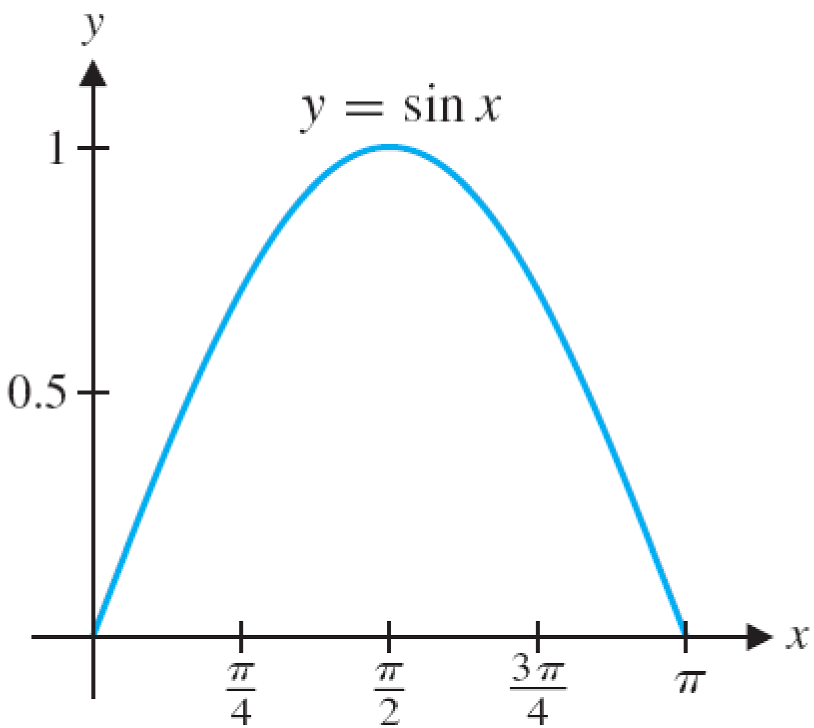
# Applications of the Definite Integral

CHAPTER 5

# 5.4 ARC LENGTH AND SURFACE AREA

## Arc Length

How could we find the *length* of the portion of the sine curve shown in the figure? (We call the length of a curve its **arc length.**)

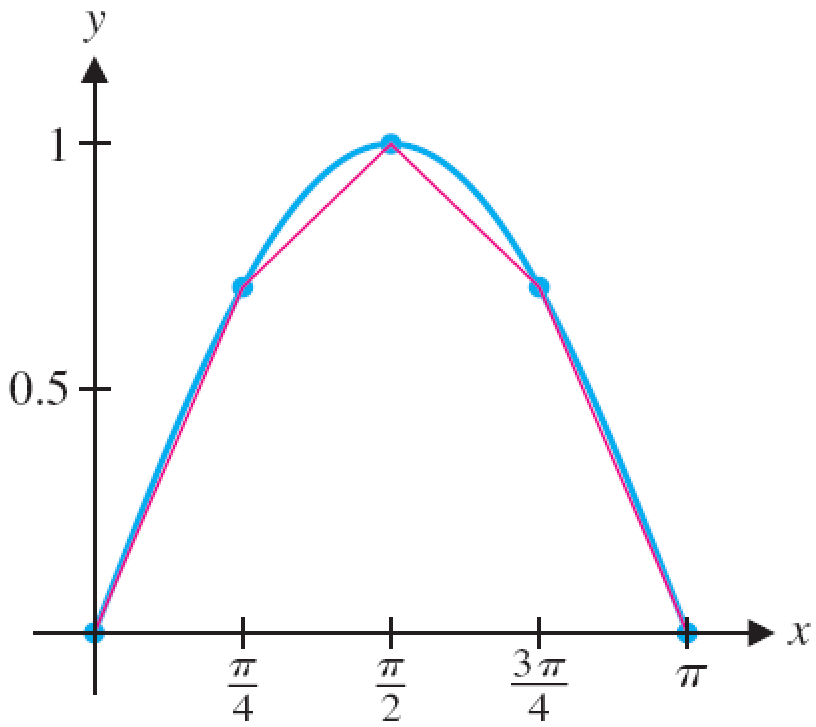


# 5.4 ARC LENGTH AND SURFACE AREA

## Arc Length

Approximate the curve with several line segments joined together.

As you would expect, the approximation of length will get closer to the actual length of the curve, as the number of line segments increases. This general idea should sound familiar.



# 5.4 ARC LENGTH AND SURFACE AREA

- Arc Length

Assume that *f* is continuous on [*a, b*] and differentiable on (*a, b*).

Begin by partitioning the interval [*a, b*] into *n* equal pieces:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

$$x_i - x_{i-1} = \Delta x = \frac{b-a}{n}, \text{ for each } i = 1, 2, \ldots, n$$

Approximate the arc length $s_i$ by the straight-line distance between two points.

$$s_i \approx d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\} = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

# 5.4 ARC LENGTH AND SURFACE AREA

- Arc Length

$$s_i \approx d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\} = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

By the Mean Value Theorem,

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

for some number $c_i \in (x_{i-1}, x_i)$. This gives us the approximation

$$\begin{aligned} s_i &\approx \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{(x_i - x_{i-1})^2 + [f'(c_i)(x_i - x_{i-1})]^2} \\ &= \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x} = \sqrt{1 + [f'(c_i)]^2}\,\Delta x. \end{aligned}$$

# 5.4 ARC LENGTH AND SURFACE AREA

- Arc Length

$$s_i \approx \sqrt{1+[f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x} = \sqrt{1+[f'(c_i)]^2}\,\Delta x$$

Adding together the lengths of these $n$ line segments,

$$s \approx \sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\,\Delta x$$

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\,\Delta x$$

# 5.4 ARC LENGTH AND SURFACE AREA

- Arc Length

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\, \Delta x$$

Recognize this as the limit of a Riemann sum for

$$\sqrt{1 + [f'(x)]^2},$$

so that the arc length is given exactly by the definite integral:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx,$$

whenever the limit exists.

# 5.4 ARC LENGTH AND SURFACE AREA

## REMARK 4.1

The formula for arc length is very simple.

Unfortunately, very few functions produce arc length integrals that can be evaluated exactly.

You should expect to use a numerical integration method on your calculator or computer to compute most arc lengths.

EXAMPLE 4.1 Using the Arc Length Formula

Find the arc length of the portion of the curve $y = \sin x$ with $0 \leq x \leq \pi$.

EXAMPLE 4.1 Using the Arc Length Formula

Find the arc length of the portion of the curve $y = \sin x$ with $0 \le x \le \pi$.

Solution

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

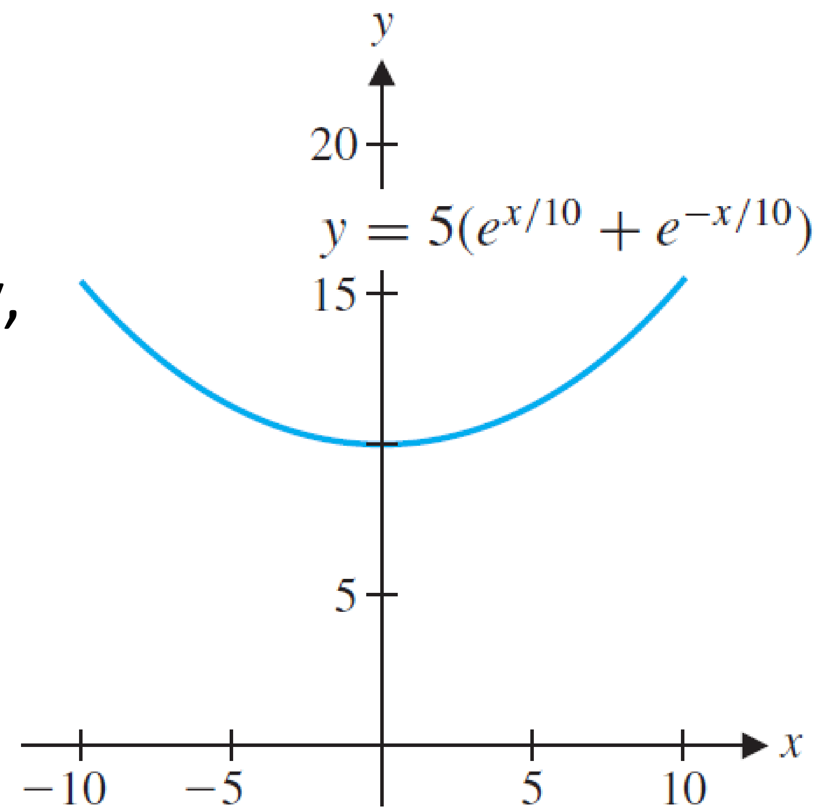
$$s = \int_0^{\pi} \sqrt{1 + (\cos x)^2}\, dx$$

Using a numerical integration method,

$$s = \int_0^{\pi} \sqrt{1 + (\cos x)^2}\, dx \approx 3.8202.$$

## EXAMPLE 4.4 Computing the Length of a Rope

A cable is hung between two poles of equal height that are 20 feet apart. A hanging cable assumes the shape of a *catenary*, the general form of which is $y = a \cosh x/a = a/2\,(e^{x/a} + e^{-x/a})$. In this case, suppose that the cable takes the shape of $y = 5(e^{x/10} + e^{-x/10})$, $-10 \le x \le 10$. How long is the cable?



## EXAMPLE 4.4 Computing the Length of a Rope

### Solution

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$s = \int_{-10}^{10} \sqrt{1 + \left(\frac{e^{x/10}}{2} - \frac{e^{-x/10}}{2}\right)^2}\, dx$$

$$= \int_{-10}^{10} \sqrt{1 + \frac{1}{4}(e^{x/5} - 2 + e^{-x/5})}\, dx$$

$$= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/5} + 2 + e^{-x/5})}\, dx$$

EXAMPLE 4.4 Computing the Length of a Rope

Solution

$$= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/5} + 2 + e^{-x/5})}\, dx$$

$$= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/10} + e^{-x/10})^2}\, dx$$

$$= \int_{-10}^{10} \frac{1}{2}(e^{x/10} + e^{-x/10})\, dx$$

$$= 5(e^{x/10} - e^{-x/10})\Big|_{x=-10}^{x=10}$$

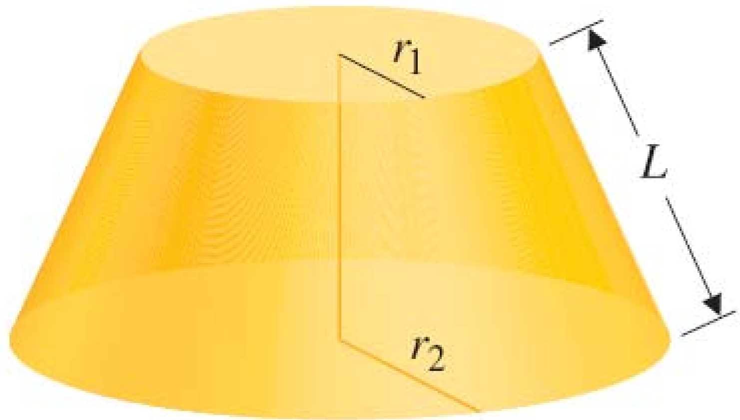
$$= 10(e - e^{-1})$$

$$\approx 23.504 \text{ feet},$$

# 5.4 ARC LENGTH AND SURFACE AREA

## Surface Area

For the **frustum** of a cone shown in the figure, the curved surface area is given by $A = \pi(r_1 + r_2)L$.

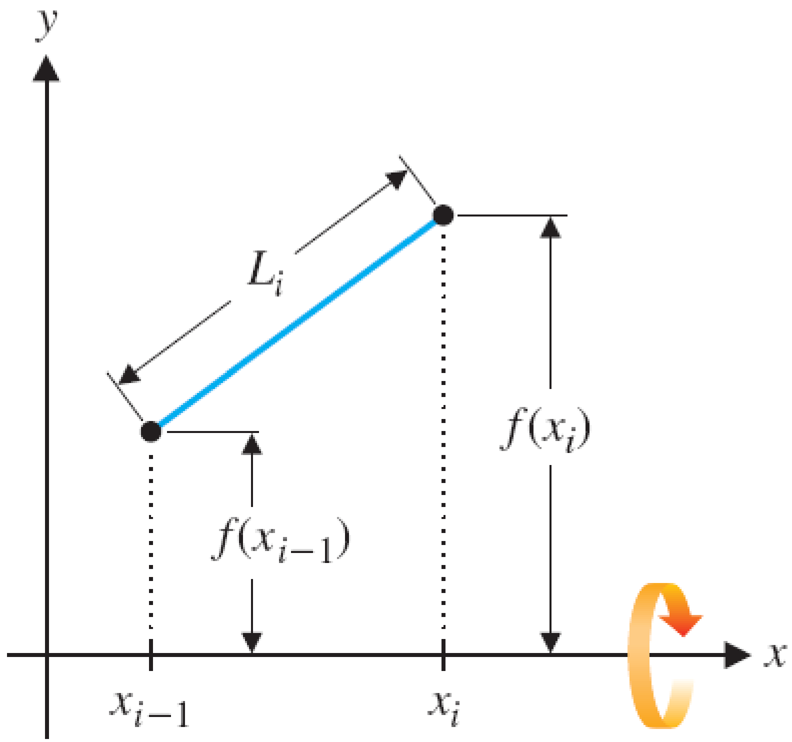


# 5.4 ARC LENGTH AND SURFACE AREA

## Surface Area

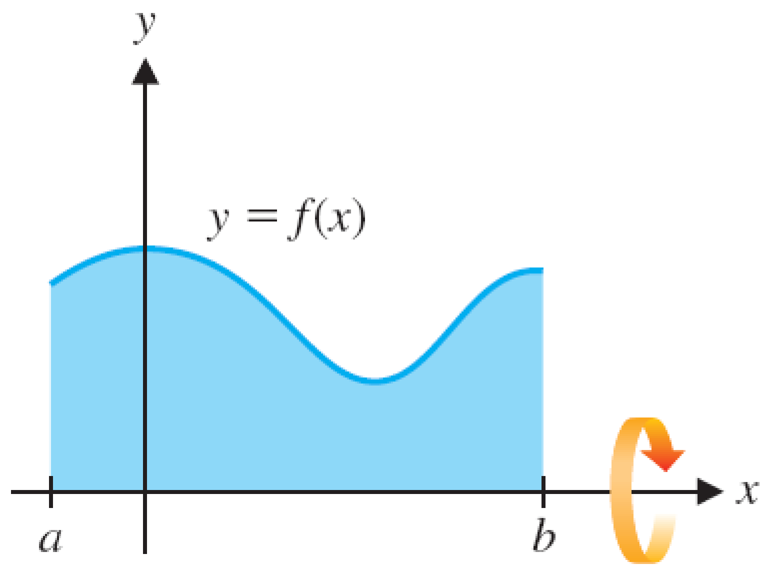
When a linear segment is rotated about a line, the resulting surface of rotation is the frustum of a cone.

We take advantage of this observation to develop a method for calculating surface areas.



## Surface Area

For the problem of finding the curved surface area of a surface of revolution, consider the case where $f(x) \geq 0$ and where $f$ is continuous on the interval $[a, b]$ and differentiable on $(a, b)$. If we revolve the graph of $y = f(x)$ about the $x$-axis on the interval $[a, b]$, we get a surface of revolution.



# 5.4 ARC LENGTH AND SURFACE AREA

## Surface Area

First partition the interval $[a, b]$ into $n$ pieces of equal size: $a = x_0 < x_1 < \cdots < x_n = b$, where

$$x_i - x_{i-1} = \Delta x = \frac{b-a}{n},$$

for each $i = 1, 2, \ldots, n$.

On each subinterval $[x_{i-1}, x_i]$, we can approximate the curve by the straight line segment joining the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

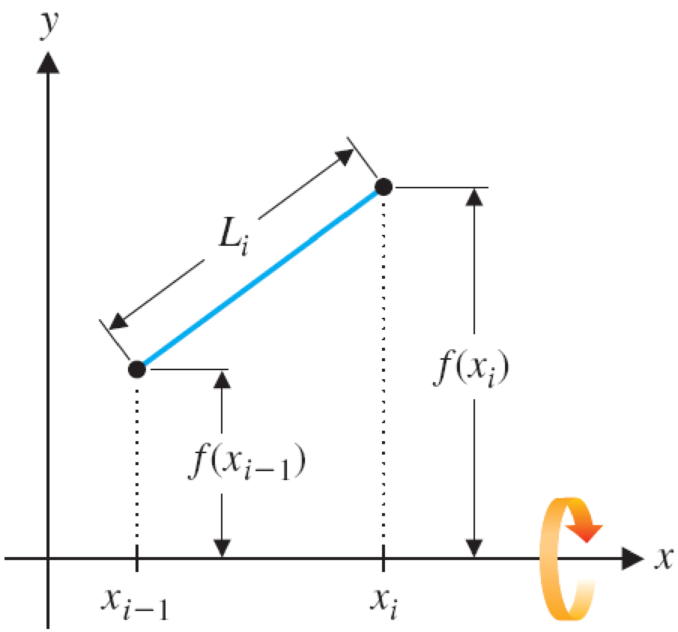
# 5.4 ARC LENGTH AND SURFACE AREA

## Surface Area

Revolving this line segment around the *x*-axis generates the frustum of a cone. The surface area of this frustum will give us an approximation to the actual surface area on the interval $[x_{i-1}, x_i]$.

Observe that the slant height of this frustum is

$$L_i = d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\}$$
$$= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$



# 5.4 ARC LENGTH AND SURFACE AREA

- Surface Area

$$L_i = d\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\}$$
$$= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

Apply the Mean Value Theorem,

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

for some number $c_i \in (x_{i-1}, x_i)$. This gives us

$$L_i = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

$$= \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x}.$$

# 5.4 ARC LENGTH AND SURFACE AREA

- Surface Area

$$L_i = \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x}$$

The surface area $S_i$ of that portion of the surface on the interval $[x_{i-1}, x_i]$ is approximately the surface area of the frustum of the cone,

$$S_i \approx \pi [f(x_i) + f(x_{i-1})] \sqrt{1 + [f'(c_i)]^2}\,\Delta x$$

$$\approx 2\pi f(c_i) \sqrt{1 + [f'(c_i)]^2}\,\Delta x,$$

since if $\Delta x$ is small,

$$f(x_i) + f(x_{i-1}) \approx 2f(c_i).$$

# 5.4 ARC LENGTH AND SURFACE AREA

## Surface Area

Repeating this argument for each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, gives an approximation to the total surface area $S$,

$$S \approx \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x$$

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx,}$$

whenever the integral exists.

## 5.4 ARC LENGTH AND SURFACE AREA

- Surface Area

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx,$$

The factor of

$$\sqrt{1 + [f'(x)]^2}\,dx$$

in the integrand corresponds to the arc length of a small section of the curve $y = f(x)$, while the factor $2\pi f(x)$ corresponds to the circumference of the solid of revolution.

## EXAMPLE 4.5 Computing Surface Area

Find the surface area of the surface generated by revolving $y = x^4$, for $0 \leq x \leq 1$, about the $x$-axis.

**Solution**

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

$$S = \int_0^1 2\pi x^4 \sqrt{1 + (4x^3)^2}\,dx$$

$$= \int_0^1 2\pi x^4 \sqrt{1 + 16x^6}\,dx \approx 3.4365$$