Chapter 6

Bernoulli's equation

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In section (5.4) we obtained the momentum equation for ideal fluids (i.e. inviscid and with constant density) in the form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} + \nabla \left(\frac{1}{2} \|\mathbf{u}\|^2 \right) = -\nabla \left(\frac{p}{\rho} \right) + \mathbf{g}.$$

So, since the constant gravity $\mathbf{g} = \nabla(\mathbf{g} \cdot \mathbf{x})$, one has

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} + \nabla \mathcal{H} = 0, \tag{6.1}$$

where

$$\mathcal{H}(\mathbf{x}, t) = \frac{p}{\rho} + \frac{1}{2} \|\mathbf{u}\|^2 - \mathbf{g} \cdot \mathbf{x} \tag{6.2}$$

is called the *Bernoulli function.*[1]

6.1 Bernoulli's theorem for steady flows

In the case of steady flows, i.e. when $\partial \mathbf{u}/\partial t = 0$, taking the scalar product of equation (6.1) with the fluid velocity, $\mathbf{u}$, gives the *Bernoulli equation*

$$(\mathbf{u} \cdot \nabla)\, \mathcal{H} = 0, \tag{6.3}$$

since $\mathbf{u} \cdot (\mathbf{u} \times \boldsymbol{\omega}) \equiv 0$.

Hence, for an ideal fluid in steady flow,

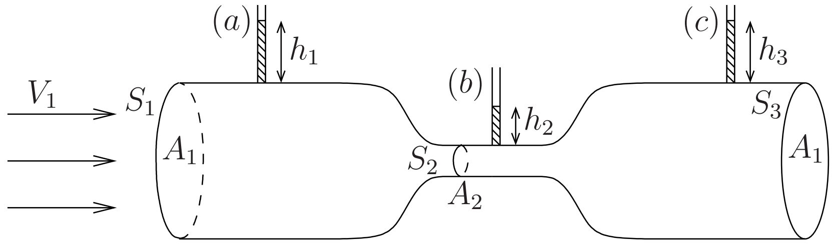
$$\mathcal{H}(\mathbf{x}) = \frac{p}{\rho} + \frac{1}{2} \|\mathbf{u}\|^2 - \mathbf{g} \cdot \mathbf{x} \tag{6.4}$$

is constant along a streamline.

[1]Not to be mistaken for Bernoulli's polynomials.

So, if a streamfunction $\psi(\mathbf{x})$ can be defined, $\mathcal{H}$ is a function of ψ ($\mathcal{H}(\mathbf{x}) \equiv \mathcal{H}(\psi)$).

Example 6.1 (The Venturi effect.)
Consider a flow through a narrow constriction of cross-section area A_2; upstream and downstream the cross-sectional area is A_1.



(Three narrow vertical tubes, (a), (b) and (c), are used to measure the pressure at different points.)

The fluid velocity is assumed inform on cross sections, S. Upstream the fluid velocity is V_1. Mass conservation implies $\displaystyle\int_S \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S$ = constant for any cross-section S, so

$$\int_{S_1} \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_{S_2} \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S \Rightarrow \int_{S_1} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_{S_2} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S \quad (\rho = \text{constant}),$$

$$\Rightarrow A_1V_1 = A_2V_2 \Rightarrow V_2 = \frac{A_1}{A_2}V_1 > V_1 \quad (\text{since } A_1 > A_2).$$

Neglecting gravity, we apply Bernoulli's equation to any streamline,

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 \quad \Rightarrow \quad p_2 = p_1 - \frac{\rho}{2}\left(V_2^2 - V_1^2\right),$$

$$\Rightarrow \quad p_2 = p_1 - \frac{\rho V_1^2}{2A_2^2}\left(A_1^2 - A_2^2\right) < p_1.$$

Thus, in the constriction the speed of the flow increases (conservation of mass) and its pressure decreases (Bernoulli's equation).

This can be measured by the thin tubes where there is fluid but no flow (i.e. fluid in hydrostatic equilibrium). If h_1 is the height of fluid in the tube (a) then

$$p_1 = p_0 + \rho g h_1 \quad (p_0 \equiv p_{\text{atm}}).$$

If h_2 is the height of fluid in the tube (b) then

$$p_2 = p_0 + \rho g h_2 \Rightarrow h_2 = \frac{p_2 - p_0}{\rho g} = \frac{p_1 - p_0}{\rho g} - \frac{V_1^2}{2gA_2^2}\left(A_1^2 - A_2^2\right),$$

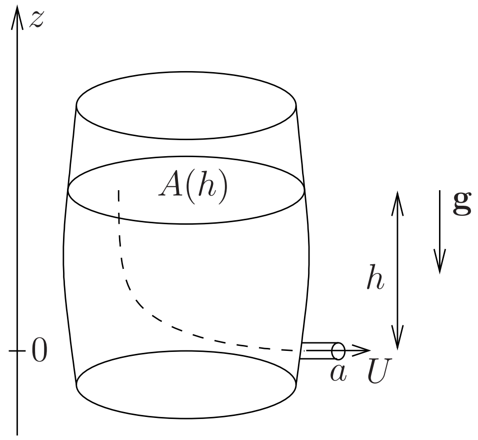
$$\Rightarrow \quad h_2 = h_1 - \frac{V_1^2}{2gA_2^2}\left(A_1^2 - A_2^2\right) < h_1.$$

In tube (c), $V_3 = V_1$ since $A_3 = A_1$ (mass conservation). So, Bernoulli's equation gives

$$p_3 + \frac{1}{2}\rho V_3^2 = p_1 + \frac{1}{2}\rho V_1^2 \Rightarrow p_3 = p_1,$$

and so $h_3 = h_1$. (In practice, h_3 will be slightly less than h_1 due to viscosity but the effect is small.)

Example 6.2 (Flow down a barrel.)
How fast does fluid flow out of a barrel?



Let h be the height of fluid level in the barrel above the outlet, which has cross-sectional area a. If $a \ll A(h)$, then the flow can be treated as approximately steady.

Mass conservation: $-A\dfrac{\mathrm{d}h}{\mathrm{d}t} = aU$ (with $U > 0$). So, if $a \ll A$ then $\left|\dfrac{\mathrm{d}h}{\mathrm{d}t}\right| \ll |U|$.

Bernoulli's theorem: consider a streamline from the surface of the fluid to the outlet,

$$p + \frac{1}{2}\rho\|\mathbf{u}\|^2 + \rho g z = \text{const}.$$

At $z = 0$: $p = p_{\text{atm}}$ and $u = U$; at $z = h$: $p = p_{\text{atm}}$ and $u = \dfrac{\mathrm{d}h}{\mathrm{d}t}$. So,

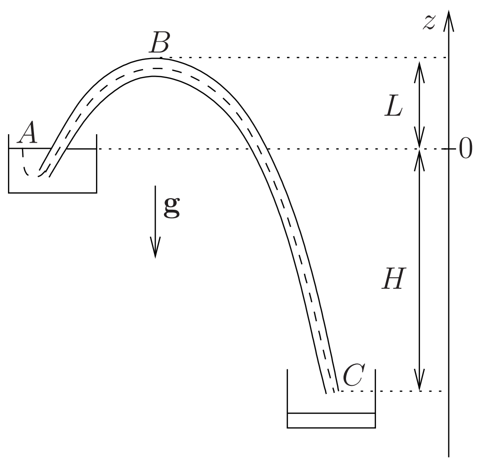
$$p_{\text{atm}} + \frac{1}{2}\rho U^2 = p_{\text{atm}} + \frac{1}{2}\rho\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + \rho g h,$$

$$\Rightarrow U^2 = \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + 2gh \Rightarrow U \simeq \sqrt{2gh} \quad \text{since} \left|\frac{\mathrm{d}h}{\mathrm{d}t}\right| \ll |U|.$$

We could have guessed this result from conservation of energy

$$\left.\begin{array}{l} \text{with } KE \simeq 0 \text{ and } PE = \rho g h \text{ at } z = h \\ \text{and } KE = \frac{1}{2}\rho U^2 \text{ and } PE = 0 \text{ at } z = 0 \end{array}\right\} \Rightarrow \frac{1}{2}\rho U^2 \simeq \rho g h.$$

Example 6.3 (Siphon.)
A technique for removing fluid from one vessel to another without pouring is to use a siphon tube.



To start the siphon we need to fill the tube with fluid, but once it is going, the fluid will continue to flow from the upper to the lower container.
In order to calculate the flow rate, we can use Bernoulli's equation along a streamline from the surface to the exit of the pipe.

At point A: $p = p_{\text{atm}}$, $z = 0$. We shall assume that the container's cross-sectional area is much larger than that of the pipe. So, $U_A \simeq 0$ (from mass conservation; see example 6.2 $-A\,\mathrm{d}h/\mathrm{d}t = aU$).

At point C: $p = p_{\text{atm}}$, $z = -H$, $u = U_c \equiv U$.

Bernoulli's equation:

$$\frac{p_{\text{atm}}}{\rho} + \underbrace{\frac{1}{2}U_A^2}_{\simeq 0} = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2}U^2 - gH \Rightarrow U \simeq \sqrt{2gH}.$$

If B is the highest point: ($U_B = U_C \equiv U$ from mass conservation)

$$\frac{p_B}{\rho} + \frac{1}{2}U^2 + gL = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2}U^2 - gH \Rightarrow p_B = p_{\text{atm}} - \rho g(L+H) < p_{\text{atm}}.$$

For $p_B > 0$, we need $H + L < \dfrac{p_{\text{atm}}}{\rho g} \approx \dfrac{10^5}{10^3 \times 10} = 10\text{m}$.

6.2 Bernoulli's theorem for potential flows

In this section we shall extend Bernoulli's theorem to the case of irrotational flows.

Recall that Euler's equation can written in the form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \mathcal{H} \quad \text{where} \quad \mathcal{H}(\mathbf{x}, t) = \frac{p}{\rho} + \frac{1}{2}\|\mathbf{u}\|^2 - \mathbf{g} \cdot \mathbf{x}.$$

If the fluid flow is irrotational, i.e. if $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$, then $\mathbf{u} \times \boldsymbol{\omega} = 0$ and $\mathbf{u} = \nabla\phi$; so, the equation above becomes

$$\nabla\left(\frac{\partial \phi}{\partial t} + \mathcal{H}\right) = 0,$$

since $\dfrac{\partial \mathbf{u}}{\partial t} = \dfrac{\partial \nabla\phi}{\partial t} = \nabla\left(\dfrac{\partial \phi}{\partial t}\right)$.

Thus, for irrotational flows,

$$\frac{\partial \phi}{\partial t} + \mathcal{H} = \frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\|\nabla\phi\|^2 - \mathbf{g} \cdot \mathbf{x} \equiv f(t) \tag{6.5}$$

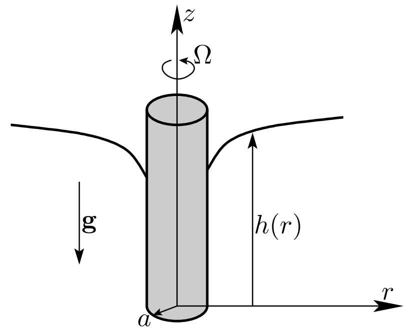
is a function of time, independent of the position, $\mathbf{x}$.

If, in addition, the flow is steady,

$$\mathcal{H} = \frac{p}{\rho} + \frac{1}{2}\|\nabla\phi\|^2 - \mathbf{g} \cdot \mathbf{x}, \tag{6.6}$$

is constant; $\mathcal{H}$ has the same value on all streamlines.

Example 6.4 (Shape of the free surface of a fluid near a rotating rod)
We consider a rod of radius a, rotating at constant angular velocity Ω, placed in a fluid. Assuming a *potential*, *axisymmetric* and *planar* fluid flow, $(u_r(r), u_\theta(r))$ in cylindrical polar coordinates, we wish to calculate the height of the free surface of the fluid near to the rod, $h(r)$. We also assume that the solid rod is an impenetrable surface on which the fluid does not slip, so that the boundary conditions for the velocity field are



$$u_r = 0 \quad \text{and} \quad u_\theta = a\Omega \quad \text{at} \quad r = a.$$

From mass conservation, one has

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r) = 0 \Leftrightarrow u_r(r) = \frac{C}{r},$$

where C is a constant of integration. However, the boundary condition $u_r = C/a = 0$ at $r = a$ implies that $C = 0$. So, $u_r = 0$ and the fluid motion is purely azimuthal.
As we assume an irrotational flow,

$$\nabla \times \mathbf{u} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_\theta)\,\hat{\mathbf{e}}_{\boldsymbol{z}} = 0 \Leftrightarrow u_\theta(r) = \frac{k}{r},$$

where k is an integration constant to be determined using the second boundary condition. At $r = a$, $u_\theta = k/a = a\Omega$ which implies that $k = a^2\Omega$. So, the fluid velocity near to the rod is

$$u_r = 0 \quad \text{and} \quad u_\theta = \frac{a^2\Omega}{r}.$$

Notice that the velocity potential, function of θ, can be determined using

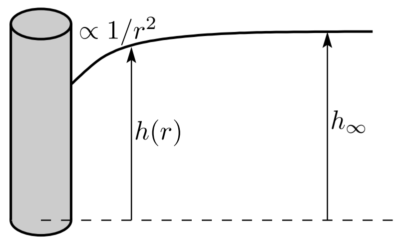
$$\mathbf{u} = \nabla\phi \Rightarrow \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \frac{a^2\Omega}{r} \Rightarrow \phi(\theta) = a^2\Omega\theta.$$

By applying Bernoulli's theorem for steady potential flows to the free surface (which is not a streamline, as streamlines are circles about the rod axis) we obtain,

$$\mathcal{H} = \underbrace{\frac{p_{\text{atm}}}{\rho} + \frac{1}{2}u_\theta^2(r) + gh(r)}_{\text{near rod}} = \underbrace{\frac{p_{\text{atm}}}{\rho} + gh_\infty}_{\text{at large } r},$$

where the constant pressure $p = p_{\text{atm}}$ is the atmospheric pressure and $\lim\limits_{r\to\infty} h(r) = h_\infty$. (Notice also that $u_\theta \propto 1/r \to 0$ as $r \to \infty$.)

Thus, the height of the free surface is



$$h(r) = h_\infty - \frac{1}{2g}u_\theta^2(r) = h_\infty - \frac{a^4\Omega^2}{2gr^2}, \qquad (6.7)$$

which shows that the free surface dips as $1/r^2$ near to the rotating rod.

Alternatively, Euler's equation could be solved directly (i.e. without involving Bernoulli's theorem) as in § 5.6 with an azimuthal flow, now potential, of the form $u_\theta = a^2\Omega/r$. We can then explain the result (6.7) in terms of centripetal acceleration; since the fluid particles move in circles, there must be an inwards central force producing the necessary centripetal acceleration (i.e. balancing the centrifugal force). Indeed, from the radial component of the momentum equation, one has

$$-\rho\frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} \Rightarrow \frac{\partial p}{\partial r} = \rho\frac{a^4\Omega^2}{r^3}.$$

However, since the fluid is in vertical hydrostatic equilibrium, the pressure satisfies

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow p(r,z) = p_{\mathrm{atm}} - \rho g(z - h(r)).$$

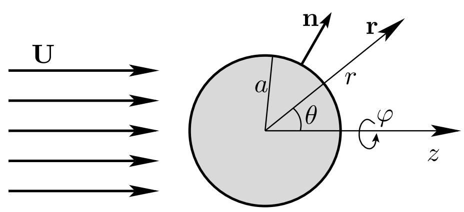
Hence, we have

$$\frac{\partial p}{\partial r} = \rho g\frac{\mathrm{d}h}{\mathrm{d}r} = \rho\frac{a^4\Omega^2}{r^3} \Rightarrow h(r) = h_\infty - \frac{a^4\Omega^2}{2gr^2},$$

as in equation (6.7).

6.3 Drag force on a sphere

We wish to calculate the pressure force exerted by a steady fluid flow on a solid sphere.



In § 4.5.1 we obtained the velocity potential of a uniform stream, $U\hat{\mathbf{e}}_{\boldsymbol{z}}$, past a stationary sphere of radius a,

$$\phi(r,z) = Uz\left(1 + \frac{a^3}{2(r^2+z^2)^{3/2}}\right),$$

in cylindrical polar coordinates (r,θ,z). In spherical polar coordinates, (r,θ,φ), this velocity potential becomes

$$\phi(r,\theta) = U\cos\theta\left(r + \frac{a^3}{2r^2}\right). \tag{6.8}$$

The non-zero components of the fluid velocity, $\mathbf{u} = \nabla\phi$, are then

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta\left(1 - \frac{a^3}{r^3}\right) \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right). \tag{6.9}$$

Hence, at $r = a$, on the solid sphere's surface, $u_r = 0$ as required by the kinematic boundary conditions and

$$u_\theta(\theta)|_{r=a} = -\frac{3}{2}U\sin\theta.$$

To express the pressure force on the sphere in terms of the fluid velocity, we use Bernoulli's theorem for steady potential flows, $\mathcal{H} = p/\rho + \|\mathbf{u}\|^2/2 = \text{constant}$, ignoring gravity. At $r = a$ the fluid pressure, $p(\theta)$, therefore satisfies

$$\frac{p(\theta)}{\rho} + \frac{1}{2}\,u_\theta^2\big|_{r=a} = \frac{p_\infty}{\rho} + \frac{1}{2}U^2,$$

where p_∞ is the pressure as $r \to \infty$.

Thus, the pressure distribution on the sphere is

$$p(\theta) = p_\infty + \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\theta\right), \tag{6.10}$$

and the total pressure force is the surface integral of $p(\theta)$ on the sphere $r = a$,

$$\mathbf{F} = -\int_S p\,\mathbf{n}\,\mathrm{d}S = -\int_0^\pi \int_0^{2\pi} p(\theta)\,\hat{\mathbf{e}}_{\boldsymbol{r}}\,a^2 \sin\theta\,\mathrm{d}\varphi\mathrm{d}\theta, \tag{6.11}$$

where $\hat{\mathbf{e}}_{\boldsymbol{r}} = \sin\theta\cos\varphi\,\hat{\mathbf{e}}_{\boldsymbol{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{e}}_{\boldsymbol{y}} + \cos\theta\,\hat{\mathbf{e}}_{\boldsymbol{z}}$.
As the flow is axisymmetric, the only non-zero component of the force should be in the axial direction, z. Indeed,

$$F_x = \mathbf{F}\cdot\hat{\mathbf{e}}_{\boldsymbol{x}} = -a^2 \int_0^{2\pi} \cos\varphi\,\mathrm{d}\varphi \int_0^\pi p(\theta)\sin^2\theta\,\mathrm{d}\theta = 0,$$

and

$$F_y = \mathbf{F}\cdot\hat{\mathbf{e}}_{\boldsymbol{y}} = -a^2 \int_0^{2\pi} \sin\varphi\,\mathrm{d}\varphi \int_0^\pi p(\theta)\sin^2\theta\,\mathrm{d}\theta = 0.$$

However, after substituting for $p(\theta)$ in

$$F_z = \mathbf{F}\cdot\hat{\mathbf{e}}_{\boldsymbol{z}} = -2\pi a^2 \int_0^\pi p(\theta)\sin\theta\cos\theta\,\mathrm{d}\theta,$$

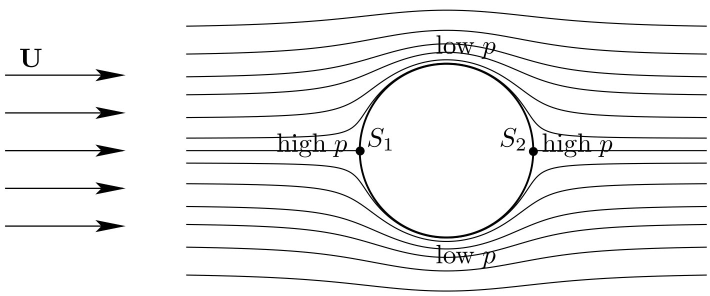
we find that

$$F_z = -2\pi a^2 \left[\left(p_\infty + \frac{1}{2}\rho U^2\right)\int_0^\pi \sin\theta\cos\theta\,\mathrm{d}\theta - \frac{9}{8}\rho U^2 \int_0^\pi \sin^3\theta\cos\theta\,\mathrm{d}\theta\right] = 0,$$

so that *the total drag force on the sphere, due to the fluid flow around it, is zero!*

D'Alembert's paradox: it can be demonstrated that the drag force on any 3-D solid body moving at uniform speed in a potential flow is zero (see, e.g., Paterson, § XI.9, p. 240).
This is not true in reality of course, as *flows past 3-D solid bodies are not potential.*

We can see why a potential flow past a sphere gives zero drag by looking at the streamlines.



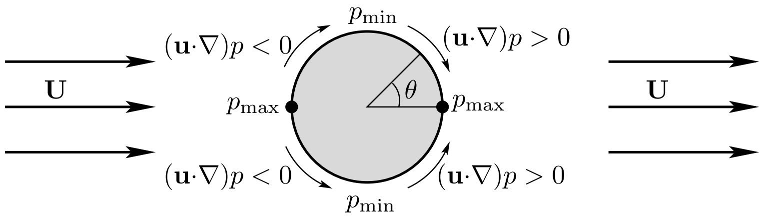
The flow is clearly fore-aft symmetric (symmetry about $z = 0$); the front (S_1) and the back (S_2) of the sphere are stagnation points at equal pressure, $P_{S_1} = P_{S_2} = p_\infty + \frac{1}{2}\rho U^2$. At the side, $u_r = 0$ and $u_\theta^2 > 0$, so from Bernoulli's theorem, the pressure there is lower than at the stagnation points but it must have the same symmetry as the flow. Notice that, from Bernoulli's theorem, the pressure does not depend on the direction of the flow, but on its speed $\|\mathbf{u}\|$ only.
However, the real flow past a sphere is not symmetric and, as a consequence, the fluid exerts a net drag force on the sphere.

6.4 Separation

The pressure distribution on the surface a solid sphere placed is a uniform stream,

$$p(\theta) = p_\infty + \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\theta\right),$$

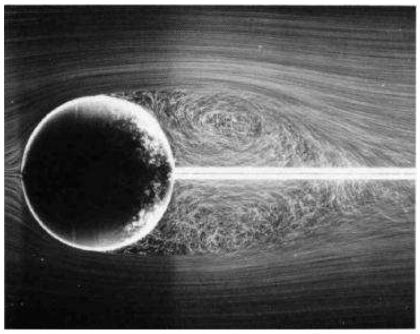
reaches its minimum, $p_{\min} = p_\infty - 5/8\ \rho U^2$, at $\theta = \pm\pi/2$. So, the pressure gradient in the direction of the flow, $(\mathbf{u}\cdot\nabla)p$, is a positive from $\theta = 0$ to $\theta = \pm\pi/2$ and negative beyond.



An *adverse pressure gradient*, $(\mathbf{u}\cdot\nabla)p > 0$ (i.e. pressure increasing in the direction of the flow along the surface), is "bad news" and causes the flow to *separate*, leaving a turbulent wake behind the sphere.
Very roughly one can estimate the pressure difference upstream and downstream as $1/2\ \rho U^2$, so that the drag force $F \propto 1/2\ \rho U^2 \times A$, where A is the cross-sectional area.
The ratio

$$C_D = \frac{F}{\frac{1}{2}\rho U^2 A} \tag{6.12}$$

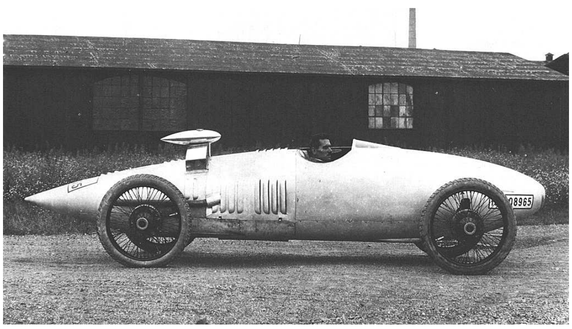
is called *drag coefficient* and depends, e.g., on the shape of the body (see Acheson §4.13, p. 150).

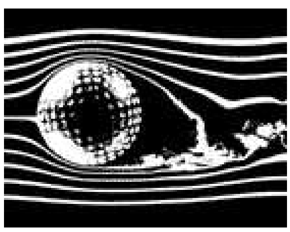


The way to reduce drag (i.e. resistance) is to reduce separation:

- Streamlining: separation occurs because of adverse pressure gradients on the surface of solid bodies. These can be reduced by using more "streamlined" shapes, that avoid diverging streamlines (e.g., aerodynamic bike helmets (time trial cyclist), ships, aeroplanes and cars).

- Surface roughness: paradoxically, a rough surface can reduce drag by reducing separation (e.g. dimple pattern of golf balls and shining of cricket ball on one side).

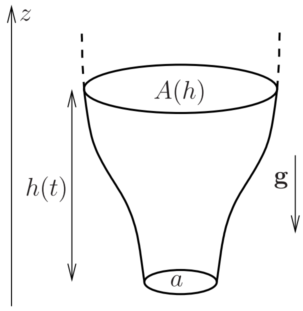




6.5 Unsteady flows

6.5.1 Flows in pipes

In example 6.2 we consider a flow out of a barrel through a small hole. Now, consider a flow out of a narrowing tube, opened to the atmosphere at both ends, where the exit is not much smaller than the cross-section (i.e. the fluid flow cannot be assumed steady).



Let $A(z)$ be the *smoothly varying* cross-sectional area of the pipe at height z, such that $A \to A_\infty$ as $z \to \infty$ and $A(0) = a$.

We assume that the flow is potential and purely in the z-direction, $u_z = \partial\phi/\partial z \equiv w$.

By conservation of mass the volume flux, $Q(t) = -w(z,t)A(z)$, must be independent of height. Hence,

$$\frac{\partial \phi}{\partial z} = w(z,t) = -\frac{Q(t)}{A(z)} \Rightarrow \phi(z,t) = \phi(0,t) - Q(t)\int_0^z \frac{\mathrm{d}\mu}{A(\mu)}.$$

(Note that we could set $\phi(0,t) = 0$ without loss of generality.) Applying Bernoulli's theorem for potential flows, $p/\rho + \|\mathbf{u}\|^2/2 + \partial\phi/\partial t - \mathbf{g}\cdot\mathbf{x} = F(t)$, at the free surface and the exit gives,

$$\begin{aligned} \text{at } z = 0, \quad & \frac{p_{\mathrm{atm}}}{\rho} + \frac{1}{2}\frac{Q^2(t)}{a^2} + \frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) = F(t), \\ \text{and at } z = h, \quad & \frac{p_{\mathrm{atm}}}{\rho} + \frac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) - \frac{\mathrm{d}Q}{\mathrm{d}t}\int_0^h \frac{\mathrm{d}z}{A(z)} + gh = F(t). \end{aligned}$$

Equating both expressions gives

$$\begin{aligned} & \frac{1}{2}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 - \frac{Q^2(t)}{a^2}\right] - \frac{\mathrm{d}Q}{\mathrm{d}t}\int_0^h \frac{\mathrm{d}z}{A(z)} + gh = 0, \\ \Leftrightarrow\ & \frac{1}{2}\left[1 - \frac{A^2(h)}{a^2}\right]\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + A(h)\frac{\mathrm{d}^2 h}{\mathrm{d}t^2}\int_0^h \frac{\mathrm{d}z}{A(z)} + gh = 0 \quad \text{since} \quad Q(t) = -A(h)\frac{\mathrm{d}h}{\mathrm{d}t}. \end{aligned}$$

The fluid height, $h(t)$, is then solution to the nonlinear second order ordinary differential equation

$$\left(A(h)\int_0^h \frac{\mathrm{d}z}{A(z)}\right)\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + \frac{1}{2}\left[1 - \frac{A^2(h)}{a^2}\right]\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + gh = 0. \tag{6.13}$$

Far from the exit this equation becomes approximately

$$h\ddot{h} + \frac{1}{2}\left(1 - \frac{A_\infty^2}{a^2}\right)\dot{h}^2 + gh = 0,$$

since, as $h \to \infty$,

$$A(h) \sim A_\infty \quad \text{and} \quad \int_0^h \frac{\mathrm{d}z}{A(z)} \sim \int_0^h \frac{\mathrm{d}z}{A_\infty} = \frac{h}{A_\infty}.$$

Using the chain rule, $\ddot{h} = \mathrm{d}\dot{h}/\mathrm{d}t = \mathrm{d}\dot{\mathrm{h}}/\mathrm{dh}\ \mathrm{d}h/\mathrm{d}t = \dot{h}\ \mathrm{d}\dot{\mathrm{h}}/\mathrm{dh}$, one finds

$$h\dot{h}\frac{\mathrm{d}\dot{h}}{\mathrm{d}h} + \frac{1}{2}\left(1 - \frac{A_\infty^2}{a^2}\right)\dot{h}^2 + gh = 0 \quad \Leftrightarrow \quad \frac{1}{2}\frac{\mathrm{d}\dot{h}^2}{\mathrm{d}h} + \frac{1}{2}\left(1 - \frac{A_\infty^2}{a^2}\right)\frac{\dot{h}^2}{h} + g = 0$$

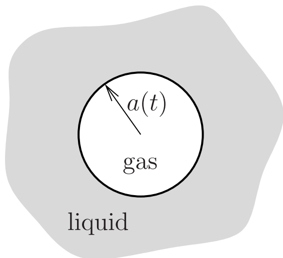
which can be written as a *linear* differential equation for $Z = \dot{h}^2/2$,

$$\frac{\mathrm{d}Z}{\mathrm{d}h} + \left(1 - \frac{A_\infty^2}{a^2}\right)\frac{Z}{h} + g = 0.$$

6.5.2 Bubble oscillations

The sound of a "babbling brook" is due to the oscillation (compression/expansion) of air bubbles entrained into the stream. The pitch of the sound depends on the size of the bubbles.

Consider a bubble of radius $a(t)$; the velocity of the fluid at the bubble surface, $u_r = \dfrac{\mathrm{d}a}{\mathrm{d}t} \equiv \dot{a}$.



We can model the oscillations of the bubble of air using a potential flow due to a point source/sink of fluid at the centre of the bubble,

$$\phi(r,t) = -\frac{k(t)}{r} \Rightarrow u_r = \frac{\partial\phi}{\partial r} = \frac{k}{r^2}.$$

The boundary condition at the bubble's surface, $r = a$, is $u_r = \dfrac{k}{a^2} = \dot{a}$. So,

$$k = \dot{a}a^2 \Rightarrow u_r = \frac{\dot{a}a^2}{r^2} \quad \text{and} \quad \phi = -\frac{\dot{a}a^2}{r} \Rightarrow \frac{\partial\phi}{\partial t} = -\frac{\ddot{a}a^2}{r} - 2\frac{a\dot{a}^2}{r}$$

Applying Bernoulli's theorem (ignoring gravity) as $r \to \infty$,

$$\frac{p}{\rho} + \frac{1}{2}\|\nabla\phi\|^2 + \frac{\partial\phi}{\partial t} = F(t) = \frac{p_\infty}{\rho} \quad (\text{as } r \to \infty, \phi \to 0 \text{ and } \|\mathbf{u}\| \to 0\text{: the fluid is stationary}).$$

At the bubble's surface,

$$\frac{p(a)}{\rho} + \frac{1}{2}\dot{a}^2 - \frac{\ddot{a}a^2}{a} - 2\frac{a\dot{a}^2}{a} = \frac{p(a)}{\rho} - \ddot{a}a - \frac{3}{2}\dot{a}^2 = F(t).$$

Combining the two expressions above, one gets

$$\frac{p(a) - p_\infty}{\rho} = \ddot{a}a + \frac{3}{2}\dot{a}^2, \tag{6.14}$$

where $p(a)$ is the fluid pressure at the bubble's surface. Now, if the gas inside the bubble of mass m is subject to adiabatic changes, its equation of state is

$$p_g = K\rho_g^\gamma \quad \text{where} \quad \rho_g = \frac{3m}{4\pi a^3},$$

and K is a constant to determine — the adiabatic index γ depends on the gas considered.

Moreover, since the bubble of gas is in balance with the surrounding fluid, continuity of pressure $p_g = p(a)$ must be satisfied at the surface $r = a(t)$.

Now, for a bubble in equilibrium, such that $a = a_0$ and $\dot{a} = \ddot{a} = 0$, equation (6.14) gives $p = p_\infty$ and, imposing pressure continuity $p_g = p$ at $r = a_0$, one gets

$$p_g = K\rho_g^\gamma = K\left(\frac{3m}{4\pi a_0^3}\right)^\gamma = p_\infty \Rightarrow K = p_\infty \left(\frac{4\pi a_0^3}{3m}\right)^\gamma.$$

So, pressure continuity at the bubble's surface $r = a(t)$ implies

$$p(a) = p_g = K\rho_g^\gamma = p_\infty \left(\frac{4\pi a_0^3}{3m}\right)^\gamma \left(\frac{3m}{4\pi a^3}\right)^\gamma = p_\infty \left(\frac{a_0}{a}\right)^{3\gamma}.$$

Then, equation (6.14) becomes

$$\frac{p_\infty}{\rho}\left(\frac{a_0^{3\gamma}}{a^{3\gamma}} - 1\right) = \ddot{a}a + \frac{3}{2}\dot{a}^2.$$

For small amplitude oscillations about the equilibrium $a(t) = a_0 + \epsilon(t)$ where $|\epsilon| \ll a_0$, so that $\dot{a} = \dot{\epsilon}$, $\ddot{a} = \ddot{\epsilon}$ and $\dot{a}^2 = \dot{\epsilon}^2 \simeq 0$; the nonlinear terms are negligible at first approximation. Thus,

$$\begin{aligned} a_0\ddot{\epsilon} &= \frac{p_\infty}{\rho}\left(\frac{a_0^{3\gamma}}{a_0^{3\gamma}\left(1 + \frac{\epsilon}{a_0}\right)^{3\gamma}} - 1\right) \simeq -3\gamma\frac{p_\infty}{\rho}\frac{\epsilon}{a_0}, \\ \Rightarrow \quad & \ddot{\epsilon} + \frac{3\gamma p_\infty}{\rho a_0^2}\epsilon = 0. \end{aligned}$$

The bubble undergo periodic small amplitude oscillations with frequency $\omega = \left(\frac{3\gamma p_\infty}{\rho a_0^2}\right)^{1/2}$. Note that the frequency scales with the inverse of the (mean) radius of the bubbles. E.g. for $\gamma = 3/2$, $p_\infty = 10^5$ Pa and $\rho = 10^3\ \mathrm{kg\,m^{-3}}$,

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi a_0}\sqrt{\frac{3\gamma p_\infty}{\rho}} \Rightarrow f \times a_0 \simeq 3\ \mathrm{kHz\ mm}.$$

For bubbles of size $a_0 = 0.2\,\mathrm{mm}$, $f \simeq 15$ kHz (G9).

6.6 Acceleration of a sphere

We have already shown that a sphere moving with a steady velocity under a potential flow has no drag force. What about an accelerating sphere?

The velocity potential for a sphere of radius a moving with velocity U in still water is

$$\phi = -\frac{Ua^3}{2r^2}\cos\theta.$$

(This flow satisfies the following boundary conditions: $\mathbf{u} = \nabla\phi \to 0$ as $r \to \infty$ together with $\mathbf{u_r} = U\cos\theta\,\hat{\mathbf{e}}_\mathbf{r}$ at $r = a$.)

Rather than calculating the pressure via Bernoulli's theorem, we calculate the work done by the forces acting on the sphere as it moves at speed U, function of time, through the fluid.

The total kinetic energy of the system sphere of mass m plus fluid is

$$\begin{aligned} T &= \frac{1}{2}mU^2 + \int_V \frac{1}{2}\rho\,(\nabla\phi)^2\,\mathrm{d}V, \\ &= \frac{1}{2}mU^2 + \frac{1}{2}\rho\int_V \left[\nabla\cdot(\phi\nabla\phi) - \phi\underbrace{\nabla^2\phi}_{0}\right]\mathrm{d}V, \quad (\text{using } \nabla\cdot(f\mathbf{A}) = \mathbf{A}\cdot\nabla f + f\nabla\cdot\mathbf{A}) \\ &= \frac{1}{2}mU^2 + \frac{1}{2}\rho\int_S \phi\nabla\phi\cdot\mathbf{n}\,\mathrm{d}S, \quad \text{by divergence theorem.} \end{aligned}$$

Here S is the surface of the sphere of radius a. So $\mathbf{n} = -\hat{\mathbf{e}}_\mathbf{r}$ and $\mathrm{d}S = a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, such that

$$\begin{aligned} T &= \frac{1}{2}mU^2 - \frac{1}{2}\rho\int_0^\pi \phi|_{r=a}\left.\frac{\partial\phi}{\partial r}\right|_{r=a} 2\pi a^2\sin\theta\,\mathrm{d}\theta, \\ &= \frac{1}{2}mU^2 + \frac{\pi a^3}{2}\rho U^2\int_0^\pi \cos^2\theta\sin\theta\,\mathrm{d}\theta, \\ &= \frac{1}{2}mU^2 + \frac{\pi a^3}{3}\rho U^2 \quad \text{since} \int_0^\pi \cos^2\theta\sin\theta\,\mathrm{d}\theta = -\frac{1}{3}\int_0^\pi \frac{\mathrm{d}\cos^3\theta}{\mathrm{d}\theta}\,\mathrm{d}\theta = \frac{2}{3}. \end{aligned}$$

So $T = \frac{1}{2}(m+M)U^2$, where $M = \frac{2}{3}\pi a^3\rho$ is called the added mass and represents the mass of fluid that must be accelerated along with the sphere.

The rate of working of the forces F acting on the sphere equals the change of kinetic energy,

$$FU = \frac{\mathrm{d}T}{\mathrm{d}t} = (m+M)U\frac{\mathrm{d}U}{\mathrm{d}t}.$$

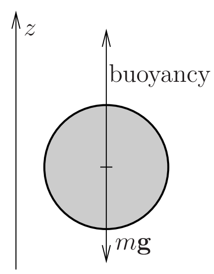
Hence, the force required to accelerate the sphere is given by

$$F = (m+M)\frac{\mathrm{d}U}{\mathrm{d}t}.$$

Thus, the acceleration of a bubble (mass m and radius a) rising under gravity (see §5.3 on Archimedes theorem) satisfies

$$F = \underbrace{\frac{4}{3}\pi a^3\rho g}_{\text{buoyancy force}}\ \underbrace{-mg}_{\text{weight}} = (2M-m)g = (m+M)\,\frac{\mathrm{d}U}{\mathrm{d}t},$$

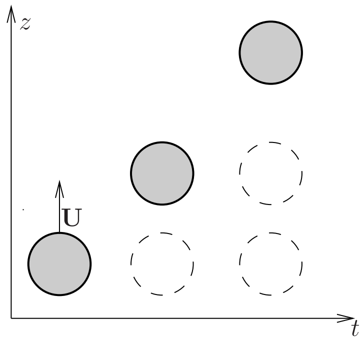
$$\Rightarrow \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{2M-m}{M+m}g = \frac{4\pi a^3\rho - 3m}{2\pi a^3\rho + 3m}g.$$



As mass density is much less for a gas than for a liquid, we can assume $m \ll M$, so that

$$\frac{\mathrm{d}U}{\mathrm{d}t} \simeq 2g.$$

Alternatively: Consider a bubble of mass m rising under gravity with speed $U = \dfrac{\mathrm{d}z}{\mathrm{d}t}$.



At height z the potential energy is

$$V = \underbrace{mgz}_{\text{weight}} - \underbrace{\frac{4}{3}\pi a^3 \rho g z}_{\text{buoyancy}}.$$

In absence of dissipative processes the total energy remains constant; hence,

$$T + V = \frac{1}{2}(m+M)U^2 + mgz - \frac{4}{3}\pi a^3 \rho g z = \text{const.}$$

Differentiating this expression with respect to time gives

$$(m+M)U\frac{\mathrm{d}U}{\mathrm{d}t} + \left(m - \frac{4}{3}\pi a^3 \rho\right) gU = 0,$$

$$\Rightarrow \quad \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{2M-m}{M+m}g = \frac{4\pi a^3\rho - 3m}{2\pi a^3 \rho + 3m}g.$$

Again, for a bubble of gas in a liquid $M \gg m$, so $\dfrac{\mathrm{d}U}{\mathrm{d}t} \simeq 2g$; the bubble accelerates at twice the gravitational acceleration.