T} & [0] & [I] \end{array}\right] \left[\begin{array}{c|c} [\phi_{AC}^S]^T & [0] \\ \hline [\phi_{AC}^r]^T & [0] \\ \hline [0] & [\phi_{BC}]^T \end{array}\right] \left\{\begin{array}{c} F_A \\ \hline F_B \end{array}\right\}$$

$$= \left[\begin{array}{c|c} -[\phi_{AC}^r]^T [\phi_{AC}^S]^{-1,T} [\phi_{AC}^S]^T + [\phi_{AC}^r]^T & [0] \\ \hline [\phi_{BC}]^T [\phi_{AC}^S]^{-1,T} [\phi_{AC}^S]^T & [\phi_{BC}]^T \end{array}\right] \left\{\begin{array}{c} F_A \\ \hline F_B \end{array}\right\}$$

$$= \left[\begin{array}{c} [0] \\ \hline [\phi_{BC}]^T\{F_A\} + [\phi_{BC}]^T\{F_B\} \end{array}\right] = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} \quad . \tag{19}$$

(NOTE: $\{F_A\} + \{F_B\} = \{0\}$ at interface)

Thus both compatibility and equilibrium conditions are satisfied at the connection interface.

E. Selection of Subsystem Modes Using Error Index

As we know, the real advantage of modal synthesis can only be achieved by partial modal coupling, that is using a small number of selected subsystem modes. However, the accuracy of the results is also decreased when fewer modes are used. Thus, a proper selection scheme is essential.

It has been found that the low (or high) frequency modes of subsystems have only dominant effect on the low (or high) frequency modes of the overall system, and the sum of all frequency squares (eigenvalues) of a system is an invariant. Thus a simple selection scheme can be set up as follows:

1. First, rearrange mode shapes and frequencies of each subsystem into an ascending order of their frequencies. Then divide the modes in each subsystem into equal number of groups.

2. Start synthesis by using modes in the first group of all subsystems. Then sum up first n eigenvalues $\lambda_{1k}(k=1,2,\cdots,n)$ obtained from synthesis and call it C_1:

$$C_1 = \sum_{k=1}^{n} \lambda_{1k} \tag{20}$$

3. More modes are used by taking successive group of modes from all subsystems. In each cycle, compute

$$C_i = \sum_{k=1}^{n} \lambda_{ik} \tag{21}$$

4. Thus the "Index of Convergence" is defined as

$$E_i = \left| \frac{C_{i+1} - C_i}{C_{i+1}} \right| , \tag{22}$$

which will eventually go below a predefined tolerance level when a sufficient number of modes are involved. Since all frequencies in each subsystem have been rearranged, the magnitude change in each synthesized frequency is in a monotonic fashion. Thus such an index will indicate the relative rate of improvement in frequencies when additional group of modes are used.

F. Conclusions

In summary, this new approach offers the following advantages and special features:

1. It is easy to interpret mathematically and physically. Such simplicity makes computer programming easier.

2. Both geometric compatibilities and force equilibrium are satisfied at interfaces between parts.

3. Mode shapes and normal frequencies from parts are the only input. Constraint modes from parts are not needed.

4. The use of free-free or constrained elastic modes from the parts depends on whether physical constraints actually exist at the interface.

5. A spectrum of system normal frequencies in the range of interest and their associated mode shapes can be computed.

6. Rigid body modes are automatically included and are treated as if they are the first few free-free elastic modes with zero frequencies. They can be included or omitted for partial modal synthesis.

Three numerical examples are given in the Appendix. The first example uses total coupling for verifying the theory of the new approach. The second example demonstrates the selection of subsystem modes for partial coupling, such that modes of the total system in the frequency range of interest can be synthesized efficiently. The third example demonstrates the synthesis scheme for obtaining higher frequency modes.

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Attachments
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Appendix

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APPENDIX

Numerical Examples

EXAMPLE 1

This example is intended to demonstrate the simplicity of the synthesis technique using the new approach, and also to show the accuracy of the results through total coupling. A comparison of synthesized results with direct solution of the total system is presented.

Given an elastic, uniform section, homogeneous, free-free beam, where:

L = total length = 40 in.,

r = radius of cross section = 1 in.,

ρ = weight density = 2 lbs. per cu. in.,

E = elastic modulus = 1×10^4 psi,

ν = Poisson ratio = 0.3,

A = area of cross section = 3.1416 sq. in.,

I = sectional moment of inertia = $\frac{1}{4} \pi r^4 = 0.7854$ in.^{4}

To compute the dynamic properties through modal synthesis, partition the beam into two identical parts, and each part into ten identical segments having a length ℓ = 2 in. The mass of each segment is lumped into its two ends, which are considered as nodes. Consider each node to have only two degrees of freedom, lateral translation and rotation. Thus each part has 22 DOF and 11 nodal points as shown in Figure 1.

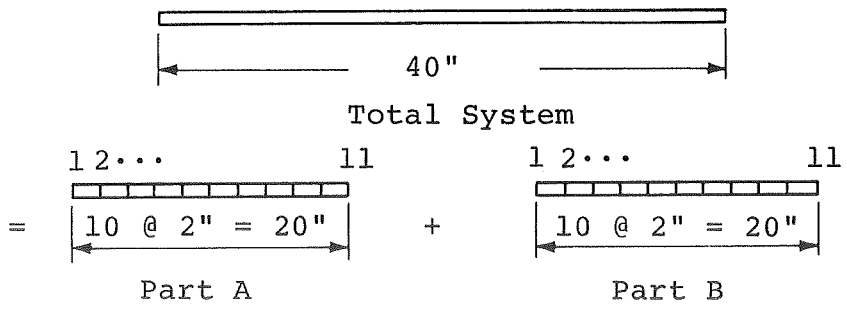

FIGURE 1

Computing frequencies and mode shapes of the parts, we have

$$[\omega_A^2] = [\omega_B^2] = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 1{,}380 & & \\ & & & 9{,}765 & \\ & & & & \ddots \\ & & & & & 6{,}154{,}535 \end{bmatrix}$$

$$[\phi_A] = [\phi_B] = \begin{bmatrix} 0.2995 & -2.9947 & 3.3432 \cdots\cdots & 1.6239 & -1.6140 \\ 0 & 0.2995 & -0.7934 \cdots\cdots & 6.7683 & -6.7255 \\ 0.2995 & -2.3955 & 1.7703 \cdots\cdots & 0.3842 & 0.3795 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.2995 & 2.9947 & 3.3432 \cdots\cdots & 1.6245 & 1.6142 \\ 0 & 0.2995 & 0.7934 \cdots\cdots & 6.7709 & -6.7266 \end{bmatrix}$$

Notice that there are two rigid body modes in each part. The interface compatibility conditions set by Equation (7) are that the displacement and rotation at node 11 of part A should be equal to the displacement and rotation at node 1 of part B. Thus the total system after synthesis has 42 DOF. According to Equations (8) and (9), $[\phi_{AC}]$ contains the last two rows of $[\phi_A]$ and $[\phi_{BC}]$ contains the first two rows of $[\phi_B]$. The $[\phi_{AC}]$ is further partitioned into a square sub-matrix $[\phi_{AC}^S]$ and the remainder $[\phi_{AC}^r]$:

$$[\phi_{AC}] = \left[\begin{array}{cc|cc} 0.2995 & 2.9947 & 3.3432 \cdots\cdots & 1.6245 \quad 1.6142 \\ 0 & 0.2995 & 0.7934 \cdots\cdots & 6.7709 \quad -6.7266 \end{array}\right]$$

$$= \left[\phi_{AC}^S \;\middle|\; \phi_{AC}^r\right]$$

$$[\phi_{BC}] = \begin{bmatrix} 0.2995 & -2.9947 & 3.3432 \cdots\cdots & 1.6239 & -1.6140 \\ 0 & 0.2995 & -0.7934 \cdots\cdots & 6.7683 & -6.7255 \end{bmatrix}$$

The 44 x 42 transformation matrix [T] for total coupling can be computed by Equation (14) as

$$[T] = \begin{bmatrix} -[\phi^S_{AC}]^{-1}[\phi^r_{AC}] & [\phi^S_{AC}]^{-1}[\phi_{BC}] \\ [I] & [0] \\ [0] & [I] \end{bmatrix}$$

$$= \begin{bmatrix} 15.33 & 34.01 & 51.85 \cdots & -220.59 & 219.12 \\ -2.65 & -4.44 & -6.12 \cdots & 22.60 & -22.46 \\ 1 & & & & \\ & 1 & & & \\ & & 1 \cdot & & \\ & & & \cdot \cdot & \\ & & & \cdot 1 & \\ & & & & 1 \end{bmatrix}$$

Then compute 42 x 42 pseudo mass and stiffness matrices as

$$[m] = [T]^T[T]$$

$$[k] = [T]^T \begin{bmatrix} \omega_A^2 & \vdots & \\ \cdots & \cdots & \cdots \\ & \vdots & \omega_B^2 \end{bmatrix} [T] \quad ,$$

and perform modal analysis for this synthesized pseudo system as shown in Equations (17) and (18).

Table 1 gives a comparison of frequencies between synthesis results and the results directly obtained from analysis of 42 DOF discretized total beam. Table 2 gives a comparison of the displacements and slopes in the first elastic mode shape (the $3^{\underline{rd}}$ mode). Mode shapes given by Reference (12) are also listed for comparison.

TABLE 1

COMPARISON OF FREQUENCIES (cps)

Mode No.	Direct Solution	Synthesis Results
1	0.00	0.00
2	0.00	0.00
3	1.53	1.54
4	4.17	4.18
5	8.09	8.09
6	13.21	13.22
7	19.48	19.48
8	26.83	26.84
9	35.20	35.20
10	44.51	44.52
15	103.0	103.0
20	172.8	172.8
25	252.6	252.6
30	315.5	315.5
35	361.1	361.1
40	382.3	382.3
42	394.8	394.8

TABLE 2

COMPARISON OF MODE SHAPES

(First Elastic Mode)

(A) TRANSLATIONS

Location of Node	Direct Solution	Synthesis Results	Reference (12)
0.0L	2.001	2.000	2.000
0.1L	1.071	1.072	1.074
0.2L	0.190	0.192	0.196
0.3L	-0.549	-0.546	-0.544
0.4L	-1.045	-1.040	-1.041
0.5L	-1.220	-1.216	-1.216
0.6L	-1.046	-1.042	-1.041
0.7L	-0.550	-0.552	-0.544
0.8L	0.189	0.184	0.196
0.9L	1.069	1.058	1.074
1.0L	1.999	1.982	2.000

(B) ROTATIONS

0.0L	-1.965	-1.973	-1.965
0.1L	-1.929	-1.939	-1.934
0.2L	-1.714	-1.737	-1.748
0.3L	-1.332	-1.332	-1.341
0.4L	-0.725	-0.725	-0.730
0.5L	0.001	0	0
0.6L	0.723	0.725	0.730
0.7L	1.331	1.332	1.341
0.8L	1.739	1.737	1.748
0.9L	1.929	1.923	1.934
1.0L	1.965	1.957	1.965

EXAMPLE 2

This example is intended to show how to select sufficient modes from parts for partial modal coupling, such that total system modes in the range of interest can be efficiently computed.

Given the same free-free beam as in Example 1 except for a hinge at midspan, if we partition the beam through this hinge and discretize the two parts the same way as Example 1, each part will also have 11 nodes and 22 DOF. The only difference is that there is only one DOF (the translation) instead of two at the interface for compatibility requirements. Since rotation of two parts at the interface may be different, the total system has 43 DOF instead of 42.

Suppose that we are interested in the lowest seven modes. The first step is to arrange modes of the two parts individually in ascending order, according to the magnitude of their frequencies. Then break the 22 modes (including 2 rigid body modes) of each part into groups:

Group No.	Modes
1	1, 2, 3, 4
2	5, 6, 7
3	8, 9, 10
4	11, 12, 13
5	14, 15, 16
6	17, 18, 19
7	20, 21, 22

Take group 1 modes from both parts for synthesis. Results will yield the 7 lowest modes of the total system. Their eigenvalues (λ_{1K}) are:

0, 0, 0, 697, 1380, 7108, 9765.

The three zeros yield three rigid body modes as shown in Figure 2.

FIGURE 2

Hence

$$C_1 = \sum_{k=1}^{7} \lambda_{1k} = 0 + 0 + 0 + 697 + 1380 + 7108 + 9765 = 18950 \quad .$$

Then add group 2 modes to group 1 for both parts, and synthesize again. Results yield the lowest 13 modes of the total system. However, we only take the same lowest 7 eigenvalues (λ_{2k}) and sum them:

$$C_2 = \sum_{k=1}^{7} \lambda_{2k} = 18745$$

Thus the first "index of convergence" is

$$E_1 = \left| \frac{C_2 - C_1}{2} \right| = \left| \frac{18745 - 18950}{18745} \right| = 109 \times 10^{-4} \quad .$$

If additional groups of modes in both parts participate in the synthesis, we have

$$C_3 = 18739.625 \qquad E_2 = 5.252 \times 10^{-4} \quad ,$$

and

$$C_4 = 18739.730 \qquad E_3 = 0.056 \times 10^{-4} \quad .$$

Notice that an increase of the same number of modes from parts yield less in accuracy improvement of the synthesis results, and at the same time the size of the coupled system gets larger and larger. For this particular case, since E_1 is already small enough, we may stop adding subsystem modes right after computing E_1. This means using the lowest 7 modes (about one third of 22) from each part for synthesis. Frequencies (cps) so obtained are compared with total coupling results (22 modes from each part) as shown in Table 3.

TABLE 3

Mode No.	7 Modes From Each Part	Total Coupling
1	0.	0.
2	0.	0.
3	0.	0.
4	4.183	4.183
5	5.913	5.914
6	13.229	13.223
7	15.727	15.727

EXAMPLE 3

For the same beam as example 2, if all the modes of the total system are required, we can perform several shifts in the synthesis. Example 2 shows that 7 modes from each part will yield satisfactory results, and since we know that n_A modes from part A and n_B modes from part B with $n_r = 1$ at the interface will yield $(n_A+n_B-n_r)$ modes of the total system after synthesis, three shifts of synthesis should be sufficient, as shown in Table 4.

TABLE 4

Synthesis Shift No.	Modes From Part A	Modes From Part B	No. of Modes Synthesized
1	1 - 7	1 - 8	14
2	8 - 15	8 - 14	14
3	15 - 22	15 - 22	15
			Total = 43

Frequencies (cps) so obtained are compared with total coupling results as shown in Table 5.

TABLE 5

(A) First Shift:

Mode No.	Partial Coupling	Total Coupling
1	0.	0.
2	0.	0.
3	0.	0.
4	4.183	4.183
5	5.913	5.914
6	13.227	13.223
7	15.727	15.727
8	26.855	26.840
9	29.712	29.713
10	44.556	44.520
11	47.304	47.304
12	65.820	65.739
13	68.089	68.090
14	90.836	89.956

(B) Second Shift:

15	91.683	91.683
16	109.220	116.537
17	117.613	117.613
18	141.034	144.646
19	145.171	145.171
20	171.596	172.764
21	172.917	172.917
22	221.105	221.105
23	221.105	221.105
24	238.746	239.333

25	239.388	239.388
26	263.263	266.025
27	266.304	266.304
28	289.750	291.879

(C) Third Shift:

29	292.520	292.520
30	295.087	315.529
31	316.616	316.616
32	319.742	336.304
33	337.862	337.862
34	341.014	353.738
35	355.725	355.725
36	358.416	367.466
37	369.736	369.736
38	371.549	377.147
39	379.287	379.287
40	379.971	382.303
41	394.780	394.780
42	394.808	394.808
43	394.836	394.836