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Best Constants for Moser-Trudinger Inequalities on High Dimensional Hyperbolic Spaces *

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Abstract

Though there have been extensive works on best constants for Moser-Trudinger inequalities in Euclidean spaces, Heisenberg groups or compact Riemannian manifolds, much less is known for sharp constants for the Moser-Trudinger inequalities on hyperbolic spaces. Earlier works only include the sharp constant for the Moser-Trudinger inequality on the two-dimensional hyperbolic disc. In this paper, we establish best constants for several types of Moser-Trudinger inequalities on high dimensional hyperbolic spaces $\mathbb{H}^n$ ($n \geq 2$). These include sharp constants for the Moser-Trudinger inequalities on both bounded and unbounded domains of the hyperbolic space $\mathbb{H}^n$ (see Theorems 1.1 and 1.2), sharp constants for the singular Moser-Trudinger inequality on unbounded domains when we impose restrictions only on the gradient norms (Theorem 1.3) or on the full hyperbolic Sobolev norms (Theorem 1.4). Our results are surprisingly general and extend most results in Euclidean spaces to hyperbolic spaces of any dimension. In particular, we have used a rearrangement-free argument in the hyperbolic spaces to establish Theorems 1.3 and 1.4 where symmetrization argument does not work to prove such sharp singular Moser-Trudinger inequalities on the entire hyperbolic space.

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1 Introduction

The Moser-Trudinger inequalities can be considered as the limiting case of Sobolev inequalities. They were established independently by Yudovič [28], Pohožaev [24] and Trudinger [26]. In 1971, Moser [22], sharpening Trudinger's inequality, proved that

Theorem A. *Let Ω be a domain with finite measure in Euclidean $n-$space $\mathbb{R}^n$, $n \geq 2$. Then there exists a sharp constant $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$(where ω_{n-1} is the area of the surface of the unit $n-$ball) such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha\,|f|^{\frac{n}{n-1}}\right)dx \leq c_0 < \infty$$

for any $\alpha \leq \alpha_n$, any $f \in C_0^\infty(\Omega)$ with $\int_\Omega |\nabla f|^n\,dx \leq 1$. This constant α_n is sharp in the sense that if $\alpha > \alpha_n$, then the above inequality can no longer hold with some c_0 independent of f.

This result has been generalized in many directions. For instance, the singular Moser-Trudinger inequality which is an interpolation of Hardy inequality and Moser-Trudinger inequality was studied by Adimurthi and Sandeep in [3]: there exists a constant $C_0 = C_0(n) > 0$ such that

$$\frac{1}{|\Omega|^{1-\frac{\alpha}{n}}}\int_\Omega \frac{\exp\left(\alpha\,|u|^{\frac{n}{n-1}}\right)}{|x|^\beta}dx \leq C_0 \tag{1.1}$$

for any $\beta \in [0,n)$, $0 \leq \alpha \leq \left(1-\frac{\beta}{n}\right)\alpha_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n\,dx \leq 1$. Moreover, this constant $\left(1-\frac{\beta}{n}\right)\alpha_n$ is sharp in the sense that if $\alpha > \left(1-\frac{\beta}{n}\right)\alpha_n$, then the above inequality can no longer hold with some C_0 independent of u.

When Ω has infinite volume, some versions of Moser-Trudinger type inequalities for unbounded domains were first proposed by D.M. Cao [8] when $n = 2$ and J.M. do Ó [12] for the general case $n \geq 2$. However, those inequalities are not sharp. These results were sharpened later by Adachi and Tanaka [1] in order to determine the best constant.

B. Ruf [23] (for the case $n = 2$), Y. Li and B. Ruf [20] (for the general case $n \geq 2$) established a critical Moser-Trudinger type inequality for unbounded domains in Euclidean spaces. It was extended further in [4] to the singular Moser-Trudinger type inequality on unbounded domains in Euclidean spaces.

Recently, there has been further progress in establishing sharp constants for both critical and subcritical Moser-Trudinger inequalities on unbounded domains in non-Euclidean setting such as on the Heisenberg groups by Lam et al. in [14], [18] which improve the earlier work on bounded domains by Cohn and Lu [9, 10] (see also [17]) or for Adams inequalities [2] on high (fractional) order Sobolev spaces by Lam and Lu in [15], [16] which improve the work of Ruf and Sani [25]. One of the key ingredients in the above works is a new approach of establishing sharp constants for Moser-Trudinger inequalities on unbounded domains and Adams inequalities without using the symmetrization argument. Indeed, optimal symmetrization principle is not available in the aforementioned circumstances.

There has also been substantial progress for the Moser-Trudinger inequality on spheres or compact Riemannian manifolds. We refer the interested reader to [6], [11], [13], [19],

[7], just to name a few. Nevertheless, much less is known for sharp constants of Moser-Trudinger inequalities in a hyperbolic space. In 2010, Mancini and Sandeep [21] established the following Moser-Trudinger inequality on a conformal disc:

Theorem B. *Let D be the unit open disc in $\mathbb{R}^2$ endowed with a conform metric $g = (\frac{2}{1-|x|^2})^2 g_e$, and $dV_g = (\frac{2}{1-|x|^2})^2 dx$ be the volume form. Then*

$$\sup_{u\in C_0^\infty(D), \int_D |\nabla u|^2 dx\leq 1} \int_D (e^{4\pi u^2}-1)dV < \infty$$

and 4π cannot be improved.

In this paper, we will establish sharp constants for Moser-Trudinger inequalities on the hyperbolic space. The hyperbolic space $\mathbb{H}^n$ ($n \geq 2$) is a complete and simply connected Riemannian manifold having constant sectional curvature equal to -1, and for a given dimensional number, any two such spaces are isometric [27]. There are several models for $\mathbb{H}^n$, the most important model being the half-space model, the ball model, and the hyperboloid or Lorentz model, with the ball model being especially useful for questions involving rotational symmetry. We will only use the ball model in this paper.

Let $B^n = \{x \in R^n : |x| < 1\}$ denote the unit open ball in the Euclidean space $\mathbb{R}^n$. The space B^n endowed with the Riemannian metric $g_{ij} = (\frac{1}{1-|x|^2})^2\delta_{ij}$ is called the ball model of the hyperbolic space $\mathbb{H}^n$. Denote the associated hyperbolic volume by $dV = (\frac{2}{1-|x|^2})^n dx$. For any measurable set $E \subset \mathbb{H}^n$, set $|E| = \int_E dV$. Let $d(0, x)$ denote the hyperbolic distance between the origin and x. It is known that $d(0, x) = \ln\frac{1+|x|}{1-|x|}$ for $x \in \mathbb{H}^n$. The hyperbolic gradient ∇_g is given by $\nabla_g = (\frac{1-|x|^2}{2})^2\nabla$.

Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. Denote $||f||_{n,\Omega} = (\int_\Omega |f|^n dV)^{\frac{1}{n}}$. Then we have the following:

$$||\nabla_g f||_{n,\Omega} = (\int_\Omega < \nabla_g f, \nabla_g f >_g^{n/2} dV)^{\frac{1}{n}} = (\int_\Omega |\nabla f|^p dx)^{\frac{1}{n}}.$$

Let $||f||_n = (\int_{\mathbb{H}^n} |f|^n dV)^{\frac{1}{n}}$. Then we have

$$||\nabla_g f||_n = (\int_{\mathbb{H}^n} < \nabla_g f, \nabla_g f >_g^{n/2} dV)^{\frac{1}{n}} = (\int_{B^n} |\nabla f|^p dx)^{\frac{1}{n}}.$$

We use $W_0^{1,n}(\Omega)$ to express the completion of $C_0^\infty(\Omega)$ under the norm

$$||u||_{W_0^{1,n}(\Omega)} = (\int_\Omega |f|^n dV + \int_\Omega |\nabla f|^n dx)^{\frac{1}{n}}.$$

We will also use $W^{1,n}(\mathbb{H}^n)$ to express the completion of $C_0^\infty(\mathbb{H}^n)$ under the norm

$$||u||_{W_0^{1,n}(\mathbb{H}^n)} = (\int_{\mathbb{H}^n} |f|^n dV + \int_{\mathbb{H}^n} |\nabla f|^n dx)^{\frac{1}{n}}.$$

It is known that the symmetrization argument is the key tool in the proof of the classical Moser-Trudinger inequalities. Now, let's recall some facts about the rearrangement in hyperbolic space [5].

Let $f : \mathbb{H}^n \to \mathbb{R}$ be such that

$$|\{x \in \mathbb{H}^n : |f(x)| > t\}| = \int_{\{x\in\mathbb{H}^n:|f(x)|>t\}} dV < +\infty$$

for every $t > 0$. Its distribution function is defined by

$$\mu_f(t) = |\{x \in \mathbb{H}^n : |f(x)| > t\}|.$$

Then its decreasing rearrangement f^* is defined by

$$f^*(s) = \sup\{t > 0, \mu_f(t) > s\}.$$

Now, define $f^\sharp : \mathbb{H}^n \to \mathbb{R}$ by

$$f^\sharp(x) = f^*(|B(0, d(0, x))|),$$

where $B(0, d(0, x))$ is the ball centered at the origin and with radius $d(0, x)$ in the hyperbolic space. Then, for every continuous increasing function $\Phi : [0,\infty) \to [0,\infty)$, we have from [5] that

$$\int_{\mathbb{H}^n} \Phi(|f|)dV = \int_{\mathbb{H}^n} \Phi(f^\sharp)dV.$$

And for any Lipschitz continuous function f,

$$\|\nabla_g f^\sharp\|_p \le \|\nabla_g f\|_p.$$

In this paper, we will first prove the sharp singular Moser-Trudinger inequality on bounded domains in the hyperbolic space of any high dimension.

Theorem 1.1 *Let $\Omega \subset \mathbb{H}^n$ be an open domain with $|\Omega| = \int_\Omega dV < +\infty$, $0 \le \beta < n$ and $0 \le \alpha \le \alpha_n(1 - \frac{\beta}{n})$, $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$. Then there exists a constant $C_\beta > 0$ such that*

$$\sup_{u\in C_0^\infty(\Omega),\|\nabla_g u\|_{n,\Omega}\le 1} \int_\Omega \frac{\exp(\alpha|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \le C_\beta \int_\Omega \frac{dV}{[d(0,x)]^\beta}.$$

The result is sharp in the sense that: if $\alpha > \alpha_n(1 - \frac{\beta}{n})$, the supreme will become infinite.

Setting $\beta = 0$ in the theorem (1.1), we can obtain the following standard Moser-Trudinger inequality on bounded domains in hyperbolic spaces of any high dimension.

Corollary 1.1 *Let $\Omega \subset \mathbb{H}^n$ be an open domain with $|\Omega| = \int_\Omega dV < +\infty$. Then there exists a constant $C_n > 0$ such that*

$$\sup_{u\in C_0^\infty(\Omega),\|\nabla_g u\|_{n,\Omega}\le 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha_n|u|^{\frac{n}{n-1}})dV \le C_n.$$

The constant α_n is sharp in the sense that if α_n is replaced by any α bigger than α_n, the supreme will become infinite.

Then we will set up the following sharp subcritical Moser-Trudinger type inequality on the entire hyperbolic space in the spirit of Adachi-Tanaka [1] when we only restrict the norm of the hyperbolic gradient of the function.

Theorem 1.2 *For any $\alpha \in (0, \alpha_n)$, there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{H}^n} [\Phi_n(\alpha(\frac{|u|}{\|\nabla_g u\|_n})^{\frac{n}{n-1}})]dV \leq C_\alpha \frac{\|u\|_n^n}{\|\nabla_g u\|_n^n},$$

for $u \in W^{1,n}(\mathbb{H}^n) \setminus \{0\}$, where $\Phi_n(x) = e^x - \sum_{j=0}^{n-2} \frac{x^j}{j!}$. Moreover, the restriction $0 < \alpha < \alpha_n$ is optimal in the sense that for $\alpha \geq \alpha_n$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset W^{1,n}(\mathbb{H}^n) \setminus \{0\}$ such that $\|\nabla_g u_k\|_n = 1$ and

$$\frac{1}{\|u_k\|_n^n} \int_{\mathbb{H}^n} [\Phi_n(\alpha(|u_k|)^{\frac{n}{n-1}})]dV \to \infty.$$

Next, we will prove the following sharp singular Adachi-Tanaka type inequality on the entire hyperbolic space which extends the result of Theorem 1.2.

Theorem 1.3 *Let $0 \leq \beta < n$. For any $\alpha \in (0, \alpha_n(1 - \frac{\beta}{n}))$, there exists a constant $C_{\alpha,\beta} > 0$ such that for any $u \in W^{1,n}(\mathbb{H}^n) \setminus \{0\}$ satisfying $\|\nabla_g u\|_n \leq 1$,*

$$\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \leq C_{\alpha,\beta} \int_{\mathbb{H}^n} \frac{|u|^n}{[d(0,x)]^\beta} dV.$$

Moreover, the restriction $0 < \alpha < \alpha_n(1 - \frac{\beta}{n})$ is also optimal.

Finally, we will establish the sharp critical singular Moser-Trudinger inequality on the entire hyperbolic space when we restrict the norms of functions to full hyperbolic Sobolev norm.

Theorem 1.4 *Let $0 \leq \beta < n, \tau > 0$. For any $\alpha \in (0, \alpha_n(1 - \frac{\beta}{n})]$, there exists a constant $C_{\alpha,\tau} > 0$ such that*

$$\sup_{u \in W^{1,n}(\mathbb{H}^n), \|\nabla_g u\|_n^n + \tau\|u\|_n^n \leq 1} \int_{\mathbb{H}^n} \frac{\Phi_n(\alpha|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \leq C_{\alpha,\tau}.$$

The constant $\alpha_n(1 - \frac{\beta}{n})$ is sharp in the sense that if $\alpha_n(1 - \frac{\beta}{n})$ is replaced by any α bigger than α_n, the supreme will become infinite.

It is worthwhile to remark that there is a crucial difference between the inequalities in Theorem 1.3 and Theorem 1.4. Indeed, the restriction on the norms of functions in Theorem 1.3 is only imposed on the gradient in hyperbolic spaces, while the restriction on the norm of functions in Theorem 1.4 is imposed on the full Sobolev norm in hyperbolic spaces. This subtlety will be evident from the proofs of these two theorems.

The organization of the paper is as follows. In Section 2, we will establish the sharp Moser-Trudinger inequality on bounded domains in hyperbolic spaces of any dimension (Theorem 1.1). Section 3 will give a sharp Adachi-Tanaka type inequality in the entire hyperbolic space when we only restrict the norm of functions to gradient norm (Theorems 1.2 and 1.3). In Section 4, we will prove the sharp singular Moser-Trudinger inequality on the entire hyperbolic space when we restrict the norm of functions to full hyperbolic Sobolev norm.

2 Sharp Moser-Trudinger inequality on bounded domains in high dimensional hyperbolic spaces

To prove Theorem 1.1, we use an idea of Moser [22]. Note that $(d(0,x))^\sharp = d(0,x)$, and for $u \geq 0$,

$$(\exp(\alpha u^{\frac{n}{n-1}}))^\sharp = \exp \alpha (u^\sharp)^{\frac{n}{n-1}},$$
$$\|\nabla_g u^\sharp\|_n \leq \|\nabla_g u\|_n \leq 1.$$

By the Hardy-Littlewood inequality and properties of the rearrangement, it suffices to show the desired inequality for u being radially symmetric, nonnegative, smooth, and compactly supported with the form $u(x) = u_0(d(0,x))$, $\Omega = \{x \in \mathbb{H}^n : d(0,x) \leq R\}$ and $u_0(R) = 0$, for some $0 < R < \infty$.

For u being radially symmetric, the desired inequality can be rewritten, in the hyperbolic polar coordinates $|x| = \tanh t/2$, as

$$\sup_{\omega_{n-1}\int_0^R |u_0'|^n (\sinh t)^{n-1} dt \leq 1} \int_0^R \frac{\exp(\alpha |u_0|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \leq C_\beta \int_0^R \frac{(\sinh t)^{n-1}}{t^\beta} dt.$$

Set $w(x) = u_0(|x|)$, then w is a smooth function with compact support in the Euclidean ball $\{|x| < R\}$. Since $t \leq \sinh t$ for $t \geq 0$, then

$$\int_{|x|<R} |\nabla w|^n dx = \omega_{n-1} \int_0^R |u_0'|^n t^{n-1} dt \leq \omega_{n-1} \int_0^R |u_0'|^n (\sinh t)^{n-1} dt \leq 1.$$

By inequality (1.1) (the singular Moser-Trudinger inequality on any bounded domain in the Euclidean space), we have

$$\int_{|x|<R} \frac{\exp(\alpha |w|^{\frac{n}{n-1}})}{|x|^\beta} dx \leq C|R|^{n-\beta}.$$

Namely,

$$\int_0^R \exp(\alpha |u_0|^{\frac{n}{n-1}}) t^{n-1-\beta} dt \leq C|R|^{n-\beta}.$$

Now we first estimate

$$\begin{aligned}
&\int_0^{R/2} \frac{\exp(\alpha |u_0|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \\
&= \int_0^{R/2} \exp(\alpha |u_0|^{\frac{n}{n-1}}) t^{n-1-\beta} (\frac{\sinh t}{t})^{n-1} dt \\
&\leq \int_0^{R/2} \exp(\alpha |u_0|^{\frac{n}{n-1}}) t^{n-1-\beta} (\frac{\sinh R/2}{R/2})^{n-1} dt \\
&\leq C R^{1-\beta} (\sinh R/2)^{n-1}.
\end{aligned}$$

Since when $R \to 0$

$$\frac{\int_0^R \frac{(\sinh t)^{n-1}}{t^\beta} dt}{R^{1-\beta} (\sinh R/2)^{n-1}} \to \frac{2^{n-1}}{n-\beta},$$

and when $R \to \infty$

$$\frac{\int_0^R \frac{(\sinh t)^{n-1}}{t^\beta} dt}{R^{1-\beta}(\sinh R/2)^{n-1}} \to \infty,$$

then

$$\sup_{\omega_{n-1}\int_0^R |u_0'|^n (\sinh t)^{n-1} dt \le 1} \int_0^{R/2} \frac{\exp(\alpha |u_0|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \le C_\beta \int_0^R \frac{(\sinh t)^{n-1}}{t^\beta} dt. \tag{2.2}$$

Next, we consider the integral over $(R/2, R)$. Since $u(R) = 0$,

$$\begin{aligned} |u_0(t)| &= |\int_t^R u_0'(s) ds| \\ &\le (\int_t^R |u_0'(s)|^n (\sinh s)^{n-1} ds)^{1/n} (\int_t^R \frac{1}{\sinh s} ds)^{\frac{n-1}{n}} \\ &\le [\ln(\frac{e^R - 1}{e^R + 1} \frac{e^t + 1}{e^t - 1})]^{\frac{n-1}{n}} (\omega_{n-1})^{-\frac{1}{n}}, \end{aligned}$$

where we use the Hölder inequality in the first inequality. So we have

$$\begin{aligned} &\int_{R/2}^R \frac{\exp(\alpha |u_0|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \\ &\le \int_{R/2}^R \frac{\exp(\alpha_n (1 - \frac{\beta}{n}) |u_0|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \\ &\le \int_{R/2}^R (\frac{e^R - 1}{e^R + 1} \frac{e^t + 1}{e^t - 1})^{n-\beta} t^{-\beta} (\sinh t)^{n-1} dt \\ &\le (\frac{e^R - 1}{e^R + 1} \frac{e^{R/2} + 1}{e^{R/2} - 1})^{n-\beta} \int_{R/2}^R t^{-\beta} (\sinh t)^{n-1} dt \\ &\le 2^{n-\beta} \int_{R/2}^R t^{-\beta} (\sinh t)^{n-1} dt. \end{aligned}$$

Then

$$\sup_{\omega_{n-1}\int_0^R |u_0'|^n (\sinh t)^{n-1} dt \le 1} \int_{R/2}^R \frac{\exp(\alpha |u_0|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \le C_\beta \int_0^R \frac{(\sinh t)^{n-1}}{t^\beta} dt \tag{2.3}$$

Therefor by (2.2) and (2.3), we get the desired inequality.

Next, we will prove the sharpness of our result. It suffices to find a sequence of function $w_k(t) : \mathbb{R} \to \mathbb{R}$, which satisfies $w_k(t) \ge 0$, $w_k'(t) \le 0$, $w_k(R) = 0$, $\omega_{n-1} \int_0^R |w_k'|^n (\sinh t)^{n-1} dt \le 1$, and

$$\int_0^R \frac{\exp(\alpha |w_k|^{\frac{n}{n-1}})}{t^\beta} (\sinh t)^{n-1} dt \to +\infty,$$

for any $\alpha > \alpha_n(1-\frac{\beta}{n})$. Now, when $R > 1$, we choose $\{w_k\}$ as follows:

$$w_k(t) = \omega_{n-1}^{-\frac{1}{n}} C_k \begin{cases} k^{\frac{n-1}{n}}, & \text{if } 0 \le t \le e^{-k}, \\ k^{\frac{n-1}{n}} \frac{-\ln t}{k}, & \text{if } e^{-k} < t \le 1, \\ 0, & \text{if } 1 < t, \end{cases}$$

where $C_k = (\frac{1}{k}\int_{e^{-k}}^{1} t^{-n}(\sinh t)^{n-1}dt)^{-\frac{1}{n}}$. Since $C_k \sim (\frac{(\sinh e^{-k})^{n-1}}{(e^{-k})^{n-1}})^{-\frac{1}{n}}$, as $k \to \infty$, then $C_k \to 1$, $\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(C_k)^{\frac{n}{n-1}} \to \frac{\alpha}{\alpha_n(1-\frac{\beta}{n})} > 1$, as $k \to \infty$. Therefore

$$\omega_{n-1}\int_0^R |w_k'|^n (\sinh t)^{n-1}dt = \int_{e^{-k}}^{1} (C_k)^n \frac{1}{k}\frac{1}{t^n}(\sinh t)^{n-1}dt = 1,$$

and as $k \to 0$,

$$\begin{aligned} &\int_0^R \frac{\exp(\alpha|w_k|^{\frac{n}{n-1}})}{t^\beta}(\sinh t)^{n-1}dt \\ &\ge \int_0^{e^{-k}} \frac{\exp(\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(n-\beta)\omega_{n-1}^{\frac{1}{n-1}}|w_k|^{\frac{n}{n-1}})}{t^\beta}(\sinh t)^{n-1}dt \\ &\sim \frac{R^{n-\beta}}{n-\beta}\exp(\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(n-\beta)k(C_k)^{\frac{n}{n-1}} - (n-\beta)k) \\ &\sim (\exp((n-\beta)k))^{\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(C_k)^{\frac{n}{n-1}}-1} \\ &\to +\infty. \end{aligned}$$

When $R \le 1$, we choose $\{w_k\}$ as follows:

$$w_k(t) = \omega_{n-1}^{-\frac{1}{n}} C_k \begin{cases} k^{\frac{n-1}{n}}, & \text{if } 0 \le t \le e^{-k}R, \\ k^{\frac{n-1}{n}} \frac{-\ln t/R}{k}, & \text{if } e^{-k}R < t \le R, \\ 0, & \text{if } R < t, \end{cases}$$

where $C_k = (\frac{1}{k}\int_{e^{-k}R}^{R} t^{-n}(\sinh t)^{n-1}dt)^{-\frac{1}{n}}\frac{1}{R}$. Since $C_k \sim (\frac{(\sinh e^{-k}R)^{n-1}}{(e^{-k}R)^{n-1}})^{-\frac{1}{n}}\frac{1}{R}$, as $k \to \infty$, then $C_k \to \frac{1}{R}$, $\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(C_k)^{\frac{n}{n-1}} \to \frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(\frac{1}{R})^{\frac{n}{n-1}} > (\frac{1}{R})^{\frac{n}{n-1}}$, as $k \to \infty$. Therefore

$$\omega_{n-1}\int_0^R |w_k'|^n (\sinh t)^{n-1}dt = \int_{e^{-k}R}^{R} (C_k)^n \frac{1}{k}\frac{R^n}{t^n}(\sinh t)^{n-1}dt = 1,$$

and

$$\begin{aligned}
&\int_0^R \frac{\exp(\alpha|w_k|^{\frac{n}{n-1}})}{t^\beta}(\sinh t)^{n-1}dt\\
&\geq \int_0^{e^{-k}R} \frac{\exp(\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(n-\beta)\omega_{n-1}^{\frac{1}{n-1}}|w_k|^{\frac{n}{n-1}})}{t^\beta}(\sinh t)^{n-1}dt\\
&= \int_0^{e^{-k}R} \frac{\exp(\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(n-\beta)k(C_k)^{\frac{n}{n-1}})}{t^\beta}(\sinh t)^{n-1}dt.
\end{aligned}$$

Since $R \leq 1$, then $(\frac{1}{R})^{\frac{n}{n-1}} \geq 1$, and $\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(C_k)^{\frac{n}{n-1}} \geq \frac{\alpha}{\alpha_n(1-\frac{\beta}{n})} > 1$. So as $k \to 0$,

$$\begin{aligned}
&\int_0^{e^{-k}R} \frac{\exp(\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(n-\beta)k(C_k)^{\frac{n}{n-1}})}{t^\beta}(\sinh t)^{n-1}dt\\
&\sim \frac{R^{n-\beta}}{n-\beta}\exp(\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(n-\beta)k(C_k)^{\frac{n}{n-1}} - (n-\beta)k)\\
&\sim (\exp((n-\beta)k))^{\frac{\alpha}{\alpha_n(1-\frac{\beta}{n})}(C_k)^{\frac{n}{n-1}}-1}\\
&\to +\infty.
\end{aligned}$$

So we have found the desired sequence and this completes the proof of the sharpness of our inequality. Thus, the proof of Theorem 1.1 is finished.

3 Sharp Moser-Trudinger inequality in the sense of Adachi and Tanaka type

To prove Theorem 1.2, we will also use the rearrangement argument. By means of symmetrization, it suffices to show the desire inequality for functions $u(x) = u_0(d(0,x))$, which are radially symmetric, nonnegative, smooth, compactly supported and $u_0(t) : [0,+\infty) \to \mathbb{R}$ is decreasing.

Following Moser' argument of the classical inequality [22], we set $w(t) = \omega_{n-1}^{\frac{1}{n}} u_0(t)$, $|x| = \tanh t/2$, then $w(t) \geq 0$, $w' \leq 0$ and $w(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Then, we have

$$\int_{\mathbb{H}^n} \Phi_n(\alpha|u|^{\frac{n}{n-1}})dV = \omega_{n-1}\int_0^\infty \Phi_n(\alpha\omega_{n-1}^{-\frac{1}{n-1}}|w|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt,$$

$$\|\nabla_g u\|_n^n = \int_0^\infty |w'|^n(\sinh t)^{n-1}dt,$$

and

$$\int_{\mathbb{H}^n} |u|^n dV = \int_0^\infty |w|^n(\sinh t)^{n-1}dt.$$

Thus, to prove the theorem, it suffices to show that for any $\beta \in (0, n)$, there exists a constant C_β such that

$$\int_0^\infty \Phi_n(\beta |w|^{\frac{n}{n-1}})(\sinh t)^{n-1} dt \le C_\beta \int_0^\infty |w|^n (\sinh t)^{n-1} dt$$

for any w satisfying $w(t) \ge 0$, $w' \le 0$, $w(t_0) = 0$ for some $t_0 \in R$ and $\int_0^\infty |w'|^n (\sinh t)^{n-1} dt \le 1$.

Set $T_0 = \sup\{t \in R : w(t) \ge 1\}$, and we know that for $t > T_0$, $0 \le w(t) < 1$ and $w(T_0) = 1$. For $t \in (T_0, \infty)$, we have $w(t) \in [0, 1)$. Since for $x \in [0, n)$ we can find a constant C_n such that $\Phi_n(x) \le C_n x^{n-1}$; thus we have

$$\int_{T_0}^{+\infty} \Phi_n(\beta |w|^{\frac{n}{n-1}})(\sinh t)^{n-1} dt \le C_n \beta^{n-1} \int_{T_0}^{+\infty} |w|^n (\sinh t)^{n-1} dt. \tag{3.4}$$

Next, we consider the integral over $(0, T_0]$. Since $w(T_0) = 1$, for $t \le T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^t w'(s) ds \\ &\le w(T_0) + \left(\int_t^{T_0} |w'(s)|^n (\sinh s)^{n-1} ds\right)^{1/n} \left(\int_t^{T_0} \frac{1}{\sinh s} ds\right)^{\frac{n-1}{n}} \\ &= 1 + \left(\ln\left(\frac{e^{T_0} - 1}{e^{T_0} + 1} \frac{e^t + 1}{e^t - 1}\right)\right)^{\frac{n-1}{n}}. \end{aligned}$$

It is well known that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ s.t.

$$1 + s^{\frac{n-1}{n}} \le ((1 + \varepsilon)s + C_\varepsilon)^{\frac{n-1}{n}}.$$

Thus, we have $|w(t)|^{\frac{n}{n-1}} \le (1 + \varepsilon) \ln\left(\frac{e^{T_0}-1}{e^{T_0}+1} \frac{e^t+1}{e^t-1}\right) + C_\varepsilon$, for $t \in (0, T_0]$. Since $\beta \in (0, n)$, we can choose $\varepsilon > 0$ so small that $\beta(1 + \varepsilon) < n$. Then

$$\begin{aligned} \int_0^{T_0} \Phi_n(\beta |w|^{\frac{n}{n-1}})(\sinh t)^{n-1} dt &\le \int_0^{T_0} e^{\beta |w|^{\frac{n}{n-1}}} (\sinh t)^{n-1} dt \\ &\le \int_0^{T_0} e^{\beta C_\varepsilon} \left(\exp\left(\ln\left(\frac{e^{T_0} - 1}{e^{T_0} + 1} \frac{e^t + 1}{e^t - 1}\right)\right)\right)^{\beta(1+\varepsilon)} (\sinh t)^{n-1} dt \\ &= e^{\beta C_\varepsilon} \left(\frac{e^{T_0} - 1}{e^{T_0} + 1}\right)^{\beta(1+\varepsilon)} \int_0^{T_0} \frac{(e^t + 1)^{\beta(1+\varepsilon)+n-1}}{(e^t - 1)^{\beta(1+\varepsilon)-n+1}} \frac{dt}{(2e^t)^{n-1}}. \end{aligned}$$

When $n > \beta(1 + \varepsilon) > n - 1$,

$$\begin{aligned} & e^{\beta C_\varepsilon} \left(\frac{e^{T_0} - 1}{e^{T_0} + 1}\right)^{\beta(1+\varepsilon)} \int_0^{T_0} \frac{(e^t + 1)^{\beta(1+\varepsilon)+n-1}}{(e^t - 1)^{\beta(1+\varepsilon)-n+1}} \frac{dt}{(2e^t)^{n-1}} \\ &\le 2e^{\beta C_\varepsilon} \left(\frac{e^{T_0} - 1}{e^{T_0} + 1}\right)^{\beta(1+\varepsilon)} \int_0^{T_0} \frac{(2e^t)^{\beta(1+\varepsilon)-1}}{(e^t - 1)^{\beta(1+\varepsilon)-n+1}} de^t \\ &\le 2e^{\beta C_\varepsilon} \left(\frac{e^{T_0} - 1}{e^{T_0} + 1}\right)^{\beta(1+\varepsilon)} (2e^{T_0})^{\beta(1+\varepsilon)-1} \int_0^{T_0} \frac{1}{(e^t - 1)^{\beta(1+\varepsilon)-n+1}} de^t \\ &\le \frac{2^{\beta(1+\varepsilon)} e^{\beta C_\varepsilon}}{n - \beta(1 + \varepsilon)} \frac{(e^{T_0} - 1)^n}{e^{T_0}}. \end{aligned}$$

When $\beta(1+\varepsilon) \le n-1$,

$$
\begin{aligned}
&\int_0^{T_0} \Phi_n(\beta|w|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \le \int_0^{T_0} e^{\beta|w|^{\frac{n}{n-1}}}(\sinh t)^{n-1}dt \\
&\le \int_0^{T_0} \exp(\frac{n-1}{1+\varepsilon}|w|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \\
&\le \int_0^{T_0} e^{\beta C_\varepsilon}(\exp(\ln(\frac{e^{T_0}-1}{e^{T_0}+1}\frac{e^t+1}{e^t-1})))^{n-1}(\sinh t)^{n-1}dt \\
&= e^{\beta C_\varepsilon}(\frac{e^{T_0}-1}{e^{T_0}+1})^{n-1}\int_0^{T_0}\frac{(e^t+1)^{n-1}}{(e^t-1)^{n-1}}\frac{(e^t-1)^{n-1}(e^t+1)^{n-1}}{(2e^t)^{n-1}}dt \\
&\le 2e^{\beta C_\varepsilon}(\frac{e^{T_0}-1}{e^{T_0}+1})^{n-1}\int_0^{T_0}(2e^t)^{n-1}dt \\
&\le 2^{n-1}e^{\beta C_\varepsilon}\frac{(e^{T_0}-1)^n}{e^{T_0}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^{T_0} |w(t)|^n(\sinh t)^{n-1}dt \ge \int_0^{T_0}(\sinh t)^{n-1}dt \\
&= \int_0^{T_0}\frac{(e^t-1)^{n-1}(e^t+1)^{n-1}}{2^{n-1}(e^t)^{n-1}}dt \\
&\ge \frac{1}{2^{n-1}}\int_0^{T_0}\frac{(e^t-1)^{n-1}}{e^t}de^t \\
&\ge \frac{1}{2^{n-1}}\frac{1}{e^{T_0}}\frac{(e^{T_0}-1)^n}{n}.
\end{aligned}
$$

Then

$$
\int_0^{T_0} \Phi_n(\beta|w|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \le C_{n,\beta}\int_0^{T_0}|w|^n(\sinh t)^{n-1}dt. \tag{3.5}
$$

Therefore, by (3.4) and (3.5), we get the desired inequality of Theorem 1.2.

Next we will show that the restriction $0 < \alpha < n\omega_{n-1}^{\frac{1}{n-1}}$ is optimal. Using Moser' idea again, repeating the argument above, we can see that it suffices to find a sequence of functions $w_k : \mathbb{R} \to \mathbb{R}$ which satisfies $w_k(t) \ge 0$, $w_k' \le 0$, $w_k(t_0) = 0$ for some $t_0 \in \mathbb{R}$, $\int_0^\infty |w_k'|^n(\sinh t)^{n-1}dt = 1$ and

$$
\int_0^{+\infty}|w_k|^n(\sinh t)^{n-1}dt \to 0,
$$

$$
\int_0^{+\infty}\Phi_n(n|w_k|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \ge \frac{1}{n}.
$$

We choose $\{w_k\}$ as follows:

$$w_k(t) = C_k \begin{cases} k^{\frac{n-1}{n}}, & \text{if } 0 \le t \le e^{-k}, \\ k^{\frac{n-1}{n}} \frac{-\ln t}{k}, & \text{if } e^{-k} \le t \le 1, \\ 0, & \text{if } 1 < t, \end{cases}$$

where $C_k = (\frac{1}{k}\int_{e^{-k}}^{1} t^{-n}(\sinh t)^{n-1}dt)^{-\frac{1}{n}}$. Since $C_k \sim (\frac{(\sinh e^{-k})^{n-1}}{e^{-(n-1)k}})^{-\frac{1}{n}}$, as $k \to \infty$, then $C_k \to 1$ and $(C_k)^{\frac{n}{n-1}}k - k \to 0$ as $k \to \infty$. Therefore

$$\int_0^{+\infty} |w_k'|^n(\sinh t)^{n-1}dt = \int_{e^{-k}}^{1} (C_k)^n \frac{1}{k}\frac{1}{t^n}(\sinh t)^{n-1}dt = 1,$$

$$\begin{aligned}
&\int_0^{+\infty} |w_k|^n(\sinh t)^{n-1}dt \\
&= \int_0^{e^{-k}} (C_k)^n k^{n-1}(\sinh t)^{n-1}dt + \int_{e^{-k}}^{1} (C_k)^n \frac{1}{k}|\ln t|^n(\sinh t)^{n-1}dt \\
&\sim \frac{k^{n-1}}{e^{nk}} + \frac{C}{k} \\
&\to 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_0^{+\infty} \Phi_n(n|w_k|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \\
&= \int_0^{e^{-k}} \Phi_n(n(C_k)^{\frac{n}{n-1}}k)(\sinh t)^{n-1}dt + \int_{e^{-k}}^{1} \Phi_n(n(C_k)^{\frac{n}{n-1}}k^{-\frac{1}{n-1}}|\ln t|^{\frac{n}{n-1}})(\sinh t)^{n-1}dt \\
&\ge \int_0^{e^{-k}} \Phi_n(n(C_k)^{\frac{n}{n-1}}k)(\sinh t)^{n-1}dt - \sum_{j=1}^{n-2}\int_{e^{-k}}^{1} \frac{(n(C_k)^{\frac{n}{n-1}}k^{-\frac{1}{n-1}}|\ln t|^{\frac{n}{n-1}})^j}{j!}(\sinh t)^{n-1}dt \\
&\sim e^{n(C_k)^{\frac{n}{n-1}}k}\frac{e^{-kn}}{n} - \sum_{j=0}^{n-2}\frac{(nC_k^{\frac{n}{n-1}}k)^j}{j!}\frac{e^{-kn}}{n} - \sum_{j=1}^{n-2}\frac{C(j)}{k^{\frac{j}{n-1}}} \\
&\to \frac{1}{n}.
\end{aligned}$$

Hence, we obtain the desired sequence. This completes the proof of Theorem 1.2.

Now, we will prove Theorem 1.3. Before doing that, we like to make some comments. We will not use the symmetrization argument here. Instead, we will use a new method, a rearrangement-free argument developed in Lam and Lu in [14, 15], to establish the sharp inequality. In fact, using this new idea, we can prove Theorem 1.3 without using the method of symmetrization.

By a standard density argument, we can suppose that $u \in C_0^\infty(\mathbb{H}^n)$, $u \ge 0$ and $\|\nabla_g u\|_n \le 1$.

Denote

$$\Omega(u) = \{x \in \mathbb{H}^n : u(x) > 1\},$$

$$I_1 = \int_{\Omega(u)} \frac{\Phi_n(\alpha(|u|)^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV,$$

and

$$I_2 = \int_{\mathbb{H}^n \setminus \Omega(u)} \frac{\Phi_n(\alpha(|u|)^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV.$$

First, we estimate I_1. Set $v(x) = u(x) - 1$ in $\Omega(u)$, then $v \in W_0^{1,n}(\Omega(u))$ and $\|\nabla_g v\|_{n,\Omega(u)} \leq 1$. Then by Theorem 1.1, we have

$$\int_{\Omega(u)} \frac{\exp(\alpha_n(1-\frac{\beta}{n})|v(x)|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \leq C_\beta \int_{\Omega(u)} \frac{dV}{[d(0,x)]^\beta}.$$

Now, put $\varepsilon = \frac{\alpha_n}{\alpha}(1-\frac{\beta}{n}) - 1 > 0$. Using the following elementary inequality:

$$(a+b)^p \leq \varepsilon b^p + (1-(1+\varepsilon)^{-\frac{1}{p-1}})^{1-p} a^p,$$

for any $a, b, \varepsilon > 0$ and $p > 1$, we have in $\Omega(u)$ that

$$|u(x)|^{\frac{n}{n-1}} = (1+v(x))^{\frac{n}{n-1}} \leq (1+\varepsilon)|v|^{\frac{n}{n-1}} + (1 - \frac{1}{(1+\varepsilon)^{n-1}})^{\frac{1}{1-n}}.$$

Set $C_\varepsilon = (1-\frac{1}{(1+\varepsilon)^{n-1}})^{\frac{1}{1-n}}$. Hence

$$\begin{aligned}
I_1 &= \int_{\Omega(u)} \frac{\Phi_n(\alpha(|u|)^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\
&\leq \int_{\Omega(u)} \frac{\exp(\alpha(|u|)^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\
&= \int_{\Omega(u)} \frac{\exp(\alpha(|v+1|)^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\
&\leq \int_{\Omega(u)} \frac{\exp(\alpha(1+\varepsilon)|v|^{\frac{n}{n-1}} + \alpha C_\varepsilon)}{[d(0,x)]^\beta} dV \\
&\leq e^{\alpha C_\varepsilon} \int_{\Omega(u)} \frac{\exp(\alpha_n(1-\frac{\beta}{n})|v|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\
&\leq C_{\alpha,\beta} \int_{\Omega(u)} \frac{dV}{[d(0,x)]^\beta} \\
&\leq C_{\alpha,\beta} \int_{\mathbb{H}^n} \frac{|u|^n}{[d(0,x)]^\beta} dV.
\end{aligned}$$

To estimate I_2, we first note that $u \le 1$ in $\mathbb{H}^n \setminus \Omega(u)$. As a consequence, we have

$$\begin{aligned} I_2 &= \int_{\mathbb{H}^n\setminus\Omega(u)} \frac{\Phi_n(\alpha(|u|)^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\ &\le \int_{\{u\le 1\}} \frac{1}{[d(0,x)]^\beta} \sum_{j=n-1}^{\infty} \frac{\alpha^j}{j!} |u|^{k\frac{n}{n-1}} dV \\ &\le \int_{\{u\le 1\}} \frac{1}{[d(0,x)]^\beta} \sum_{j=n-1}^{\infty} \frac{\alpha^j}{j!} |u|^n dV \\ &\le e^\alpha \int_{\mathbb{H}^n} \frac{|u|^n}{[d(0,x)]^\beta} dV. \end{aligned}$$

Finally, noting that $\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV = I_1 + I_2$, we get the desired inequality.

To prove the restriction $0 < \alpha < \alpha_n(1-\frac{\beta}{n})$ is optimal, we can use the following sequence of functions:

$$w_k(x) = \omega_{n-1}^{-\frac{1}{n}} C_k \begin{cases} k^{\frac{n-\beta-1}{n-\beta}}, & \text{if } 0 \le d(0,x) \le e^{-k}, \\ k^{\frac{n-\beta-1}{n-\beta}} \frac{-\ln d(0,x)}{k}, & \text{if } e^{-k} \le d(0,x) \le 1, \\ 0, & \text{if } 1 < d(0,x), \end{cases}$$

where $C_k = (k^{\frac{-n}{n-\beta}} \int_{e^{-k}}^1 t^{-n} (\sinh t)^{n-1} dt)^{-\frac{1}{n}}$. Doing almost the same calculation as we did in the proof of Theorems 1.1 and 1.2, we can obtain that the result is optimal. This completes the proof of Theorem 1.3.

4 Sharp singular Moser-Trudinger inequality on the entire hyperbolic space

Now, we will prove Theorem 1.4. It suffices to prove that for any β, τ satisfying $0 \le \beta < n$, and $\tau > 0$, there exists a constant $C_{\beta,\tau}$ such that for any $u \in C_0^\infty(\mathbb{H}^n)$, $u \ge 0$ and $\|\nabla_g u\|_n^n + \tau \|u\|_n^n \le 1$, there holds

$$\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n(1-\frac{\beta}{n})|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \le C_{\beta,\tau}.$$

Set

$$A(u) = 2^{-\frac{1}{n(n-1)}} \tau^{\frac{1}{n}} \|u\|_n,$$

$$\Omega(u) = \{x \in B^n : u(x) > A(u)\}.$$

Then, it's clear that

$$A(u) < 1.$$

Moreover

$$\int_{\mathbb{H}^n} |u|^n dv \ge \int_{\Omega(u)} |u|^n dV > 2^{-\frac{1}{n-1}} \tau \|u\|_n^n |\Omega(u)|,$$

so we have

$$|\Omega(u)| \le 2^{\frac{1}{n-1}} \frac{1}{\tau}.$$

Now, we write

$$\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha_n(1-\frac{\beta}{n})|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV = I_1 + I_2,$$

where

$$I_1 = \int_{\Omega(u)} \frac{\Phi_n(\alpha_n(1-\frac{\beta}{n})|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV$$

and

$$I_2 = \int_{\mathbb{H}^n\setminus\Omega(u)} \frac{\Phi_n(\alpha_n(1-\frac{\beta}{n})|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV.$$

First, we estimate I_2. Since $\{B^n \setminus \Omega(u)\} \subset \{u(x) < 1\}$, we see that

$$\begin{aligned}
I_2 &\le \int_{\{u\le 1\}} \frac{1}{[d(0,x)]^\beta} \sum_{j=n-1}^{\infty} \frac{\alpha_n^j(1-\frac{\beta}{n})^j}{j!} |u|^{k\frac{n}{n-1}} dV \\
&\le \sum_{j=n-1}^{\infty} \frac{\alpha_n^j(1-\frac{\beta}{n})^j}{j!} \int_{\{u\le 1\}} \frac{1}{[d(0,x)]^\beta} |u|^n dV \\
&= C_\beta\Big(\int_{\{u\le 1, d(0,x)\le 1\}} \frac{1}{[d(0,x)]^\beta}|u|^n dV + \int_{\{u\le 1, d(0,x)>1\}} \frac{1}{[d(0,x)]^\beta}|u|^n dV\Big) \\
&\le C_\beta\Big(\int_{\{d(0,x)\le 1\}} \frac{1}{[d(0,x)]^\beta} dV + \int_{\{u\le 1, d(0,x)>1\}} |u|^n dV\Big) \\
&\le C_{\beta,\tau}.
\end{aligned}$$

Next, to estimate I_1, we set $v(x) = u(x) - A(u)$ in $\Omega(u)$, then $v \in W_0^{1,n}(\Omega(u))$. Moreover

$$\begin{aligned}
|u|^{\frac{n}{n-1}} &\le (|v| + A(u))^{\frac{n}{n-1}} \\
&\le |v|^{\frac{n}{n-1}} + \frac{n}{n-1} 2^{\frac{n}{n-1}-1}(|v|^{\frac{n}{n-1}-1}A(u) + |A(u)|^{\frac{n}{n-1}}) \\
&\le |v|^{\frac{n}{n-1}} + \frac{n}{n-1} 2^{\frac{n}{n-1}-1} \frac{|v|^{\frac{n}{n-1}}|A(u)|^n}{n} + \frac{n}{n-1} 2^{\frac{n}{n-1}-1}\Big(\frac{n-1}{n} + |A(u)|^{\frac{n}{n-1}}\Big) \\
&= |v|^{\frac{n}{n-1}}\Big(1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n\Big) + C_n,
\end{aligned}$$

where we use Young's inequality and the following elementary inequality: for all $q \le 1$ and $a, b \ge 0$,

$$(a+b)^q \le a^q + q2^{q-1}(a^{q-1}b + b^q).$$

Let $w(x) = (1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n)^{\frac{n-1}{n}} v(x)$ in $\Omega(u)$. Then $w \in W_0^{1,n}(\Omega(u))$ and $|u|^{\frac{n}{n-1}} \le |w|^{\frac{n}{n-1}} + C_n$. Moreover we have

$$\nabla_g w = \Big(1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n\Big)^{\frac{n-1}{n}} \nabla_g v.$$

Then

$$\int_{\Omega(u)} |\nabla w|^n dx = (1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n)^{n-1} \int_{\Omega(u)} |\nabla u|^n dx$$
$$\leq (1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n)^{n-1}(1 - \tau \int_{\mathbb{H}^n} |u|^n dV).$$

Thus

$$(\int_{\Omega(u)} |\nabla w|^n dx)^{\frac{1}{n-1}} \leq (1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n)(1 - \tau \int_{\mathbb{H}^n} |u|^n dV)^{\frac{1}{n-1}}$$
$$\leq (1 + \frac{2^{\frac{1}{n-1}}}{n-1}|A(u)|^n)(1 - \frac{\tau}{n-1} \int_{\mathbb{H}^n} |u|^n dV)$$
$$\leq (1 + \frac{\tau}{n-1} \int_{\mathbb{H}^n} |u|^n dV)(1 - \frac{\tau}{n-1} \int_{\mathbb{H}^n} |u|^n dV)$$
$$\leq 1,$$

here we use the inequality $(1-x)^q \leq 1 - qx$ for all $0 \leq x \leq 1$, $0 < q \leq 1$. Thus, we can use Theorem 1.1 to estimate I_1:

$$I_1 \leq \int_{\Omega(u)} \frac{\exp(\alpha_n(1-\frac{\beta}{n})|u|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV$$
$$\leq \int_{\Omega(u)} \frac{\exp(\alpha_n(1-\frac{\beta}{n})|w|^{\frac{n}{n-1}} + C_n)}{[d(0,x)]^\beta} dV$$
$$\leq C_\beta \int_{\Omega(u)} \frac{dV}{[d(0,x)]^\beta}$$
$$= C_\beta(\int_{\Omega(u)\cap\{d(0,x)>1\}} \frac{dV}{[d(0,x)]^\beta} + \int_{\Omega(u)\cap\{d(0,x)\leq 1\}} \frac{dV}{[d(0,x)]^\beta})$$
$$\leq C_\beta(|\Omega(u)| + \int_{\{d(0,x)\leq 1\}} \frac{dV}{[d(0,x)]^\beta})$$
$$\leq C_{\beta,\tau}.$$

By the estimates of I_1 and I_2, we obtain the inequality and then complete the proof of the first part of Theorem 1.4.

Next, we will show that the inequality in Theorem 1.4 is sharp. Namely, we will show that the inequality in Theorem 1.4 does not hold if $\alpha > \alpha_n(1-\beta)$. We choose $\{u_k\}_{k=1}^\infty$ as follows:

$$u_k(x) = \omega_{n-1}^{-\frac{1}{n}} C_k \begin{cases} k^{\frac{n-\beta-1}{n-\beta}}, & \text{if } 0 \leq d(0,x) \leq e^{-k}, \\ k^{\frac{n-\beta-1}{n-\beta}} \frac{-\ln[d(0,x)]}{k}, & \text{if } e^{-k} \leq d(0,x) \leq 1, \\ 0, & \text{if } 1 < d(0,x), \end{cases}$$

where $C_k = (k^{\frac{-n}{n-\beta}} \int_{e^{-k}}^1 t^{-n}(\sinh t)^{n-1} dt)^{-\frac{1}{n}}$. Since $C_k \sim (\frac{(\sinh e^{-k})^{n-1}}{e^{-(n-1)k}})^{-\frac{1}{n}} k^{\frac{\beta}{n(n-\beta)}}$, as $k \to \infty$, then $C_k \to k^{\frac{\beta}{n(n-\beta)}}$, as $k \to \infty$. Then, by calculation

$$\|\nabla_g u_k\|_n^n = 1,$$

and

$$\int_{\mathbb{H}^n} |u_k|^n dV = O(\frac{1}{k}).$$

Set $\|u_k\|_{n,\tau} = (\|\nabla_g u_k\|_n^n + \tau \int_{B^n} |u_k|^n dV)^{\frac{1}{n}}$, and $\tilde{u}_k = \frac{u_k}{\|u_k\|_{n,\tau}}$. Now, $\tilde{u}_k \in W^{1,n}(\mathbb{H}^n)$, $\|\nabla_g \tilde{u}_k\|_n^n + \tau\|\tilde{u}_k\|_n^n = 1$. It follows that

$$\begin{aligned}
&\int_{\mathbb{H}^n} \frac{\Phi_n(\alpha|\tilde{u}_k|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\
&\geq \int_{d(0,x)\leq e^{-k}} \frac{\Phi_n(\alpha|\tilde{u}_k|^{\frac{n}{n-1}})}{[d(0,x)]^\beta} dV \\
&= \omega_{n-1}\Phi_n(\frac{\alpha(\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-\beta-1}{n-\beta}})^{\frac{n}{n-1}}}{(\|u_k\|_{n,\tau})^{\frac{n}{n-1}}}) \int_0^{e^{-k}} \frac{(\sinh t)^{n-1}}{t^\beta} dt \\
&\sim \Phi_n(\frac{\alpha(\omega_{n-1}^{-\frac{1}{n}} C_k k^{\frac{n-\beta-1}{n-\beta}})^{\frac{n}{n-1}}}{(\|u_k\|_{n,\tau})^{\frac{n}{n-1}}})(e^{-k})^{n-\beta} \\
&\sim \Phi_n(\frac{\frac{\alpha}{\alpha_n} nk}{(\|u_k\|_{n,\tau})^{\frac{n}{n-1}}})(e^{-k})^{n-\beta} \\
&\sim \exp(kn(\frac{\alpha}{\alpha_n(\|u_k\|_{n,\tau})^{\frac{n}{n-1}}} + \frac{\beta}{n} - 1)).
\end{aligned}$$

Since $\alpha > \alpha_n(1 - \frac{\beta}{n})$ and $\|u_k\|_{n,\tau} = 1 + \tau O(\frac{1}{k})$, then the last term in the above inequality tends to infinity as $k \to \infty$. So we complete the proof of Theorem 1.4.

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