Bi-Quadratic Optimization over Unit Spheres and Semidefinite Programming Relaxations

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Abstract. This paper studies the so-called bi-quadratic optimization over unit spheres

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} & \sum\limits_{1\le i,k\le n,\, 1\le j,l\le m} b_{ijkl}x_iy_jx_ky_l \\ \text{subject to} & \|x\|=1,\ \|y\|=1. \end{array}$$

We show that this problem is NP-hard and there is no polynomial time algorithm returning a positive relative approximation bound. After that, we present various approximation methods based on semidefinite programming (SDP) relaxations. Our theoretical results are: For general bi-quadratic forms, we develop a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm; for bi-quadratic forms that are square-free, we get a relative approximation bound $\frac{1}{nm}$; when $\min\{n,m\}$ is a constant, we present two polynomial time approximation schemes (PTASs) which are based on sum of squares (SOS) relaxation hierarchy and grid sampling of the standard simplex. For practical computational purposes, we propose the first order SOS relaxation, a convex quadratic SDP relaxation and a simple minimum eigenvalue method, and give their quality analyses. Some illustrative numerical examples are also given.

Key Words. Bi-quadratic optimization, semidefinite programming, approximate solution, sum of squares, polynomial time approximation scheme

1 Introduction

Consider the bi-quadratic polynomial optimization of the form

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} & b(x,y) = \sum\limits_{1\le i,k\le n,\, 1\le j,l\le m} b_{ijkl}x_iy_jx_ky_l \\ \text{subject to} & \|x\|=1,\ \|y\|=1, \end{array} \tag{1.1}$$

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where $\|\cdot\|$ denotes the standard 2-norm in Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Without loss of generality, we can assume the coefficients b_{ijkl} satisfy the symmetric property: $b_{ijkl} = b_{kjil} = b_{ilkj}$ for $i, k = 1, \cdots, n$ and $j, l = 1, \cdots, m$. Denote $\mathcal{A} := (b_{ijkl})$. Then $\mathcal{A}$ is a fourth order partially symmetric tensor.

Throughout this paper, $\mathbb{R}^n$ denotes the space of real n-dimensional column vectors, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices, and ^{T} denotes transpose. $S_{n,m} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \|x\| = \|y\| = 1\}$ denotes the unit bi-sphere. For $x \in \mathbb{R}^n$, x_j denotes the j-th component of x. For any matrix A and fourth order tensor $\mathcal{A}$, $\|A\|_F$ and $\|\mathcal{A}\|_F$ denote the Frobenius norms of A and $\mathcal{A}$ respectively, i.e.,

$$\|A\|_F = \left(\mathrm{Tr}(A^T A)\right)^{1/2}, \quad \|\mathcal{A}\|_F = \left(\sum_{1 \le i,k \le n, 1 \le j,l \le m} b_{ijkl}^2\right)^{1/2},$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. For $A \in \mathcal{S}^n$, $A \succeq 0$ (resp. $A \succ 0$) means that A is positive semidefinite (resp. positive definite). $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathcal{S}^n$. I stands for the identity matrix in an appropriate dimension.

Problem (1.1) arises from the strong ellipticity condition problem in solid mechanics (for $n = m = 3$) [18, 19, 32, 34, 39] and the entanglement problem in quantum physics. The entanglement problem is to determine whether a quantum state is separable or inseparable (entangled), or to check whether an $mn \times mn$ symmetric matrix $A \succeq 0$ can be decomposed as a convex combination of tensor products of n and m dimensional vectors [6]. It has fundamental importance in quantum science and has attracted much attention since the pioneer work of Einstein, Podolsky and Rosen [10] and Schrödinger [33]. The entanglement problem was proved to be NP-hard by Gurvits [15].

Bi-quadratic optimization (1.1) also has another application. Suppose that (x^*, y^*) is a global minimizer and $p_{\min}$ is the minimum objective value of (1.1). Let $p_{\max}$ be the maximum objective value of (1.1) under the same sphere constraints and $(\bar{x}, \bar{y})$ be a global maximizer. If $|p_{\min}| \ge |p_{\max}|$, then $p_{\min} \cdot \left(x^*(y^*)^T\right) \otimes \left(x^*(y^*)^T\right)$ is the best rank-one approximation to the tensor $\mathcal{A}$. If $|p_{\max}| > |p_{\min}|$, then $p_{\max} \cdot \left(\bar{x}\bar{y}^T\right) \otimes \left(\bar{x}\bar{y}^T\right)$ is the best rank-one approximation to $\mathcal{A}$; see [30] for details. The best rank-one approximation problem has wide applications in signal and image processing, wireless communication systems, data analysis, higher-order statistics, as well as independent component analysis [3, 5, 7, 8, 14, 20, 27, 40].

If we fix $x \in \mathbb{R}^n$ in (1.1), then we have a quadratic optimization problem

$$\min_{y \in \mathbb{R}^m} y^T B(x) y \;\text{ subject to } \|y\| = 1, \tag{1.2}$$

where $B(x) = \left(\sum_{i,k=1}^n b_{ijkl} x_i x_k\right)_{1 \le j,l \le m}$ is an $m \times m$ symmetric matrix. Similarly, if we fix $y \in \mathbb{R}^m$, then we have a quadratic optimization problem

$$\min_{x \in \mathbb{R}^n} x^T C(y) x \;\text{ subject to } \|x\| = 1, \tag{1.3}$$

where $C(y) = \left(\sum_{j,l=1}^{m} b_{ijkl} y_j y_l\right)_{1\le i,k\le n}$ is an $n \times n$ symmetric matrix. Since problem (1.1) is so closely related to quadratic optimization, we call it a *bi-quadratic optimization problem*, or a *bi-quadratic program.* By using the symbol $\mathcal{A}$, $B(x)$ and $C(y)$ can also be written as $\mathcal{A}xx^T$ and $yy^T\mathcal{A}$, respectively. So, it is clear that

$$b(x, y) = (\mathcal{A}xx^T) \bullet (yy^T) = (yy^T\mathcal{A}) \bullet (xx^T),$$

where $X \bullet Y$ stands for usual matrix inner product, i.e., $X \bullet Y = \mathrm{Tr}(X^TY)$.

Contributions In Section 2, we show the problem (1.1) is NP-hard. Thus, it is not expected to find a polynomial time algorithm to solve (1.1) for general bi-quadratic form $b(x, y)$. Actually, we have proved a stronger result: there does not exist any polynomial time approximation algorithm that returns an upper bound having the same sign as the optimal value, unless P=NP.

In Section 3, we propose various approximation methods to solve (1.1) using semidefinite programming (SDP) and analyze their approximation qualities. For general bi-quadratic forms $b(x, y)$, we give a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm. For $b(x, y)$ that is square-free (contains no quartic term with x_i^2 or y_j^2 for any i and j), we given an SDP relaxation with relative approximation bound $\frac{1}{nm}$. In case that $\min\{n, m\}$ is a constant, we give two polynomial time approximation schemes (PTASs); one is based on sum of squares (SOS) relaxation hierarchy, and the other is based on grid sampling of the standard simplex originally used by Bomze and de Klerk [1].

In Section 4, for practical computational purposes, we propose the first order SOS relaxation, a convex quadratic SDP relaxation, a simple minimum eigenvalue relaxation method; and give the quality analyses of the three methods with certain rounding procedures.

Some illustrative numerical examples are given in Section 5. We conclude and list a few open problems in the final section.

2 Complexity analysis: hardness results

Since $b(x, y)$ is a continuous function and the feasible set of (1.1) is compact, the problem (1.1) has a global minimizer (x^*, y^*). When either x or y is fixed, the problem is then reduced to an eigenvalue problem and hence can be solved in polynomial time. However, when x and y are both variables, (1.1) is a non-convex optimization problem, since its objective is bi-quadratic and nonconvex. How difficult is to solve (1.1) globally? In this section, we show that the problem (1.1) is NP-hard to solve. We can even prove a stronger result: there does not exist any polynomial time approximation algorithm that returns an upper bound having the same sign as the optimal value, unless P=NP.

We first define a quality measure of approximation ratio:

Definition 2.1. *Let $\mathfrak{A}$ be a polynomial time (in n and m) approximation algorithm to solve (1.1). For an instance of (1.1), we say $\mathfrak{A}$ has a relative approximation bound*

$C = C(\mathfrak{A}, b) \in (0,1]$ *for it, if the algorithm* $\mathfrak{A}$ *can find an upper bound* p *for (1.1) such that*

$$\begin{cases} C \cdot p \leq & p_{\min} & \leq p, & \text{if } p_{\min} \geq 0, \\ p_{\min} \leq & p & \leq C \cdot p_{\min}, & \text{if } p_{\min} < 0, \end{cases} \tag{2.1}$$

where $p_{\min}$ *is the optimal value of (1.1).*

In this definition, the closer C to 1, the better the approximation algorithm will be.

2.1 Hardness of bi-quadratic optimization

Our main result in this section is the following:

Theorem 2.2. *(i) The following problem is NP-hard: Given any bi-quadratic objective function* $b(x, y)$ *of (1.1), find the minimum value* $p_{\min}$ *of* $b(x, y)$ *over the bi-sphere* $S_{n,m}$*.*
(ii) Unless P=NP, there does not exist a polynomial time approximation algorithm $\mathfrak{A}$ *for (1.1) to get a positive relative approximation bound for every instance of (1.1).*

Proof. (i) We show the NP-hardness when the bi-quadratic forms are restricted to be square-free and that $n = m$. To see this point, let $G = (V, E)$ be a graph with V being the set of n vertices and E being its edge set. Then define a square-free bi-quadratic form associated with G as

$$b_G(x, y) := -2 \sum_{(i,j) \in E} x_i x_j y_i y_j.$$

Let $\Delta_n = \{x \in \mathbb{R}^n_+ : x_1 + \cdots + x_n = 1\}$ be the standard simplex. Then we have that

$$\min_{(x,y) \in S_{n,m}} b_G(x, y) = -\max_{\|x\|=1} \sum_{(i,j) \in E} 2x_i^2 x_j^2 = -\max_{x \in \Delta_n} \sum_{(i,j) \in E} 2x_i x_j = -1 + \frac{1}{\alpha(G)},$$

due to a theorem of Motzkin and Straus [24]. Here, $\alpha(G)$ is the stability number of the graph G, i.e., the cardinality of the maximum independent set of G. Therefore, to compute the minimum of $b_G(x, y)$ over the bi-sphere is NP-hard, since it is known to be NP-hard to compute $\alpha(G)$.

(ii) We prove this is impossible when $n = m$. Given any integer vector a, define bi-quadratic form

$$b_a(x, y) = (a^T x)^2 (a^T y)^2 + \left(1 - \frac{1}{n}\right) \|x\|^2 \cdot \|y\|^2 - 2 \sum_{1 \leq i < j \leq n} x_i x_j y_i y_j. \tag{2.2}$$

In the rest of the proof, we restrict (x, y) to be in $S_{n,m}$. Then we have

$$2 \sum_{1 \leq i < j \leq n} x_i x_j y_i y_j \leq \sum_{1 \leq i < j \leq n} x_i^2 x_j^2 + \sum_{1 \leq i < j \leq n} y_i^2 y_j^2 \leq 1 - \frac{1}{2} \left(\sum_{i=1}^n x_i^4 + \sum_{i=1}^n y_i^4 \right) \leq 1 - \frac{1}{n}.$$

In the above, all the inequalities become equalities if and only if $x = \pm y$ has the form $\frac{1}{\sqrt{n}}(\pm 1, \cdots, \pm 1)$. Obviously,

$$(a^T x)^2 (a^T y)^2 \geq 0$$

and the inequality becomes an equality if and only if at least one of $a^T x$ and $a^T y$ is equal to zero. So we can see that $p_{\min} \geq 0$ and the equality holds if and only if the integer vector a can be partitioned into two parts of equal sum, which is known to be NP-hard.

Now we prove (ii) by contradiction. Assume such an algorithm $\mathfrak{A}$ exists. Then for every integer vector a, we apply the algorithm $\mathfrak{A}$ to the bi-quadratic form $b_a(x, y)$ defined in (2.2) and would get a bound p and $0 < C = C(\mathfrak{A}, a) \leq 1$ such that

$$C \cdot p \leq p_{\min} \leq p.$$

Then we can see $p_{\min} = 0$ if and only if $p = 0$. This implies we can decide whether an arbitrary integer vector would be partitioned into two parts of equal sums in polynomial time, which is impossible unless P=NP. $\square$

The proof of item (i) of Theorem 2.2 indicates a stronger result: the problem (1.1) remains to be NP-hard when the bi-quadratic forms are restricted to be square-free and $n = m$. The item (ii) of Theorem 2.2 says that there exists no problem-data dependent or independent positive relative approximation quality bound (the relation (2.1)) for (1.1). However, there is a problem-data independent positive relative approximation quality bound when the bi-quadratic forms are restricted to be square-free. This will be shown in Theorem 3.3.

2.2 Hardness of bi-linear SDP relaxation

Now we propose a natural bi-linear SDP relaxation for (1.1) and discuss its quality. It is easy to see that problem (1.1) can be written as

$$\begin{array}{rl} p_{\min} := \min\limits_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} & (\mathcal{A} x x^T) \bullet (y y^T) \\ \text{subject to} & \mathrm{Tr}(x x^T) = 1, \\ & \mathrm{Tr}(y y^T) = 1, \end{array} \tag{2.3}$$

which is equivalent to

$$\begin{array}{rl} \min\limits_{X, Y} & (\mathcal{A} X) \bullet Y \\ \text{subject to} & \mathrm{Tr}(X) = 1, X \succeq 0, \\ & \mathrm{Tr}(Y) = 1, Y \succeq 0, \\ & \mathrm{rank}(X) = 1, \mathrm{rank}(Y) = 1. \end{array} \tag{2.4}$$

Here $X \in \mathcal{S}^n$, $Y \in \mathcal{S}^m$ and $\mathcal{A}X$ is an $m \times m$ matrix with

$$(\mathcal{A}X)_{jl} = \sum_{i,k=1}^{n} b_{ijkl} X_{ik}, \quad j, l = 1, 2, \cdots, m.$$

Thus, a bi-linear SDP relaxation of (1.1) is

$$\begin{array}{rl} p_{sdp} := \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y \\ \text{subject to} & \mathrm{Tr}(X) = 1, X \succeq 0, \\ & \mathrm{Tr}(Y) = 1, Y \succeq 0. \end{array} \tag{2.5}$$

We denote by p_{sdp} the optimal value of (2.5). It is clear that $p_{sdp} \le p_{\min}$.

We now consider how to generate an optimal solution (x^*, y^*) of the original problem (1.1) from an optimal solution pair (X^*, Y^*) of the bi-linear SDP problem (2.5). To this aim, we state a matrix decomposition result first.

Lemma 2.3. *(Sturm and Zhang [35]) Let $X \in \mathcal{S}^n_+$ be a positive semidefinite matrix of rank r. Let $G \in \mathcal{S}^n$ be such that $G \bullet X \ge 0$. Then, one can always find $x^1, \cdots, x^r \in \mathbb{R}^n$ in polynomial time such that $X = \sum_{i=1}^r x^i (x^i)^T$ and*

$$G \bullet x^i (x^i)^T = G \bullet X / r \quad \text{for} \quad i = 1, \cdots, r.$$

Theorem 2.4. *The bi-quadratic optimization (1.1) and bi-linear SDP (2.5) are equivalent, that is, (1.1) and (2.5) have the same optimal value and one optimal solution pair of (1.1) can be obtained from the optimal solution pair of (2.5).*

Proof. Let (X^*, Y^*) be an optimal solution matrix pair of (2.5). Without loss of generality, we assume that X^* and Y^* have full ranks n and m, respectively. Then, by Lemma 2.3, one can find the decompositions of X^* and Y^* such that

$$X^* = \sum_{i=1}^n \bar{x}^i (\bar{x}^i)^T, \quad \|\bar{x}^i\|^2 = I \bullet X^*/n = 1/n, \quad \forall\ i$$

and

$$Y^* = \sum_{j=1}^m \bar{y}^j (\bar{y}^j)^T, \quad \|\bar{y}^j\|^2 = I \bullet Y^*/m = 1/m, \quad \forall\ j.$$

There must exist an index, say 1, such that $(\mathcal{A}X^*) \bullet \bar{y}^1 (\bar{y}^1)^T \le p_{sdp}/m$, since

$$p_{sdp} = (\mathcal{A}X^*) \bullet Y^* = (\mathcal{A}X^*) \bullet \left(\sum_{j=1}^m \bar{y}^j (\bar{y}^j)^T \right).$$

Let $y^* = \sqrt{m}\bar{y}^1$. Then we must have

$$(\mathcal{A}X^*) \bullet y^* (y^*)^T \le p_{sdp}, \ \|y^*\|^2 = 1.$$

Continue this process on X^*. There must be an index, say 1, such that

$$(\mathcal{A}\bar{x}^1 (\bar{x}^1)^T) \bullet y^* (y^*)^T \le p_{sdp}/n.$$

Let $x^* = \sqrt{n}\bar{x}^1$, we must have

$$(\mathcal{A}x^* (x^*)^T) \bullet y^* (y^*)^T \le p_{sdp}, \ \|x^*\|^2 = 1, \|y^*\|^2 = 1.$$

That is, (x^*, y^*) is a feasible solution pair for the original problem (1.1) so that

$$p_{\min} \le (\mathcal{A}x^*(x^*)^T) \bullet y^*(y^*)^T \le p_{sdp} \le p_{\min},$$

which implies that $p_{\min} = p_{sdp} = (\mathcal{A}x^*(x^*)^T) \bullet y^*(y^*)^T$. We complete the proof. $\square$

Theorem 2.4 shows that we can obtain a solution of (1.1) in polynomial time from a solution of (2.5). Therefore, (2.5) must be still hard to solve.

Corollary 2.5. *It is NP-hard to solve the bi-linear SDP relaxation (2.5).*

Proof. Theorem 2.4 shows that the bi-quadratic optimization (1.1) and its bi-linear SDP relaxation (2.5) have the same optimal value. From Theorem 2.2, we know (1.1) is NP-hard, which immediately implies the relaxation (2.5) is also NP-hard. $\square$

Our result is in contrast to the bi-linear optimization over two vector simplexes:

$$\min_{u\in\mathbb{R}^n, v\in\mathbb{R}^m} \quad u^T A v \ \text{ subject to } \quad \sum_{i=1}^{n} u_i = 1, \sum_{j=1}^{m} v_j = 1, \quad u \ge 0,\, v \ge 0.$$

The above problem is solvable in polynomial time by simply choosing the minimum element in the matrix A.

3 Approximation quality bounds

Theorem 2.2 shows that the bi-quadratic optimization (1.1) is NP-hard, and finding an approximate solution with positive relative approximation bound is also NP-hard. But this does not exclude the approximatability when the bi-quadratic form $b(x, y)$ in (1.1) has special structures. This section will give various approximation results when $b(x, y)$ is general or has special features. SDP relaxation methods are important on approximating quadratic optimization problems and has received much attention recently, e.g., [11], [13], [16], [22], [36] and [38]. Our approximation results are also based on SDP relaxations.

To present our results, we begin with another quality measure of approximation ratio:

Definition 3.1. *Let $1 > \epsilon \ge 0$ and $\mathfrak{A}$ be an approximation algorithm for (1.1). We say $\mathfrak{A}$ is a $(1-\epsilon)$-approximation algorithm for (1.1) if for any instance of (1.1) the algorithm $\mathfrak{A}$ returns a feasible pair $(\bar{x}, \bar{y})$ to (1.1) such that*

$$b(\bar{x}, \bar{y}) - p_{\min} \le \epsilon(p_{\max} - p_{\min}).$$

Recall that $p_{\min}$ (resp. $p_{\max}$) is the minimum (resp. maximum) value of the objective in (1.1). We say (1.1) has a polynomial time approximation scheme (PTAS) if for every $1 > \epsilon > 0$, there exists a $(1-\epsilon)$-approximation algorithm.

One can see that Definition 3.1 is weaker than Definition 2.1 but standard. If $p_{\max} = 0$, then the two definitions coincide each other with $C = 1 - \epsilon$.

We will consider the general bi-quadratic form $b(x, y)$ first and give a $\frac{1}{2\max\{m,n\}^2}$-approximation algorithm for (1.1). When $b(x, y)$ has only squared terms in x or y, we will show (1.1) can be solved in polynomial time. When $b(x, y)$ is square-free, we will show (1.1) has an SDP relaxation with a relative approximation bound $\frac{1}{nm}$ under Definition 2.1. When $\min\{n, m\}$ is a constant, we present two PTASs for (1.1).

3.1 SDP approximation bounds based on ellipsoids

Let $p_{\min}$ and $p_{\max}$ be the minimum and the maximum objective values of (1.1) under the unit ball conditions. Let $\mathcal{A}$ be the fourth order partially symmetric tensor defined in Introduction. A bi-linear SDP relaxation of (1.1) is (2.5). Theorem 2.4 actually indicates that this relaxation is tight, namely, given any (X, Y) feasible for (2.5), one can in polynomial time find feasible solution pairs $(x', y'), (x'', y'')$ of (1.1) such that

$$b(x', y') \le (\mathcal{A}X) \bullet Y, \quad b(x'', y'') \ge (\mathcal{A}X) \bullet Y.$$

The bi-linear SDP relaxation (2.5) can be rewritten as

$$\begin{array}{lll} p_{\min} := \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y + \frac{1}{n}(\mathcal{A}I_n) \bullet Y + \frac{1}{m}(\mathcal{A}X) \bullet I_m + \frac{1}{mn}(\mathcal{A}I_n) \bullet I_m \\ \text{subject to} & \mathrm{Tr}(X) = 0, X + \frac{1}{n}I_n \succeq 0 \\ & \mathrm{Tr}(Y) = 0, Y + \frac{1}{m}I_m \succeq 0 \end{array} \tag{3.1}$$

after some linear transformations $X := X - \frac{1}{n}I_n$ and $Y := Y - \frac{1}{m}I_m$.

The objective function in (3.1) contains linear and constant terms, which are all zeros when the bi-quadratic form $b(x, y)$ is square-free. The constant term $\bar{p} := \frac{1}{mn}(\mathcal{A}I_n) \bullet I_m$ is the objective value of (2.5) for the feasible pair $(\frac{1}{n}I_n, \frac{1}{m}I_m)$. Thus, we know

$$p_{\min} \le \bar{p} \le p_{\max}.$$

We denote

$$\phi(X, Y) = (\mathcal{A}X) \bullet Y + \frac{1}{n}(\mathcal{A}I_n) \bullet Y + \frac{1}{m}(\mathcal{A}X) \bullet I_m.$$

Note that the following relation holds for matrices in $\mathcal{S}^n$:

$$\left\{X : \begin{array}{c} \mathrm{Tr}(X) = 0 \\ \|X\|_F \le \frac{1}{n} \end{array}\right\} \subseteq \left\{X : \begin{array}{c} \mathrm{Tr}(X) = 0 \\ X \succeq -\frac{1}{n}I_n \end{array}\right\} \subseteq \left\{X : \begin{array}{c} \mathrm{Tr}(X) = 0 \\ \|X\|_F \le \sqrt{1 - \frac{1}{n}} \end{array}\right\}. \tag{3.2}$$

For any scalars $\lambda > 0$ and $\mu > 0$, denote $\Omega(\lambda, \mu)$ for the optimization problem:

$$\begin{array}{rl} p(\lambda, \mu) := \min\limits_{X,Y} & \phi(X, Y) \\ \text{subject to} & \mathrm{Tr}(X) = \mathrm{Tr}(Y) = 0, \\ & \|X\|_F \le \lambda,\ \|Y\|_F \le \mu. \end{array} \tag{3.3}$$

This is a non-homogeneous quadratic optimization over two ellipsoidal constraints. It can be viewed as using an ellipsoidal set to approximate the affine conic feasible set of (2.5), which was first used in Ye [37] and by Fu et al. [13] for polyhedral constrained non-convex quadratic optimization, and more recently by Luo and Zhang [23] for homogeneous quartic polynomial optimization. Note again the relationship between the optimal values

$$p(1,1) \le p_{\min} - \bar{p} \le p\left(\frac{1}{n}, \frac{1}{m}\right) \le p\left(\frac{1}{\max\{m,n\}}, \frac{1}{\max\{m,n\}}\right).$$

For any optimal pair (X^*, Y^*) of (3.3), the linear sum $\frac{1}{n}(\mathcal{A}I_n) \bullet Y^* + \frac{1}{m}(\mathcal{A}X^*) \bullet I_m$ must be non-positive, otherwise we can replace (X^*, Y^*) by $(-X^*, -Y^*)$ to get a smaller objective value. Hence, we have the relation

$$p(1,1) \le p\left(\frac{1}{n}, \frac{1}{m}\right) \le p\left(\frac{1}{\max\{m,n\}}, \frac{1}{\max\{m,n\}}\right) \le \frac{1}{\max\{m,n\}^2} p(1,1).$$

Thus, if one can compute a feasible pair $(\bar{X}, \bar{Y})$ for $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ such that $\phi(\bar{X}, \bar{Y}) \le \alpha p\left(\frac{1}{n}, \frac{1}{m}\right)$, then

$$\phi(\bar{X}, \bar{Y}) \le \frac{\alpha}{\max\{m,n\}^2} p(1,1) \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - \bar{p}).$$

Taking $X = \bar{X} + \frac{1}{n} I_n$ and $Y = \bar{Y} + \frac{1}{m} I_m$, we have

$$(\mathcal{A}X) \bullet Y - \bar{p} \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - \bar{p}).$$

From the proof of Theorem 2.4, one can, in polynomial time, compute a solution (x', y') feasible to (1.1) such that

$$b(x', y') - \bar{p} \le (\mathcal{A}X) \bullet Y - \bar{p} \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - \bar{p}).$$

Since $\bar{p} \le p_{\max}$, this also implies $b(x', y') - p_{\max} \le \frac{\alpha}{\max\{m,n\}^2}(p_{\min} - p_{\max})$ and

$$b(x', y') - p_{\min} \le \left(1 - \frac{\alpha}{\max\{m,n\}^2}\right)(p_{\max} - p_{\min}).$$

In other words, we are able to get a $\frac{1}{2\max\{m,n\}^2}$-approximation for (1.1) if $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ has a feasible solution of relative approximation bound $\alpha = \frac{1}{2}$.

Theorem 3.2. *The SDP relaxation $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ can return a solution (x', y') for bi-quadratic optimization (1.1) such that*

$$b(x', y') - p_{\min} \le \left(1 - \frac{1}{2\max\{m,n\}^2}\right)(p_{\max} - p_{\min}).$$

Proof. From the above discussion, we know it suffices to show that $\Omega(\frac{1}{n}, \frac{1}{m})$ allows a solution of relative approximation bound $\alpha = \frac{1}{2}$. To see this, note that (3.3) can be equivalently formulated as the quadratic optimization problem

$$\begin{array}{ll} \min\limits_{z \in \mathbb{R}^N} & q(z) := z^T Q z + 2c^T z \\ \text{subject to} & z^T A_1 z \le 1, \\ & z^T A_2 z \le 1, \end{array} \tag{3.4}$$

where $A_1, A_2 \succeq 0$ and $A_1 + A_2 \succ 0$, $N = \frac{1}{2}[n(n+1) + m(m+1)] - 2$ and Q is symmetric. Denote its minimal value by $q_{\min}$. Then $q_{\min} \le 0$ as $z = 0$ is a feasible solution of (3.4). A standard SDP relaxation for the above problem is

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & Q \bullet W + 2c^T z \\ \text{subject to} & A_1 \bullet W \le 1,\ A_2 \bullet W \le 1, \\ & \begin{pmatrix} 1 & z^T \\ z & W \end{pmatrix} \succeq 0. \end{array}$$

This SDP has three constraints, so that an optimal $\begin{pmatrix} 1 & (z^*)^T \\ z^* & W^* \end{pmatrix}$ can be computed in polynomial time such that its rank equals 2 (e.g., see [38]). Hence the Schur complement $W^* - z^*(z^*)^T$ must be rank one and one can write

$$W^* = z^*(z^*)^T + w^*(w^*)^T$$

for some $z^* \in \mathbb{R}^n$. Let us choose w^* such that $c^T w^* \le 0$ (otherwise, we choose $-w^*$ as w^*). Note that both z^* and w^* are feasible for (3.4), because both A_1 and A_2 are positive semidefinite. Now we have

$$q(z^*) = Q \bullet z^*(z^*)^T + 2c^T z^*, \quad q(w^*) = Q \bullet w^*(w^*)^T + 2c^T w^*.$$

Adding these two, together with $c^T w^* \le 0$, we have

$$q(z^*) + q(w^*) = Q \bullet \left(z^*(z^*)^T + w^*(w^*)^T\right) + 2c^T(z^* + w^*) \le Q \bullet W^* + 2c^T z^* = q_{\min},$$

which implies

$$\min\{q(z^*), q(w^*)\} \le \frac{1}{2} q_{\min}.$$

That is, either z^* or w^* is a solution of relative approximation bound $\alpha = \frac{1}{2}$ for (3.4). □

Theorem 3.2 establishes an approximation bound for general bi-quadratic form $b(x, y)$. When $b(x, y)$ has special features, better results are possible.

Theorem 3.3. *For bi-quadratic optimization (1.1), we have:*

(i) *If $b(x, y)$ in (1.1) is square-free, then the SDP relaxation $\Omega\left(\frac{1}{n}, \frac{1}{m}\right)$ can be solved in polynomial time and*

$$p_{\min} \le p\left(\frac{1}{n}, \frac{1}{m}\right) \le \frac{1}{nm} p_{\min}.$$

(ii) *The bi-quadratic optimization (1.1) can be solved in polynomial time if $b(x, y)$ has only squared terms in x, or has only squared terms in y.*

Proof. (i) When $b(x, y)$ is square-free, $\bar{p} = 0$ and $\phi(X, Y)$ is homogeneous, so that $p(\frac{1}{n}, \frac{1}{m}) = \frac{1}{nm} p(1, 1)$. Then $p(1, 1) \le p_{\min} \le p(\frac{1}{n}, \frac{1}{m})$ immediately implies the inequalities in (i). On the other hand, when $b(x, y)$ is square-free, we point out that problem (3.3) is polynomial time solvable, since by eliminating variables it can be reduced to maximizing a quadratic form over two homogeneous quadratic inequalities. The latter problem can be solved by the S-lemma; see Ye and Zhang [38].

(ii) Now we consider the special case that $b(x, y)$ in (1.1) has only squared terms in x or has only squared terms in y. Assume the latter case. Then (1.1) has the form

$$\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \sum_{1 \le i,k \le n,\, 1 \le j \le m} b_{ijkj} x_i x_k y_j^2 \ \text{ subject to } \quad \|x\|^2 = 1,\ \|y\|^2 = 1$$

$$= \min_{x \in \mathbb{R}^n,\, \|x\|^2 = 1} \ \min_{1 \le j \le m} \sum_{1 \le i,k \le n} b_{ijkj} x_i x_k = \min_{1 \le j \le m} \ \min_{x \in \mathbb{R}^n,\, \|x\|^2 = 1} \sum_{1 \le i,k \le n} b_{ijkj} x_i x_k$$

$$= \min_{1 \le j \le m} \lambda_{\min}(B_j),$$

where for $j = 1, \cdots, m$, $\lambda_{\min}(B_j)$ is the smallest eigenvalue of the symmetric $n \times n$ matrix $B_j = (b_{ijkj})_{1 \le i,k \le n}$. Since we may find the smallest eigenvalue of a symmetric $n \times n$ matrix in polynomial time, this case can be solved in polynomial time. Similarly one can solve the case that $b_{ijkl} = 0$ whenever $i \ne k$ in polynomial time. $\square$

3.2 A partial PTAS for (1.1) based on sum of squares

Let $B(x)$ be the symmetric matrix in (1.2). Then the original bi-quadratic optimization (1.1) can be equivalently formulated as

$$\begin{array}{rl} p_{\min} := \max & \gamma \\ \text{subject to} & B(x) - \gamma(x^T x) I_m \succeq 0,\ \forall x \in \mathbb{R}^n. \end{array} \tag{3.5}$$

A sequence of SDP relaxations based on sum of squares (SOS) can be applied to solve the problem (3.5). Recently SOS techniques have received much attention in solving nonconvex polynomial optimization problems [9, 17, 21, 28, 25]. Usually a hierarchy of SDP relaxations based on SOS can be applied to obtain a sequence of lower bounds converge to the optimal value of polynomial optimization problems. A general convergence rate was given by Nie and Schweighofer [26].

Let $N \ge 0$ be an integer. Consider the following N-th order SOS relaxation

$$\begin{array}{rl} p_N := \max & \gamma \\ \text{subject to} & (x^T x)^N \big(B(x) - \gamma(x^T x) I_m \big) \text{ is SOS }. \end{array} \tag{3.6}$$

For a symmetric matrix polynomial $F(x)$, we say $F(x)$ is SOS if there exists some matrix polynomial $G(x)$ such that $F(x) = G(x)^T G(x)$. Obviously, for any integer N, p_N is a lower bound of $p_{\min}$. When $N = 0$, the dual of the relaxation (3.6) is the problem (4.2) of the next section. The convergence result is as follows.

Theorem 3.4. *For any $N \geq \frac{3n}{\log 2} - \frac{1}{2}n - 2$, it holds*

$$0 \leq \frac{p_{\min} - p_N}{p_{\max} - p_{\min}} \leq \frac{6n}{(2N + n + 4)\log 2 - 6n},$$

where $p_{\max}$ is the maximum of $b(x, y)$ over the bi-sphere $S_{n,m}$.

SOS methods have been applied to minimize forms (homogeneous scalar polynomials) over unit spheres. Faybusovich [11] proved a quality bound like in Theorem 3.4 for minimizing general even forms over unit spheres, using a result of Reznick [31] on degree bounds of representing positive definite forms by using sum of squares. To prove Theorem 3.4, we need generalize that result of degree bounds to positive definite matrix forms (homogeneous matrix polynomials). That is the following lemma.

Lemma 3.5. *Let $F(x)$ be a homogeneous symmetric matrix polynomial of degree $2d$ such that $F(x) \succ 0$ for any $x \neq 0$. Let*

$$c(F) = \max_{\|\xi\|=1} \frac{\max_{\|x\|=1} \xi^T F(x)\xi}{\min_{\|x\|=1} \xi^T F(x)\xi}.$$

Then for any integer N such that

$$N \geq \frac{nd(2d-1)}{(2\log 2)} c(F) - \frac{n + 2d}{2},$$

the matrix polynomial $(\sum_i x_i^2)^N F(x)$ is SOS.

Proof. We generalize the proof in Section 7 of Reznick [31] for scalar forms to matrix forms. Write $F(x) = \sum_i F_i f_i(x)$ where F_i are matrices and $f_k(x)$ are scalar homogeneous polynomials. Let $G(x) = x_1^2 + \cdots + x_n^2$. For any polynomial $p(x)$, the differential operator $p(\partial)$ is defined by replacing each x_j by $\frac{\partial}{\partial x_j}$, e.g., $G(\partial) = \Delta$ is the Laplacian operator. The matrix differential operator $F(\partial)$ is defined to be $\sum_k F_k f_k(\partial)$. For every polynomial h of degree $2d$, it holds

$$h(\partial)G^N = \Phi_N(h)G^{N-2d}, \quad \text{where} \quad \Phi_N(h) = \sum_{k \geq 0} \frac{(N)_{d-k}}{2^{2k-d} d!} \Delta^k(h) G^k.$$

Here $(N)_t = N(N-1)\cdots(N-(t-1))$. The above two identity implies that

$$h(\partial)G^N = h(\partial)\left(\sum_{k=1}^{N} \lambda_k (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N}\right),$$

$$\Phi_N(h)G^{N-2d} = (2N)_d \sum_{k=1}^{N} \lambda_k h(\alpha_{k1}, \cdots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N-2d}.$$

If we choose $h = \Phi_N^{-1}(f_i)$, then we have

$$f_i(x)G^{N-2d} = (2N)_d \sum_{k=1}^{N} \lambda_k \Phi_N^{-1}(f_i)(\alpha_{k1}, \cdots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N-2d}.$$

Therefore, it holds

$$F(x)G^{N-2d} = (2N)_d \sum_{k=1}^{N} \lambda_k \sum_i F_i \Phi_N^{-1}(f_i)(\alpha_{k1}, \cdots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2N-2d}. \quad (3.7)$$

For any polynomial p, $\Phi_N^{-1}(p)$ has the formula

$$\Phi_N^{-1}(p) = \frac{1}{(N)_d 2^d}\left(p - \frac{\Delta(p)G}{2(n+2N-2)} + \frac{\Delta^2(p)G^2}{8(n+2N-2)(n+2N-4)} - \cdots\right).$$

Hence

$$\sum_i F_i \Phi_N^{-1}(f_i) = \frac{1}{(N)_d 2^d}\left(F(x) - \frac{\Delta(F)G}{2(n+2N-2)} + \frac{\Delta^2(F)G^2}{8(n+2N-2)(n+2N-4)} - \cdots\right).$$

Obviously, it holds

$$\lim_{N\to\infty} (N)_d 2^d \sum_i F_i \Phi_N^{-1}(f_i(x)) = F(x).$$

When $F(x) \succ 0$, we can choose N big enough such that $\sum_i F_i \Phi_N^{-1}(f_i(x)) \succ 0$.

For any vector ξ with $\|\xi\| = 1$, it holds

$$\xi^T \left(\sum_i F_i \Phi_N^{-1}(f_i)\right)\xi = \frac{1}{(N)_d 2^d}\left(\xi^T F\xi - \frac{\Delta(\xi^T F\xi)G}{2(n+2N-2)} + \frac{\Delta^2(\xi^T F\xi)G^2}{8(n+2N-2)(n+2N-4)} - \cdots\right).$$

By the Theorem in Section 7 in [31], when

$$N \geq \frac{nd(2d-1)}{(2\log 2)} \frac{\max_{\|x\|=1} \xi^T F(x)\xi}{\min_{\|x\|=1} \xi^T F(x)\xi} - \frac{n+2d}{2},$$

$\xi^T \left(\sum_i F_i \Phi_N^{-1}(f_i)\right)\xi$ is positive. Choose a uniform N for all $\|\xi\| = 1$. For

$$N \geq \frac{nd(2d-1)}{(2\log 2)} c(F) - \frac{n+2d}{2},$$

we have $\sum_i F_i \Phi_N^{-1}(f_i(x)) \succ 0$. So $(\sum_i x_i^2)^N F(x)$ is SOS by (3.7). $\square$

Proof of Theorem 3.4. Note that we have the inequality

$$p_{\min} I_m \preceq B(x) \preceq p_{\max} I_m, \quad \forall\, x \in \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Let $\gamma < p_{\min}$. Then it holds

$$(p_{\min} - \gamma) I_m \preceq B(x) - \gamma(x^T x) I_m \preceq (p_{\max} - \gamma) I_m, \quad \forall x \in \{x \in \mathbb{R}^n : \|x\| = 1\},$$

and hence

$$c(B(x) - \gamma(x^T x) I_m) \leq \frac{p_{\max} - \gamma}{p_{\min} - \gamma}.$$

Now fix one $N > \frac{3n}{\log 2} - \frac{1}{2}n - 2$, and choose

$$\gamma_N = p_{\min} - \frac{6n(p_{\max} - p_{\min})}{(2N + n + 4)\log 2 - 6n}.$$

Then we can verify that

$$N = \frac{3n}{(\log 2)} \frac{p_{\max} - \gamma_N}{p_{\min} - \gamma_N} - \frac{n+4}{2}.$$

By Lemma 3.5, we know $(x^T x)^N (B(x) - \gamma_N (x^T x) I_m)$ is SOS. By definition of p_N, we know p_N satisfies the inequality claimed by Theorem 3.4. $\square$

Let $C(y)$ be the symmetric quadratic matrix defined in (1.3). Then the equivalent formulation (1.3) of (1.1) can be formulated as

$$\begin{array}{rl} p_{\min} := \max & \gamma \\ \text{subject to} & C(y) - \gamma(y^T y) I_n \succeq 0, \ \forall y \in \mathbb{R}^m. \end{array} \tag{3.8}$$

Similarly, a sequence of convergent SDP relaxations using sum of squares can be applied to solve the problem (3.8), as we have done for (3.5). Let $N \geq 0$ be an integer. The N-th order SOS relaxation for (3.8) is

$$\begin{array}{rl} \tilde{p}_N := \max & \gamma \\ \text{subject to} & (y^T y)^N \big(C(y) - \gamma(y^T y) I_n\big) \text{ is SOS }. \end{array} \tag{3.9}$$

Obviously, for any integer N, $\tilde{p}_N$ is a lower bound of $p_{\min}$. When $N = 0$, the dual of the relaxation (3.9) is also the same as (4.2). A similar convergence result is as follows.

Theorem 3.6. *For any $N \geq \frac{3m}{\log 2} - \frac{1}{2}m - 2$, it holds*

$$0 \leq \frac{p_{\min} - \tilde{p}_N}{p_{\max} - p_{\min}} \leq \frac{6m}{(2N + m + 4)\log 2 - 6m},$$

where $p_{\max}$ is the maximum of $b(x, y)$ over the bi-sphere $S_{n,m}$.

Note that when $\min\{n, m\}$ and N are fixed, either the SOS relaxation (3.6) or (3.9) can be solved in polynomial time. Thus Theorems 3.4 and 3.6 imply the following corollary:

Corollary 3.7. *If $\min\{n, m\}$ is fixed, there exists a PTAS based on SOS relaxations (3.6) or (3.9) for solving (1.1) .*

3.3 Another partial PTAS for (1.1) based on grid sampling on simplex

Now consider the bi-quadratic optimization of the special form

$$\begin{array}{lll} p_{\min} := \min\limits_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} & \sum\limits_{1\le i,k\le n,\, 1\le j,l\le m} b_{ijkl}x_iy_jx_ky_l & \\ \text{subject to} & \|x\| = 1,\ \|y\| = 1, & \\ & y \ge 0. & \end{array} \tag{3.10}$$

The difference of (3.10) from the original bi-quadratic optimization (1.1) is that (3.10) requires $y \ge 0$. In this case, one can choose $y \in \mathbb{R}^m_+$ to be from grid points $\left\{0, \sqrt{\frac{1}{d}}, \cdots, \sqrt{\frac{d-1}{d}}, 1\right\}$ such that $y_1^2 + \cdots + y_m^2 = 1$, for some given integer d. They represent uniform grid points on the partial sphere $\{y \in \mathbb{R}^m_+ : \|y\| = 1\}$. The total number of such feasible grid points is $\binom{m+d-1}{d}$ which is polynomial in m for any fixed integer $d \ge 1$.

For each feasible grid point $\hat{y}$, one can solve the minimum eigenvalue problem

$$p_{\hat{y}} := \min_{x\in\mathbb{R}^n} \sum_{1\le i,k\le n} \sum_{1\le j,l\le m} b_{ijkl}x_i\hat{y}_jx_k\hat{y}_l \quad \text{subject to} \quad \|x\| = 1.$$

The above problem can be solved in polynomial time for each fixed $\hat{y}$. Then, one can choose $\hat{y}$ among these grid points such that $p_{\hat{y}}$ is the smallest, which gives a $(1 - \frac{1}{d})$-approximation solution to (3.10) (see Bomze and de Klerk [1]). Thus we have:

Theorem 3.8. *There is a PTAS for solving problem (3.10).*

Similarly, if in problem (3.10) the constraint $y \in \mathbb{R}^m_+$ is replaced by $x \in \mathbb{R}^n_+$, then a similar PTAS exists. So, for the original bi-quadratic optimization (1.1), if we know in advance the sign of optimal vector x^* or y^*, then the PTAS above can be modified slightly to solve (1.1). For instance, when all the coefficients of the bi-quadratic form are non-positive, the optimal x^* and y^* must be nonnegative, and hence a PTAS exists.

Note that the number of sign patterns for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are at most 2^n and 2^m respectively. If $\min\{n, m\}$ is fixed, then we can get a PTAS for (1.1) by solving subproblems of the form (3.10) at most $2^{\min\{n,m\}}$ times. Hence, this presents a PTAS for solving (1.1) when $\min\{n, m\}$ is fixed.

Corollary 3.9. *If $\min\{m, n\}$ is fixed, there exists a PTAS based on the grid sampling on simplex for solving (1.1).*

4 Some practical semidefinite relaxations

Section 2 proved the NP-hardness of the bi-quadratic optimization (1.1). Section 3 presented some approximation results. The methods provided there are more for the purpose of theoretical analysis and might not be quite efficient for practical problems. In this section, we give some more practical semidefinite relaxation methods. They are based on the first order SOS relaxation and a convex quadratic SDP relaxation.

4.1 First order SOS relaxation and the minimum eigenvalue method

Note that the bi-linear SDP (2.5) can be equivalently formulated as

$$\begin{array}{rl} p_{\min} := \min & \sum\limits_{1\le i,k\le n,\, 1\le j,l\le m} b_{ijkl} X_{ik} Y_{jl} \\ \text{subject to} & \mathrm{Tr}(X\otimes Y) = 1, \\ & X\otimes Y \succeq 0. \end{array} \tag{4.1}$$

Here $\otimes$ denotes the standard Kronecker product. In (4.1), define $m\times m$ matrices $B^{(i,k)} = (b_{ijkl})_{1\le j,l\le m}$. Then we can further relax the above bi-linear SDP (4.1) as the linear SDP

$$\begin{array}{rl} p_{sos} := \min\limits_{Z} & \sum\limits_{1\le i,k\le n} B^{(i,k)} \bullet Z^{(i,k)} \\ \text{subject to} & \sum\limits_{i=1}^{n} \mathrm{Tr}(Z^{(i,i)}) = 1, \\ & Z^{(i,k)} = Z^{(k,i)}, (Z^{(i,k)})^T = Z^{(i,k)}, \forall\,(i,k), \\ & Z := \left(Z^{(i,j)}\right)_{1\le i,j\le n} \succeq 0. \end{array} \tag{4.2}$$

Obviously, the optimal value p_{sos} of (4.2) is a lower bound for the minimum value $p_{\min}$ of (1.1). The dual of the SDP relaxation (4.2) can be shown to have the form

$$\begin{array}{rl} \max\limits_{\gamma,W} & \gamma \\ \text{subject to} & B = W + \gamma I_{nm} \\ & W^{(i,k)} = W^{(k,i)},\ (W^{(i,k)})^T = W^{(i,k)}, \forall\,(i,k), \\ & W := \left(W^{(i,j)}\right)_{1\le i,j\le n} \succeq 0, \end{array} \tag{4.3}$$

where the matrix B is defined as $B = \left(B^{(i,j)}\right)_{1\le i,j\le n}$.

Theorem 4.1. *The relaxation (4.2) has the following properties:*

(i) For any feasible γ in (4.3), the difference $b(x,y) - \gamma x^T x \cdot y^T y$ is a sum of squares, i.e., there exist matrices $A_1, \cdots, A_K \in \mathbb{R}^{n\times m} (K \le nm)$ such that

$$b(x,y) - \gamma \cdot x^T x \cdot y^T y = \sum_{k=1}^{K} (x^T A_k y)^2.$$

In particular, the difference $b(x,y) - p_{sos} \cdot x^T x \cdot y^T y$ is a sum of squares.

(ii) It holds that $\lambda_{\min}(B) \le p_{sos} \le p_{\min}$.

(iii) If $\min\{n,m\} = 2$, then $p_{\min} = p_{sos}$.

Proof. (i) Let (γ, W) be a feasible pair for (4.3). Then we have the relation

$$(x\otimes y)^T B (x\otimes y) = (x\otimes y)^T W (x\otimes y) + \gamma \|(x\otimes y)\|^2.$$

Hence we get the polynomial identity

$$b(x, y) - \gamma \cdot x^T x \cdot y^T y = (x \otimes y)^T W (x \otimes y).$$

Since $W \succeq 0$, there exists a matrix $L \in \mathbb{R}^{nm \times K}$ such that $W = LL^T$. Here K is the rank of W. For every $k = 1, \ldots, K$, let A_k be a matrix such that the vectorization of A_k equals the k-th column of L. Thus the first part of (i) is proved.

Since the feasible set of (4.2) has nonempty interior, the optimal value of the dual (4.3) is attainable and must equal p_{sos}. Hence there exists some W^* such that (p_{sos}, W^*) is feasible for (4.3). So the second part of (i) can be implied by the first part of (i).

(ii) The second inequality is obvious. In SDP relaxation (4.2), if we do not require any off-diagonal block of Z to be symmetric, then it can be further relaxed to

$$\min \quad B \bullet Z \quad \text{subject to} \quad \mathrm{Tr}(Z) = 1, \quad Z \succeq 0. \tag{4.4}$$

The optimal value above is exactly $\lambda_{\min}(B)$. Then we can see $\lambda_{\min}(B) \leq p_{sos}$.

(iii) By definition of $p_{\min}$, we know $b(x, y) - p_{\min} \cdot x^T x \cdot y^T y$ is a nonnegative bi-quadratic form. When $n = 2$ or $m = 2$, Calderón [2] showed that every nonnegative bi-quadratic form $b(x, y)$ must be a sum of squares. So we have $p_{\min} \leq p_{sos}$ by the definition p_{sos}. By (ii), we know $p_{\min} = p_{sos}$. □

From (i) of Theorem 4.1, we can see that the dual problem (4.3) is actually the first one ($N = 0$) in the hierarchy defined in (3.6). Hence $p_{sos} = p_0$. Once the SDP relaxation (4.2) is solved, we get a lower bound p_{sos} and an optimal matrix $Z^* \succeq 0$. When Z^* has rank one, the block-symmetric structures of Z^* imply that there are some vectors $x^* \in \mathbb{R}^n, y^* \in \mathbb{R}^m$ such that $Z^* = (x^* \otimes y^*)(x^* \otimes y^*)^T$, and hence (x^*, y^*) is one global optimizer for (1.1). Now we consider the general case that

$$Z^* = \lambda_1 z^1 (z^1)^T + \cdots + \lambda_r z^r (z^r)^T$$

for orthonormal vectors $z^1, \ldots, z^r$ and scalars $\lambda_1 \geq 0, \ldots, \lambda_r \geq 0$ with $\lambda_1 + \cdots + \lambda_r = 1$. For each z^i, we pack it back into an $m \times n$ matrix $U_i = \mathrm{mat}(z^i)$ by columns, i.e., the m elements in the j-th column of U_i consist of $z^i_{(j-1)m+1}, \cdots, z^i_{jm}$ of z^i. Then find the Singular Value Decomposition (SVD)

$$U_i = \sigma_{i,1} u^{i,1} (v^{i,1})^T + \cdots + \sigma_{i,k_i} u^{i,k_i} (v^{i,k_i})^T.$$

From the set of all pairs $(v^{i,p}, u^{j,q})$ obtained above, choose one pair (x^*, y^*) such that

$$b(x^*, y^*) = \min_{\substack{1 \leq i,j \leq r \\ 1 \leq p \leq k_i,\, 1 \leq q \leq k_j}} b(v^{i,p}, u^{j,q}).$$

The performance of the above pair selection process is as follows:

Theorem 4.2. *Let $\lambda_{\max}(B)$ be the largest eigenvalue of the symmetric matrix B in (4.2). From the first order SOS relaxation, together with the pair selection process described above, one can generate a feasible pair (x^*, y^*) to (1.1) such that*

$$\lambda_{\max}(B) - b(x^*, y^*) \geq \frac{1}{\min\{n, m\}} (\lambda_{\max}(B) - p_{\min}).$$

Proof. From the rank one decomposition of optimal matrix Z^*, there exists one z^i, say z^1, such that

$$(z^1)^T B z^1 \le p_{sos} \quad \text{and} \quad \|z^1\|^2 = 1. \tag{4.5}$$

Note that $z^1(z^1)^T$ may not have the desired block-symmetry any more. We can pack z^1 back into an $m \times n$ matrix $U_1 = \text{mat}(z^1)$ by columns. The rank of U_1 is k_1 ($\le \min\{\text{m}, \text{n}\}$). $\|z^1\| = 1$ implies $\sigma_{1,1}^2 + \cdots + \sigma_{1,k_1}^2 = 1$. Hence, from (4.5), we have

$$\begin{aligned}
&\lambda_{\max}(B) - p_{sos} \\
&\le \text{vec}(U_1)^T \left(\lambda_{\max}(B) I_{mn} - B\right) \text{vec}(U_1) \\
&= \left(\sum_{j=1}^{k_1} \sigma_{1,j} \text{vec}\left(u^{1,j}(v^{1,j})^T\right) \right)^T \left(\lambda_{\max}(B) I_{mn} - B\right) \left(\sum_{j=1}^{k_1} \sigma_{1,j} \text{vec}\left(u^{1,j}(v^{1,j})^T\right) \right) \\
&\le k_1 \cdot \left(\sum_{j=1}^{k_1} \sigma_{1,j}^2 \text{vec}\left(u^{1,j}(v^{1,j})^T\right)^T \left(\lambda_{\max}(B) I_{mn} - B\right) \text{vec}\left(u^{1,j}(v^{1,j})^T\right) \right) \\
&= k_1 \cdot \left(\lambda_{\max}(B) - \sum_{j=1}^{k_1} \sigma_{1,j}^2 b\left(v^{1,j}, u^{1,j}\right) \right),
\end{aligned}$$

where the first inequality comes from (4.5), and the second inequality comes from $\lambda_{\max}(B) I_{nm} - B \succeq 0$. From $\sum_{j=1}^{k_1} \sigma_{1,j}^2 = 1$, we must have one j, say $j = 1$ such that

$$\lambda_{\max}(B) - b\left(v^{1,1}, u^{1,1}\right) \ge \frac{1}{k_1}(\lambda_{\max}(B) - p_{sos}) \ge \frac{1}{\min\{m,n\}}(\lambda_{\max}(B) - p_{sos}),$$

that is, $(v^{1,1}, u^{1,1})$ is an approximate solution to the original problem (1.1) such that

$$\lambda_{\max}(B) - b\left(v^{1,1}, u^{1,1}\right) \ge \frac{\lambda_{\max}(B) - p_{sos}}{\min\{m,n\}} \ge \frac{\lambda_{\max}(B) - p_{\min}}{\min\{m,n\}},$$

where the second inequality comes from $p_{sos} \le p_{\min}$. From the selection of the pair (x^*, y^*), we immediately get the claim of the theorem. □

Note that the approximation result here depends only on $\min\{m, n\}$, which is probably why the first-order SOS relaxation (4.2) is more effective than other SDP relaxation methods like (4.7) in practice.

One may solve the linear SDP (4.2) without the block symmetry constraints, that is, solve (4.4) instead by computing a minimum-eigenvalue eigenvector of B and proceed with the SVD rounding. Then a similar analysis gives the approximation result:

$$\lambda_{\max}(B) - b\left(x^*, y^*\right) \ge \frac{1}{\min\{m,n\}}(\lambda_{\max}(B) - \lambda_{\min}(B)).$$

4.2 A convex quadratic SDP Relaxation

In this subsection, we present another method for estimating the optimal value $p_{\min}$ of (1.1). This method generates a lower bound of $p_{\min}$ from the solution pair $(\bar{X}, \bar{Y})$ of a convex SDP relaxation of (1.1). At the same time, we obtain also an approximate solution of (1.1).

Note that the bi-quadratic optimization (1.1) is equivalent to

$$\begin{array}{ll} \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y + \alpha \{X \bullet X + Y \bullet Y\} \\ \text{subject to} & \mathrm{Tr}(X) = 1, X \succeq 0, \\ & \mathrm{Tr}(Y) = 1, Y \succeq 0, \\ & \mathrm{rank}(X) = 1, \mathrm{rank}(Y) = 1, \end{array} \tag{4.6}$$

for any constant $\alpha > 0$. Thus, we consider the natural quadratic SDP relaxation

$$\begin{array}{ll} p_{csdp}(\alpha) := \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y + \alpha \{X \bullet X + Y \bullet Y\} \\ \text{subject to} & \mathrm{Tr}(X) = 1, X \succeq 0, \\ & \mathrm{Tr}(Y) = 1, Y \succeq 0, \end{array} \tag{4.7}$$

where $\alpha > 0$ is large enough such that (4.7) is convex. Denoted by $\hat{b}(X,Y)$ the objective function in (4.7). In fact, $\hat{b}(X,Y)$ can be written as

$$\hat{b}(X,Y) = \left(\mathrm{vec}(X)^T, \mathrm{vec}(Y)^T\right) (F(\mathcal{A}) + \alpha I) \begin{pmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{pmatrix},$$

where the operator "vec" and $F(\mathcal{A})$ is defined as

$$\mathrm{vec}(X) = \left(X_{11}, \sqrt{2}X_{12}, \cdots, \sqrt{2}X_{1n}, X_{22}, \sqrt{2}X_{23}, \cdots, \sqrt{2}X_{n-1,n}, X_{nn}\right)^T,$$

$$F(\mathcal{A}) = \frac{1}{2} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

Here, A is a $\frac{1}{2}n(n+1) \times \frac{1}{2}m(m+1)$ matrix such that $(\mathcal{A}X) \bullet Y = \mathrm{vec}(X)^T A \mathrm{vec}(Y)$. It is well known that $\hat{b}(X,Y)$ is convex if and only if $F(\mathcal{A}) + \alpha I \succeq 0$, which is equivalent to that $4\alpha^2 I - A^T A \succeq 0$. Therefore, we may choose $\alpha \geq \frac{1}{2}\|A\|_2$ to guarantee the convexity of (4.7), where $\|A\|_2 = (\lambda_{\max}(A^T A))^{1/2}$.

Note that the convex quadratic SDP (4.7) is equivalent to the standard linear SDP

$$\begin{array}{ll} \min\limits_{X,Y,W} & \begin{pmatrix} 0 & 0 \\ 0 & F(\mathcal{A}) + \alpha I \end{pmatrix} \bullet W \\ \text{subject to} & \mathrm{Tr}(X) = 1, \mathrm{Tr}(Y) = 1, \\ & W := \begin{pmatrix} 1 & \mathrm{vec}(X)^T \; \mathrm{vec}(Y)^T \\ \mathrm{vec}(X) & \\ \mathrm{vec}(Y) & Z \end{pmatrix} \succeq 0, \\ & X \succeq 0, Y \succeq 0. \end{array} \tag{4.8}$$

We mention that (4.8) is relatively easier to solve than (4.2), because the numbers of equality constraints in (4.8) and (4.2) are $\mathcal{O}(n^2 + m^2)$ and $\mathcal{O}(n^2 m^2)$ respectively. This is also observed in the numerical results.

Once the convex quadratic SDP (4.7) is solved, we can extract an approximate solution pair $(\bar{x}, \bar{y})$ of (1.1) as follows. Let $(\bar{X}, \bar{Y})$ be an optimal solution pair of (4.7) with α. By eigenvalue decomposition, one knows that

$$\bar{X} = \bar{\lambda}_1 \bar{x}^1 \left(\bar{x}^1\right)^T + \cdots + \bar{\lambda}_r \bar{x}^r \left(\bar{x}^r\right)^T, \quad \bar{Y} = \bar{\mu}_1 \bar{y}^1 \left(\bar{y}^1\right)^T + \cdots + \bar{\mu}_s \bar{y}^s \left(\bar{y}^s\right)^T.$$

Here, $\bar{x}^1, \cdots, \bar{x}^r$ and $\bar{y}^1, \cdots, \bar{y}^s$ are the orthonormal eigenvectors of $\bar{X}$ and $\bar{Y}$ with respect to positive eigenvalues $\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_r > 0$ and $\bar{\mu}_1 \geq \cdots \geq \bar{\mu}_s > 0$, respectively. Let $(\bar{x}, \bar{y})$ be a vector pair satisfying

$$b\left(\bar{x}, \bar{y}\right) = \min\left\{b\left(\bar{x}^i, \bar{y}^j\right) \ : \ 1 \leq i \leq r, 1 \leq j \leq s\right\}.$$

For any $\alpha \geq \frac{1}{2}\|A\|_2$ and $(\bar{x}, \bar{y})$ generated above, $b(\bar{x}, \bar{y})$ is an upper bound for $p_{\min}$. A lower bound for (1.1) is readily given by $p_{csdp} := p_{csdp}(\alpha) - 2\alpha$, since (4.7) is an SDP relaxation of (4.6) which is equivalent to the original problem (1.1) but its optimal value is larger than that of (1.1) by 2α.

The quality of convex SDP relaxation (4.7) and the extraction process described above is given below.

Theorem 4.3. *The approximate solution $(\bar{x}, \bar{y})$ of problem (1.1) generated as above from the optimal solution of the convex quadratic SDP relaxation (4.7) satisfies*

$$b(\bar{x}, \bar{y}) - p_{\min} \ \leq \ \alpha\left(2 - \frac{1}{n} - \frac{1}{m}\right), \tag{4.9}$$

where α is a number satisfying $\alpha \geq \frac{1}{2}\|A\|_2$.

Proof. Since $(\bar{X}, \bar{Y})$ is an optimal solution of (4.7), there exist $\bar{\zeta}, \bar{\eta} \in \mathbb{R}$ such that the following system holds

$$\begin{cases} \mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I \succeq 0, \\ \bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I \succeq 0, \\ (\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I) \bullet \bar{Y} = 0, \\ (\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{X} = 0. \end{cases} \tag{4.10}$$

Since $\mathrm{Tr}(\bar{X}) = 1$ and $\mathrm{Tr}(\bar{Y}) = 1$, from the third and the fourth equations of (4.10), we have

$$\bar{\zeta} = (\mathcal{A}\bar{X}) \bullet \bar{Y} + 2\alpha\bar{Y} \bullet \bar{Y} \quad \text{and} \quad \bar{\eta} = (\bar{Y}\mathcal{A}) \bullet \bar{X} + 2\alpha\bar{X} \bullet \bar{X},$$

which imply that

$$(\bar{\zeta} + \bar{\eta})/2 = p_{csdp}(\alpha). \tag{4.11}$$

Moreover, it is readily to see that

$$(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{x}^1(\bar{x}^1)^T = 0, \ (\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I) \bullet \bar{y}^1(\bar{y}^1)^T = 0.$$

By this, we have

$$\begin{aligned} \sum_{j=1}^{s} \bar{\mu}_j \left(\bar{y}^j(\bar{y}^j)^T\mathcal{A}\right) \bullet \bar{x}^1(\bar{x}^1)^T &= \bar{\eta} - 2\alpha\bar{\lambda}_1, \\ \sum_{i=1}^{r} \bar{\lambda}_i \left(\mathcal{A}\bar{x}^i(\bar{x}^i)^T\right) \bullet \bar{y}^1(\bar{y}^1)^T &= \bar{\zeta} - 2\alpha\bar{\mu}_1. \end{aligned} \tag{4.12}$$

From the definition of $(\bar{x}, \bar{y})$, it is clear that

$$b(\bar{x}, \bar{y}) \le b\left(\bar{x}^i, \bar{y}^1\right) = \left(\mathcal{A}\bar{x}^i(\bar{x}^i)^T\right) \bullet \bar{y}^1(\bar{y}^1)^T, \quad b(\bar{x}, \bar{y}) \le b\left(\bar{x}^1, \bar{y}^j\right) = \left(\bar{y}^j(\bar{y}^j)^T\mathcal{A}\right) \bullet \bar{x}^1(\bar{x}^1)^T,$$

which imply, together with (4.12), that

$$b(\bar{x}, \bar{y}) \le \bar{\zeta} - 2\alpha\bar{\mu}_1 \quad \text{and} \quad b(\bar{x}, \bar{y}) \le \bar{\eta} - 2\alpha\bar{\lambda}_1, \tag{4.13}$$

since $\sum_{i=1}^{r} \bar{\lambda}_i = 1$ and $\sum_{j=1}^{s} \bar{\mu}_j = 1$. By (4.11) and (4.13), we have

$$b(\bar{x}, \bar{y}) \le p_{csdp}(\alpha) - \alpha(\bar{\lambda}_1 + \bar{\mu}_1), \tag{4.14}$$

which implies, together with $p_{\min} \le b(\bar{x}, \bar{y})$ and $p_{csdp}(\alpha) - 2\alpha \le p_{\min}$, that

$$b(\bar{x}, \bar{y}) - p_{\min} \le \alpha\left(2 - \bar{\lambda}_1 - \bar{\mu}_1\right).$$

By this and the fact that $\bar{\lambda}_1 \ge 1/s \ge 1/n$ and $\bar{\mu}_1 \ge 1/r \ge 1/m$, we obtain the desired result and complete the proof. □

We should point out, for the convex quadratic SDP (4.7) to approximate the biquadratic optimization (1.1) efficiently, the constant $\alpha > 0$ in (4.7) cannot be too large. This will be shown in Theorem 4.4. In general, the obtained lower bound for (1.1) by solving (4.7) is better when α near $\frac{1}{2}\|A\|_2$ is chosen.

Theorem 4.4. *Assume that $b(x, y) \ge 0$ for every (x, y), i.e., $\mathcal{A}X \in \mathcal{S}_+^m$ whenever $X \in \mathcal{S}_+^n$ and $Y\mathcal{A} \in \mathcal{S}_+^n$ whenever $Y \in \mathcal{S}_+^m$. If $(\bar{X}, \bar{Y})$ is an optimal solution of (4.7) with*

$$\alpha > \frac{1}{2}\max\{n-1, m-1\}\|\mathcal{A}\|_F, \tag{4.15}$$

then we have

$$\operatorname{rank}(\bar{X}) = n \quad \text{and} \quad \operatorname{rank}(\bar{Y}) = m. \tag{4.16}$$

Proof. Since $(\bar{X}, \bar{Y})$ is an optimal solution of (4.7), there exist $\bar{\zeta}, \bar{\eta} \in \mathbb{R}$ such that

$$\begin{cases} \mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I \succeq 0, \\ \bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I \succeq 0, \\ (\mathcal{A}\bar{X} + 2\alpha\bar{Y} - \bar{\zeta}I) \bullet \bar{Y} = 0, \\ (\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{X} = 0. \end{cases} \tag{4.17}$$

Let $\operatorname{rank}(\bar{X}) = r$ and $\operatorname{rank}(\bar{Y}) = s$. It is clear that $r \ge 1$ and $s \ge 1$ because $\operatorname{Tr}(\bar{X}) = 1$ and $\operatorname{Tr}(\bar{Y}) = 1$, respectively. Moreover, since $\operatorname{Tr}(\bar{X}) = 1$, by Lemma 2.3, there exist $\bar{x}^i \in \mathbb{R}^n$ $(i = 1, \cdots, r)$, such that

$$\bar{X} = \sum_{i=1}^{r} \bar{x}^i(\bar{x}^i)^T, \; I \bullet \bar{x}^i(\bar{x}^i)^T = 1/r, \quad \text{for} \quad i = 1, \cdots, r.$$

Consequently,

$$(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I) \bullet \bar{x}^i(\bar{x}^i)^T = 0, \quad \text{for} \quad i = 1, \cdots, r. \tag{4.18}$$

On the other hand, from the second expression in (4.17), we have that $\mathrm{Tr}(\bar{Y}\mathcal{A}) + 2\alpha\mathrm{Tr}(\bar{X}) - \bar{\eta}\mathrm{Tr}(I) \geq 0$, which implies

$$\bar{\eta}n \leq 2\alpha + \|\mathcal{A}\|_F, \tag{4.19}$$

since $\mathrm{Tr}(\bar{Y}\mathcal{A}) \leq \|\mathcal{A}\|_F\|\bar{Y}\|_F \leq \|\mathcal{A}\|_F$. Moreover, we have that for every k,

$$\begin{aligned}
&\left(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I\right) \bullet \bar{x}^k(\bar{x}^k)^T \\
&\geq \left(2\alpha\sum_{i=1}^{r} \bar{x}^i(\bar{x}^i)^T - \bar{\eta}I\right) \bullet \bar{x}^k(\bar{x}^k)^T \\
&\geq \left(2\alpha\bar{x}^k(\bar{x}^k)^T - \bar{\eta}I\right) \bullet \bar{x}^k(\bar{x}^k)^T \\
&= \frac{1}{r}\left(2\alpha\frac{1}{r} - \bar{\eta}\right),
\end{aligned} \tag{4.20}$$

where the first inequality comes from the assumption that $Y\mathcal{A}$ is positive semidefinite for any $Y \in \mathcal{S}^m_+$, and the second inequality comes from the fact that $xx^T \bullet \tilde{x}\tilde{x}^T \geq 0$ for any $x, \tilde{x} \in \mathbb{R}^n$.

Now we prove the conclusion for $\bar{X}$ by contradiction. Suppose that $\mathrm{rank}(\bar{X}) = r < n$. Then, it is readily to see that $\frac{1}{r} \geq \frac{1}{n-1}$, which implies, together with (4.19), that

$$2\alpha\frac{1}{r} - \bar{\eta} \geq 2\alpha\frac{1}{n-1} - \frac{2\alpha}{n} - \frac{\|\mathcal{A}\|_F}{n} = \frac{1}{n}\left(2\alpha\frac{1}{n-1} - \|\mathcal{A}\|_F\right) > 0 \tag{4.21}$$

where the final inequality comes from (4.15). (4.21) shows, together with (4.20), that for any $i = 1, \cdots, r$,

$$\left(\bar{Y}\mathcal{A} + 2\alpha\bar{X} - \bar{\eta}I\right) \bullet \bar{x}^i(\bar{x}^i)^T > 0,$$

which contradicts (4.18). Therefore, it holds that $\mathrm{rank}(\bar{X}) = n$. The conclusion for $\bar{Y}$ can be proved similarly. The proof of theorem is completed. $\square$

5 Illustrative numerical results

This section reports some numerical results on the computational performances of the first order SOS relaxation (4.2), the convex SDP relaxation (4.7), and the minimum eigenvalue method (4.4). For the first order SOS method, we solve the SDP (4.2) to find a lower bound p_{sos} and an optimal solution Z^*, and then apply the SVD rounding procedure described in front of Theorem 4.2 to get an approximate solution pair (x^*, y^*) of (1.1). For the convex quadratic SDP method, we choose $\alpha = \frac{1}{2}\|A\|_2$ and solve the SDP (4.8) to get the optimal solution pair $(\bar{X}, \bar{Y})$. Then, follow the rounding procedure described in front of Theorem 4.3 to get an approximate solution pair $(\bar{x}, \bar{y})$ of (1.1) and a lower bound $p_{csdp} := p_{csdp}(\alpha) - 2\alpha$. For the minimum eigenvalue method, we first compute the minimal eigenvalue $\lambda_{\min}(B)$ and the corresponding eigenvector $\hat{z}$ by solving (4.4). Then we apply the same SVD rounding decomposition on the matrix $\hat{U} = \mathrm{mat}(\hat{z})$ to obtain an approximate solution $(\hat{x}, \hat{y})$ of (1.1).

All the numerical computations here were done by using a Intel Core 2 Duo 2.4GHz computer with 2GB of RAM, and all the SDP problems were solved by the SDP software SDPA-M (Version 6.2.0) [12].

Example 5.1. *Consider the bi-quadratic optimization*

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^3, y\in\mathbb{R}^3} & x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 + 2(x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2) \\ & -2x_1x_2y_1y_2 - 2x_1x_3y_1y_3 - 2x_2x_3y_2y_3 \\ \text{subject to} & \|x\|^2 = 1,\ \|y\|^2 = 1. \end{array}$$

First, we use the first order SOS relaxation (4.2) to find a lower bound of $p_{\min}$ *and then extract an approximate solution for it. It can be shown [4] that* $p_{\min} = 0$ *and the objective bi-quadratic form is not SOS. From the given fourth order tensor* $\mathcal{A}$*, it can be verified that the coefficient matrix* B *in (4.2) has* $\lambda_{\max}(B) = 2.118$ *and* $\lambda_{\min}(B) = -0.118$*. By solving (4.2), we get* $p_{sos} = -0.0972$*. It is clear that* $\lambda_{\min}(B) < p_{sos} < p_{\min}$*. Now, we extract an approximate solution of the original problem from* Z^* *by applying the SVD rounding procedure, and get* $x^* = (-1, 0, 0)^T$ *and* $y^* = (0, 0, -1)^T$*. Note that* $b(x^*, y^*) = 0$ *attains the exact minimum objective value.*

Second, we use the convex quadratic SDP relaxation (4.6) to solve the problem. Choose $\alpha = \frac{1}{2}\|A\|_2 = 1.5$*. It is not difficult to obtain the optimal value* $p_{csdp}(\alpha) = 2$ *of (4.6) and an optimal matrix pair*

$$\bar{X} = \bar{Y} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Hence, we obtain a lower bound -1 *for the minimum* $p_{\min}$*. Moreover, after rounding we obtain an approximation solution pair* $\bar{x} = (-1, 0, 0)^T$ *and* $\bar{y} = (0, 0, 1)^T$*, which also attains the exact minimum objective value.*

Furthermore, based upon the $\lambda_{\min}(B)$ *and its eigenvector* $\hat{z}$*, we extract the same exact solutions* $\hat{x} = (-1, 0, 0)^T$ *and* $\hat{y} = (0, 0, -1)^T$*.*

Example 5.2. *Consider the bi-quadratic optimization*

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^6, y\in\mathbb{R}^6} & \sum\limits_{i=1}^{5} x_ix_{i+1}y_iy_{i+1} \\ \text{subject to} & \|x\|^2 = 1,\ \|y\|^2 = 1. \end{array}$$

First, we use the first order SDP relaxation (4.2) to solve the problem. It can be verified that $\lambda_{\min}(B) = -0.4505$ *and* $\lambda_{\max}(B) = 0.4505$*. We obtain* $p_{sos} = -0.25$ *and a corresponding optimal* Z^**. Then, by applying the SVD rounding procedure, we extract an approximate solution from* Z^*

$$x^* = (0, 0, 0, 0, -0.7066, -0.7076)^T, \quad y^* = (0, 0, 0, 0.0001, -0.7077, 0.7065)^T$$

such that $b(x^*, y^*) = -0.2500$.

Second, we use convex quadratic SDP relaxation to solve the problem. Choosing $\alpha = \frac{1}{2}\|A\|_2 = 1/4$, *we obtain a lower bound* $p_{csdp} = -0.4167$ *and optimal matrices*

$$\bar{X} = \bar{Y} = \begin{pmatrix} 0.1667 & -0.0016 & 0 & 0 & 0 & 0 \\ -0.0016 & 0.1667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1667 \end{pmatrix}.$$

Hence, we obtain a lower bound -0.4167 *for the minimum* $p_{\min}$. *From the rounding procedure, we obtain an approximate solution with objective value* -0.2500 *as follows*

$$\bar{x} = (-0.7071, -0.7071, 0, 0, 0, 0)^T, \quad \bar{y} = (0.7071, -0.7071, 0, 0, 0, 0)^T.$$

Third, from the eigenvector $\hat{z}$ *corresponding to* $\lambda_{\min}(B)$, *we extract an approximate solution*

$$\hat{x} = (0, 0, 1, 0, 0, 0)^T, \quad \hat{y} = (0, 0, 1, 0, 0, 0)^T$$

such that $b(\hat{x}, \hat{y}) = 0$, *which does not attain the minimum objective value.*

Example 5.3. *Consider bi-quadratic optimization*

$$\begin{array}{cl} \min\limits_{x\in\mathbb{R}^9, y\in\mathbb{R}^{12}} & \sum\limits_{1\le i,k\le 9,\, 1\le j,l\le 12} x_i y_j x_k y_l \\ \text{subject to} & \|x\|^2 = 1,\ \|y\|^2 = 1. \end{array}$$

It can be verified that $\lambda_{\min}(B) = 0$ *and* $\lambda_{\max}(B) = 108$. *By solving the first order SOS relaxation (4.2), we obtain* $p_{sos} = 0$ *and extract a pair* (x^*, y^*) *with objective value* 0*:*

$$\begin{array}{rcl} x^* & = & (0.5144, 0.1874, 0.6634, 0.4207, 0.1573, \\ & & -0.1194, 0.1873, -0.0378, 0.0877)^T, \\ y^* & = & (-0.2207, 0.1225, -0.4439, 0.3975, 0.3158, -0.0189, \\ & & -0.5694, 0.0357, 0.3487, -0.1163, 0.0055, 0.1434)^T. \end{array}$$

For the convex SDP method, choosing $\alpha = \frac{1}{2}\|A\|_2 = 54$ *and solving the SDP (4.8), we get a lower bound* -97.0909 *and extract a pair* $(\bar{x}, \bar{y})$ *with objective value* 0*:*

$$\begin{array}{rcl} \bar{x} & = & (-0.8445, -0.0163, 0.1397, 0.1302, 0.0452, \\ & & -0.0028, 0.4257, 0.0198, -0.2576)^T, \\ \bar{y} & = & (0.0464, 0.0617, 0.4152, -0.0305, -0.1712, 0.0110, \\ & & 0.0655, -0.0409, 0.0778, 0.5078, -0.6675, -0.2754)^T. \end{array}$$

Dim	first order SOS (4.2)			convex quadratic SDP (4.7)			minimum Eig. M. (4.4)		
	Low.B.	$b(x^*, y^*)$	Cpu	Low.B	$b(\bar{x}, \bar{y})$	Cpu	Low.B	$b(\hat{x}, \hat{y})$	Cpu
$(6, 7)$	-260.31	-260.31	0.39	-417.54	-257.50	0.12	-311.88	-251.17	0.01
$(5, 8)$	-119.14	-119.14	0.21	-246.91	-45.71	0.09	-146.28	-116.42	0.01
$(7, 8)$	-268.10	-268.10	1.03	-337.78	-263.78	0.28	-309.76	-262.41	0.03
$(7, 9)$	-565.19	-565.19	1.79	-587.20	-564.91	0.43	-605.64	-563.64	0.04
$(8, 9)$	-526.71	-526.71	3.45	-593.00	-525.31	0.57	-591.49	-521.98	0.04
(9, 9)	-609.39	-609.39	6.45	-783.48	-602.14	0.84	-695.60	-597.92	0.06
(10, 10)	-752.19	-752.19	18.81	-1003.22	-739.71	1.59	-880.63	-738.41	0.10
(11, 11)	-362.66	-362.66	54.46	-980.97	-342.77	2.84	-444.13	-334.49	0.17
(12, 12)	-499.41	-499.41	142.18	-982.55	-483.11	4.29	-623.55	-474.42	0.31
(13, 13)	–	–	–	-491.36	-7.06	5.15	-35.48	-14.12	0.48
(14, 14)	–	–	–	-509.56	-66.72	7.59	-82.96	-73.14	0.67
(20, 20)	–	–	–	-1360.31	-220.52	75.23	-250.54	-231.30	5.43
(50, 50)	–	–	–	–	–	–	-9.34	-4.76	2.14
(100, 100)	–	–	–	–	–	–	-9.28	-8.88	28.54
(150, 150)	–	–	–	–	–	–	-13.26	-11.30	190.45
(200, 300)	–	–	–	–	–	–	-8.17	-6.46	1256.63
(300, 300)	–	–	–	–	–	–	-8.41	-6.32	1678.65
(300, 600)	–	–	–	–	–	–	-8.20	-7.06	17826.25

Table 1: Computational results for random examples

For the minimum eigenvalue method, we also extract an approximate solution with objective value 0*:*

$$
\begin{aligned}
\hat{x} &= (0.0972, 0.2778, -0.3277, -0.1575, -0.6329 \\
&\quad -0.1641, -0.0369, 0.5925, 0.0366)^T, \\
\hat{y} &= (-0.3893, 0.2134, 0.1229, 0.0646, 0.3544, -0.3069 \\
&\quad -0.0483, 0.0915, -0.3173, 0.6016, -0.2807, -0.1088)^T.
\end{aligned}
$$

Finally we test some dense and sparse random examples for relatively larger dimension (n, m). The coefficients of the bi-quadratic form $b(x, y)$ in (1.1) are generated randomly by normal distribution. For (n, m) with $(6, 7) - (20, 20)$, the coefficients of the bi-quadratic form $b(x, y)$ are dense, while for (n, m) beyond 50, they are sparse. Again, the first order SOS relaxation (4.2), the convex quadratic SDP relaxation (4.7) and the minimum eigenvalue method (4.4) are applied to solving these randomly generated bi-quadratic optimization problems. The computational results are summarized in Table 1, where "Dim" stands for the dimension pair (n, m), "Low.B." denotes the computed lower bound p_{sos}, p_{csdp} or $\lambda_{\min}(B)$, and "Cpu" the consumed CPU time in seconds.

From Table 1, we see that the first order SOS relaxation (4.2) provides a better lower bound than both the convex quadratic SDP relaxation (4.7) and the minimum eigenvalue method (4.4), while the latter two consume less CPU time, especially for large-scale problems. This is because (4.2) has $\mathcal{O}(m^2n^2)$ equality constraints, (4.7) has only $\mathcal{O}(m^2 + n^2)$ equality constraints, and (4.4) is just a problem of finding the minimum eigenvalue and the corresponding eigenvector of B. For $(n, m) = (13, 13)$, $(14, 14)$ and $(20, 20)$, we obtain a lower bound and an approximate solution $(\bar{x}, \bar{y})$ from solving (4.7). For $(n, m) = (50, 50)$ and beyond, we are only able to obtain the eigenvector $\hat{z}$ corresponding to $\lambda_{\min}(B)$ and an approximate solution $(\hat{x}, \hat{y})$ from solving (4.4), due to the memory limit when solving the SDP problems. It seems that there is a trade-off on choosing among the relaxation methods: the first order SOS relaxation (4.2), the convex quadratic SDP relaxation (4.7), and the minimum eigenvalue method (4.4).

6 Conclusion and open problems

This paper discusses minimizing bi-quadratic forms over unit spheres. We proved this problem is NP-hard. Based on semidefinite programming relaxation, we developed several approximation algorithms with guaranteed approximation bounds. When $\min\{m, n\}$ is a constant, we established two PTASs for (1.1). We also proposed three practical computational methods: first order SOS relaxation, the minimum eigenvalue method and convex quadratic SDP relaxation. Preliminary computational results indicate that they are all promising. It seems that the minimum eigenvalue method with the SVD rounding procedure is the most time efficient and still generates good quality solutions.

Theorem 4.1 (iii) shows that when $\min\{m, n\} = 2$, (1.1) is polynomial time solvable. When $\min\{m, n\}$ is a constant bigger than 2, is (1.1) still polynomial time solvable? Is there a PTAS for solving (1.1) for general bi-quadratic form $b(x, y)$? Does (1.1) have a PTAS when $b(x, y)$ is restricted to be square-free? In Theorem 3.2, can we improve the approximation bound to $\mathcal{O}(\frac{1}{mn})$? To the best knowledge of the authors, all such questions are open.

One natural generalization of bi-quadratic optimization (1.1) is

$$\begin{array}{cl} \min & b(x, y) \\ \text{subject to} & x^T A_i x \leq 1, \; i = 1, \ldots, m_1, \\ & y^T B_j y \leq 1, \; j = 1, \ldots, m_2. \end{array} \tag{6.1}$$

Here $b(x, y)$ is still a bi-quadratic form and A_i, B_j are constant symmetric matrices. We can see that (1.1) is a special case of (6.1). Hence problem (6.1) is also NP-hard. Are our approximation results in Section 3 applicable to approximating (6.1)?

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