

BEST PRACTICES IN MATH COURSE SEQUENCING AND INTEGRATED MATH

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In the following report, Hanover Research discusses best practices in secondary math course sequencing and recommended sequencing approaches. Special attention is given to an integrated course sequencing approach, research that supports this model, and its implications for teachers.

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EXECUTIVE SUMMARY AND KEY FINDINGS

INTRODUCTION

In 2010, the California (CA) State Board of Education adopted its own version of the Common Core State Standards (CCSS-M) – a shift in state policy that has had implications for how districts align and sequence courses to meet standards for learning. While the CA Education Code does not require districts to implement specific academic content standards, the CA Department of Education (CDE) “strongly recommends their local use” and, as of 2015, “virtually all” high school students in the state took math classes aligned with the new standards.¹ In the state’s guidance document, *Mathematics Framework for California Public Schools*, the CDE notes that the state’s implementation of the CA CCSS-M will involve many transitions, including “new instructional approaches, new instructional materials, professional support for teachers, and technology readiness.”² In addition to these changes, the redesigned standards require that middle and high schools *reconsider and adjust traditional course pathways*.³

To support public school districts in their consideration of secondary math course pathways, Hanover Research (Hanover) examines best practices and recommended approaches to math course sequencing for all students as well as supports for struggling learners.⁴ The report consists of two sections:

- **Section I: Math Course Sequences Models and Supports** provides an overview of two model approaches to sequencing: the *traditional* and the *integrated* approach. This section also explores universal sequencing best practices, sequencing adjustments, and instructional strategies to support students with difficulties learning math.
- **Section II: Taking an Integrated Approach** provides additional information about the integrated approach to math course sequencing, including pathway variations, recommendations and research supporting an integrated approach, and implications for teachers.

¹ [1] “California Common Core State Standards - Mathematics.” California State Board of Education, 2014. p. i. <http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf> [2] “CCSS FAQs.” California Department of Education, 2016. <http://www.cde.ca.gov/re/cc/ccssfaqs.asp> [3] Harlow, J. “California Districts Moving to New ‘Integrated’ High School Math Pathway.” *EdSource*, October 27, 2015. <https://edsources.org/2015/california-districts-moving-to-new-integrated-high-school-math-pathway/89288>

² “Mathematics Framework Kindergarten Through Grade Twelve.” California Department of Education, 2015. pp. 825–826. https://www.mydigitalchalkboard.org/cognoti/content/file/resources/documents/b7/b771959c/b771959c9673c5764ed5c7222d6d009e8221158f/MathFrameworks_chalkboard.pdf

³ *Ibid.*

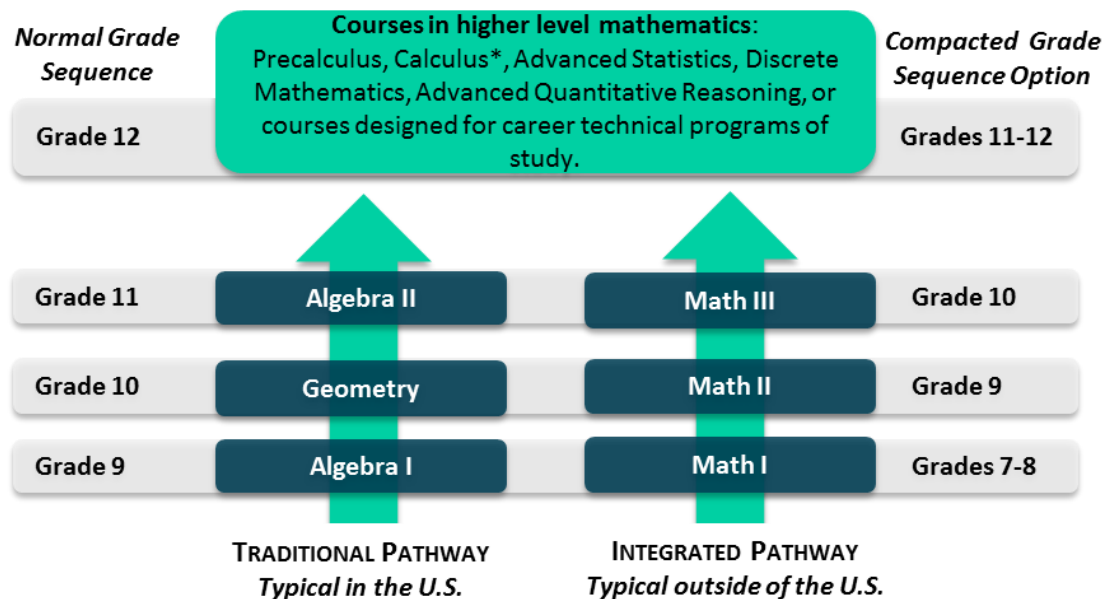
⁴ In the following report, all specific math courses are capitalized (e.g. Algebra I, Geometry, etc.). Math subjects or concepts referred to more generally are not capitalized (e.g. algebra, geometry, etc.).

KEY FINDINGS

COURSE SEQUENCING DESIGN

- Districts adopting the CA CCSS-M must decide between *traditional* and *integrated* approaches to high school level math course sequencing.** The former maintains separate courses for Algebra I, Geometry, and Algebra II, while the latter combines aspects of each of these domains (as well as other math topics) into three integrated courses (e.g., Math I, II, and III). The CCSS Initiative and California Department of Education (CDE) do not state a preference for either approach, both of which cover the same standards. In the typical sequences, Grade 9 students take either Algebra I or Math I and reach Precalculus by Grade 12 (Figure ES.1, *Normal Grade Sequence*).
- Districts should include accelerated pathways open to all qualified students when designing course sequences.** Accelerated pathways should include the same content as non-compacted courses and not begin before Grade 7. Multiple entry measures that include standardized test scores can help ensure equitable access to advanced pathways and ensure that selected students are likely to succeed. The CDE recommends that districts compact three courses into two years, rather than one course into two years. Acceleration may occur at the middle and/or high school levels (Figure ES.1, *Compacted Grade Sequence Option*).

Figure ES.1: Traditional and Integrated High School Math Pathways



*Note: Calculus follows Precalculus and is a fifth course, in most cases. Students typically would need to follow the compacted grade sequence described above, or a variation of this model.

Source: CCSS Initiative⁵

⁵ Figure adapted from: Ibid., p. 4.

- **Districts should make sequencing adjustments to support struggling students.** Districts may also provide courses sequencing options that expand content across multiple years (e.g., Math I or Algebra I expanded across Grades 9-10) or within the year through increased instructional time (e.g., through support classes, tutoring, and/or extended class time). Math teachers should use evidence-based instructional strategies, including explicit instruction and use of multiple instructional examples, as well as targeted strategies for supporting students in learning algebraic concepts, working with fractions, and developing problem-solving skills, as appropriate.

ADOPTING AN INTEGRATED APPROACH

- **Many educators and experts support adopting an integrated approach, which research suggests can positively impact student achievement.** An integrated approach emphasizes the connections and interrelationships between math domains and is the common approach internationally, including in high-achieving countries. Recent studies suggest that students who receive an integrated curriculum outperform peers in traditional high school math classes (i.e., Algebra I, Geometry, and Algebra II). Many mathematicians and other experts advocate for an integrated approach, including the now former president of the National Council of Teachers of Mathematics (NCTM) who referred to the traditional secondary math sequence as “outmoded.”⁶
- **An integrated approach may require secondary math teachers to provide instruction in new or less familiar domains.** To teach Math I, II, or III, former Algebra I and II teachers will need to incorporate aspects of geometry into their instruction and curriculum, while Geometry teachers will need to incorporate algebra content standards. However, educators note that as adopting the CCSS-M already requires curricular and instructional changes, the transition period to the new standards is an ideal time to adopt an integrated approach. Districts commonly implement an integrated approach over multiple years, establish teacher support before the transition, and provide on-going and on-demand professional development to facilitate a smooth transition.

⁶ Shaughnessy J., M. “An Opportune Time to Consider Integrated Mathematics.” National Council of Teachers of Mathematics, 2011. http://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/J_-Michael-Shaughnessy/An-Opportune-Time-to-Consider-Integrated-Mathematics/

SECTION I: MATH COURSE SEQUENCE MODELS AND SUPPORTS

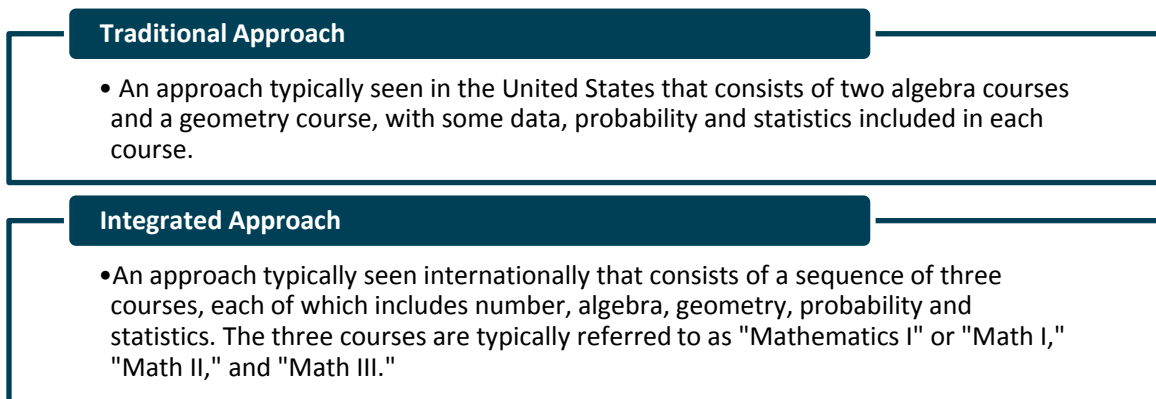
The following section discusses model course pathways for middle and high schools that implement California’s Common Core State Standards (CCSS-M), as well as best practices to guide their design. This section also outlines sequencing modifications and instructional strategies to support students who struggle in math.

MODEL COURSE PATHWAYS

OVERVIEW OF TRADITIONAL AND INTEGRATED APPROACHES

The CCSS Initiative highlights two approaches to high school level math course sequencing: a **traditional approach**, which maintains separate courses for Algebra I, Algebra II, and Geometry, and an **integrated approach**, which combines aspects of each subject (in addition to other math topics) into three, sequential courses. At the high school level, the CCSS-M are organized by conceptual category, rather than grade level (as they are for the elementary grades), and as such, districts have flexibility in designing math course sequences that cover these standards. While the CCSS Initiative identifies model pathways for high school level math—each of which incorporates all college and career readiness standards—it emphasizes that “the pathways and courses are models, not mandates,” and represent starting points from which states and districts can develop their own course sequence.⁷ Aligned with suggestions for sequencing made by the California Department of Education (CDE), Figure 1.1 describes the traditional and integrated approaches below.⁸

Figure 1.1: Approaches to High School Math Course Sequencing



Source: CCSS Initiative⁹

⁷ “Common Core State Standards for Mathematics - Appendix A.” Common Core State Standards Initiative. pp. 2–3. <http://www.corestandards.org/>

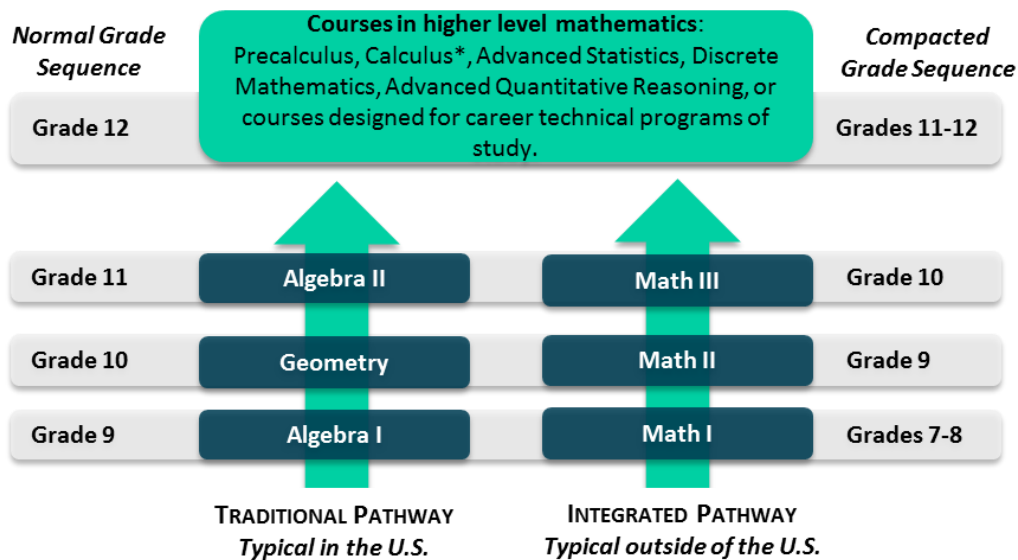
⁸ “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 407.

⁹ Figure text quoted verbatim with minor changes from: “Common Core State Standards for Mathematics - Appendix A,” Op. cit., pp. 3–4.

In a standard sequence of high school math classes aligned with the national and CA CCSS-M, students in Grade 9 take either Algebra I or Math I, depending on their district’s approach. By the end of Grade 11, students should be finished with all three traditional or integrated math courses and prepared to take Precalculus or another higher-level math course. While the standards do not require that high school students take a fourth math course, the CCSS Initiative recommends that students “continue to take mathematics courses throughout their high school career.”¹⁰ The CDE correspondingly outlines standards for four advanced-level courses: Precalculus, Statistics and Probability, Calculus, and Advanced Placement Probability and Statistics.¹¹

Districts may also compress traditional and integrated classes into fewer years to allow students to reach Calculus and other advanced-level math courses by the end of high school. One model option is to integrate high school level classes into the middle school curriculum. For example, the CCSS Initiative highlights a compacted version of the traditional pathway in which students in Grades 7 and 8 complete the math content of Grade 7, Grade 8, and Algebra I (i.e., Grade 8 Algebra). Similarly, a compacted version of the integrated pathway allows students in Grades 7-9 to complete the math content of Grade 7, Grade 8, and Math I (i.e., Grade 8 Math I). Both compacted models omit no content and allow students to reach Calculus by Grade 12.¹² Figure 1.2 outlines the normal and compacted grade sequences for the traditional and integrated high school math pathways.

Figure 1.2: Traditional and Integrated High School Math Pathways



*Note: Calculus follows Precalculus and is a fifth course, in most cases. Students typically would need to follow the compacted grade sequence described above, or a variation of this model.
Source: CCSS Initiative¹³

¹⁰ [1] Ibid., p. 3. [2] “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 407.

¹¹ Ibid.

¹² “Common Core State Standards for Mathematics - Appendix A,” Op. cit., p. 3.

¹³ Figure adapted from: Ibid., p. 4.

While California does not track which of the two approaches districts implement, a recent survey of the state’s largest 30 districts by enrollment indicates that half are maintaining the traditional pathway, while the remainder have embraced the integrated pathway.¹⁴ The following subsections further describe and contrast the two model approaches for sequencing high school level math courses.

TRADITIONAL PATHWAY

Because the traditional pathway is familiar to most U.S. districts, adapting the traditional pathway to the CA CCSS-M may require fewer instructional and curricular changes (in comparison to adopting the integrated pathway). Proponents of maintaining the traditional pathway cite “a lack of new instructional materials to implement the integrated pathway,” as well as “a feeling that the existing course sequence works” for their decisions.¹⁵ This may especially be the case among high-achieving districts in which most students already pursue post-secondary education opportunities.¹⁶ Thus, districts may be able to continue using previous textbooks supplemented with materials aligned with the CCSS-M, rather than purchase completely new materials, when aligning course curricula and sequencing with the CCSS-M. For instance, Los Angeles Unified School District’s secondary math coordinator cited financial constraints as a factor in the district’s decision to maintain the traditional pathway.¹⁷

While maintaining a traditional math sequencing pathway can minimize the need for new instructional materials, teachers likely still require leadership’s support to adapt the traditional pathway to incorporate new standards. For Grades K-8, shifts in the CCSS-M require teachers to adjust instruction to ensure that students meet new standards, which “require students to demonstrate a deeper understanding of math concepts.”¹⁸ Many California educators report receiving limited training on implementing the new standards, and consequently, teachers express a need for targeted professional development. In response, some districts have started training teachers to provide instruction that incorporates new learning standards.¹⁹ Similarly, teachers of high school level math courses may require training and support to adapt their instruction and curriculum to the CA CCSS-M.²⁰ Overall, the CCSS Initiative cites the following “key shifts” introduced by the new math standards across grade levels, which may require curricular and instructional changes for teachers:²¹

¹⁴ Harlow, Op. cit.

¹⁵ Ibid.

¹⁶ Ibid.

¹⁷ Ibid.

¹⁸ Mongeau, L. “Common Core Standards Bring Dramatic Changes to Elementary School Math.” *EdSource*, January 20, 2014. <https://edsource.org/2014/common-core-standards-bring-dramatic-changes-to-elementary-school-math-2/63665>

¹⁹ Ibid.

²⁰ Harlow, Op. cit.

²¹ Bullets adapted (bolded text quoted verbatim) from: “Key Shifts in Mathematics.” Common Core State Standards Initiative. <http://www.corestandards.org/other-resources/key-shifts-in-mathematics/>

- **Greater focus on fewer topics.** For example, under the new standards, Grade 7 Math focuses on “ratios and proportional relationships, and arithmetic of rational numbers,” while Grade 8 Math focuses on linear algebra and linear functions.
- **Coherence: Linking topics and thinking across grades.** The standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years.
- **Rigor: Pursue conceptual understanding, procedural skills and fluency, and application with equal intensity.** The new standards call for educators to “pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skills and fluency, and application.”

TRADITIONAL PATHWAY MODELS

As described above, under the new CCSS-M standards, Algebra I is the default course in Grade 9; Geometry, the default course in Grade 10; and Algebra II, the default course in Grade 11. **Important to note is that increases in the rigor of these courses’ standards and their sequencing in high school has implications for what students must learn in Grade 8 prior to high school entrance.** Before the adoption of CA’s new standards, students may have completed Algebra I in Grade 8. With the adoption of the CCSS-M, however, the CDE notes that increased Grade 8 standards will cause students to complete Algebra I in Grade 9. Thus, most students will complete the CA CCSS-M for Grade 8, which “are significantly more rigorous than the Algebra I course that many students took in eighth grade,” while in Grade 8.²² At large, these Grade 8 standards include content that was previously included in the Algebra I course, including algebraic foundations such as “more in-depth study of linear relationships and equations, a more formal treatment of functions, and the exploration of irrational numbers.”²³ Consequently, the new Algebra I course in Grade 9 is also more rigorous than the previous Algebra I course, as it is designed to build on the new, rigorous Grade 8 standards.²⁴

Due to more rigorous standards, districts adapting a traditional pathway to the CA CCSS-M must reconsider modified opportunities for acceleration in middle school. A compacted traditional model combines Algebra I in Grades 7 and 8. While this compacted traditional pathway may resemble districts’ previous accelerated options for high-achieving students, the increased rigor of the new standards may make previous accelerated options ill-suited for some students.²⁵ As the CDE explains:²⁶

Because of the rigor that has been added to the CA CCSS[-]M for grade eight, course sequencing needs to be recalibrated to ensure students are able to master the additional content. Specifically, today’s students, who are similar to those who

²² “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., pp. 825–827.

²³ Ibid., p. 827.

²⁴ Ibid.

²⁵ Ibid., p. 830.

²⁶ Quoted verbatim from: Ibid., p. 827.

previously may have been able to master an Algebra I course in grade eight, may find the new CA CCSS[-]M for grade-eight content significantly more difficult.

In addition to the accelerated pathway that compacts Algebra I into Grades 7 and 8, districts can offer modified options at the high school level for advanced students. These options notably “delay decisions about which students to accelerate while still allowing access to advanced mathematics in grade twelve.”²⁷ Figure 1.3 summarizes acceleration strategies for a traditional pathway below, each of which may be suited towards particular student needs.

Figure 1.3: Acceleration Options – Traditional Approach

- ✓ **Students could “double up”** by enrolling in the Geometry course during the same year that they take Algebra I or Algebra II.
- ✓ Allow students in schools with block scheduling to **take a mathematics course in both semesters of the same academic year.**
- ✓ **Offer summer courses** that are designed to provide the equivalent experience of a full course in all aspects, including attention to the Standards for Mathematical Practice.
- ✓ **Create different compaction ratios**, compressing four years of high school content into three years, beginning in ninth grade.
- ✓ **Create a hybrid Algebra II/Precalculus course** that allows students to go straight to Calculus in grade twelve
- ✓ **Standards that focus on a sub-topic such as trigonometry or statistics could be pulled out and taken alongside the traditional or integrated courses** so that students would only need to “double up” for one semester.

Source: CDE²⁸

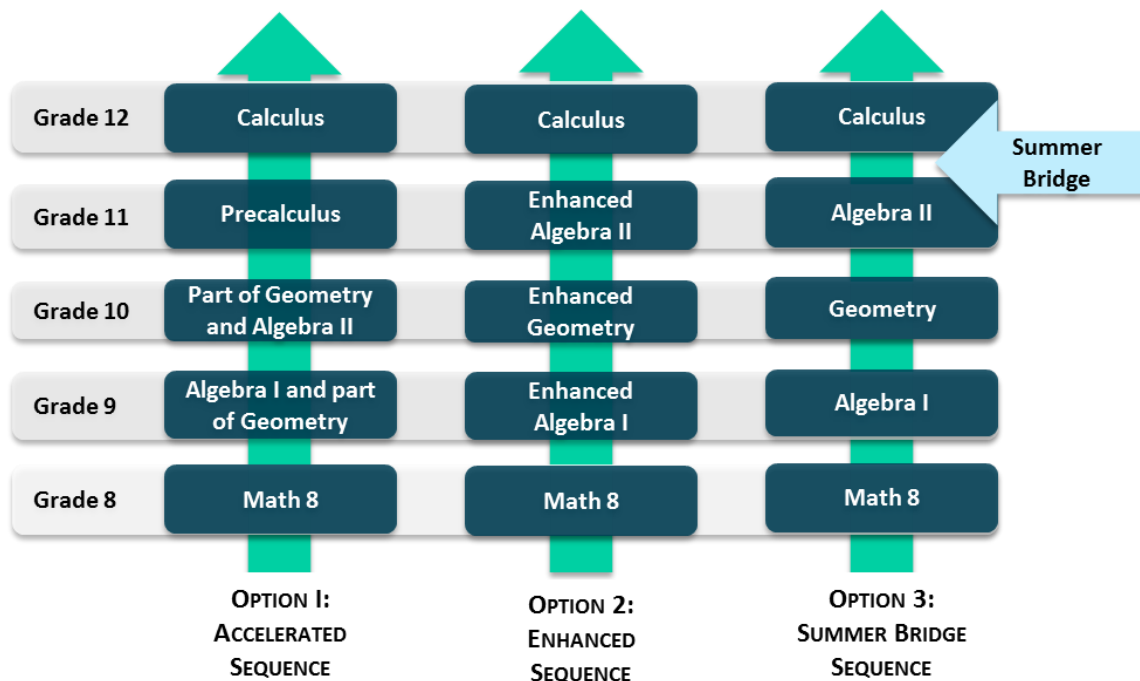
In total, the CDE outlines three accelerated high school pathways to adapt the traditional approach to high school level math. These are referred to as the “accelerated,” “enhanced,” and “summer bridge” sequences (see Figure 1.4 on the following page). Beyond these recommended pathways, California districts have flexibility in creating pathways that best prepare their students for college-level math. For example, in 2013, Oakland Unified School District (Oakland USD) staff recommended that the district offer a course that combines Algebra II and Math Analysis (similar to a Precalculus course) to allow students to take AP Calculus in Grade 12.²⁹

²⁷ Ibid., p. 831.

²⁸ Figure bullets quoted verbatim from: Ibid., pp. 831–832.

²⁹ “Oakland and San Francisco Create Course Pathways Through.” Strategic Education Research Partnership, 2014. p. 7. http://serpinstitute.org/assets/daro_serp_ccss_and_acceleration.pdf

Figure 1.4: Accelerated High School Pathways – Traditional Approach



Note that while Calculus is listed as the course for Grade 12 students, students may elect to take another advanced-level class.

Source: CDE³⁰

INTEGRATED PATHWAY

An integrated pathway blends algebra and geometry content, as well as content from probability and statistics, in a sequence of three courses, typically referred to as Math I, Math II, and Math III.³¹ A common approach internationally, “the integrated sequence is meant to take math learning out of silos and teach students how to bridge connections among topics,” according to an article published in *Education Week*.³² Student survey data from the 2013 National Assessment of Educational Progress (NAEP), however, indicates that less than five percent of U.S. students took an integrated math course in Grades 10-12 in the preceding years, though a growing number of districts are adopting the integrated approach as they transition to the CCSS.³³ Ineffective statewide mandates requiring districts to adopt an integrated approach to high school math, broader politics surrounding the CCSS, and insufficient support at the local level may partly explain why such few students take integrated math. These factors are explored in further detail in Figure 1.5 on the following page.

³⁰ Figure adapted from: “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., pp. 834–836.

³¹ “Common Core State Standards for Mathematics - Appendix A,” Op. cit., p. 3.

³² Will, M. “In Transition to Common Core, Some High Schools Turn to ‘Integrated’ Math.” *Education Week*, November 10, 2014. <http://www.edweek.org/ew/articles/2014/11/12/12cc-integratedmath.h34.html>

³³ [1] Loveless, T. “Implementing Common Core: Curriculum Part 2.” The Brookings Institute, 2014. <https://www.brookings.edu/research/implementing-common-core-curriculum-part-2/> [2] Will, Op. cit.

Figure 1.5: Statewide Adoptions of the Integrated Approach to High School Math



When adopting the CCSS, several states initially mandated that districts take the integrated approach to high school level math. Most of these states have since partially reversed course, allowing districts to choose between the two approaches. Notably, however, no state mandates that districts universally take the traditional approach.

- **North Carolina.** In 2013, the state mandated that all middle and high schools would adopt the integrated approach.³⁴ However, in 2016, the state’s senate passed a bill requiring secondary schools to offer both integrated and traditional approaches. a local newspaper notes that “the bill seems to have been motivated in large part by the specter of the controversial national education standards initiative known as Common Core,” rather than due to the merits of each approach.³⁵
- **West Virginia.** West Virginia originally mandated that districts adopt the integrated approach but reversed course in 2015, when as many as one-fifth of districts returned to the traditional approach.³⁶
- **Utah.** Of the states that initially mandated that districts adopt the integrated approach to high school math, only Utah has not reversed course.³⁷
- **Georgia.** Before the creation of the CCSS, Georgia made a statewide effort to implement integrated math across districts in the mid-2000s. However, this effort resulted in local pushback and many districts returned to the traditional pathway.³⁸

Source: See in-figure citations.

While adopting the integrated pathway involves more intensive curricular redesign and professional development, proponents argue that districts adopting the CCSS-M already face these challenges to some extent, regardless of the approach they choose. Therefore, **some educators note that the shift to the new CCSS standards is a logical time to also switch to an integrated approach to math.** As a chairman of the math department in a Massachusetts district observed, “if we stayed with a traditional pathway, we’d have to rewrite curriculum anyway to fit the new standards.”³⁹ Some educators also argue that switching to an integrated approach would increase curricular alignment with the CCSS-M; staying with the traditional approach, alternatively, may increase the likelihood that educators continue teaching as normal and align instruction less faithfully with the new standards.⁴⁰

³⁴ Robelen, E. “N.C. Moving to Integrated High School Math With Common Core.” *Education Week - Curriculum Matters*, July 3, 2013. http://blogs.edweek.org/edweek/curriculum/2013/07/nc_moves_to_integrated_high_sc.html?cmp=SOC-SHR-FB

³⁵ Ellinwood, M. “NC Senate’s Math Bill Doesn’t Add up.” *NC Policy Watch*, June 21, 2016. <http://www.ncpolicywatch.com/2016/06/21/nc-senates-math-bill-doesnt-add-up/>

³⁶ Loewus, L.H. “Many West Virginia Districts Returning to Traditional Math Courses.” *Education Week - Curriculum Matters*, June 16, 2015. http://blogs.edweek.org/edweek/curriculum/2015/06/many_west_virginia_districts_returning_to_traditional_math_courses.html?cmp=SOC-SHR-FB

³⁷ “Utah Core Standards.” Utah Education Network. <http://www.uen.org/core/>

³⁸ Loveless, Op. cit.

³⁹ Will, Op. cit.

⁴⁰ Ibid.

INTEGRATED PATHWAY MODELS

Accelerated options and other variations of an integrated pathway mirror those discussed for a traditional pathway (see Figure 1.2, for example, which outlines an accelerated option that compacts courses in middle school for both integrated and traditional pathways). Districts may compact courses during the middle and/or high school grades to allow high-achieving students to complete Math I, II, and III such that students can complete Calculus and/or other advanced-level math courses beyond Precalculus by the end of Grade 12. Section II of this report provides further detail on variations of an integrated pathway.

UNIVERSAL MATH COURSE SEQUENCING BEST PRACTICES


Regardless of a chosen approach to high school level math, districts should follow a set of universal best practices in math course scheduling to guide students into appropriate level courses and promote equity in course access.

APPROPRIATELY CHALLENGING STUDENTS

Much debate around math course sequencing focuses on *how* and *when* to integrate algebra into the course sequence to appropriately challenge students as they prepare for more advanced math in the later secondary grades.⁴¹ Many view Algebra I as a “gatekeeper course” due to its role in funneling students into other high school classes as well as into college and careers. In fact, some advocate for an Algebra-for-All policy in which all students take Algebra I in middle school (typically in Grade 8).⁴² Despite these well-meaning policies, universal Algebra I programs have not always been met with success, and at times, have shown to negatively impact both low- and high-achieving students (see Figure 1.6).

Figure 1.6: Weighing the Effects of a Universal Grade 8 Algebra

In “The Misplaced Math Student: Lost in Eighth-Grade Algebra,” researcher Robert Loveless argues that placing all students in Algebra I in Grade 8 does a disservice to two groups of students: (1) “misplaced students” who are several years behind in math education and (2) well-prepared students whose learning is stunted as teachers struggle to serve the needs of the misplaced students who may be years behind.⁴³



This concern was borne out in Chicago Public Schools, where a watered-down Algebra I curriculum contributed to an environment in which “math grades declined, math failures increased, absenteeism rose among average- and higher-skilled students, and graduation and college-going rates declined.”⁴⁴

Source: See in-figure citations.

⁴¹ Senechal, J. “Middle Grades Math: Assessing the Debate Over the When and How of Algebra.” Metropolitan Educational Research Consortium, 2014. p. 3. http://www.merc.soe.vcu.edu/wp-content/uploads/sites/3387/2015/03/Algebra-for-All_Final-Updated-6-1-15.pdf

⁴² [1] “The Final Report of the National Mathematics Advisory Panel.” U.S. Department of Education, 2008. p. xiii. <http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf> [2] Ibid.

⁴³ Loveless, T. “The Misplaced Math Student: Lost in 8th Grade Algebra.” *The Brown Center Report on American Education*, September 2008. p. 13. <https://www.brookings.edu/research/the-misplaced-math-student-lost-in-eighth-grade-algebra/>

⁴⁴ Allensworth et al. “College Prep for All? What We’ve Learned from Chicago’s Efforts.” *Consortium on Chicago School Research at the University of Chicago*, August 2010. p. 8. <http://files.eric.ed.gov/fulltext/ED512288.pdf>

Experts recommend that many students wait until Grade 9 to take Algebra I, under a traditional approach, or Math I, under an integrated approach.

For example, in 2013, the then president of the National Council of Teachers of Mathematics (NCTM) stated that “[schools] do more harm than good by placing students in a formal algebra course before they are ready, and few students are truly ready to understand the important concepts of algebra before eighth grade. Many students should wait until ninth grade.”⁴⁵ These recommendations align with the CA CCSS-M, which intends for most students to wait until Grade 9 to take Algebra I or Math I, partly due to the lack of success around universal Algebra I in middle school.⁴⁶

According to the CDE, districts must consider: “When, and under what circumstances, will placing students in the grade-eight CA CCSS[-]M course transfer to greater mathematics understanding throughout high school?”⁴⁷

When developing accelerated pathways and placement criteria, districts should:

- (1) ensure that the curriculum remains comprehensive, and*
- (2) not design pathways that separate students before Grade 7*

As discussed, however, districts should—and frequently do—provide modified pathways to allow capable high-achieving students to complete advanced math courses, such as Calculus, by the end of high school. This remains the case under the CA CCSS-M, although few students may be suited for acceleration. As the CDE states:⁴⁸

Although the CA CCSS[-]M are more rigorous than California’s previous standards for mathematics, there will still be some students who are able to move through the mathematics quickly. Those students may choose to take an accelerated or enhanced mathematics program beginning in eighth grade (or even earlier) so they can take college-level mathematics in high school.

DESIGNING ACCELERATED PATHWAYS

When developing accelerated pathways and placement criteria, districts should (1) ensure that the curriculum remains comprehensive, and (2) not design pathways that separate students before Grade 7. In addition, the CDE recommends that districts use multiple criteria to determine whether a student would benefit from acceleration, encourage all students to take four years of math, and provide advanced-level courses apart from Calculus for high-achieving students (see Figure 1.7 on the following page).

⁴⁵ Gojak, L. “Algebra: Not ‘If’ but ‘When.’” National Council of Teachers of Mathematics, 2013. http://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Linda-M_-Gojak/Algebra_-_Not_-_If_-_but_-_When_/

⁴⁶ [1] “Common Core State Standards for Mathematics - Appendix A,” Op. cit., p. 3. [2] “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 407.

⁴⁷ “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 828.

⁴⁸ Quoted verbatim from: Ibid., p. 830.

Figure 1.7: Recommendations to Guide Accelerated Pathway Design and Student Placement

RECOMMENDATION	DESCRIPTION
Avoid Omitting Content	Compacted courses should include the same Common Core State Standards as the noncompacted courses. When accelerated pathways are considered, it is recommended that three years of material be compacted into two years, rather than compacting two years into one.
Wait to Accelerate	Decisions to accelerate students into the Common Core State Standards for higher mathematics before ninth grade should not be rushed. Premature placement of students into an accelerated pathway should be avoided at all costs. In order to ensure that students are developmentally ready for accelerated content, it is not recommended to compact the standards before grade seven.
Accelerate Based on Evidence	Decisions to accelerate students into higher mathematics before ninth grade should be based on solid evidence of student learning. Before a student is placed on an accelerated pathway, serious efforts must be made to consider solid evidence of the student’s conceptual understanding, knowledge of procedural skills, fluency, and ability to apply mathematics.
Provide Four Years of Challenging Math	A menu of challenging options should be available for students after their third year of mathematics—and all students should be strongly encouraged to take mathematics in all years of high school. In addition to Calculus, Advanced courses may also include Statistics, Discrete Mathematics, or Mathematical Decision Making via mathematical modeling.

Source: CDE⁴⁹

Districts should identify students for accelerated pathways using multiple criteria, including teacher recommendations and standardized achievement measures, to determine their ability to succeed in advanced placements. The CDE notes that it is “essential that multiple measures are used to determine a student’s readiness for acceleration” and recommends developing a system that gathers evidence used to determine student placement.⁵⁰ Measures should cover multiple math domains, assess the CCSS-M Standards for Mathematical Practice, and “include constructed responses [...] used to determine a student’s conceptual understanding.”⁵¹ The Utah State Board of Education, for example, advises that schools use “three valid assessments to demonstrate readiness” to enter a compacted course sequence in Grade 7 or later.⁵² Similarly, WestEd recommends that student identification for algebra is “based on a careful review of the student’s record to date in mastering prealgebraic concepts, measured in several ways, including:”⁵³

- Prior-year CST [i.e., standardized end-of-year test] scores,
- Teacher recommendations,
- Results from district-administered benchmark assessments, and
- Consultation with parents and counselors.

⁴⁹ Figure descriptions quoted verbatim with minor changes from: Ibid., p. 829.

⁵⁰ Ibid., p. 830.

⁵¹ Ibid.

⁵² “High Ability Students and the Utah Mathematics Core Standard.” Utah State Board of Education. <https://www.schools.utah.gov/file/1db0bc25-2342-4c54-b0f2-56e1cc603f9e>

⁵³ Bullets quoted verbatim from: Finkelstein, N. et al. “College Bound in Middle School & High School? How Math Course Sequence Matter.” Center for the Future of Teaching and Learning at WestEd, 2012. p. ix. https://www.wested.org/wp-content/files_mf/139931976631921CFTL_MathPatterns_Main_Report.pdf

Educators may also assemble a portfolio of student work to determine readiness for acceleration, or use a commercial or other type of placement assessment. As one assessment option, the CDE highlights the Mathematics Diagnostic Testing Project (MDTP), “a statewide effort involving the California State University, the University of California, California Community Colleges, and California K–12 mathematics teachers” that helps assess whether students are ready for a range of courses, from Prealgebra through Calculus.⁵⁴ Regardless of the criteria considered, a cross-sectional team of teachers and instructional leadership members should review final decisions.⁵⁵

Lastly, when designing accelerated pathways, districts should plan for students to take typical end-of-course (EOC) assessments associated with all compacted content. For example, the Utah State Board of Education advises that districts develop a timeline so that accelerated students take all appropriate Student Assessment of Growth and Excellence (SAGE) assessments, the state’s CCSS-aligned assessment system. This may involve students taking multiple assessments in a single year. For example, students taking Grade 7 Math, Grade 8 Math, and Math I across Grades 7-8 would need to take two EOC assessments in one of the years (likely Grade 8) to assess student’s mastery of Grade 8 Math and Math I standards.⁵⁶

ENSURING EQUITABLE OUTCOMES

Districts should examine whether placement policies associated with math course sequencing lead to inequitable outcomes for certain student subgroups. The Education Trust notes that traditional measures for identifying and selecting students for advanced-level courses (e.g., recommendations from teachers and counselors, success in prerequisite courses, and minimum grade point averages) may perpetuate the exclusion of underrepresented students by creating “rigid barriers simply based on tradition or for the benefit of teachers.”⁵⁷ Similarly, The Brookings Institute (Brookings) notes that “when courses are assigned through subjective systems based on teacher or counselor recommendations, inequitable access can result, with qualified black and Hispanic students less likely to be placed into advanced courses than their similarly qualified white counterparts.”⁵⁸

⁵⁴ “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 831.

⁵⁵ Ibid., p. 825.

⁵⁶ “High Ability Students and the Utah Mathematics Core Standard,” Op. cit.

⁵⁷ Theokas, C. and R. Saaris. “Finding America’s Missing AP and IB Students.” The Education Trust, 2013. pp. 8–9. https://edtrust.org/wp-content/uploads/2013/10/Missing_Students.pdf

⁵⁸ Dougherty, S. et al. “Does More Rigorous Middle School Math Coursework Change Students’ College Readiness?” The Brookings Institute, 2017. <https://www.brookings.edu/blog/brown-center-chalkboard/2017/04/27/does-more-rigorous-middle-school-math-coursework-change-students-college-readiness/>



To identify existing inequities, the NCTM suggests that districts conduct a review of policies that “include an examination of the use and impact of tracking, protocols for student placement in mathematics, the availability of opportunities for both remediation and enrichment, and student outcomes, including persistence within the pre-K–12 mathematics pipeline over time.”⁵⁹

Considering multiple measures that include scores on standardized achievement tests can improve equitable outcomes in course placement. The CDE stresses that “serious efforts must be made to consider solid, objective evidence of student learning in order to avoid unwittingly depriving particular groups of students of opportunities.”⁶⁰ Moreover, educators need to weight students’ short-term readiness in math with long-term ability to succeed in a more advanced pathway.⁶¹ Both the Broad Foundation and the National Governors’ Association recommend that schools use standardized test scores to identify students who are likely to succeed in advanced-level courses and who may otherwise not participate. The Broad Foundation states that “the use of a predictive formula based on a standardized test score is an easy and relatively unbiased way to identify additional students likely to succeed in an academically advanced curriculum.” Similarly, a 2016 report published by the Education Commission of the States recommends that schools advise students toward AP courses “based on objective metrics [such as] performance on assessments aligned to college- and career-readiness standards.”⁶² In addition, an equitable and effective assessment-based identification system requires districts to cover the cost of the assessment and provide time for all students to take it (e.g., the associated test for an AP math course).⁶³

SUPPORT FOR STRUGGLING LEARNERS

In the remainder of this section, Hanover describes sequencing adjustments and supports, as well as instructional strategies, to support struggling learners and ensure that all students master the core requirements of the CA CCSS-M.

SEQUENCING ADJUSTMENTS AND SUPPORTS

Effective math course sequencing enables students to master grade-level content and progress successfully to higher-level courses that build on prior knowledge and skills.⁶⁴ Learning math is progressive, where new knowledge and skills build on previously mastered concepts and foundations. Course sequencing should therefore emphasize coherence across grade levels and allow students to solidify prior learning.⁶⁵ However, **research indicates that**

⁵⁹ “Access and Equity in Mathematics Education.” National Council of Teachers of Mathematics, 2014. <http://www.nctm.org/Standards-and-Positions/Position-Statements/Access-and-Equity-in-Mathematics-Education/>

⁶⁰ “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 828.

⁶¹ Ibid.

⁶² Zinth, J. “Policy Analysis - Advanced Placement: Model Policy Components.” p. 5. <http://www.ecs.org/ec-content/uploads/AP-Model-Policy-Components.pdf>

⁶³ “Expanding Advanced Placement (AP*) Access.” The Broad Foundation, 2010. pp. 21–24. <https://issuu.com/aisdgifted/docs/1344-expandingapaccess>

⁶⁴ For example, see: “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 827.

⁶⁵ “Key Shifts in Mathematics,” Op. cit.

typical secondary math course pathways frequently fail to foster continued proficiency and effective learning across grade levels for lower achieving students. For example, in 2012, WestEd published a study that involved approximately 24,000 California students across 12 K-12 districts who were enrolled in Grade 7 in 2004-05 and completed Grade 12 in their district in 2009-10. The authors found that students who did not score at the “proficient” level or higher on the Grade 7 California Standards Test (CST) in math were unlikely to test at the proficient level on the Algebra I and II CSTs in high school. While 34 percent of all students tested at the proficient level on the Algebra I CST by the end of Grade 11, the majority of these students had already tested at this level by the end of Grade 8.⁶⁶

Districts should consider readjusting course sequences in which students frequently fail and repeat core courses. Course sequencing in which many students repeat courses is both problematic for student learning and an inefficient use of resources. According to the results of the WestEd study, one-third of students repeated Algebra I between Grade 7 and 12, and less than a quarter of those students attained proficiency on the Algebra I CST. As such, the authors concluded that “current course sequences are typically not cost effective” and recommend that districts consider costs associated with time, teacher allocation, and student placement when analyzing course sequencing.⁶⁷ The CDE aims to partially address high rates of course repetition through the implementation of the CA CCSS-M by promoting deeper understanding and having students take Algebra I or Math I in Grade 9, instead of Grade 8. To prevent students from failing courses in middle and high school, the CDE also notes that districts consider vertical alignment starting with elementary school math.⁶⁸

EXPANDING CONTENT

To support student learning, districts may provide math course sequencing options that expand content through additional class time, either within the year or across multiple years. The CDE highlights this practice at some California districts, which have “developed course structures that allow mathematics content to be reinforced over multiple years through expansion [...]”⁶⁹ For example, a district might offer an Algebra I or Math I course over two years, rather than the one year.⁷⁰ Additional instruction over the summer is another option for supporting students as they master grade level content. In total, the CCSS Initiative identifies the following options for making adjustments within the school year to reinforce conceptual mastery:⁷¹

- Providing students a “math support” class during the school day;
- After-school tutoring; and
- Extended class time (or blocking of classes) in mathematics.

⁶⁶ Finkelstein et al., *Op. cit.*, pp. vii–viii.

⁶⁷ *Ibid.*, pp. viii–x.

⁶⁸ “Mathematics Framework Kindergarten Through Grade Twelve,” *Op. cit.*, p. 832.

⁶⁹ *Ibid.*

⁷⁰ Displayed course sequence adapted from: “Course Descriptions (IUSD).” Irvine Unified School District.

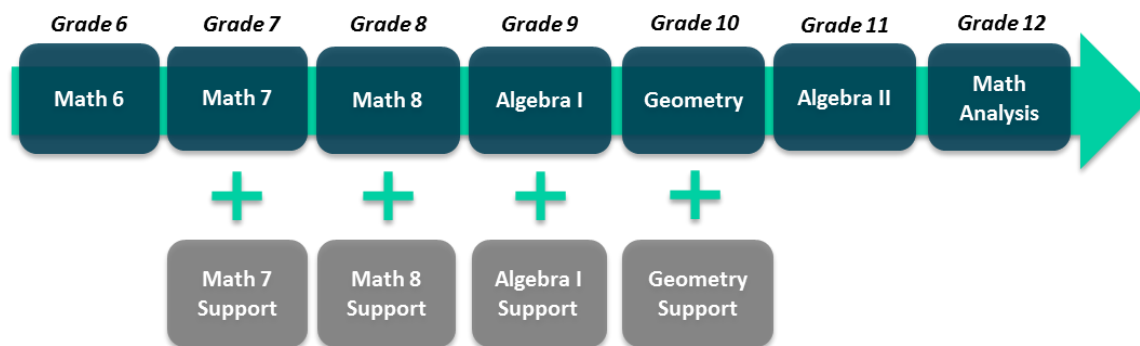
<https://iusd.org/about/departments/education-services/academics/stem/mathematics/course-descriptions>

⁷¹ Bullets “Common Core State Standards for Mathematics - Appendix A,” *Op. cit.*

As an example of sequencing adjustments, in a 2013 proposal, Oakland USD proposed making math support classes available for students starting in Math 7. Oakland USD staff noted that “recognizing that some students may need additional support in order to succeed, school sites must implement a system of support, including offering daily support courses for students and their families to choose for varying periods of time.”⁷² With this in mind, Oakland USD outlined a course sequence that includes supplementary support course options for Math 7, Math 8, Algebra I, and Geometry (illustrated in Figure 1.8). The district guidelines for these support courses include the following:⁷³

- Support courses are taught by a math teacher leader or an experienced math teacher who has demonstrated success with diverse learners;
- Pedagogy and curriculum in support courses are student-centered and focus on conceptual understanding, including opportunities to engage with a variety of mathematical tools. An on-line component, or “blended learning”, is highly recommended;
- Teachers of support courses have collaboration opportunities both within their school’s math team and also with other support course teachers; and
- Whenever possible, students taking support courses will be taught by the same teacher of the grade-level core course they are enrolled in.

Figure 1.8: Traditional Math Sequence with Support Course Options



Source: Strategic Education Research Partnership⁷⁴

INSTRUCTIONAL STRATEGIES

The transition to the CCSS-M poses additional challenges for students who struggle in math. Under the more rigorous standards, students who are already behind grade-level in math will likely lack some of the prerequisite knowledge and skills necessary for students to meet new grade-level standards. Accordingly, in early surveys following the adoption of the CCSS-M, the majority of teachers noted feeling unprepared to teach these standards to high-needs

⁷² “Oakland and San Francisco Create Course Pathways Through,” Op. cit., p. 8.

⁷³ Bullets quoted verbatim from: Ibid., pp. 8–9.

⁷⁴ Figure adapted from: Ibid., p. 8.

students.⁷⁵ This is cause for concern, as research indicates that students with below grade-level skills in elementary and middle school can become “trapped in a trajectory of failure” if they fail to receive high-quality instruction that supports their needs.⁷⁶

To support struggling students in math, research suggests that teachers should (1) provide explicit instruction regularly, and (2) use multiple instructional examples. In 2008, the Center on Instruction published a guide to effective instructional practices and recommendations to support struggling learners in math based on findings of a meta-analysis of high-quality studies (i.e., randomized control trials and high-quality quasi-experimental designs). The authors note that while other practices may also be effective, insufficient high-quality evidence existed to recommend their use.⁷⁷ Figure 1.9 presents a summary of these recommendations below and on the following page.

Figure 1.9: Recommendation to Guide Math Instruction for Struggling Students

RECOMMENDATION	DESCRIPTION
<p>Teach students using explicit instruction on a regular basis</p>	<p><i>When teaching a new procedure or concept, teachers should begin by modeling and/or thinking aloud and working through several examples. Explicit instruction includes teaching components such as:</i></p> <ul style="list-style-type: none"> ▪ Clear modeling of the solution specific to the problem, ▪ Thinking the specific steps aloud during modeling, ▪ Presenting multiple examples of the problem and applying the solution to the problems, and ▪ Providing immediate corrective feedback to the students on their accuracy.
<p>Teach students using multiple instructional examples</p>	<p><i>The underlying intent [of this strategy] is to expose students to many of the possible variations and at the same time highlight the common but critical features of seemingly disparate problems.</i></p> <ul style="list-style-type: none"> ▪ Multiple examples can be presented in a specified sequence or pattern such as concrete to abstract, easy to hard, and simple to complex. For example, fractions and algebraic equations can be taught first with concrete examples, then with pictorial representations, and finally in an abstract manner.
<p>Have students verbalize decisions and solutions to a math problem</p>	<p><i>Encouraging students to verbalize, or think-aloud, their decisions and solutions to a math problem is an essential aspect of scaffolded instruction.</i></p> <ul style="list-style-type: none"> ▪ Student verbalizations can be problem-specific or generic. Students can verbalize the specific steps that lead to the solution of the problem (e.g., I need to divide by two to get half) or they can verbalize generic heuristic steps that are common to problems (e.g., Now I need to check my answer), as well as in a solution format or in a self-questioning/answer format.

⁷⁵ Rebera, A. “Math Teachers Strive to Bring Core to At-Risk Students.” *Education Week*, March 13, 2013. http://www.edweek.org/tm/articles/2013/03/13/ccio_math_at-risk.html

⁷⁶ [1] Dougherty, C. “Catching Up to College and Career Readiness: The Challenge Is Greater for At-Risk Students.” ACT Research and Policy, 2014. p. 3. <http://files.eric.ed.gov/fulltext/ED546855.pdf> [2] Clements, D. and J. Sarama. “Equity and Mathematics Education.” James C. Kennedy Institute for Educational Success, 2015. pp. 5–7. https://www.du.edu/kennedyinstitute/media/documents/equity_2_kennedy_institute_pages.pdf

⁷⁷ Jayanthi, M., R. Gersten, and S. Baker. “Mathematics Instruction for Students with Learning Disabilities or Difficulty Learning Mathematics: A Guide for Teachers.” Center on Instruction, 2008. pp. 1–2. <http://www.centeroninstruction.org/files/Mathematics%20Instruction%20LD%20Guide%20for%20Teachers.pdf>

RECOMMENDATION	DESCRIPTION
<p><i>Teach students to visually represent the information in the math problem</i></p>	<p><i>When used systematically, visuals have (e.g., drawings, graphic representations) positive benefits on students' mathematic performance.</i></p> <ul style="list-style-type: none"> ▪ Visuals are more effective when combined with explicit instruction. ▪ Students benefit more when they use a visual representation prescribed by the teacher rather than one that they self-select. ▪ Visuals that are designed specifically to address a particular problem type are more effective than those that are not problem specific. ▪ Visual representations are more beneficial if not only the teacher, but both the teacher and the students use the visuals.
<p><i>Teach students to solve problems using multiple/ heuristic strategies</i></p>	<p><i>A heuristic is a method or strategy that exemplifies a generic approach for solving a problem. Instruction in heuristics, unlike direct instruction, is not problem-specific. Heuristics can be used in organizing information and solving a range of math problems.</i></p> <ul style="list-style-type: none"> ▪ A heuristic strategy can include steps such as “Read the problem. Highlight the key words. Solve the problems. Check your work.”
<p><i>Provide ongoing formative assessment data and feedback to teachers</i></p>	<p><i>Teachers can administer assessments to their group of students and then a computer can provide them with data depicting students' current mathematics abilities.</i></p> <ul style="list-style-type: none"> ▪ Greater benefits on student performance will be observed if teachers are provided with not only performance feedback information but also instructional tips and suggestions that can help teachers decide what to teach, when to introduce the next skill, and how to group/pair students.
<p><i>Provide peer-assisted instruction to students</i></p>	<p><i>Students with learning disabilities sometimes receive some type of peer assistance or one-on-one tutoring in areas in which they need help.</i></p> <ul style="list-style-type: none"> ▪ Cross-age peer tutoring appears to be more beneficial than within-class peer-assisted learning for students with learning disabilities.

Source: Center on Instruction⁷⁸

While some instructional strategies will vary by math content, this report’s **Appendix** highlights a series of recommendations from What Works Clearinghouse (WWC) for supporting elementary and middle school students in problem-solving and working with fractions. The WWC is an office within the Institute of Education Sciences at the U.S. Department of Education and regularly publishes practitioner guides on specific topics that identify the most impactful strategies to help students learn based on empirical studies meeting stringent standards of evidence.⁷⁹ The strategies listed in this Appendix may be implemented to help ensure that students matriculate into high school with the foundational skills required for learning secondary math.

INSTRUCTIONAL STRATEGIES FOR ALGEBRA

Algebra and integrated math course teachers should use algebra-specific instructional strategies to support all students. As the first math subject that requires “extensive abstract thinking,” algebra presents an added challenge for many students. In a 2015 guide, the WWC

⁷⁸ Figure text quoted verbatim with minor changes from: Ibid., pp. 5–11.

⁷⁹ “What We Do.” What Works Clearinghouse. <http://ies.ed.gov/ncee/wwc/WhatWeDo>

identifies and recommends that algebra teachers incorporate the evidence-based instructional practices listed in Figure 1.10.⁸⁰

Figure 1.10: Recommended Instructional Strategies – Algebra Instruction

RECOMMENDATION	STEP 1	STEP 2	STEP 3
<i>Use solved problems to engage students in analyzing algebraic reasoning and strategies</i>	Have students discuss solved problem structures and solutions to make connections among strategies and reasoning.	Select solved problems that reflect the lesson’s instructional aim, including problems that illustrate common errors.	Use whole-class discussions, small-group work, and independent practice activities to introduce, elaborate on, and practice working with solved problems.
<i>Teach students to utilize the structure of algebraic representations</i>	Promote the use of language that reflects mathematical structure.	Encourage students to use reflective questioning to notice structure as they solve problems.	Teach students that different algebraic representations can convey different information about an algebra problem.
<i>Teach students to intentionally choose from alternative algebraic strategies when solving problems</i>	Teach students to recognize and generate strategies for solving problems.	Encourage students to articulate the reasoning behind their choice of strategy and the mathematical validity of their strategy when solving problems.	Have students evaluate and compare different strategies for solving problems.

Source: What Works Clearinghouse⁸¹

⁸⁰ Star, J. et al. “Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students.” What Works Clearinghouse, 2015. p. 1. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/wwc_algebra_040715.pdf

⁸¹ Figure text quoted verbatim from: Ibid., p. 2.

SECTION II: TAKING AN INTEGRATED APPROACH

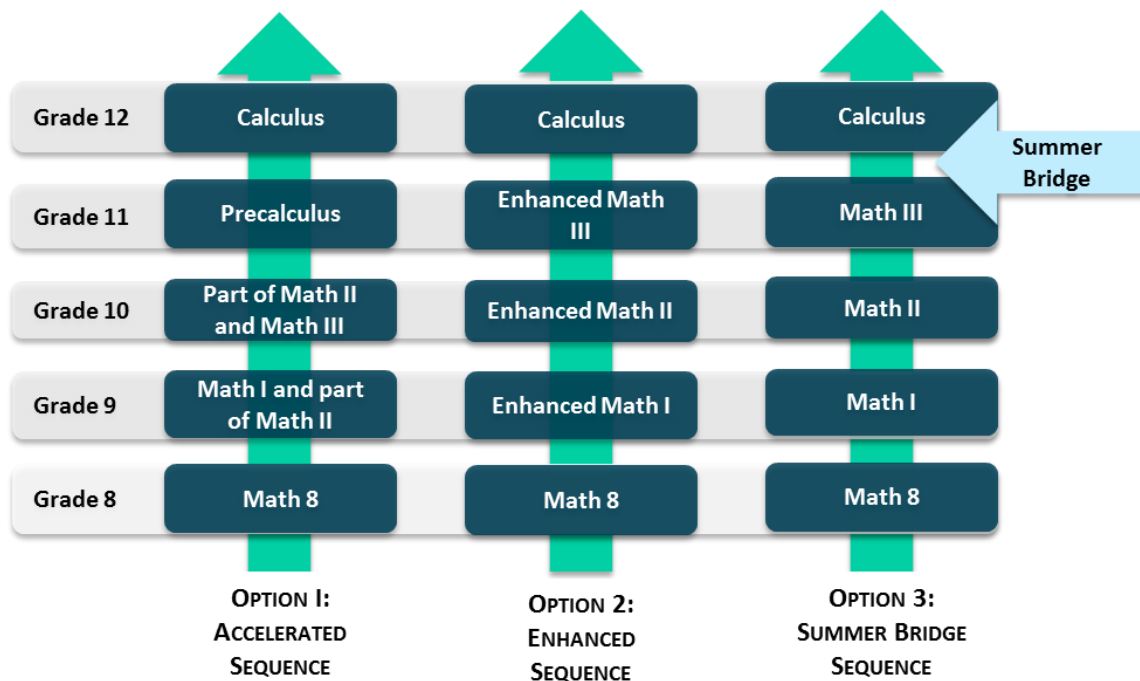
This section presents variations of an integrated approach to high school level math, discusses the rationale for choosing an integrated over a traditional approach, and outlines implications for secondary teachers switching to an integrated approach.

VARIATIONS OF AN INTEGRATED APPROACH

MODIFICATIONS FOR ACCELERATION

Districts may adapt an integrated pathway to accommodate students that are ready for acceleration by compacting courses across multiple years. As Figure 1.2 in Section I demonstrates, under the standard integrated approach, students take Math I in Grade 9, Math II in Grade 10, and Math III in Grade 11, with the opportunity to take a fourth math course in Grade 12 (e.g., Precalculus). To provide students with a pathway to take Calculus or another advanced-level math course in Grade 12, districts may choose to offer an accelerated middle school pathway, in which Grade 7-8 students cover content from Grade 7, Grade 8, and Math I in two years.⁸² Districts may also offer students opportunities to accelerate their sequencing during high school, as shown in Figure 2.1, below.

Figure 2.1: Accelerated High School Pathways – Integrated Approach



Note that while Calculus is listed as the course for Grade 12 students, students may elect to take another advanced-level class.

Source: CDE⁸³

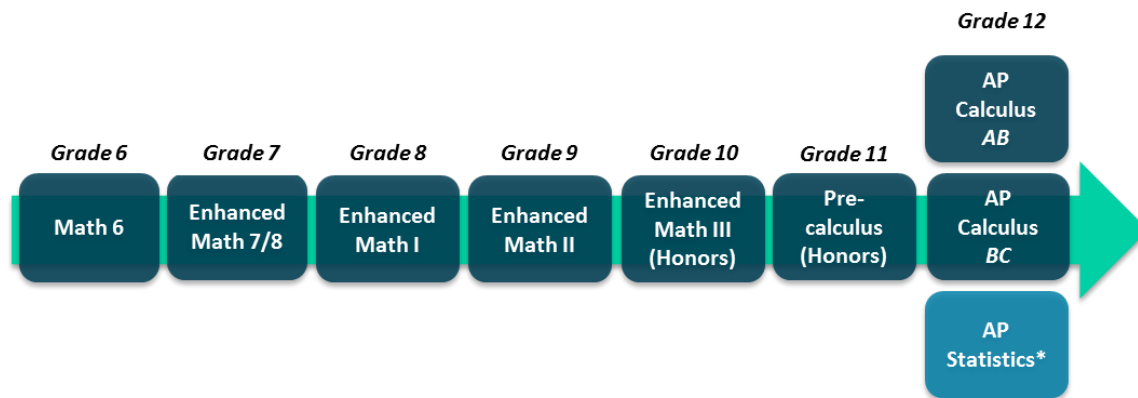
⁸² “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., pp. 831–833.

⁸³ Figure adapted from: Ibid., pp. 834–836.

In accelerated pathways, students typically do not take Calculus before Grade 12.⁸⁴ For example, the “Summer Bridge Sequence” teaches the Precalculus curriculum during a course taken in the summer, which allows students to take Calculus in their senior year. Similarly, in the “Enhanced Sequence,” students receive the Precalculus curriculum across multiple years (e.g., integrated across Enhanced Math II and III), which allows them to reach Calculus by Grade 12.⁸⁵ While the Utah State Board of Education notes that, “in rare circumstances, a [local education agency] may telescope mathematics courses to allow an especially advanced student to take Calculus before their senior year,” it cautions that, “extreme care must be taken to properly identify and verify that these students are eligible and ready for such acceleration.”⁸⁶

Accelerated pathways may also compact and expand course content in middle and high school. For example, Irvine Unified School District (Irvine USD), which implements an integrated approach, offers “Enhanced Math 7/8” for Grade 7 students and Enhanced Math I for Grade 8 students (despite their names, these courses spread content from the three courses across both years, rather than compacting Grade 7 and Grade 8 content into a single year). At the high school level, accelerated students can continue to take Enhanced Math II and III in Grades 9-10, Honors Precalculus in Grade 11, and then AP Calculus and/or Statistics in Grade 12. High school course content is expanded, rather than compacted. For example, Enhanced Math II includes (+) standards (standards outside of the core curriculum, as defined in the CCSS-M), as well as content from Math III and Pre-Calculus (Figure 2.2).⁸⁷

Figure 2.2: Accelerated Pathway – Integrated Approach (Irvine USD)



*Note: Students are typically allowed to take one math course each year. However, students may take AP Statistics, along with another advanced-level math course (e.g., AP Calculus AB/BC).
Source: Irvine USD⁸⁸

⁸⁴ “High Ability Students and the Utah Mathematics Core Standard,” Op. cit.

⁸⁵ “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 832.

⁸⁶ “High Ability Students and the Utah Mathematics Core Standard,” Op. cit.

⁸⁷ “Course Descriptions (IUSD),” Op. cit.

⁸⁸ Figure adapted from: [1] Ibid. [2] “Irvine High School Course of Study.” Irvine Unified School District, 2017. p. 42.
https://iusd.org/sites/default/files/documents/pdfs/ihs_program_of_studies_2017-18_4-19-17.pdf

Similarly, the Utah State Board of Education outlines an acceleration option that expands content in Grade 7 and 8 and compacts content in Grades 9-11, integrating Precalculus into Math I, II, and III. This allows students to reach AP Calculus in Grade 12. The state Board of Education notes that “this option is supported by the Mathematics Task Force, consisting of Utah mathematics educators from public and higher education, policy makers, and other stakeholders.”⁸⁹ In this pathway variation, students in the accelerated path may exit to the regular Math I, II, or III courses without repeating coursework.⁹⁰

While not a recommended option, districts may also choose to compact two years of integrated math into a single year. For example, the Tennessee Department of Education highlights two “doubling up” options where students either take Math I and II in Grade 9 or Math II and III in Grade 10.⁹¹ However, this goes against the CDE’s recommendation regarding accelerated pathway design:⁹²

When accelerated pathways are considered, it is recommended that three years of material be compacted into two years, rather than compacting two years into one. The rationale is that mathematical concepts are likely to be omitted when two years of material are squeezed into one. This practice is to be avoided, as the standards have been carefully developed to define clear learning progressions through the major mathematical domains.

MODIFICATIONS FOR ADDITIONAL SUPPORT

Districts may modify the typical integrated course sequence by expanding content across multiple years to support struggling learners. For example, in addition to offering an accelerated sequence, Irvine USD provides high school students the option of taking Math I over two years (i.e., Grades 9-10) in courses designated “Math I AB and Math I CD.” In this sequence, students would complete Math III in Grade 12 (see Figure 2.3 on the following page). The district describes the Math I courses as follows:⁹³

This two-year Math I course will build on and extend skills learned in middle school while developing mastery and understanding of fundamental algebraic and geometric concepts, properties and skills. Students will explore the content of Math I over a two-year period with a focus on conceptual understanding and symbolic reasoning. The Mathematical Practice Standards apply throughout the two-year course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations.

⁸⁹ “High Ability Students and the Utah Mathematics Core Standard,” Op. cit., pp. 1–2.

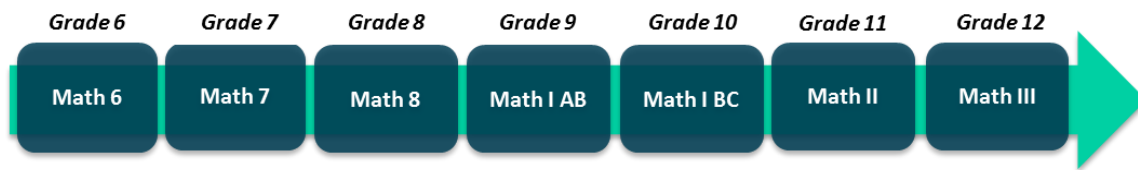
⁹⁰ Ibid.

⁹¹ “Making Decisions about Mathematics Course Sequences and Accelerating Students.” Tennessee Department of Education. p. 4. https://tn.gov/assets/entities/education/attachments/std_math_making-decisions-sequences-accelerating.pdf

⁹² Quoted verbatim from: “Mathematics Framework Kindergarten Through Grade Twelve,” Op. cit., p. 829.

⁹³ Quoted verbatim from: “Course Descriptions (IUSD),” Op. cit.

Figure 2.3: Decelerated Pathway - Integrated Approach (Irvine USD)



Source: Irvine USD⁹⁴

RATIONALE FOR CHOOSING AN INTEGRATED APPROACH

Advocates note that an integrated approach to instruction in algebra and geometry offers several benefits over the traditional pathway. They argue that an integrated approach encourages students to recognize the connections and interrelationships between multiple math domains, while allowing students to “continue to systematically build proficiency in each domain (Algebra, Geometry, and Statistics/Probability) each year.”⁹⁵ In contrast, the traditional approach typically requires students to reach proficiency in each domain within a single year. Some educators also find that the integrated approach encourages deeper thinking about math practices and problems, even in comparison to students who are three years ahead of them under the traditional approach.⁹⁶

Given these perceived benefits, many educators and experts support adopting an integrated approach to traditional algebra and geometry courses. For example, in a 2011 statement, the then president of the NCTM referred to traditional high school math sequencing as “an outmoded approach in a 21st century educational system” and advocated for an

integrated approach to better prepare students for college.⁹⁷ Similarly, a group of senior mathematicians, teachers, statisticians, and curriculum developers convened by the non-profit Consortium for Mathematics and Its Applications note that an integrated approach is common internationally (including countries whose students routinely outperform U.S. students in math) and argue that, “a broad and integrated vision of high school mathematics would serve our students better than the narrow and compartmentalized structure of traditional programs.”⁹⁸

“A broad and integrated vision of high school mathematics would serve our students better than the narrow and compartmentalized structure of traditional programs.”

⁹⁴ Displayed course sequence adapted from: Ibid.

⁹⁵ “High School Mathematics Pathways: Helping Schools and Districts Make an Informed Decision about High School Mathematics.” Indiana Department of Education.
<http://www.doe.in.gov/sites/default/files/curriculum/highschoolmathematicspathways.pdf>

⁹⁶ Will, Op. cit.

⁹⁷ Shaughnessy J., Op. cit.

⁹⁸ Strauss, V. “The Future of High School Math Education.” *Washington Post*, December 6, 2013.

<https://www.washingtonpost.com/news/answer-sheet/wp/2013/12/06/the-future-of-high-school-math-education/>

In a 2014 report for Brookings, senior researcher Tom Loveless argued that the CCSS Initiative implicitly supports an integrated approach by presenting it as an equally effective option to the traditional approach.⁹⁹

By treating both approaches as if they have equal standing—regardless of the overwhelming relative popularity of the traditional sequence—the CCSS cannot help but be regarded as prying open a window for integrated math courses...Neutrality, in this case, is a tacit endorsement.

SUPPORTING RESEARCH

Recent research supports adopting an integrated approach to high school math. In 2008, the U.S. Department of Education’s National Mathematics Advisory Panel noted that a literature search “did not produce studies that clearly examined whether an integrated approach or a single-subject sequence is more effective for algebra and more advanced mathematics course work.”¹⁰⁰ Since the publication of this report, however, more recent evidence indicates that high school students receiving an integrated math curriculum achieve at higher levels than their peers receiving a traditional curriculum. Below, Figure 2.4 summarizes this evidence.¹⁰¹

Figure 2.4: Overview of Studies Evaluating Integrated Math

AUTHORS (DATE)	DESIGN	POPULATION	LENGTH	FINDINGS
Schoen and Hirsh (2003)	Quasi-experimental	1,050	2 years	Students receiving the Core-Plus Mathematics Project integrated curriculum scored significantly higher on multiple achievement metrics than students using a traditional curriculum.
Chavez et al. (2015)*	Quasi-experimental	2,200	1 year	Students who received the Core-Plus III integrated curriculum scored significantly higher on end-of-year outcome measures than students taking an Algebra II course.
Tarr et al. (2013)*	Quasi-experimental	3,258	1 year	Students who received the Core-Plus II integrated curriculum scored significantly higher on standardized achievement tests than students taking a Geometry course.
Grouws et al. (2013)*	Quasi-experimental	2,161	1 year	Students who received the Core-Plus I integrated curriculum scored significantly higher on three achievement tests than students taking an Algebra I course.

*Note: Phases of the Comparing Options in Secondary Mathematics: Investigating Curricula (COSMIC) project supported by the National Science Foundation. The COSMIC project is a longitudinal study of the impact of high school math textbooks and curricula on student learning. Most students participating in studies by Chavez et al. and Tarr et al. were also included in prior COSMIC study(ies); the total length of the COSMIC project was three years. Information about the COSMIC project can be found at <http://cosmic.missouri.edu/>.

⁹⁹ Quoted verbatim from: Loveless, Op. cit.

¹⁰⁰ “The Final Report of the National Mathematics Advisory Panel.” U.S. Department of Education, 2008. p. 22. <https://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf>

¹⁰¹ Note that this list of studies is not necessarily comprehensive.

The following subsections describe the studies presented in this figure in additional detail. Note that while each compares the effects of an integrated approach against a traditional approach, *no studies specifically examine the effects of an integrated approach aligned with the CCSS-M*. Rather, studies consider the Core-Plus Mathematics Project (CPMP) curriculum, an integrated curriculum that interweaves “strands of algebra and functions, geometry and trigonometry, statistics and probability, and discrete mathematics” into three years of high school math.¹⁰²

Schoen and Hirsh (2003)

In *Standards-based School Mathematics Curricula: What Are They? What Do Students Learn?* published in 2003, authors Schoen and Hirsh compared students enrolled in traditional high school math courses with those receiving the CPMP curriculum.¹⁰³ Meeting WWC evidence standards,¹⁰⁴ the study used a quasi-experiment design with student matched-pairs to assess achievement outcomes for 1,050 students who were in either a Core-Plus Mathematics classroom or a traditional classroom. The authors found that the CPMP curriculum had a positive and statistically significant effect “on the Iowa Tests of Educational Development mathematics subtest in ninth grade, all three subtests of the Course 1 CPMP Posttest in ninth grade, and contextual algebra and coordinate geometry subtests of the Course 2 CPMP Posttest in tenth grade.”¹⁰⁵ The CPMP curriculum also led to a non-significant effect on SAT scores, which the WWC nevertheless notes is “large enough to be considered substantively important.”¹⁰⁶

COSMIC PROJECT

The following three studies are associated with the Comparing Options in Secondary Mathematics: Investigating Curriculum (COSMIC) project, which evaluates high school students’ learning under integrated and traditional approaches to math instruction and curriculum organization.¹⁰⁷ The project is funded by the National Science Foundation (NSF).¹⁰⁸

Each study considered students enrolled in 11 high schools located in six school districts across five geographically disparate states. Each school used both the integrated and

¹⁰² Schoen, H. and C. Hirsch. “The Core-Plus Mathematics Project: Perspectives and Student Achievement.” In *Standards-Based School Mathematics Curricula: What Are They? What Do Students Learn?*, edited by S. Senk and D. Thompson, Lawrence Erlbaum Associates, Inc., 2003. pp. 2–3.
http://www.cpmponline.org/pdfs/CPMP_Chapter_in_Senk_03.pdf

¹⁰³ Ibid.

¹⁰⁴ According to the What Works Clearinghouse, “Meets standards with reservations is the middle possible rating for a group design rating reviewed by the WWC. Studies receiving this rating provide a lower degree of confidence that an observed effect was caused by the intervention. Strong quasi-experimental designs may receive this rating.” See: <https://ies.ed.gov/ncee/wwc/Glossary>

¹⁰⁵ Schoen and Hirsch, Op. cit., p. 3.

¹⁰⁶ Ibid.

¹⁰⁷ “Comparing Options in Secondary Mathematics: Investigating Curriculum.” The University of Chicago.
<https://arc.uchicago.edu/reese/projects/comparing-options-secondary-mathematics-investigating-curriculum>

¹⁰⁸ Tarr, J., D. Ross, et al. “Identification of Student- and Teacher-Level Variables in Modeling Variation of Mathematics Achievement Data.” American Education Research Association, 2010. p. 2.
<http://files.eric.ed.gov/fulltext/ED510315.pdf>

traditional subject-specific curricula. The authors note that this selection strengthens their findings by eliminating “the variability in the number of number of days of instruction between curriculum treatments,” as well as by “control[ing] for other important contextual factors including homework and technology policies, organization and length of class periods, professional development provided, student body demographics, and other school characteristics.”¹⁰⁹ The authors collected data spanning the 2006-07 and 2008-09 academic years. In each school year, the authors examined the effect of an integrated math curriculum versus the effect of a subject-specific math curriculum on student achievement.¹¹⁰

Grouws et al. (2013)

The first phase of the COSMIC project examined student achievement after more than 2,000 students across 10 of the considered schools completed an integrated curriculum (i.e., Core-Plus Mathematics Course I) or a subject-specific curriculum (i.e., Algebra I), as the first course in the high school math sequence. The majority (78 percent) of participating students were white and 30 percent qualified for free or reduced price lunch. The authors used four measures of student learning, a test of prior achievement and three EOC tests, when examining the effects of the curricula. Two of the EOC tests were developed specifically for the study and assessed common objectives, problem solving, and reasoning. The third EOC test was the Iowa Test of Educational Development (ITED), a nationally standardized test.¹¹¹

The authors found that students receiving the integrated curriculum outperformed peers receiving the traditional curriculum on all three EOC tests, with effects sizes ranging from 0.17 to 0.45. The authors posited that “the focus on integration of content increases the likelihood of learning with meaning by developing ideas in contexts in which the relationship between different concepts can be an integral part of the instructional focus.”¹¹²

Tarr et al. (2013)

The second phase of the COSMIC project examined student achievement after more than 3,000 students completed the second year of the math sequence. Roughly one-third of students received the integrated curriculum (i.e., Core-Plus Mathematics Course II), while two-thirds of students received the subject specific curriculum (i.e., Geometry).¹¹³ Most

¹⁰⁹ Chavez, O. et al. “Third-Year High School Mathematics Curriculum: Effects of Content Organization and Curriculum Implementation.” *International Journal of Science and Mathematics Education*, 13:1, 2015. p. 102; 117. Accessed via Springer Link.

¹¹⁰ Tarr, Ross, et al., Op. cit., p. 2.

¹¹¹ Grouws, D. et al. “Curriculum and Implementation Effects on High School Students’ Mathematics Learning From Curricula Representing Subject-Specific and Integrated Content Organizations.” *Journal for Research in Mathematics Education*, 44:2, 2013. pp. 431–434.
https://www.researchgate.net/publication/258201166_Curriculum_and_Implementation_Effects_on_High_School_Students%27_Mathematics_Learning_From_Curricula_Representing_Subject-Specific_and_Integrated_Content_Organizations

¹¹² Ibid., p. 452; 459.

¹¹³ Tarr, J., D. Grouws, et al. “The Effects of Content Organization and Curriculum Implementation on Students’ Mathematics Learning in Second-Year High School Courses.” *Journal for Research in Mathematics Education*, 44:4, 2013. pp. 697–699.
<https://www.mheonline.com/research/assets/products/621bf66ddb7c962a/CorePlusMathematicsjrme201307.pdf>

students were in their second year of high school when taking these level courses (Grade 10), though some students completed these courses in other grade levels (approximately 20 percent of students in each group completed the course in Grade 9, and less than 10 percent of students in each group completed the course in Grades 11 and 12). African American, Hispanic, and Individualized Education Plan (IEP) students were more likely to enroll in Geometry than in Math II.¹¹⁴

The authors found that students in the integrated program “scored significantly higher” on the ITED designed for their grade level.¹¹⁵ However, students across programs performed similarly on the two other end-of-course outcome measures, which “were developed to be more sensitive to the study curricula.”¹¹⁶ While the authors acknowledged the need for research focused on an integrated curriculum aligned with the CCSS-M specifically, results of the study led them to conclude “that the Core-Plus integrated program can yield greater student learning than approaches embodied by the subject-specific textbooks.”¹¹⁷

Chavez et al. (2015)

In Chavez et al.’s study – the third in the COSMIC series – the authors examined outcomes of over 2,200 high school students from 10 of the schools in the COSMIC project. Students were enrolled either in Algebra II or Core-Plus Mathematics Course 3 and most participated in the two earlier COSMIC analyses. Similar to the earlier studies, the authors used a quasi-experimental design to investigate curriculum effects measured by two EOC outcomes while controlling for prior achievement and curriculum implementation.¹¹⁸ The authors’ models indicate that students receiving the integrated curriculum “scored significantly higher than those in the subject-specific curriculum on the common objectives test.”¹¹⁹

IMPLICATIONS FOR TEACHERS

While any adoption of the CCSS-M requires instructional and curricular adjustment, **under an integrated approach, secondary math teachers may need to provide instruction in new or less familiar domains.** For example, former Algebra I and II teachers will need to incorporate aspects of geometry into their curriculum and instruction; conversely, former Geometry teachers will need to provide instruction in algebra in alignment with the Math I, II, and III curricula. Geometry, Algebra I, and II teachers with significant experience teaching under a traditional approach may require additional supports, in comparison with teachers who previously taught both algebra- and geometry-focused courses.¹²⁰

Moreover, as the CA CCSS-M are more rigorous than previous standards, teachers may need to scaffold instruction to address skill and knowledge gaps as students transition into the new

¹¹⁴ Ibid., p. 699.

¹¹⁵ Ibid., p. 696; 722.

¹¹⁶ Ibid., p. 722.

¹¹⁷ Ibid.

¹¹⁸ Chavez et al., Op. cit., pp. 102–103.

¹¹⁹ Ibid., p. 97.

¹²⁰ Clark, J. “Teachers Grapple With Changes To High School Math.” *North Carolina Public Radio*, December 2, 2015. <http://wunc.org/post/teachers-grapple-changes-high-school-math#stream/0>

content standards and expectations.¹²¹ Increased rigor may also pose a challenge for teachers of compacted courses, which compress these standards into a shorter timeframe.¹²² Figure 2.5, on the following page, compares the content in Math I and Algebra I under the CA CCSS-M on the following page to illustrate the differences in these standards. Standards in **red font** are those that are only addressed in one course or the other.

Districts commonly implement the new approach across multiple years to smooth the transition. A secondary math teacher and department chair at Soledad Unified School District recommends adopting an integrated approach across multiple years by introducing Math I to Grade 9 students and implementing Math II and III as this class progresses. Schools should provide on-going and on-demand professional development, as well as dedicated time for weekly teacher preparation.¹²³ Districts may also focus on garnering teacher support before the transition, as Glendale Unified School District did through the 2016-17 school year, preceding its adoption of an integrated approach in the following school year.¹²⁴

¹²¹ Dilger, L. "Shifting the Paradigm in High School Math: How We Implemented Common Core in the Classroom." Schoology, 2017. <https://www.schoology.com/blog/shifting-paradigm-high-school-math-how-we-implemented-common-core-classroom>

¹²² "Mathematics Framework Kindergarten Through Grade Twelve," Op. cit., pp. 832–833.

¹²³ Dilger, Op. cit.

¹²⁴ [1] Corrigan, K. "Glendale Unified Is Changing How Math Is Taught to Students. Here's How." *Los Angeles Times*, February 28, 2017. <http://www.latimes.com/socal/glendale-news-press/news/tn-gnp-me-math-plan-20170228-story.html> [2] "Integrated Math." Glendale Unified School District. <https://www.gusd.net/Page/1843>

Figure 2.5: Overview Standards – Integrated Math I and Algebra I

MATH I	ALGEBRA I
NUMBER AND QUANTITY	
Quantities <ul style="list-style-type: none"> Reason quantitatively and use units to solve problems. 	The real number system <ul style="list-style-type: none"> Extend the properties of exponents to rational exponents. Use properties of rational and irrational numbers. Quantities <ul style="list-style-type: none"> Reason quantitatively and use units to solve problems.
FUNCTIONS	
Interpreting functions <ul style="list-style-type: none"> Understand the concept of a function and use function notation. Interpret functions that arise in applications in terms of the context. Analyze functions using different representations. Building functions <ul style="list-style-type: none"> Build a function that models a relationship between two quantities. Build new functions from existing functions. 	Linear, quadratic, and exponential models <ul style="list-style-type: none"> Construct and compare linear, quadratic, and exponential models and solve problems. Interpret expressions for functions in terms of the situation they model.
STATISTICS AND PROBABILITY	
Interpreting categorical and quantitative data <ul style="list-style-type: none"> Summarize, represent, and interpret data on a single count or measurement variable. Summarize, represent, and interpret data on two categorical and quantitative variables. Interpret linear models. 	
ALGEBRA	
Seeing structure in expressions <ul style="list-style-type: none"> Interpret the structure of expressions. Creating equations <ul style="list-style-type: none"> Create equations that describe numbers or relationships. Reasoning with equations and inequalities <ul style="list-style-type: none"> Understand solving equations as a process of reasoning and explain the reasoning. Solve equations and inequalities in one variable. Solve systems of equations. Represent and solve equations and inequalities graphically. 	Seeing structure in expressions <ul style="list-style-type: none"> Interpret the structure of expressions. Write expressions in equivalent forms to solve problems. Arithmetic with Polynomials and Rational Expressions <ul style="list-style-type: none"> Perform arithmetic operations on polynomials. Creating Equations <ul style="list-style-type: none"> Create equations that describe numbers or relationships Reasoning with Equations and Inequalities <ul style="list-style-type: none"> Understand solving equations as a process of reasoning and explain the reasoning. Solve equations and inequalities in one variable. Solve systems of equations.
GEOMETRY	
Congruence <ul style="list-style-type: none"> Experiment with transformations in the plane. Understand congruence in terms of rigid motions. Make geometric constructions. Expressing geometric properties with equations <ul style="list-style-type: none"> Use coordinates to prove simple geometric theorems algebraically. 	N/A

Source: CA State Board of Education¹²⁵

¹²⁵ (On the following page) Figure text quoted verbatim from: “California Common Core State Standards - Mathematics.” California State Board of Education, 2014. pp. 62-63; 88-89. <http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf>

APPENDIX

The figures in this appendix highlight recommendations from three WWC practitioner guides related to math instruction and supports for students in elementary and middle school. The figures also include implementation steps and an estimate of the strength of the evidence supporting each recommendation.¹²⁶ In total, the following guides are outlined in each respective figure:

- **Improving Mathematical Problem Solving in Grades 4 Through 8** (Figure A.1)
- **Developing Effective Fractions Instruction for Kindergarten Through 8th Grade** (Figure A.2)
- **Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools** (Figure A.3)

¹²⁶ For a description of the WWC's levels of evidence, see: Siegler, R. et al. "Developing Effective Fractions Instruction for Kindergarten Through 8th Grade." Institute of Education Sciences, 2010. pp. 4–5.
https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/fractions_pg_093010.pdf

Figure A.1: WWC Panel Recommendations - Improving Mathematical Problem Solving in Grades 4 Through 8

RECOMMENDATION	IMPLEMENTATION STEPS	STRENGTH OF EVIDENCE
1. Prepare problems and use them in whole-class instruction.	a) Include both routine and non-routine problems in problem-solving activities. b) Ensure that students will understand the problem by addressing issues students might encounter with the problem’s context or language. c) Consider students’ knowledge of mathematical content when planning lessons.	Minimal
2. Assist students in monitoring and reflecting on the problem-solving process.	d) Provide students with a list of prompts to help them monitor and reflect during the problem-solving process. e) Model how to monitor and reflect on the problem-solving process. f) Use student thinking about a problem to develop students’ ability to monitor and reflect.	Strong
3. Teach students how to use visual representations.	g) Select visual representations that are appropriate for students and the problems they are solving. h) Use think-alouds and discussions to teach students how to represent problems visually. i) Show students how to convert the visually represented information into mathematical notation.	Strong
4. Expose students to multiple problem-solving strategies.	j) Provide instruction in multiple strategies. k) Provide opportunities for students to compare multiple strategies in worked examples. l) Ask students to generate and share multiple strategies for solving a problem.	Moderate
5. Help students recognize and articulate mathematical concepts and notation.	m) Describe relevant mathematical concepts and notation, and relate them to the problem-solving activity. n) Ask students to explain each step used to solve a problem in a worked example. o) Help students make sense of algebraic notation.	Moderate

Source: What Works Clearinghouse¹²⁷

¹²⁷ Figure summarized verbatim from: “Improving Mathematical Problem Solving in Grades 4 Through 8.” What Works Clearinghouse, 2012. https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/mps_pg_052212.pdf#page=16

Figure A.2: WWC Panel Recommendations - Developing Effective Fractions Instruction for Kindergarten Through 8th Grade

RECOMMENDATION	IMPLEMENTATION STEPS	STRENGTH OF EVIDENCE
<p>1. Build on students’ informal understanding of sharing and proportionality to develop initial fraction concepts.</p>	<ul style="list-style-type: none"> a) Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects. b) Extend equal-sharing activities to develop students’ understanding of ordering and equivalence of fractions. c) Build on students’ informal understanding to develop more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions. 	<p>Minimal</p>
<p>2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.</p>	<ul style="list-style-type: none"> d) Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share. e) Provide opportunities for students to locate and compare fractions on number lines. f) Use number lines to improve students’ understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions. g) Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students’ ability to translate among these forms. 	<p>Moderate</p>
<p>3. Help students understand why procedures for computations with fractions make sense.</p>	<ul style="list-style-type: none"> h) Use area models, number lines, and other visual representations to improve students’ understanding of formal computational procedures. i) Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions. j) Address common misconceptions regarding computational procedures with fractions. k) Present real-world contexts with plausible numbers for problems that involve computing with fractions. 	<p>Moderate</p>
<p>4. Develop students’ conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.</p>	<ul style="list-style-type: none"> l) Develop students’ understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students’ developing strategies for solving ratio, rate, and proportion problems. m) Encourage students to use visual representations to solve ratio, rate, and proportion problems. n) Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems. 	<p>Minimal</p>
<p>5. Professional development programs should place a high priority on improving teachers’ understanding of fractions and of how to teach them.</p>	<ul style="list-style-type: none"> o) Build teachers’ depth of understanding of fractions and computational procedures involving fractions. p) Prepare teachers to use varied pictorial and concrete representations of fractions and fraction operations. q) Develop teachers’ ability to assess students’ understandings and misunderstandings of fractions. 	<p>Minimal</p>

Source: What Works Clearinghouse¹²⁸

¹²⁸ Figure summarized verbatim from: Siegler et al., Op. cit., p. 1.

Figure A.3: WWC Panel Recommendations - Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools

RECOMMENDATION	IMPLEMENTATION STEPS	STRENGTH OF EVIDENCE
<p>1. Screen all students to identify those at risk for potential mathematics difficulties and provide interventions to students identified as at risk.</p>	<ul style="list-style-type: none"> a) As a district or school sets up a screening system, have a team evaluate potential screening measures. The team should select measures that are efficient and reasonably reliable and that demonstrate predictive validity. Screening should occur in the beginning and middle of the year. b) Select screening measures based on the content they cover, with an emphasis on critical instructional objectives for each grade. c) In grades 4 through 8, use screening data in combination with state testing results d) Use the same screening tool across a district to enable analyzing results across schools 	<p>Moderate</p>
<p>2. Instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers in kindergarten through grade 5 and on rational numbers in grades 4 through 8.</p>	<ul style="list-style-type: none"> e) For students in kindergarten through grade 5, tier 2 and tier 3 interventions should focus almost exclusively on properties of whole numbers and operations. Some older students struggling with whole numbers and operations would also benefit from in-depth coverage of these topics. f) For tier 2 and tier 3 students in grades 4 through 8, interventions should focus on in-depth coverage of rational numbers as well as advanced topics in whole number arithmetic (such as long division). g) Districts should appoint committees, including experts in mathematics instruction and mathematicians with knowledge of elementary and middle school mathematics curricula, to ensure that specific criteria are covered in-depth in the curriculum they adopt. 	<p>Minimal</p>
<p>3. Instruction during the intervention should be explicit and systematic.</p>	<ul style="list-style-type: none"> h) Ensure that instructional materials are systematic and explicit. In particular, they should include numerous clear models of easy and difficult problems, with accompanying teacher think-alouds. i) Provide students with opportunities to solve problems in a group and communicate problem-solving strategies. j) Ensure that instructional materials include cumulative review in each session. 	<p>Strong</p>
<p>4. Interventions should include instruction on solving word problems that is based on common underlying structures.</p>	<ul style="list-style-type: none"> k) Teach students about the structure of various problem types, how to categorize problems based on structure, and how to determine appropriate solutions for each problem type. l) Teach students to recognize the common underlying structure between familiar and unfamiliar problems and to transfer known solution methods from familiar to unfamiliar problems. 	<p>Strong</p>

RECOMMENDATION	IMPLEMENTATION STEPS	STRENGTH OF EVIDENCE
<p>5. Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.</p>	<p>m) Use visual representations such as number lines, arrays, and strip diagrams. n) If visuals are not sufficient for developing accurate abstract thought and answers, use concrete manipulatives first. Although this can also be done with students in upper elementary and middle school grades, use of manipulatives with older students should be expeditious because the goal is to move toward understanding of—and facility with—visual representations, and finally, to the abstract.</p>	<p>Moderate</p>
<p>6. Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.</p>	<p>o) Provide about 10 minutes per session of instruction to build quick retrieval of basic arithmetic facts. Consider using technology, flash cards, and other materials for extensive practice to facilitate automatic retrieval. p) For students in kindergarten through grade 2, explicitly teach strategies for efficient counting to improve the retrieval of mathematics facts. q) Teach students in grades 2 through 8 how to use their knowledge of properties, such as commutative, associative, and distributive law, to derive facts in their heads.</p>	<p>Moderate</p>
<p>7. Monitor the progress of students receiving supplemental instruction and other students who are at risk.</p>	<p>r) Monitor the progress of tier 2, tier 3, and borderline tier 1 students at least once a month using grade-appropriate general outcome measures. s) Use curriculum-embedded assessments in interventions to determine whether students are learning from the intervention. These measures can be used as often as every day or as infrequently as once every other week. t) Use progress monitoring data to regroup students when necessary.</p>	<p>Minimal</p>
<p>8. Include motivational strategies in tier 2 and tier 3 interventions.</p>	<p>u) Reinforce or praise students for their effort and for attending to and being engaged in the lesson. v) Consider rewarding student accomplishments. w) Allow students to chart their progress and to set goals for improvement.</p>	<p>Minimal</p>

Source: What Works Clearinghouse¹²⁹

¹²⁹ Figure summarized verbatim from: “Improving Mathematical Problem Solving in Grades 4 Through 8.” Op., cit.

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Arlington, VA 22203

P 202.559.0500 F 866.808.6585

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