BINOMIAL EXPANSIONS

EXAM QUESTIONS

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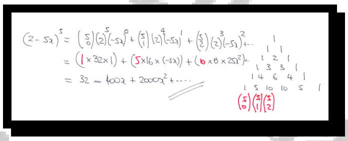
Question 1 **(**)**

Find, without any calculating aid, the first three terms in the expansion of $(2-5x)^5$, in ascending powers of x.

$\boxed{}$, $\boxed{32-400x+2000x^2-...}$



$(2-5x)^5 = \binom{5}{0}(2)^5(-5x)^0 + \binom{5}{1}(2)^4(-5x)^1 + \binom{5}{2}(2)^3(-5x)^2 + \dots$

$= (1\times32\times1) + (5\times16\times(-5x)) + (10\times8\times25x^2) + \dots$

$= 32 - 400x + 2000x^2 + \dots$

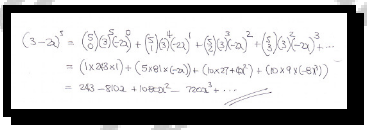
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\binom{5}{0}\binom{5}{1}\binom{5}{2}$

Question 2 **(**)**

Expand $(3-2x)^5$ in ascending powers of x, up and including the term in x^3.

$\boxed{}$, $\boxed{243-810x+1080x^2-720x^3+...}$



$(3-2x)^5 = \binom{5}{0}(3)^5(-2x)^0 + \binom{5}{1}(3)^4(-2x)^1 + \binom{5}{2}(3)^3(-2x)^2 + \binom{5}{3}(3)^2(-2x)^3 + \dots$

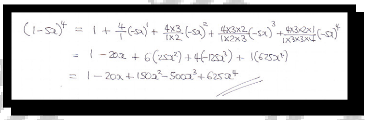
$= (1\times243\times1) + (5\times81\times(-2x)) + (10\times27+4x^2) + (10\times9\times(-8x^3))$

$= 243 - 810x + 1080x^2 - 720x^3 + \dots$

Question 3 **(**)**

Find the binomial expansion of $(1-5x)^4$ in ascending powers of x.

$\boxed{}$, $\boxed{1-20x+150x^2-500x^3+625x^4}$



$(1-5x)^4 = 1 + \frac{4}{1}(-5x)^1 + \frac{4\times3}{1\times2}(-5x)^2 + \frac{4\times3\times2}{1\times2\times3}(-5x)^3 + \frac{4\times3\times2\times1}{1\times3\times3\times4}(-5x)^4$

$= 1 - 20x + 6(25x^2) + 4(-125x^3) + 1(625x^4)$

$= 1 - 20x + 150x^2 - 500x^3 + 625x^4$

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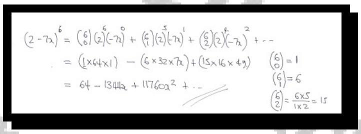
Question 4 **(**)**

Find, without any calculating aid, the first three terms in the expansion of $(2-7x)^6$, in ascending powers of x.

$$\boxed{64-1344x+11760x^2+\dots}$$



$$(2-7x)^6 = \binom{6}{0}(2)^6(-7x)^0 + \binom{6}{1}(2)^5(-7x)^1 + \binom{6}{2}(2)^4(-7x)^2 + \dots$$
$$= (1\times 64\times 1) - (6\times 32\times 7x) + (15\times 16\times 49)$$
$$= 64 - 1344x + 11760x^2 + \dots$$

$\binom{6}{0}=1$

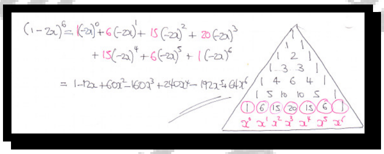
$\binom{6}{1}=6$

$\binom{6}{2}=\frac{6\times 5}{1\times 2}=15$

Question 5 **(**)**

Find the binomial expansion of $(1-2x)^6$ in ascending powers of x.

$$\boxed{1-12x+60x^2-160x^3+240x^4-192x^5+64x^6}$$



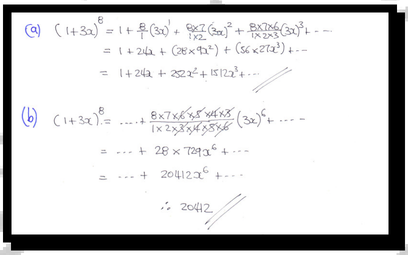
$$(1-2x)^6 = 1(-2x)^0 + 6(-2x)^1 + 15(-2x)^2 + 20(-2x)^3$$
$$+ 15(-2x)^4 + 6(-2x)^5 + 1(-2x)^6$$
$$= 1-12x+60x^2-160x^3+240x^4-192x^5+64x^6$$

Question 6 **(**)**

a) Find the first four terms, in ascending powers of x, in the binomial expansion of $(1+3x)^8$.

b) Determine the coefficient of x^6 in the binomial expansion of $(1+3x)^8$.

$\boxed{}$, $\boxed{1+24x+252x^2+1512x^3+\dots}$, $\boxed{20412}$



(a) $$(1+3x)^8 = 1 + \frac{8}{1}(3x)^1 + \frac{8\times 7}{1\times 2}(3x)^2 + \frac{8\times 7\times 6}{1\times 2\times 3}(3x)^3 + \dots$$
$$= 1 + 24x + (28\times 9x^2) + (56\times 27x^3) + \dots$$
$$= 1 + 24x + 252x^2 + 1512x^3 + \dots$$

(b) $$(1+3x)^8 = \dots + \frac{8\times 7\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}}{1\times 2\times \cancel{3}\times \cancel{4}\times \cancel{5}\times \cancel{6}}(3x)^6 + \dots$$
$$= \dots + 28\times 729x^6 + \dots$$
$$= \dots + 20412x^6 + \dots$$
$$\therefore 20412$$

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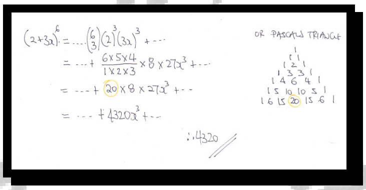
Question 7 **(**+)**

Find, without using a calculator, the coefficient of x^3 in the expansion of $(2+3x)^6$.

$\boxed{4320}$



$(2+3x)^6 = \ldots \binom{6}{3}(2)^3(3x)^3 + \ldots$

$= \ldots + \frac{6\times5\times4}{1\times2\times3}\times8\times27x^3 + \ldots$

$= \ldots + 20\times8\times27x^3 + \ldots$

$= \ldots + 4320x^3 + \ldots$

$\therefore 4320$

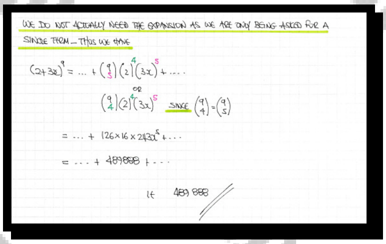
OR PASCALS TRIANGLE

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Question 8 **(**+)**

Find the coefficient of x^5 in the binomial expansion of $(2+3x)^9$.

$\boxed{}$, $\boxed{489888}$



WE DO NOT ACTUALLY NEED THE EXPANSION AS WE ARE ONLY BEING ASKED FOR A SINGLE TERM – THUS WE HAVE

$(2+3x)^9 = \ldots + \binom{9}{5}(2)^4(3x)^5 + \ldots$

OR

$\binom{9}{4}(2)^4(3x)^5$ SINCE $\binom{9}{4} = \binom{9}{5}$

$= \ldots + 126\times16\times243x^5 + \ldots$

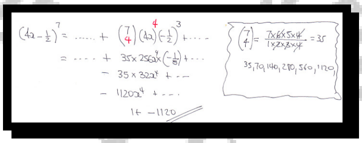
$= \ldots + 489888 + \ldots$

It 489 888

Question 9 **(**+)**

Find, without using a calculator, the coefficient of x^4 in the expansion of $\left(4x-\frac{1}{2}\right)^7$.

$\boxed{-1120}$



$\left(4x-\frac{1}{2}\right)^7 = \ldots\ldots + \binom{7}{4}(4x)^4\left(-\frac{1}{2}\right)^3 + \ldots$

$= \ldots + 35\times256x^4\left(-\frac{1}{8}\right) + \ldots$

$- 35\times32x^4 + \ldots$

$- 1120x^4 + \ldots$

It -1120

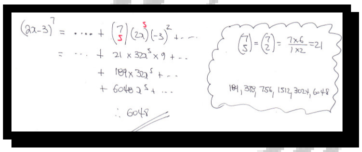
$\binom{7}{4} = \frac{7\times\cancel{6}\times5\times\cancel{4}}{1\times\cancel{2}\times\cancel{3}\times\cancel{4}} = 35$

35, 70, 140, 280, 560, 1120,

Question 10 **(**+)**

Find, without using a calculator, the coefficient of x^5 in the expansion of $(2x-3)^7$.

$\boxed{6048}$



$(2x-3)^7 = \dots + \binom{7}{5}(2x)^5(-3)^2 + \dots$

$= \dots + 21 \times 32x^5 \times 9 + \dots$

$+ 189 \times 32x^5 + \dots$

$+ 6048x^5 + \dots$

$\therefore 6048$

$\binom{7}{5} = \binom{7}{2} = \frac{7\times 6}{1\times 2} = 21$

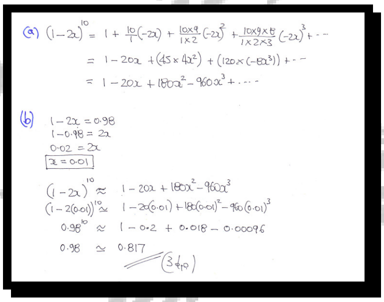
189, 378, 756, 1512, 3024, 6048

Question 11 **(**+)**

a) Find the first four terms, in ascending powers of x, in the binomial expansion of $(1-2x)^{10}$.

b) Use the answer of part (a) with a suitable value of x to find an approximate value for 0.98^{10}, giving the answer correct to three decimal places.

$\boxed{}$, $\boxed{1-20x+180x^2-960x^3+\dots}$, $\boxed{\approx 0.817}$



(a) $(1-2x)^{10} = 1 + \frac{10}{1}(-2x) + \frac{10\times 9}{1\times 2}(-2x)^2 + \frac{10\times 9\times 8}{1\times 2\times 3}(-2x)^3 + \dots$

$= 1 - 20x + (45 \times 4x^2) + (120 \times (-8x^3)) + \dots$

$= 1 - 20x + 180x^2 - 960x^3 + \dots$

(b) $1-2x = 0.98$

$1 - 0.98 = 2x$

$0.02 = 2x$

$\boxed{x = 0.01}$

$(1-2x)^{10} \approx 1 - 20x + 180x^2 - 960x^3$

$(1-2(0.01))^{10} \simeq 1 - 20(0.01) + 180(0.01)^2 - 960(0.01)^3$

$0.98^{10} \approx 1 - 0.2 + 0.018 - 0.00096$

$0.98 \simeq 0.817$ (3 d.p.)

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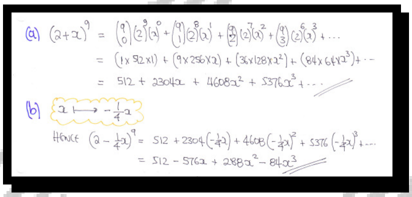
Question 12 **(**+)**

a) Find, in ascending powers of x, the first four terms in the binomial expansion of $(2+x)^9$.

b) By using the answer of part **(a)**, or otherwise, find the first four terms in the binomial expansion of $\left(2-\frac{1}{4}x\right)^9$.

$\boxed{}$, $\boxed{(2+x)^9 = 512 + 2304x + 4608x^2 + 5376x^3 + \ldots}$,

$\boxed{\left(2-\frac{1}{4}x\right)^9 = 512 - 576x + 288x^2 - 84x^3 + \ldots}$



(a) $(2+x)^9 = \binom{9}{0}(2)^9(x)^0 + \binom{9}{1}(2)^8(x)^1 + \binom{9}{2}(2)^7(x)^2 + \binom{9}{3}(2)^6(x)^3 + \ldots$

$= (1 \times 512 \times 1) + (9 \times 256 \times x) + (36 \times 128 \times x^2) + (84 \times 64 \times x^3) + \ldots$

$= 512 + 2304x + 4608x^2 + 5376x^3 + \ldots$

(b) $x \longmapsto -\frac{1}{4}x$

HENCE $\left(2-\frac{1}{4}x\right)^9 = 512 + 2304\left(-\frac{1}{4}x\right) + 4608\left(-\frac{1}{4}x\right)^2 + 5376\left(-\frac{1}{4}x\right)^3 + \ldots$

$= 512 - 576x + 288x^2 - 84x^3$

Question 13 **(**+)**

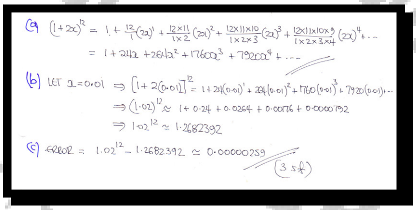
a) Find the first **five** terms, in ascending powers of x, in the binomial expansion of $(1+2x)^{12}$.

b) Use the answer of part **(a)** with a suitable value of x to find an approximate value for 1.02^{12}.

c) Determine the error in this approximation.

$\boxed{}$, $\boxed{1 + 24x + 264x^2 + 1760x^3 + 7920x^4 + \ldots}$, $\boxed{1.02^{12} \approx 1.2682392}$,

$\boxed{\text{error} \approx 0.00000259}$



(a) $(1+2x)^{12} = 1 + \frac{12}{1}(2x)^1 + \frac{12 \times 11}{1 \times 2}(2x)^2 + \frac{12 \times 11 \times 10}{1 \times 2 \times 3}(2x)^3 + \frac{12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4}(2x)^4 + \ldots$

$= 1 + 24x + 264x^2 + 1760x^3 + 7920x^4 + \ldots$

(b) LET $x = 0.01$ $\Rightarrow [1+2(0.01)]^{12} = 1 + 24(0.01)^1 + 264(0.01)^2 + 1760(0.01)^3 + 7920(0.01) + \ldots$

$\Rightarrow (1.02)^{12} \approx 1 + 0.24 + 0.0264 + 0.00176 + 0.0000792$

$\Rightarrow 1.02^{12} \simeq 1.2682392$

(c) ERROR $= 1.02^{12} - 1.2682392 \simeq 0.00000259$

(3 s.f.)

Question 14 (+)**

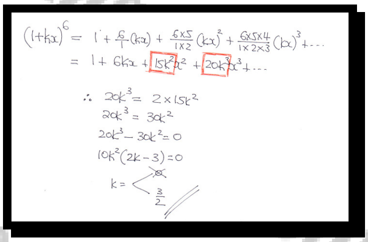
In the binomial expansion of

$$(1+kx)^6,$$

where k is constant, the coefficient of x^3 is twice as large as the coefficient of x^2.

Find the value of k.

$\boxed{}$, $\boxed{k=\frac{3}{2}}$



$$(1+kx)^6 = 1+\frac{6}{1}(kx)+\frac{6\times5}{1\times2}(kx)^2+\frac{6\times5\times4}{1\times2\times3}(kx)^3+\dots$$

$$= 1+6kx+\boxed{15k^2}x^2+\boxed{20k^3}x^3+\dots$$

$$\therefore\ 20k^3 = 2\times15k^2$$

$$20k^3 = 30k^2$$

$$20k^3-30k^2=0$$

$$10k^2(2k-3)=0$$

$$k = \left\langle \begin{array}{l} \cancel{0} \\ \frac{3}{2} \end{array}\right.$$

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Question 15 (***)

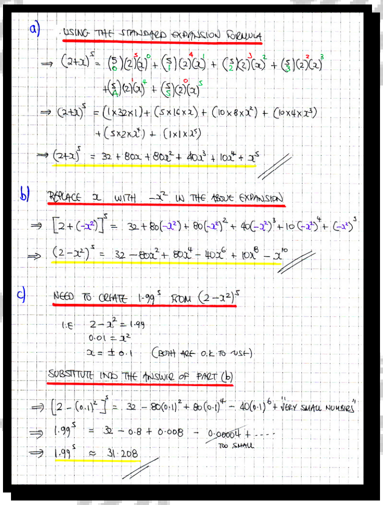
a) Find, in ascending powers of x, the binomial expansion of $(2+x)^5$.

b) By using the expression obtained in part **(a)**, or otherwise, find the binomial expansion of $\left(2-x^2\right)^5$.

c) Use the expression obtained in part **(b)** to estimate, correct to 3 decimal places, the value of 1.99^5.

$\boxed{(2+x)^5 = 32+80x+80x^2+40x^3+10x^4+x^5}$,

$\boxed{\left(2-x^2\right)^5 = 32-80x^2+80x^4-40x^6+10x^8-x^{10}}$, $\boxed{1.99^5 \approx 31.208}$



a) USING THE STANDARD EXPANSION FORMULA

$\Rightarrow (2+x)^5 = \binom{5}{0}(2)^5(x)^0 + \binom{5}{1}(2)^4(x)^1 + \binom{5}{2}(2)^3(x)^2 + \binom{5}{3}(2)^2(x)^3 + \binom{5}{4}(2)^1(x)^4 + \binom{5}{5}(2)^0(x)^5$

$\Rightarrow (2+x)^5 = (1\times32\times1)+(5\times16\times x)+(10\times8\times x^2)+(10\times4\times x^3) + (5\times2\times x^4)+(1\times1\times x^5)$

$\Rightarrow (2+x)^5 = 32+80x+80x^2+40x^3+10x^4+x^5$ //

b) REPLACE x WITH $-x^2$ IN THE ABOVE EXPANSION

$\Rightarrow \left[2+(-x^2)\right]^5 = 32+80(-x^2)+80(-x^2)^2+40(-x^2)^3+10(-x^2)^4+(-x^2)^5$

$\Rightarrow (2-x^2)^5 = 32-80x^2+80x^4-40x^6+10x^8-x^{10}$ //

c) NEED TO CREATE 1.99^5 FROM $(2-x^2)^5$

I.E $2-x^2=1.99$

$0.01=x^2$

$x=\pm0.1$ (BOTH ARE O.K TO USE)

SUBSTITUTE INTO THE ANSWER OF PART (b)

$\Rightarrow \left[2-(0.1)^2\right]^5 = 32-80(0.1)^2+80(0.1)^4-40(0.1)^6+$ "VERY SMALL NUMBERS"

$\Rightarrow 1.99^5 = 32-0.8+0.008-\underset{\text{TOO SMALL}}{\cancel{0.00004}}+\dots$

$\Rightarrow 1.99^5 \approx 31.208$ //

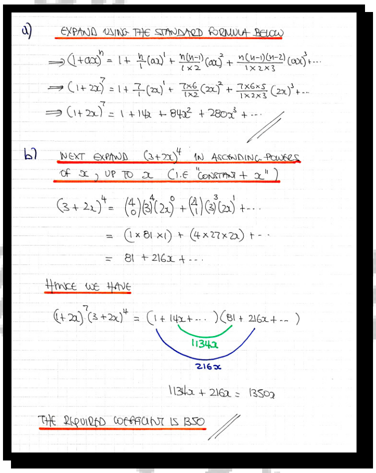
Question 16 (***)

a) Find the first four terms, in ascending powers of x, of the binomial expansion of $(1+2x)^7$.

b) Hence determine the coefficient of x in the expansion of

$$(1+2x)^7(3+2x)^4.$$

$\boxed{}$, $\boxed{1+14x+84x^2+280x^3+\ldots}$, $\boxed{1350}$



a) EXPAND USING THE STANDARD FORMULA BELOW

$$\Rightarrow (1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1\times2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times2\times3}(ax)^3 + \ldots$$

$$\Rightarrow (1+2x)^7 = 1 + \frac{7}{1}(2x)^1 + \frac{7\times6}{1\times2}(2x)^2 + \frac{7\times6\times5}{1\times2\times3}(2x)^3 + \ldots$$

$$\Rightarrow (1+2x)^7 = 1 + 14x + 84x^2 + 280x^3 + \ldots$$

b) NEXT EXPAND $(3+2x)^4$ IN ASCENDING POWERS OF x, UP TO x (I.E "CONSTANT + x")

$$(3+2x)^4 = \binom{4}{0}(3)^4(2x)^0 + \binom{4}{1}(3)^3(2x)^1 + \ldots$$

$$= (1\times81\times1) + (4\times27\times2x) + \ldots$$

$$= 81 + 216x + \ldots$$

HENCE WE HAVE

$$(1+2x)^7(3+2x)^4 = (1+14x+\ldots)(81+216x+\ldots)$$

1134x

216x

$$1134x + 216x = 1350x$$

THE REQUIRED COEFFICIENT IS 1350

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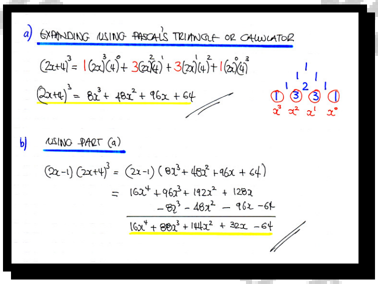
Question 17 (***)

a) Find the binomial expansion of $(2x+4)^3$, in descending powers of x.

b) Hence determine the expansion of

$$(2x-1)(2x+4)^3.$$

$\boxed{\text{MP1}}$, $\boxed{8x^3+48x^2+96x+64}$, $\boxed{16x^4+88x^3+144x^2+32x-64}$



a) EXPANDING USING PASCAL'S TRIANGLE OR CALCULATOR

$(2x+4)^3 = 1(2x)^3(4)^0 + 3(2x)^2(4)^1 + 3(2x)^1(4)^2 + 1(2x)^0(4)^3$

$(2x+4)^3 = 8x^3 + 48x^2 + 96x + 64$

$$\begin{matrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ ① & & ③ & & ③ & & ① \\ x^3 & & x^2 & & x^1 & & x^0 \end{matrix}$$

b) USING PART (a)

$(2x-1)(2x+4)^3 = (2x-1)(8x^3+48x^2+96x+64)$

$= 16x^4 + 96x^3 + 192x^2 + 128x$

$\quad - 8x^3 - 48x^2 - 96x - 64$

$16x^4 + 88x^3 + 144x^2 + 32x - 64$

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Question 18 (***)

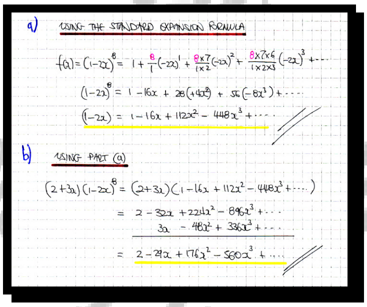
$$f(x) = (1-2x)^8$$

a) Find the first four terms in the expansion of $f(x)$, in ascending powers of x.

b) Hence determine, in ascending powers of x, the first four terms in the expansion of

$$(2+3x)(1-2x)^8 .$$

$\boxed{}$, $\boxed{1-16x+112x^2-448x^3+\dots}$, $\boxed{2-29x+176x^2-560x^3+\dots}$



a) USING THE STANDARD EXPANSION FORMULA

$f(x) = (1-2x)^8 = 1 + \frac{8}{1}(-2x)^1 + \frac{8\times7}{1\times2}(-2x)^2 + \frac{8\times7\times6}{1\times2\times3}(-2x)^3 + \dots$

$(1-2x)^8 = 1 - 16x + 28(+4x^2) + 56(-8x^3) + \dots$

$(1-2x) = 1 - 16x + 112x^2 - 448x^3 + \dots$

b) USING PART (a)

$(2+3x)(1-2x)^8 = (2+3x)(1-16x+112x^2-448x^3+\dots)$

$= 2 - 32x + 224x^2 - 896x^3 + \dots$

$\quad 3x - 48x^2 + 336x^3 + \dots$

$= 2 - 29x + 176x^2 - 560x^3 + \dots$

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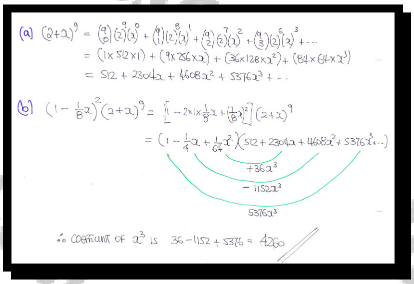
Question 19 (***)

a) Find, in ascending powers of x, the first four terms in the binomial expansion of $(2+x)^9$.

b) Hence find the coefficient of x^3 in the expansion of

$$\left(1-\tfrac{1}{8}x\right)^2(2+x)^9 .$$

$\boxed{\quad}$, $\boxed{(2+x)^9 = 512 + 2304x + 4608x^2 + 5376x^3 + \ldots}$, $\boxed{\left[x^3\right] = 4260}$



(a) $(2+x)^9 = \binom{9}{0}(2)^9(x)^0 + \binom{9}{1}(2)^8(x)^1 + \binom{9}{2}(2)^7(x)^2 + \binom{9}{3}(2)^6(x)^3 + \ldots$

$= (1\times512\times1) + (9\times256\times x) + (36\times128\times x^2) + (84\times64\times x^3)$

$= 512 + 2304x + 4608x^2 + 5376x^3 + \ldots$

(b) $\left(1-\frac{1}{8}x\right)^2(2+x)^9 = \left[1-2\times1\times\frac{1}{8}x+\left(\frac{1}{8}x\right)^2\right](2+x)^9$

$= \left(1-\frac{1}{4}x+\frac{1}{64}x^2\right)\left(512+2304x+4608x^2+5376x^3+\ldots\right)$

$+36x^3$

$-1152x^3$

$5376x^3$

∴ COEFFICIENT OF x^3 IS $36-1152+5376 = 4260$

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Question 20 **(***)**

$$f(x) = (2+x)^4$$

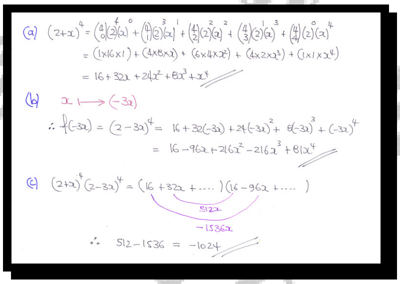
a) Find the expansion of $f(x)$, in ascending powers of x.

b) Deduce the expansion of $(2-3x)^4$, also in ascending powers of x.

c) Determine the coefficient of x in the expansion of

$$(2+x)^4(2-3x)^4.$$

$\boxed{f(x) = 16 + 32x + 24x^2 + 8x^3 + x^4}$, $\boxed{16 - 96x + 216x^2 - 216x^3 + 81x^4}$, $\boxed{-1024}$



(a) $(2+x)^4 = \binom{4}{0}(2)^4(x)^0 + \binom{4}{1}(2)^3(x)^1 + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4$

$= (1\times16\times1) + (4\times8\times x) + (6\times4\times x^2) + (4\times2\times x^3) + (1\times1\times x^4)$

$= 16 + 32x + 24x^2 + 8x^3 + x^4$

(b) $x \longmapsto (-3x)$

$\therefore f(-3x) = (2-3x)^4 = 16 + 32(-3x) + 24(-3x)^2 + 8(-3x)^3 + (-3x)^4$

$= 16 - 96x + 216x^2 - 216x^3 + 81x^4$

(c) $(2+x)^4(2-3x)^4 = (16 + 32x + \dots)(16 - 96x + \dots)$

$512x$

$-1536x$

$\therefore\ 512 - 1536 = -1024$

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Question 21 (***)

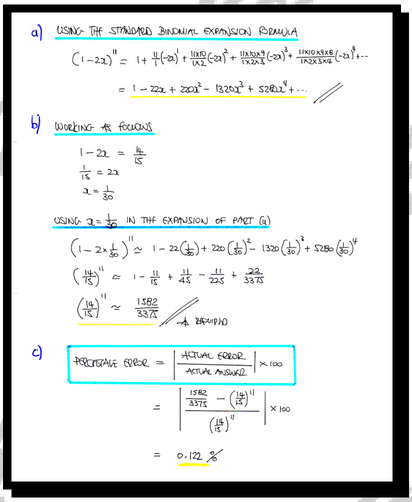
a) Find the first **five** terms, in ascending powers of x, in the binomial expansion of $(1-2x)^{11}$.

b) Use the answer of part **(a)** with a suitable value of x to show that

$$\left(\frac{14}{15}\right)^{11} \approx \frac{1582}{3375}.$$

c) Determine the percentage error in the approximation of part **(b)**.

$\boxed{\quad\quad}$, $\boxed{1-22x+220x^2-1320x^3+5280x^4+\ldots}$, $\boxed{\%\text{ error} \approx 0.122\%}$



a) USING THE STANDARD BINOMIAL EXPANSION FORMULA

$$(1-2x)^{11} = 1+\frac{11}{1}(-2x)^1+\frac{11\times10}{1\times2}(-2x)^2+\frac{11\times10\times9}{1\times2\times3}(-2x)^3+\frac{11\times10\times9\times8}{1\times2\times3\times4}(-2x)^4+\ldots$$

$$= 1-22x+220x^2-1320x^3+5280x^4+\ldots \quad //$$

b) WORKING AS FOLLOWS

$$1-2x = \frac{14}{15}$$

$$\frac{1}{15} = 2x$$

$$x = \frac{1}{30}$$

USING $x=\frac{1}{30}$ IN THE EXPANSION OF PART (a)

$$\left(1-2\times\frac{1}{30}\right)^{11} \simeq 1-22\left(\frac{1}{30}\right)+220\left(\frac{1}{30}\right)^2-1320\left(\frac{1}{30}\right)^3+5280\left(\frac{1}{30}\right)^4$$

$$\left(\frac{14}{15}\right)^{11} \simeq 1-\frac{11}{15}+\frac{11}{45}-\frac{11}{225}+\frac{22}{3375}$$

$$\left(\frac{14}{15}\right)^{11} \simeq \frac{1582}{3375} \quad //$$

AS REQUIRED

c)

$$\boxed{\text{PERCENTAGE ERROR} = \left|\frac{\text{ACTUAL ERROR}}{\text{ACTUAL ANSWER}}\right| \times 100}$$

$$= \left|\frac{\frac{1582}{3375}-\left(\frac{14}{15}\right)^{11}}{\left(\frac{14}{15}\right)^{11}}\right| \times 100$$

$$= 0.122\ \%$$

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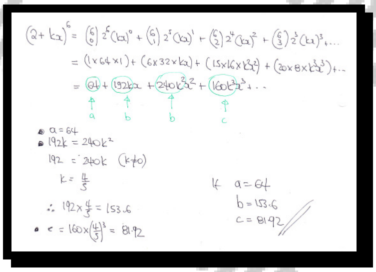
Question 22 **(***+)**

It is given that if k is a non zero constant, then

$$(2+kx)^6 \equiv a+bx+bx^2+cx^3+\ldots$$

Determine the value of each of the constants a, b and c.

$\boxed{C2E}$, $\boxed{a=64}$, $\boxed{b=153.6}$, $\boxed{c=81.92}$



$(2+kx)^6 = \binom{6}{0}2^6(kx)^0 + \binom{6}{1}2^5(kx)^1 + \binom{6}{2}2^4(kx)^2 + \binom{6}{3}2^3(kx)^3 + \ldots$

$= (1\times64\times1) + (6\times32\times kx) + (15\times16\times k^2x^2) + (20\times8\times k^3x^3) + \ldots$

$= \underset{\overset{\uparrow}{a}}{\boxed{64}} + \underset{\overset{\uparrow}{b}}{\boxed{192k}}x + \underset{\overset{\uparrow}{b}}{\boxed{240k^2}}x^2 + \underset{\overset{\uparrow}{c}}{\boxed{160k^3}}x^3 + \ldots$

- $a = 64$
- $192k = 240k^2$

$192 = 240k \quad (k\neq0)$

$k = \frac{4}{5}$

$\therefore 192\times\frac{4}{5} = 153.6$

- $c = 160\times\left(\frac{4}{5}\right)^3 = 81.92$

If $a = 64$

$b = 153.6$

$c = 81.92$

Question 23 **(***+)**

In the binomial expansion of

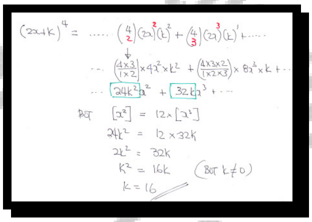
$$(2x+k)^4,$$

where k is a non zero constant,

the coefficient of x^2 is 12 times as large as the coefficient of x^3.

Find the value of k.

$\boxed{C2Y}$, $\boxed{k=16}$



$(2x+k)^4 = \ldots\ldots \binom{4}{2}(2x)^2(k)^2 + \binom{4}{3}(2x)^3(k)^1 + \ldots\ldots$

$\downarrow$

$\ldots \left(\frac{4\times3}{1\times2}\right)\times4x^2\times k^2 + \left(\frac{4\times3\times2}{1\times2\times3}\right)\times8x^3\times k + \ldots$

$\ldots \boxed{24k^2}x^2 + \boxed{32k}x^3 + \ldots$

BUT $[x^2] = 12\times[x^3]$

$24k^2 = 12\times32k$

$2k^2 = 32k$

$k^2 = 16k \quad$ (BUT $k\neq0$)

$k = 16$

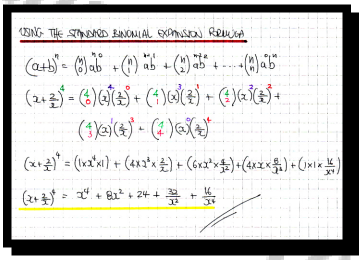
Question 24 (***+)

Find the binomial expansion of

$$\left(x+\frac{2}{x}\right)^4,\ x \neq 0,$$

simplifying each term of the expansion.

$$\boxed{},\ \boxed{x^4+8x^2+24+\frac{32}{x^2}+\frac{16}{x^4}}$$



USING THE STANDARD BINOMIAL EXPANSION FORMULA

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n}a^0 b^n$$

$$\left(x+\frac{2}{x}\right)^4 = \binom{4}{0}(x)^4\left(\frac{2}{x}\right)^0 + \binom{4}{1}(x)^3\left(\frac{2}{x}\right)^1 + \binom{4}{2}(x)^2\left(\frac{2}{x}\right)^2 + \binom{4}{3}(x)^1\left(\frac{2}{x}\right)^3 + \binom{4}{4}(x)^0\left(\frac{2}{x}\right)^4$$

$$\left(x+\frac{2}{x}\right)^4 = (1\times x^4\times 1) + \left(4\times x^3\times\frac{2}{x}\right) + \left(6\times x^2\times\frac{4}{x^2}\right) + \left(4\times x\times\frac{8}{x^3}\right) + \left(1\times 1\times\frac{16}{x^4}\right)$$

$$\left(x+\frac{2}{x}\right)^4 = x^4 + 8x^2 + 24 + \frac{32}{x^2} + \frac{16}{x^4}$$

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Question 25 **(***+)**

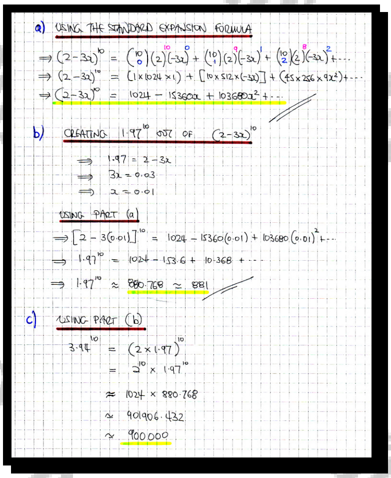
a) Determine, in ascending powers of x, the first three terms in the binomial expansion of $(2-3x)^{10}$.

b) Use the first three terms in the binomial expansion of $(2-3x)^{10}$, with a suitable value for x, to find an approximation for 1.97^{10} .

c) Use the answer of part **(b)** to estimate, correct to 2 significant figures, the value of 3.94^{10}.

$\boxed{\text{MP1-F}}$, $\boxed{(2-3x)^{10}=1024-15360x+103680x^2+\ldots}$, $\boxed{1.97^{10}\approx 880.768\approx 881}$,

$\boxed{3.94^{10}\approx 900000}$



a) USING THE STANDARD EXPANSION FORMULA

$$\Rightarrow (2-3x)^{10} = \binom{10}{0}(2)^{10}(-3x)^{0} + \binom{10}{1}(2)^{9}(-3x)^{1} + \binom{10}{2}(2)^{8}(-3x)^{2}+\ldots$$

$$\Rightarrow (2-3x)^{10} = (1\times 1024\times 1) + [10\times 512\times(-3x)] + (45\times 256\times 9x^2)+\ldots$$

$$\Rightarrow (2-3x)^{10} = 1024 - 15360x + 103680x^2+\ldots$$

b) CREATING 1.97^{10} OUT OF $(2-3x)^{10}$

$$\Rightarrow 1.97 = 2-3x$$

$$\Rightarrow 3x = 0.03$$

$$\Rightarrow x = 0.01$$

USING PART (a)

$$\Rightarrow [2-3(0.01)]^{10} = 1024 - 15360(0.01) + 103680(0.01)^2+\ldots$$

$$\Rightarrow 1.97^{10} = 1024 - 153.6 + 10.368 +\ldots$$

$$\Rightarrow 1.97^{10} \approx 880.768 \simeq 881$$

c) USING PART (b)

$$3.94^{10} = (2\times 1.97)^{10}$$

$$= 2^{10}\times 1.97^{10}$$

$$\approx 1024\times 880.768$$

$$\simeq 901906.432$$

$$\simeq 900000$$

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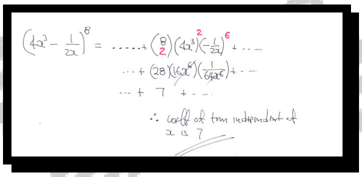
Question 26 **(***+)**

Find the term which is independent of x, in the binomial expansion of

$$\left(4x^3 - \frac{1}{2x}\right)^8, \ x \neq 0 .$$

$\boxed{7}$



$\left(4x^3 - \frac{1}{2x}\right)^8 = \ldots\ldots + \binom{8}{2}\left(4x^3\right)^2\left(-\frac{1}{2x}\right)^6 + \ldots$

$\ldots + (28)\left(16x^6\right)\left(\frac{1}{64x^6}\right) + \ldots$

$\ldots + 7 + \ldots$

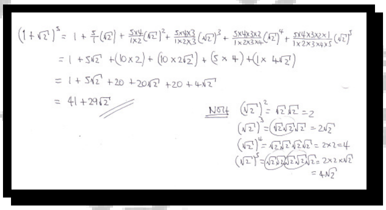
$\therefore$ coeff of term independent of x is 7

Question 27 **(***+)**

$$\left(1+\sqrt{2}\right)^5 \equiv a + b\sqrt{2} .$$

Determine the value of each of the constants a and b.

$\boxed{a = 41, \ b = 29}$



$\left(1+\sqrt{2}\right)^5 = 1 + \frac{5}{1}\left(\sqrt{2}\right) + \frac{5\times4}{1\times2}\left(\sqrt{2}\right)^2 + \frac{5\times4\times3}{1\times2\times3}\left(\sqrt{2}\right)^3 + \frac{5\times4\times3\times2}{1\times2\times3\times4}\left(\sqrt{2}\right)^4 + \frac{5\times4\times3\times2\times1}{1\times2\times3\times4\times5}\left(\sqrt{2}\right)^5$

$= 1 + 5\sqrt{2} + (10\times2) + (10\times2\sqrt{2}) + (5\times4) + (1\times4\sqrt{2})$

$= 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2}$

$= 41 + 29\sqrt{2}$

NOTE $\left(\sqrt{2}\right)^2 = \sqrt{2}\sqrt{2} = 2$

$\left(\sqrt{2}\right)^3 = \sqrt{2}\sqrt{2}\sqrt{2} = 2\sqrt{2}$

$\left(\sqrt{2}\right)^4 = \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} = 2\times2 = 4$

$\left(\sqrt{2}\right)^5 = \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} = 2\times2\times\sqrt{2}$

$= 4\sqrt{2}$

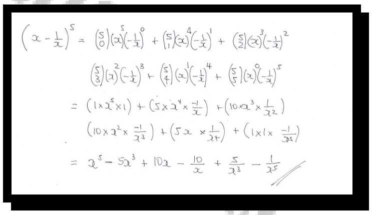
Question 28 **(***+)**

Find the binomial expansion of

$$\left(x-\frac{1}{x}\right)^5,\ x\neq 0,$$

simplifying each term of the expansion.

$$\boxed{x^5-5x^3+10x-\frac{10}{x}+\frac{5}{x^3}-\frac{1}{x^5}}$$



$$\left(x-\frac{1}{x}\right)^5 = \binom{5}{0}(x)^5\left(-\frac{1}{x}\right)^0 + \binom{5}{1}(x)^4\left(-\frac{1}{x}\right)^1 + \binom{5}{2}(x)^3\left(-\frac{1}{x}\right)^2$$

$$\binom{5}{3}(x)^2\left(-\frac{1}{x}\right)^3 + \binom{5}{4}(x)^1\left(-\frac{1}{x}\right)^4 + \binom{5}{5}(x)^0\left(-\frac{1}{x}\right)^5$$

$$= \left(1\times x^5\times 1\right) + \left(5\times x^4\times \frac{-1}{x}\right) + \left(10\times x^3\times \frac{1}{x^2}\right)$$

$$\left(10\times x^2\times \frac{-1}{x^3}\right) + \left(5x \times \frac{1}{x^4}\right) + \left(1\times 1\times \frac{-1}{x^5}\right)$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

Question 29 **(***+)**

In the binomial expansion of

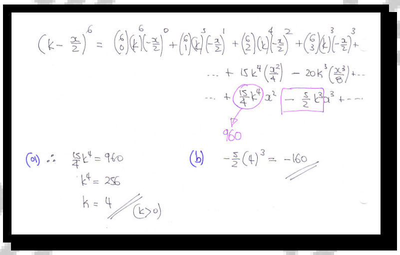
$$\left(k-\frac{x}{2}\right)^6,$$

where k is a positive constant, one of the terms is $960x^2$.

a) Find the value of k.

b) Determine the coefficient of x^3.

$\boxed{k=4}$, $\boxed{-160}$



$$\left(k-\frac{x}{2}\right)^6 = \binom{6}{0}(k)^6\left(-\frac{x}{2}\right)^0 + \binom{6}{1}(k)^5\left(-\frac{x}{2}\right)^1 + \binom{6}{2}(k)^4\left(-\frac{x}{2}\right)^2 + \binom{6}{3}(k)^3\left(-\frac{x}{2}\right)^3 + $$

$$\dots + 15k^4\left(\frac{x^2}{4}\right) - 20k^3\left(\frac{x^3}{8}\right) + \dots$$

$$\dots + \underbrace{\frac{15}{4}k^4}_{960}x^2 \boxed{-\frac{5}{2}k^3}x^3 + \dots$$

(a) $\therefore\ \frac{15}{4}k^4 = 960$

$k^4 = 256$

$k = 4$ $(k>0)$

(b) $-\frac{5}{2}(4)^3 = -160$

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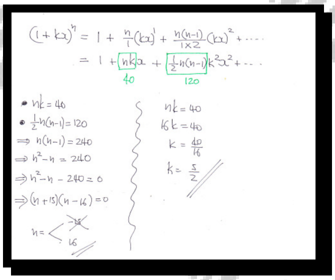
Question 30 **(***+)**

Given that k is a non zero constant and n is a positive integer, then

$$(1+kx)^n \equiv 1+40x+120k^2x^2+\dots$$

Find the value of k and the value of n.

$\boxed{}$, $\boxed{k=\frac{5}{2}}$, $\boxed{n=16}$



$$(1+kx)^n = 1+\frac{n}{1}(kx)^1+\frac{n(n-1)}{1\times 2}(kx)^2+\dots$$

$$= 1+\underbrace{\boxed{nk}}_{40}x+\underbrace{\boxed{\tfrac{1}{2}n(n-1)}}_{120}k^2x^2+\dots$$

- $nk=40$
- $\frac{1}{2}n(n-1)=120$

$\Rightarrow n(n-1)=240$

$\Rightarrow n^2-n=240$

$\Rightarrow n^2-n-240=0$

$\Rightarrow (n+15)(n-16)=0$

$n = \cancel{-15}$, 16 //

$nk=40$

$16k=40$

$k=\frac{40}{16}$

$k=\frac{5}{2}$ //

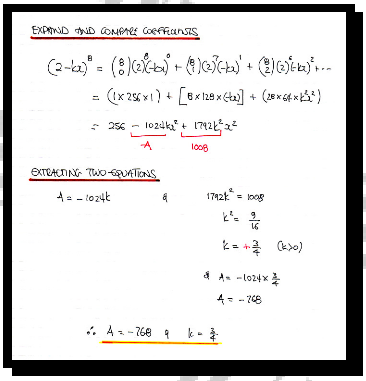
Question 31 **(***+)**

Given that k and A are constants with $k>0$, then

$$(2-kx)^8 \equiv 256+Ax+1008x^2+\dots$$

Find the value of k and the value of A.

$\boxed{\text{SYN-I}}$, $\boxed{k=\frac{3}{4}}$, $\boxed{A=-768}$



EXPAND AND COMPARE COEFFICIENTS

$$(2-kx)^8 = \binom{8}{0}(2)^8(-kx)^0+\binom{8}{1}(2)^7(-kx)^1+\binom{8}{2}(2)^6(-kx)^2+\dots$$

$$= (1\times256\times1)+[8\times128\times(-kx)]+(28\times64\times k^2x^2)$$

$$= 256\underbrace{-1024kx^2}_{A}+\underbrace{1792k^2x^2}_{1008}$$

EXTRACTING TWO-EQUATIONS

$A=-1024k$ &

$1792k^2=1008$

$k^2=\frac{9}{16}$

$k=+\frac{3}{4}$ $(k>0)$

& $A=-1024\times\frac{3}{4}$

$A=-768$

$\therefore A=-768$ & $k=\frac{3}{4}$

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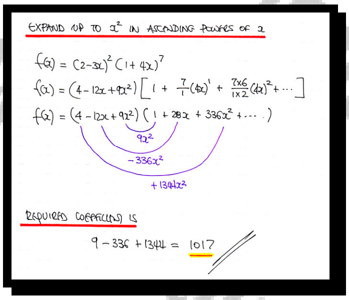
Question 32 **(***+)**

$$f(x) = (2-3x)^2(1+4x)^7.$$

Find the coefficient of x^2 in the polynomial expansion of $f(x)$.

$\boxed{\text{SYN-C}}$, $\boxed{1017}$



EXPAND UP TO x^2 IN ASCENDING POWERS OF x

$f(x) = (2-3x)^2(1+4x)^7$

$f(x) = (4-12x+9x^2)\left[1+\frac{7}{1}(4x)^1+\frac{7\times6}{1\times2}(4x)^2+\dots\right]$

$f(x) = (4-12x+9x^2)(1+28x+336x^2+\dots)$

$9x^2$

$-336x^2$

$+1344x^2$

REQUIRED COEFFICIENT IS

$9-336+1344 = 1017$

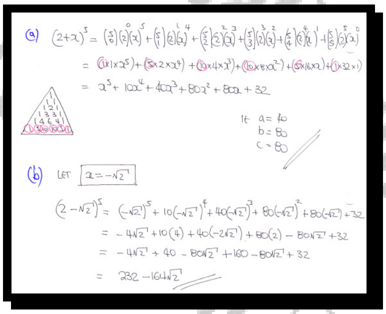
Question 33 **(***+)**

$$(2+x)^5 = x^5+10x^4+ax^3+bx^2+cx+32.$$

a) Find the value of each of the constants a, b and c.

b) Hence, or otherwise, simplify $\left(2-\sqrt{2}\right)^5$, giving the final answer in the form $p+q\sqrt{2}$, where p and q are constants.

$\boxed{a=40,\ b=80,\ c=80}$, $\boxed{232-164\sqrt{2}}$



(a) $(2+x)^5 = \binom{5}{0}(2)^0(x)^5+\binom{5}{1}(2)^1(x)^4+\binom{5}{2}(2)^2(x)^3+\binom{5}{3}(2)^3(x)^2+\binom{5}{4}(2)^4(x)^1+\binom{5}{5}(2)^5(x)^0$

$= (1\times1\times x^5)+(5\times2\times x^4)+(10\times4\times x^3)+(10\times8\times x^2)+(5\times16\times x)+(1\times32\times1)$

$= x^5+10x^4+40x^3+80x^2+80x+32$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

∴ $a=40$
$b=80$
$c=80$

(b) LET $\boxed{x=-\sqrt{2}}$

$(2-\sqrt{2})^5 = (-\sqrt{2})^5+10(-\sqrt{2})^4+40(-\sqrt{2})^3+80(-\sqrt{2})^2+80(-\sqrt{2})+32$

$= -4\sqrt{2}+10(4)+40(-2\sqrt{2})+80(2)-80\sqrt{2}+32$

$= -4\sqrt{2}+40-80\sqrt{2}+160-80\sqrt{2}+32$

$= 232-164\sqrt{2}$

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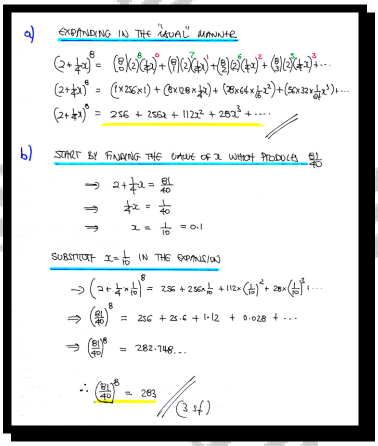
Question 34 (***+)

$$f(x) = \left(2 + \tfrac{1}{4}x\right)^8 .$$

a) Find the first four terms in ascending powers of x in the expansion of $f(x)$.

b) Use the expansion found in part (a) to find an approximation, to 3 significant figures, for $\left(\frac{81}{40}\right)^8$.

$\boxed{f(x) = 256 + 256x + 112x^2 + 28x^3 + \ldots}$, $\boxed{\left(\frac{81}{40}\right)^8 \approx 283}$



a) EXPANDING IN THE "USUAL" MANNER

$(2+\frac{1}{4}x)^8 = \binom{8}{0}(2)^8(\frac{1}{4}x)^0 + \binom{8}{1}(2)^7(\frac{1}{4}x)^1 + \binom{8}{2}(2)^6(\frac{1}{4}x)^2 + \binom{8}{3}(2)^5(\frac{1}{4}x)^3 + \ldots$

$(2+\frac{1}{4}x)^8 = (1\times256\times1) + (8\times128\times\frac{1}{4}x) + (28\times64\times\frac{1}{16}x^2) + (56\times32\times\frac{1}{64}x^3) + \ldots$

$(2+\frac{1}{4}x)^8 = 256 + 256x + 112x^2 + 28x^3 + \ldots$

b) START BY FINDING THE VALUE OF x WHICH PRODUCES $\frac{81}{40}$

$\Rightarrow 2+\frac{1}{4}x = \frac{81}{40}$

$\Rightarrow \frac{1}{4}x = \frac{1}{40}$

$\Rightarrow x = \frac{1}{10} = 0.1$

SUBSTITUTE $x=\frac{1}{10}$ IN THE EXPANSION

$\Rightarrow \left(2+\frac{1}{4}\times\frac{1}{10}\right)^8 = 256 + 256\times\frac{1}{10} + 112\times\left(\frac{1}{10}\right)^2 + 28\times\left(\frac{1}{10}\right)^3 + \ldots$

$\Rightarrow \left(\frac{81}{40}\right)^8 = 256 + 25.6 + 1.12 + 0.028 + \ldots$

$\Rightarrow \left(\frac{81}{40}\right)^8 = 282.748\ldots$

$\therefore \left(\frac{81}{40}\right)^8 = 283$ (3 sf)

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Question 35 (***+)

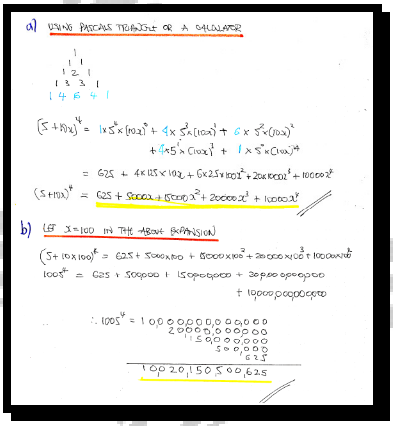
a) Find the binomial expansion of

$$(5+10x)^4 .$$

b) Hence, by using the answer of part (a) with a suitable value of x find the exact value of 1005^4 .

You may not give the answer in standard index form.

$\boxed{\quad\quad}$, $\boxed{625+5000x+15000x^2+20000x^3+10000x^4}$, $\boxed{1,020,150,500,625}$



a) USING PASCALS TRIANGLE OR A CALCULATOR

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$(5+10x)^4 = 1\times5^4\times(10x)^0 + 4\times5^3\times(10x)^1 + 6\times5^2\times(10x)^2 + 4\times5^1\times(10x)^3 + 1\times5^0\times(10x)^4$

$= 625 + 4\times125\times10x + 6\times25\times100x^2 + 20\times1000x^3 + 10000x^4$

$(5+10x)^4 = 625 + 5000x + 15000x^2 + 20000x^3 + 10000x^4$ //

b) LET $x=100$ IN THE ABOVE EXPANSION

$(5+10\times100)^4 = 625 + 5000\times100 + 15000\times100^2 + 20000\times100^3 + 10000\times100^4$

$1005^4 = 625 + 500,000 + 150,000,000 + 20,000,000,000 + 1,000,000,000,000$

$\therefore 1005^4 =$
1,000,000,000,000
20,000,000,000
150,000,000
500,000
625

1,020,150,500,625 //

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Question 36 (***+)

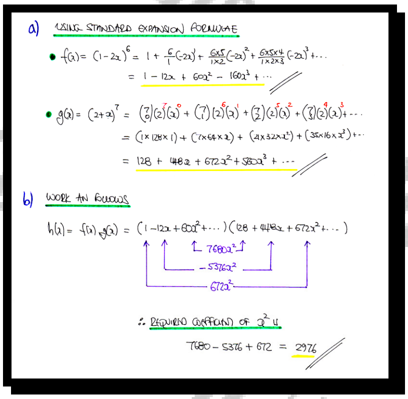
$$f(x)=(1-2x)^6, \qquad g(x)=(2+x)^7, \qquad h(x)=f(x)\,g(x).$$

a) Find the first four terms in ascending powers of x in the binomial expansion of $f(x)$ and in the binomial expansion of $g(x)$.

b) Hence determine the coefficient of x^2 in the binomial expansion of $h(x)$.

$\boxed{}$, $\boxed{f(x)=1-12x+60x^2-160x^3+\dots}$,

$\boxed{g(x)=128+448x+672x^2+560x^3+\dots}$, $\boxed{2976}$



a) USING STANDARD EXPANSION FORMULAE

- $f(x) = (1-2x)^6 = 1 + \frac{6}{1}(-2x)^1 + \frac{6\times5}{1\times2}(-2x)^2 + \frac{6\times5\times4}{1\times2\times3}(-2x)^3 + \dots$

  $= 1 - 12x + 60x^2 - 160x^3 + \dots$

- $g(x) = (2+x)^7 = \binom{7}{0}(2)^7(x)^0 + \binom{7}{1}(2)^6(x)^1 + \binom{7}{2}(2)^5(x)^2 + \binom{7}{3}(2)^4(x)^3 + \dots$

  $= (1\times128\times1) + (7\times64\times x) + (21\times32\times x^2) + (35\times16\times x^3) + \dots$

  $= 128 + 448x + 672x^2 + 560x^3 + \dots$

b) WORK AN BELOWS

$h(x) = f(x)\,g(x) = (1 - 12x + 60x^2 + \dots)(128 + 448x + 672x^2 + \dots)$

$7680x^2$

$-5376x^2$

$672x^2$

$\therefore$ REQUIRED COEFFICIENT OF x^2 IS

$7680 - 5376 + 672 = 2976$

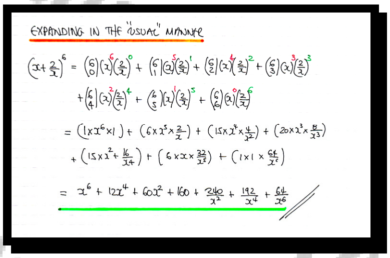
Question 37 **(***+)**

Find the binomial expansion of

$$\left(x+\frac{2}{x}\right)^6,\ x\neq 0,$$

simplifying each term of the expansion.

☐ , $\boxed{x^6+12x^4+60x^2+160+\frac{240}{x^2}+\frac{192}{x^4}+\frac{64}{x^6}}$



EXPANDING IN THE "USUAL" MANNER

$$\left(x+\frac{2}{x}\right)^6 = \binom{6}{0}(x)^6\left(\frac{2}{x}\right)^0 + \binom{6}{1}(x)^5\left(\frac{2}{x}\right)^1 + \binom{6}{2}(x)^4\left(\frac{2}{x}\right)^2 + \binom{6}{3}(x)^3\left(\frac{2}{x}\right)^3$$

$$+\binom{6}{4}(x)^2\left(\frac{2}{x}\right)^4 + \binom{6}{5}(x)^1\left(\frac{2}{x}\right)^5 + \binom{6}{6}(x)^0\left(\frac{2}{x}\right)^6$$

$$= (1\times x^6\times 1) + \left(6\times x^5\times\frac{2}{x}\right) + \left(15\times x^4\times\frac{4}{x^2}\right) + \left(20\times x^3\times\frac{8}{x^3}\right)$$

$$+\left(15\times x^2+\frac{16}{x^4}\right) + \left(6\times x\times\frac{32}{x^5}\right) + \left(1\times 1\times\frac{64}{x^6}\right)$$

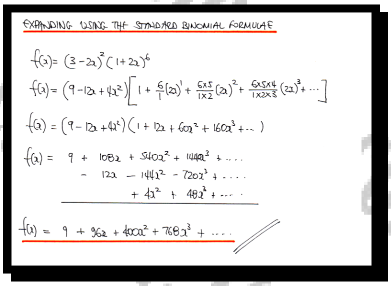
$$= x^6+12x^4+60x^2+160+\frac{240}{x^2}+\frac{192}{x^4}+\frac{64}{x^6}$$

Question 38 **(***+)**

$$f(x)=(3-2x)^2(1+2x)^6.$$

Find the binomial expansion of $f(x)$ in ascending powers of x, up and including the term in x^3.

☐ , $\boxed{9+96x+400x^2+768x^3+\ldots}$



EXPANDING USING THE STANDARD BINOMIAL FORMULAE

$$f(x)=(3-2x)^2(1+2x)^6$$

$$f(x)=(9-12x+4x^2)\left[1+\frac{6}{1}(2x)^1+\frac{6\times5}{1\times2}(2x)^2+\frac{6\times5\times4}{1\times2\times3}(2x)^3+\cdots\right]$$

$$f(x)=(9-12x+4x^2)(1+12x+60x^2+160x^3+\cdots)$$

$$f(x)= 9+108x+540x^2+1440x^3+\ldots$$
$$-12x-144x^2-720x^3+\ldots$$
$$+4x^2+48x^3+\ldots$$

$$f(x)=9+96x+400x^2+768x^3+\ldots$$

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Question 39 **(***+)**

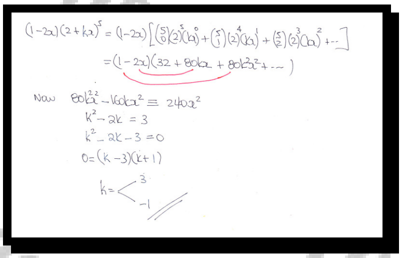
It is given that

$$(1-2x)(2+kx)^5 \equiv A + Bx + 240x^2 + \ldots,$$

where k, A and B are constants.

Determine the possible values of k

☐ , $\boxed{k = -1,\ 3}$



$$(1-2x)(2+kx)^5 = (1-2x)\left[\binom{5}{0}(2)^5(kx)^0 + \binom{5}{1}(2)^4(kx)^1 + \binom{5}{2}(2)^3(kx)^2 + \ldots\right]$$

$$= (1-2x)(32 + 80kx + 80k^2x^2 + \ldots)$$

Now $80k^2x^2 - 160kx^2 \equiv 240x^2$

$k^2 - 2k = 3$

$k^2 - 2k - 3 = 0$

$0 = (k-3)(k+1)$

$k = \begin{cases} 3 \\ -1 \end{cases}$

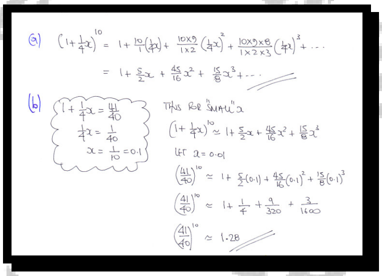
Question 40 **(***+)**

a) Find the binomial expansion of $\left(1+\frac{1}{4}x\right)^{10}$ in ascending powers of x up and including the term in x^3, simplifying fully each coefficient.

b) Use the expansion of part **(a)** to show that

$$\left(\frac{41}{40}\right)^{10} \approx 1.28\,.$$

☐ , $\boxed{1 + \frac{5}{2}x + \frac{45}{16}x^2 + \frac{15}{8}x^3 + \ldots}$



(a) $\left(1+\frac{1}{4}x\right)^{10} = 1 + \frac{10}{1}\left(\frac{1}{4}x\right) + \frac{10\times9}{1\times2}\left(\frac{1}{4}x\right)^2 + \frac{10\times9\times8}{1\times2\times3}\left(\frac{1}{4}x\right)^3 + \ldots$

$= 1 + \frac{5}{2}x + \frac{45}{16}x^2 + \frac{15}{8}x^3 + \ldots$

(b) $1 + \frac{1}{4}x = \frac{41}{40}$

$\frac{1}{4}x = \frac{1}{40}$

$x = \frac{1}{10} = 0.1$

THUS FOR "SMALL" x

$\left(1+\frac{1}{4}x\right)^{10} \simeq 1 + \frac{5}{2}x + \frac{45}{16}x^2 + \frac{15}{8}x^3$

LET $x = 0.01$

$\left(\frac{41}{40}\right)^{10} \simeq 1 + \frac{5}{2}(0.1) + \frac{45}{16}(0.1)^2 + \frac{15}{8}(0.1)^3$

$\left(\frac{41}{40}\right)^{10} \simeq 1 + \frac{1}{4} + \frac{9}{320} + \frac{3}{1600}$

$\left(\frac{41}{40}\right)^{10} \simeq 1.28$

Question 41 (***+)

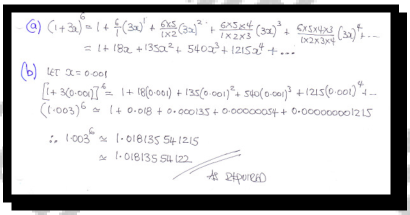
$$f(x) = (1+3x)^6 .$$

a) Find, in ascending powers of x, the binomial expansion of $f(x)$ up and including the term in x^4.

b) Use the expansion found in part **(a)** to show that

$$(1.003)^6 \approx 1.01813554122 .$$

$$\boxed{f(x) = 1 + 18x + 135x^2 + 540x^3 + 1215x^4 + \ldots}$$



(a) $(1+3x)^6 = 1 + \frac{6}{1}(3x)^1 + \frac{6\times5}{1\times2}(3x)^2 + \frac{6\times5\times4}{1\times2\times3}(3x)^3 + \frac{6\times5\times4\times3}{1\times2\times3\times4}(3x)^4 + \ldots$

$= 1 + 18x + 135x^2 + 540x^3 + 1215x^4 + \ldots$

(b) LET $x = 0.001$

$[1+3(0.001)]^6 = 1 + 18(0.001) + 135(0.001)^2 + 540(0.001)^3 + 1215(0.001)^4 + \ldots$

$(1.003)^6 \simeq 1 + 0.018 + 0.000135 + 0.00000054 + 0.000000001215$

$\therefore 1.003^6 \simeq 1.018135541215$

$\simeq 1.01813554122$

AS REQUIRED

Question 42 (***+)

a) Find the binomial expansion of $(3+2x)^4$.

b) State the binomial expansion of $(3-2x)^4$.

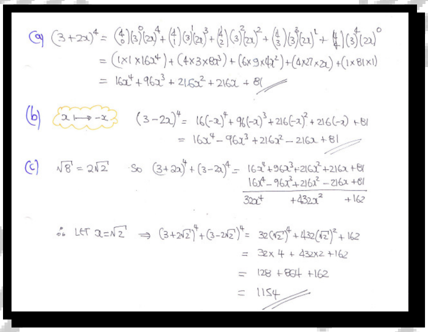
c) Use the answers of part (a) and (b) to find

$$\left(3+\sqrt{8}\right)^4+\left(3-\sqrt{8}\right)^4.$$

No credit will be given for any other type of evaluation.

$$\boxed{(3+2x)^4=16x^4+96x^3+216x^2+216x+81},$$

$$\boxed{(3-2x)^4=16x^4-96x^3+216x^2-216x+81},\ \boxed{\left(3+\sqrt{8}\right)^4+\left(3-\sqrt{8}\right)^4=1154}$$



(a) $(3+2x)^4=\binom{4}{0}(3)^0(2x)^4+\binom{4}{1}(3)^1(2x)^3+\binom{4}{2}(3)^2(2x)^2+\binom{4}{3}(3)^3(2x)^1+\binom{4}{4}(3)^4(2x)^0$

$=(1\times1\times16x^4)+(4\times3\times8x^3)+(6\times9\times4x^2)+(4\times27\times2x)+(1\times81\times1)$

$=16x^4+96x^3+216x^2+216x+81$

(b) $x\longmapsto -x$ $\quad(3-2x)^4=16(-x)^4+96(-x)^3+216(-x)^2+216(-x)+81$

$=16x^4-96x^3+216x^2-216x+81$

(c) $\sqrt{8}=2\sqrt{2}$ So $(3+2x)^4+(3-2x)^4=$

$$\begin{array}{r} 16x^4+96x^3+216x^2+216x+81\\ 16x^4-96x^3+216x^2-216x+81\\ \hline 32x^4\qquad\quad+432x^2\qquad\quad+162 \end{array}$$

$\therefore$ LET $x=\sqrt{2}\Rightarrow(3+2\sqrt{2})^4+(3-2\sqrt{2})^4=32(\sqrt{2})^4+432(\sqrt{2})^2+162$

$=32\times4+432\times2+162$

$=128+864+162$

$=1154$

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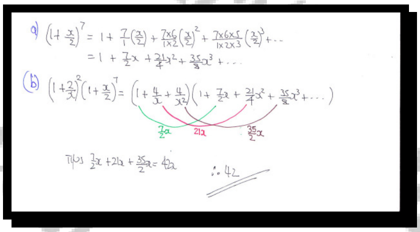
Question 43 (***+)

a) Find the first four terms, in ascending powers of x, of the binomial expansion of $\left(1+\frac{x}{2}\right)^7$, giving each coefficient in exact simplified form.

b) Hence determine the coefficient of x in the expansion of

$$\left(1+\frac{2}{x}\right)^2\left(1+\frac{x}{2}\right)^7 .$$

$\boxed{}$, $\boxed{1+\frac{7}{2}x+\frac{21}{4}x^2+\frac{35}{8}x^3+\ldots}$, $\boxed{[x]=42}$



a) $\left(1+\frac{x}{2}\right)^7 = 1+\frac{7}{1}\left(\frac{x}{2}\right)+\frac{7\times6}{1\times2}\left(\frac{x}{2}\right)^2+\frac{7\times6\times5}{1\times2\times3}\left(\frac{x}{2}\right)^3+\ldots$

$= 1+\frac{7}{2}x+\frac{21}{4}x^2+\frac{35}{8}x^3+\ldots$

(b) $\left(1+\frac{2}{x}\right)^2\left(1+\frac{x}{2}\right)^7 = \left(1+\frac{4}{x}+\frac{4}{x^2}\right)\left(1+\frac{7}{2}x+\frac{21}{4}x^2+\frac{35}{8}x^3+\ldots\right)$

$\frac{7}{2}x$ $21x$ $\frac{35}{2}x$

Thus $\frac{7}{2}x+21x+\frac{35}{2}x=42x$

$\therefore 42$

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Question 44 (***+)

$$f(x) = (1+2x)^7$$

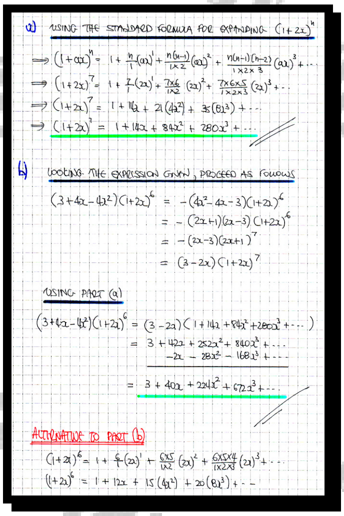
a) Determine the first four terms, in ascending powers of x, in the binomial expansion of $f(x)$.

b) Hence, or otherwise, find the first four terms in the expansion of

$$\left(3+4x-4x^2\right)(1+2x)^6,$$

giving the answer in ascending powers of x.

$\boxed{}$, $\boxed{1+14x+84x^2+280x^3+\ldots}$, $\boxed{3+40x+224x^2+672x^3+\ldots}$



a) USING THE STANDARD FORMULA FOR EXPANDING $(1+2x)^n$

$\Rightarrow (1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1\times 2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(ax)^3 + \ldots$

$\Rightarrow (1+2x)^7 = 1 + \frac{7}{1}(2x)^1 + \frac{7\times 6}{1\times 2}(2x)^2 + \frac{7\times 6\times 5}{1\times 2\times 3}(2x)^3 + \ldots$

$\Rightarrow (1+2x)^7 = 1 + 14x + 21(4x^2) + 35(8x^3) + \ldots$

$\Rightarrow (1+2x)^7 = 1 + 14x + 84x^2 + 280x^3 + \ldots$

b) LOOKING THE EXPRESSION GIVEN, PROCEED AS FOLLOWS

$(3+4x-4x^2)(1+2x)^6 = -(4x^2-4x-3)(1+2x)^6$

$= -(2x+1)(2x-3)(1+2x)^6$

$= -(2x-3)(2x+1)^7$

$= (3-2x)(1+2x)^7$

USING PART (a)

$(3+4x-4x^2)(1+2x)^6 = (3-2x)(1+14x+84x^2+280x^3+\ldots)$

$= 3 + 42x + 252x^2 + 840x^3 + \ldots$

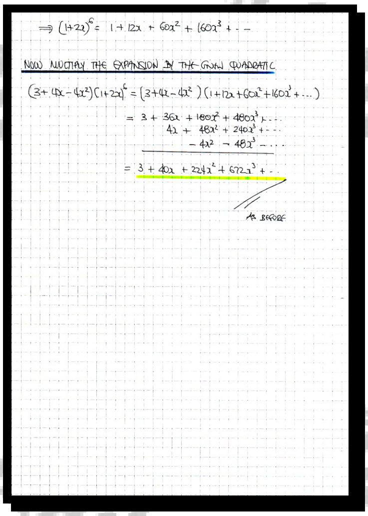
$\quad -2x - 28x^2 - 168x^3 + \ldots$

$= 3 + 40x + 224x^2 + 672x^3 + \ldots$

ALTERNATIVE TO PART (b)

$(1+2x)^6 = 1 + \frac{6}{1}(2x)^1 + \frac{6\times 5}{1\times 2}(2x)^2 + \frac{6\times 5\times 4}{1\times 2\times 3}(2x)^3 + \ldots$

$(1+2x)^6 = 1 + 12x + 15(4x^2) + 20(8x^3) + \ldots$



$\Rightarrow (1+2x)^6 = 1 + 12x + 60x^2 + 160x^3 + \ldots$

NOW MULTIPLY THE EXPANSION BY THE GIVEN QUADRATIC

$(3+4x-4x^2)(1+2x)^6 = (3+4x-4x^2)(1+12x+60x^2+160x^3+\ldots)$

$= 3 + 36x + 180x^2 + 480x^3 + \ldots$

$\quad 4x + 48x^2 + 240x^3 + \ldots$

$\quad -4x^2 - 48x^3 - \ldots$

$= 3 + 40x + 224x^2 + 672x^3 + \ldots$

AS BEFORE

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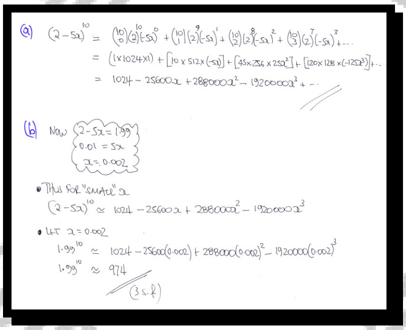
Question 45 (***+)

a) Find the binomial expansion of $(2-5x)^{10}$ in ascending powers of x up and including the term in x^3, simplifying fully each coefficient.

b) Use the expansion of part (a) to show that

$$(1.99)^{10} \approx 974 .$$

$$\boxed{1024 - 25600x + 288000x^2 - 1920000x^3 + \ldots}$$



(a) $(2-5x)^{10} = \binom{10}{0}(2)^{10}(-5x)^0 + \binom{10}{1}(2)^9(-5x)^1 + \binom{10}{2}(2)^8(-5x)^2 + \binom{10}{3}(2)^7(-5x)^3 + \ldots$

$= (1 \times 1024 \times 1) + [10 \times 512 \times (-5x)] + [45 \times 256 \times 25x^2] + [120 \times 128 \times (-125x^3)] + \ldots$

$= 1024 - 25600x + 288000x^2 - 1920000x^3 + \ldots$

(b) Now $2-5x = 1.99$
$0.01 = 5x$
$x = 0.002$

- Thus for "small" x
$(2-5x)^{10} \simeq 1024 - 25600x + 288000x^2 - 1920000x^3$
- Let $x = 0.002$
$1.99^{10} \simeq 1024 - 25600(0.002) + 288000(0.002)^2 - 1920000(0.002)^3$
$1.99^{10} \approx 974$
(3 s.f.)

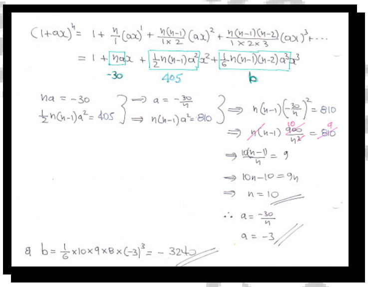
Question 46 (****)

$$(1+ax)^n = 1 - 30x + 405x^2 + bx^3 + \ldots,$$

where a and b are constants, and n is a positive integer.

Determine the value of n, a and b.

$\boxed{\text{C2M}}$, $\boxed{n=10}$, $\boxed{a=-3}$, $\boxed{b=-3240}$



$(1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1 \times 2}(ax)^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}(ax)^3 + \ldots$

$= 1 + \underbrace{\boxed{na}}_{-30}x + \underbrace{\boxed{\tfrac{1}{2}n(n-1)a^2}}_{405}x^2 + \underbrace{\boxed{\tfrac{1}{6}n(n-1)(n-2)a^3}}_{b}x^3$

$na = -30 \Rightarrow a = -\frac{30}{n}$

$\frac{1}{2}n(n-1)a^2 = 405 \Rightarrow n(n-1)a^2 = 810$

$\Rightarrow n(n-1)\left(-\frac{30}{n}\right)^2 = 810$

$\Rightarrow \cancel{n}(n-1)\frac{\overset{10}{\cancel{900}}}{n^{\cancel{2}}} = \overset{9}{\cancel{810}}$

$\Rightarrow \frac{10(n-1)}{n} = 9$

$\Rightarrow 10n - 10 = 9n$

$\Rightarrow n = 10$

$\therefore a = \frac{-30}{n}$

$a = -3$

& $b = \frac{1}{6} \times 10 \times 9 \times 8 \times (-3)^3 = -3240$

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Question 47 (****)

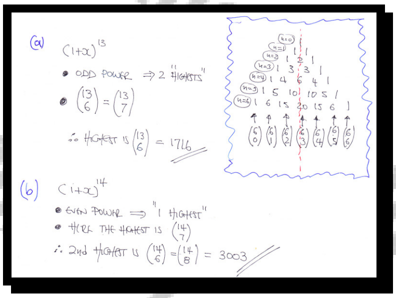
$$f(x) \equiv (1+x)^n,\ x \in \mathbb{R},\ n \in \mathbb{N}.$$

Determine showing a clear complete method the coefficient …

a) … of the **highest** power of x in the binomial expansion of $f(x)$, when $n = 13$.

b) … of the **second highest** power of x in the binomial expansion of $f(x)$, when $n = 14$.

$\boxed{\text{C2}}$, $\boxed{1716}$, $\boxed{3003}$



(a) $(1+x)^{13}$

- ODD POWER $\Rightarrow$ 2 "HIGHESTS"
- $\binom{13}{6} = \binom{13}{7}$

$\therefore$ HIGHEST IS $\binom{13}{6} = 1716$

$n=0$ 1
$n=1$ 1 1
$n=2$ 1 2 1
$n=3$ 1 3 3 1
$n=4$ 1 4 6 4 1
$n=5$ 1 5 10 10 5 1
$n=6$ 1 6 15 20 15 6 1

$\binom{6}{0}$ $\binom{6}{1}$ $\binom{6}{2}$ $\binom{6}{3}$ $\binom{6}{4}$ $\binom{6}{5}$ $\binom{6}{6}$

(b) $(1+x)^{14}$

- EVEN POWER $\Rightarrow$ "1 HIGHEST"
- HERE THE HIGHEST IS $\binom{14}{7}$

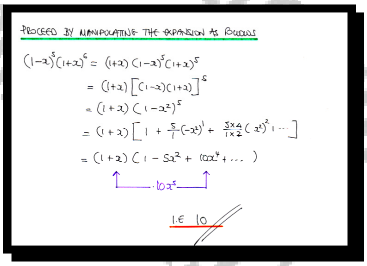
$\therefore$ 2nd HIGHEST IS $\binom{14}{6} = \binom{14}{8} = 3003$

Question 48 (****)

Find the coefficient of x^5 in the binomial expansion of

$$(1-x)^5(1+x)^6.$$

$\boxed{\text{SYN-O}}$, $\boxed{\left[x^5\right] = 10}$



PROCEED BY MANIPULATING THE EXPANSION AS FOLLOWS

$$(1-x)^5(1+x)^6 = (1+x)(1-x)^5(1+x)^5$$
$$= (1+x)\left[(1-x)(1+x)\right]^5$$
$$= (1+x)(1-x^2)^5$$
$$= (1+x)\left[1 + \frac{5}{1}(-x^2)^1 + \frac{5\times4}{1\times2}(-x^2)^2 + \cdots\right]$$
$$= (1+x)(1 - 5x^2 + 10x^4 + \dots)$$

$10x^5$

I.E 10

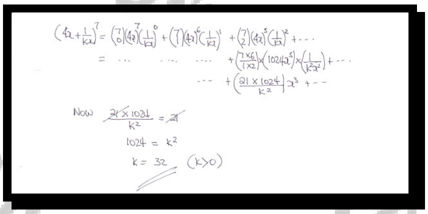
Question 49 **(****)**

$$f(x) = \left(4x + \frac{1}{kx}\right)^7,$$

where k is a positive constant.

Given the coefficient of x^3 in the binomial expansion of $f(x)$ is 21, determine the value of k.

$\boxed{}$, $\boxed{k = 32}$



$$\left(4x+\frac{1}{kx}\right)^7 = \binom{7}{0}(4x)^7\left(\frac{1}{kx}\right)^0 + \binom{7}{1}(4x)^6\left(\frac{1}{kx}\right)^1 + \binom{7}{2}(4x)^5\left(\frac{1}{kx}\right)^2 + \dots$$

$$= \dots \quad \dots \quad \dots \quad + \left(\frac{7\times 6}{1\times 2}\right)\times\left(1024x^5\right)\times\left(\frac{1}{k^2x^2}\right) + \dots$$

$$\dots + \left(\frac{21\times 1024}{k^2}\right)x^3 + \dots$$

Now $\dfrac{\cancel{21}\times 1024}{k^2} = \cancel{21}$

$1024 = k^2$

$k = 32$ $(k>0)$

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Question 50 **(****)**

$$f(x) = (1+2x)^5, \ x \in \mathbb{R}.$$

a) Find the binomial expansion of $f(x)$.

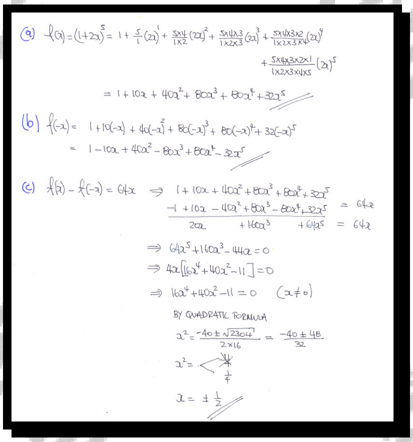
b) Hence state the binomial expansion of $f(-x)$.

c) Find the two non zero solutions of the equation

$$f(x) - f(-x) = 64x.$$

$\boxed{}$, $\boxed{f(x) = 1+10x+40x^2+80x^3+80x^4+32x^5}$,

$\boxed{f(-x) = 1-10x+40x^2-80x^3+80x^4-32x^5}$ $\boxed{x = \pm\frac{1}{2}}$



(a) $f(x) = (1+2x)^5 = 1 + \frac{5}{1}(2x)^1 + \frac{5\times4}{1\times2}(2x)^2 + \frac{5\times4\times3}{1\times2\times3}(2x)^3 + \frac{5\times4\times3\times2}{1\times2\times3\times4}(2x)^4 + \frac{5\times4\times3\times2\times1}{1\times2\times3\times4\times5}(2x)^5$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

(b) $f(-x) = 1 + 10(-x) + 40(-x)^2 + 80(-x)^3 + 80(-x)^4 + 32(-x)^5$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

(c) $f(x) - f(-x) = 64x \implies$

$$\begin{array}{rrrrrrl} 1 & +10x & +40x^2 & +80x^3 & +80x^4 & +32x^5 & \\ -1 & +10x & -40x^2 & +80x^3 & -80x^4 & +32x^5 & = 64x \\ \hline & 20x & & +160x^3 & & +64x^5 & = 64x \end{array}$$

$\implies 64x^5 + 160x^3 - 44x = 0$

$\implies 4x[16x^4 + 40x^2 - 11] = 0$

$\implies 16x^4 + 40x^2 - 11 = 0 \quad (x \neq 0)$

BY QUADRATIC FORMULA

$x^2 = \frac{-40 \pm \sqrt{2304}}{2\times16} = \frac{-40 \pm 48}{32}$

$x^2 = \left\langle \begin{array}{l} \cancel{-\frac{11}{4}} \\ \frac{1}{4} \end{array} \right.$

$x = \pm\frac{1}{2}$

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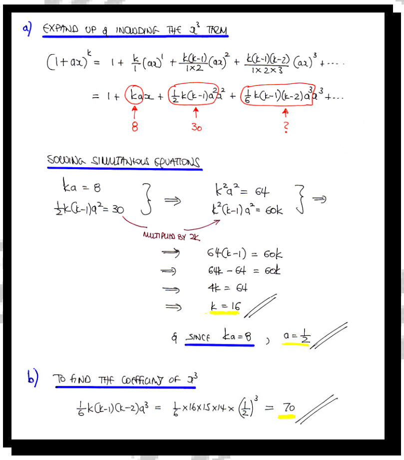
Question 51 **(****)**

In the binomial expansion of $(1+ax)^k$, where a and k are non zero constants, the coefficient of x is 8 and the coefficient of x^2 is 30.

a) Determine the value of a and the value of k.

b) Find the coefficient of x^3.

$\boxed{\quad}$, $\boxed{a=\frac{1}{2}}$, $\boxed{k=16}$, $\boxed{70}$



a) EXPAND UP & INCLUDING THE x^3 TERM

$$(1+ax)^k = 1 + \frac{k}{1}(ax)^1 + \frac{k(k-1)}{1\times 2}(ax)^2 + \frac{k(k-1)(k-2)}{1\times 2\times 3}(ax)^3 + \dots$$

$$= 1 + \underset{\substack{\uparrow\\ 8}}{\boxed{ka}}x + \underset{\substack{\uparrow\\ 30}}{\boxed{\tfrac{1}{2}k(k-1)a^2}}x^2 + \underset{\substack{\uparrow\\ ?}}{\boxed{\tfrac{1}{6}k(k-1)(k-2)a^3}}x^3 + \dots$$

SOLVING SIMULTANEOUS EQUATIONS

$$\left.\begin{array}{l} ka = 8 \\ \frac{1}{2}k(k-1)a^2 = 30 \end{array}\right\} \Rightarrow \left.\begin{array}{l} k^2a^2 = 64 \\ k^2(k-1)a^2 = 60k \end{array}\right\} \Rightarrow$$

MULTIPLIED BY 2k

$$\Rightarrow 64(k-1) = 60k$$
$$\Rightarrow 64k - 64 = 60k$$
$$\Rightarrow 4k = 64$$
$$\Rightarrow k = 16$$

& SINCE $ka = 8$, $a = \frac{1}{2}$

b) TO FIND THE COEFFICIENT OF x^3

$$\frac{1}{6}k(k-1)(k-2)a^3 = \frac{1}{6}\times 16\times 15\times 14\times\left(\frac{1}{2}\right)^3 = 70$$

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Question 52 (****)

$$f(x) \equiv (k+x)^n, \; x \in \mathbb{R},$$

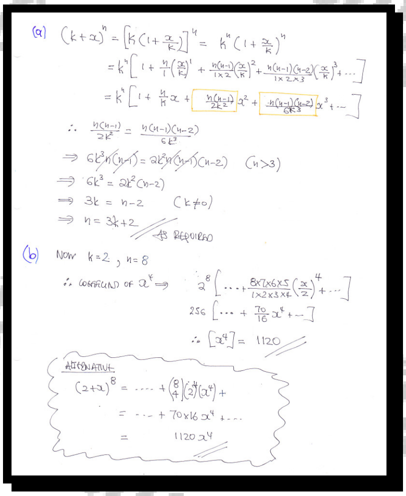
where k and n are constants such that $k \in \mathbb{R}$, $k \neq 0$, $n \in \mathbb{N}$, $n > 3$.

a) Given the coefficients of x^2 and x^3 in the binomial expansion of $f(x)$ are equal, show clearly that

$$n = 3k + 2.$$

b) Given further that $k = 2$, determine the coefficient of x^4 in the binomial expansion of $f(x)$.

$\boxed{}$, $\boxed{\left[x^4\right] = 1120}$



(a) $(k+x)^n = \left[k\left(1+\frac{x}{k}\right)\right]^n = k^n\left(1+\frac{x}{k}\right)^n$

$= k^n\left[1 + \frac{n}{1}\left(\frac{x}{k}\right)^1 + \frac{n(n-1)}{1\times2}\left(\frac{x}{k}\right)^2 + \frac{n(n-1)(n-2)}{1\times2\times3}\left(\frac{x}{k}\right)^3 + \dots\right]$

$= k^n\left[1 + \frac{n}{k}x + \boxed{\frac{n(n-1)}{2k^2}}x^2 + \boxed{\frac{n(n-1)(n-2)}{6k^3}}x^3 + \dots\right]$

$\therefore \frac{n(n-1)}{2k^2} = \frac{n(n-1)(n-2)}{6k^3}$

$\Rightarrow 6k^3\cancel{n}\cancel{(n-1)} = 2k^2\cancel{n}\cancel{(n-1)}(n-2) \quad (n>3)$

$\Rightarrow 6k^3 = 2k^2(n-2)$

$\Rightarrow 3k = n-2 \quad (k \neq 0)$

$\Rightarrow n = 3k+2$ // AS REQUIRED

(b) Now $k=2$, $n=8$

$\therefore$ COEFFICIENT OF $x^4 \Rightarrow$ $2^8\left[\dots + \frac{8\times7\times6\times5}{1\times2\times3\times4}\left(\frac{x}{2}\right)^4 + \dots\right]$

$256\left[\dots + \frac{70}{16}x^4 + \dots\right]$

$\therefore \left[x^4\right] = 1120$ //

ALTERNATIVE

$(2+x)^8 = \dots + \binom{8}{4}(2)^4(x^4) +$

$= \dots + 70\times16\,x^4 + \dots$

$= 1120\,x^4$ //

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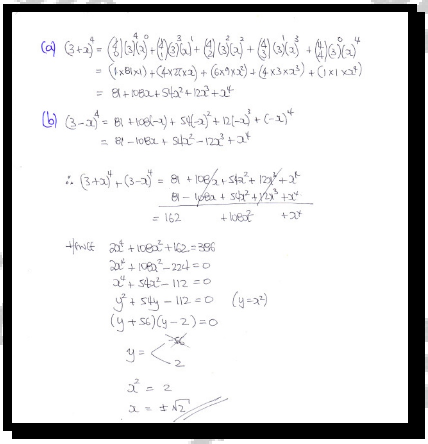
Question 53 **(****)**

a) Find the binomial expansion of $(3+x)^4$, simplifying fully each coefficient.

b) Hence solve the equation

$$(3+x)^4+(3-x)^4=386.$$

$\boxed{81+108x+54x^2+12x^3+x^4}$, $\boxed{x=\pm\sqrt{2}}$



(a) $(3+x)^4=\binom{4}{0}(3)^4(x)^0+\binom{4}{1}(3)^3(x)^1+\binom{4}{2}(3)^2(x)^2+\binom{4}{3}(3)^1(x)^3+\binom{4}{4}(3)^0(x)^4$

$=(1\times81\times1)+(4\times27\times x)+(6\times9\times x^2)+(4\times3\times x^3)+(1\times1\times x^4)$

$=81+108x+54x^2+12x^3+x^4$

(b) $(3-x)^4=81+108(-x)+54(-x)^2+12(-x)^3+(-x)^4$

$=81-108x+54x^2-12x^3+x^4$

$\therefore (3+x)^4+(3-x)^4=81+\cancel{108x}+54x^2+\cancel{12x^3}+x^4$

$\quad 81-\cancel{108x}+54x^2+\cancel{12x^3}+x^4$

$=162 \quad +108x^2 \quad +x^4$

Hence $2x^4+108x^2+162=386$

$2x^4+108x^2-224=0$

$x^4+54x^2-112=0$

$y^2+54y-112=0 \quad (y=x^2)$

$(y+56)(y-2)=0$

$y=\begin{cases}\cancel{-56}\\ 2\end{cases}$

$x^2=2$

$x=\pm\sqrt{2}$

Question 54 (****)

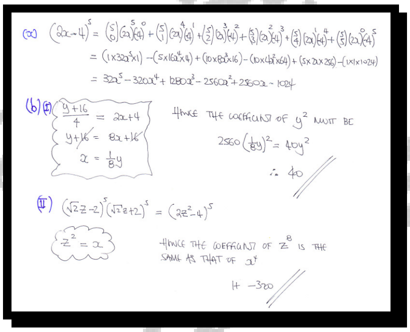
a) Find the binomial expansion of $(2x-4)^5$, simplifying fully each coefficient.

b) Hence find the coefficient of …

i. … y^2 in the binomial expansion of $\left(\frac{y+16}{4}\right)^5$.

ii. … z^8 in the binomial expansion of $\left(\sqrt{2}z-2\right)^5\left(\sqrt{2}z+2\right)^5$.

$\boxed{}$, $\boxed{32x^5-320x^4+1280x^3-2560x^2+2560x-1024}$, $\boxed{\left[y^2\right]=40}$, $\boxed{\left[z^8\right]=-320}$


(a) $$(2x-4)^5 = \binom{5}{0}(2x)^5(-4)^0+\binom{5}{1}(2x)^4(-4)^1+\binom{5}{2}(2x)^3(-4)^2+\binom{5}{3}(2x)^2(-4)^3+\binom{5}{4}(2x)^1(-4)^4+\binom{5}{5}(2x)^0(-4)^5$$
$$= (1\times32x^5\times1)-(5\times16x^4\times4)+(10\times8x^3\times16)-(10\times4x^2\times64)+(5\times2x\times256)-(1\times1\times1024)$$
$$= 32x^5-320x^4+1280x^3-2560x^2+2560x-1024$$

(b)(I) $\frac{y+16}{4} = 2x+4$
$y+\cancel{16} = 8x+\cancel{16}$
$x = \frac{1}{8}y$

HENCE THE COEFFICIENT OF y^2 MUST BE
$2560\left(\frac{1}{8}y\right)^2 = 40y^2$
$\therefore 40$

(II) $(\sqrt{2}z-2)^5(\sqrt{2}z+2)^5 = (2z^2-4)^5$

$z^2 = x$

HENCE THE COEFFICIENT OF z^8 IS THE SAME AS THAT OF x^4

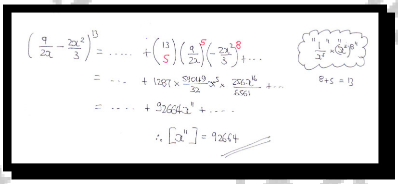
IT -320

Question 55 (****)

Find the coefficient of x^{11} in the binomial expansion of

$$\left(\frac{9}{2x}-\frac{2x^2}{3}\right)^{13}.$$

$\boxed{}$, $\boxed{\left[x^{11}\right]=92664}$


$$\left(\frac{9}{2x}-\frac{2x^2}{3}\right)^{13} = \dots\dots + \binom{13}{5}\left(\frac{9}{2x}\right)^5\left(-\frac{2x^2}{3}\right)^8 + \dots$$
$$= \dots + 1287\times\frac{59049}{32}x^5\times\frac{256x^{16}}{6561}+\dots$$
$$= \dots + 92664x^{11}+\dots$$
$$\therefore \left[x^{11}\right] = 92664$$

$"\frac{1}{x^5}" \times "(x^2)^8"$
$8+5=13$

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Question 56 (****)

If $k > 0$ and n is a positive integer, then

$$(1+kx)^n \equiv 1+\frac{7}{2}x+Bx^2+Bx^3+\ldots,$$

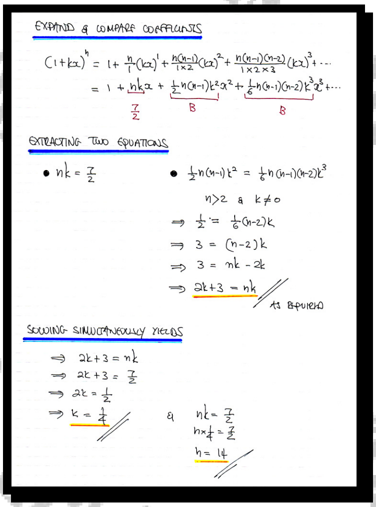
where B is a non zero constant.

By considering the coefficients of x^2 and x^3, show that

$$nk = 2k+3,$$

and hence find the value of n and the value of k.

$\boxed{}$, $\boxed{n=14}$, $\boxed{k=\frac{1}{4}}$



EXPAND & COMPARE COEFFICIENTS

$$(1+kx)^n = 1+\frac{n}{1}(kx)^1+\frac{n(n-1)}{1\times2}(kx)^2+\frac{n(n-1)(n-2)}{1\times2\times3}(kx)^3+\ldots$$

$$= 1+\underbrace{nk}_{\frac{7}{2}}x+\underbrace{\tfrac{1}{2}n(n-1)k^2}_{B}x^2+\underbrace{\tfrac{1}{6}n(n-1)(n-2)k^3}_{B}x^3+\ldots$$

EXTRACTING TWO EQUATIONS

- $nk = \frac{7}{2}$
- $\frac{1}{2}n(n-1)k^2 = \frac{1}{6}n(n-1)(n-2)k^3$

$n>2$ & $k \neq 0$

$\Rightarrow \frac{1}{2} = \frac{1}{6}(n-2)k$

$\Rightarrow 3 = (n-2)k$

$\Rightarrow 3 = nk-2k$

$\Rightarrow 2k+3 = nk$ // AS REQUIRED

SOLVING SIMULTANEOUSLY YIELDS

$\Rightarrow 2k+3 = nk$

$\Rightarrow 2k+3 = \frac{7}{2}$

$\Rightarrow 2k = \frac{1}{2}$

$\Rightarrow k = \frac{1}{4}$ //

& $nk = \frac{7}{2}$

$n \times \frac{1}{4} = \frac{7}{2}$

$n = 14$ //

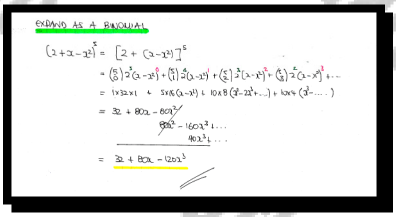
Question 57 (****)

Show that if x is numerically small

$$\left(2+x-x^2\right)^5 \approx A+Bx+Cx^3$$

where A, B and C are integers to be found.

$\boxed{A=32}$, $\boxed{B=80}$, $\boxed{C=-120}$



EXPAND AS A BINOMIAL

$$(2+x-x^2)^5 = [2+(x-x^2)]^5$$

$$= \binom{5}{0}2^5(x-x^2)^0+\binom{5}{1}2^4(x-x^2)^1+\binom{5}{2}2^3(x-x^2)^2+\binom{5}{3}2^2(x-x^2)^3+\dots$$

$$= 1\times32\times1 + 5\times16(x-x^2)+10\times8(x^2-2x^3+\dots)+10\times4(x^3-\dots)$$

$$= 32+80x-\cancel{80x^2}$$

$$\cancel{80x^2}-160x^3+\dots$$

$$40x^3+\dots$$

$$= 32+80x-120x^3$$

Question 58 (****)

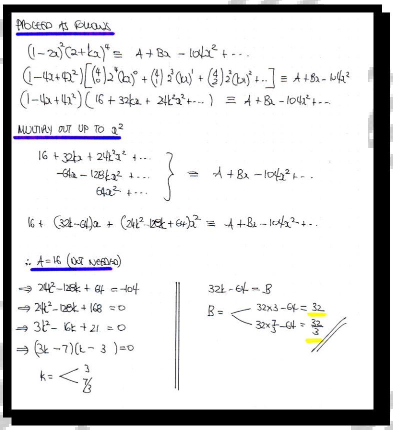
It is given that

$$(1-2x)^2(2+kx)^4 \equiv A+Bx-104x^2+\dots,$$

where k, A and B are non zero constants.

Determine the possible values of B.

$\boxed{B=32 \ \cup \ \frac{32}{3}}$



PROCEED AS FOLLOWS

$$(1-2x)^2(2+kx)^4 \equiv A+Bx-104x^2+\dots$$

$$(1-4x+4x^2)\left[\binom{4}{0}2^4(kx)^0+\binom{4}{1}2^3(kx)^1+\binom{4}{2}2^2(kx)^2+\dots\right] \equiv A+Bx-104x^2$$

$$(1-4x+4x^2)(16+32kx+24k^2x^2+\dots) \equiv A+Bx-104x^2+\dots$$

MULTIPLY OUT UP TO x^2

$$\left.\begin{array}{r} 16+32kx+24k^2x^2+\dots \\ -64x-128kx^2+\dots \\ 64x^2+\dots \end{array}\right\} \equiv A+Bx-104x^2+\dots$$

$$16+(32k-64)x+(24k^2-128k+64)x^2 \equiv A+Bx-104x^2+\dots$$

$\therefore A=16$ (NOT NEEDED)

$$\Rightarrow 24k^2-128k+64=-104$$

$$\Rightarrow 24k^2-128k+168=0$$

$$\Rightarrow 3k^2-16k+21=0$$

$$\Rightarrow (3k-7)(k-3)=0$$

$$k = \begin{cases} 3 \\ 7/3 \end{cases}$$

$$32k-64=B$$

$$B = \begin{cases} 32\times3-64=32 \\ 32\times\frac{7}{3}-64=\frac{32}{3} \end{cases}$$

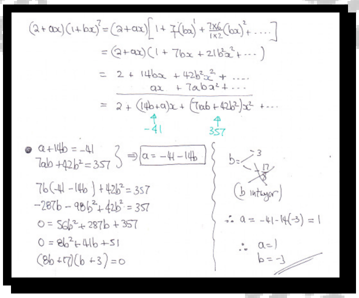
Question 59 (****)

$$(2+ax)(1+bx)^7 = 2-41x+357x^2+\ldots,$$

where a and b are integers.

Show that $b=-3$ and find the value of a.

$\boxed{}$, $\boxed{a=1}$



$(2+ax)(1+bx)^7 = (2+ax)\left[1+\frac{7}{1}(bx)^1+\frac{7\times6}{1\times2}(bx)^2+\ldots\right]$

$= (2+ax)(1+7bx+21b^2x^2+\ldots)$

$= 2+14bx+42b^2x^2+\ldots$

$\quad\;\; ax + 7abx^2+\ldots$

$= 2+(14b+a)x+(7ab+42b^2)x^2+\ldots$

$\uparrow -41 \qquad \uparrow 357$

$\left.\begin{array}{l} a+14b=-41 \\ 7ab+42b^2=357 \end{array}\right\} \Rightarrow \boxed{a=-41-14b}$

$7b(-41-14b)+42b^2=357$

$-287b-98b^2+42b^2=357$

$0=56b^2+287b+357$

$0=8b^2+41b+51$

$(8b+17)(b+3)=0$

$b = -3$, $\cancel{-\frac{17}{8}}$

(b integer)

$\therefore a=-41-14(-3)=1$

$\therefore a=1$

$b=-3$

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Question 60 (****+)

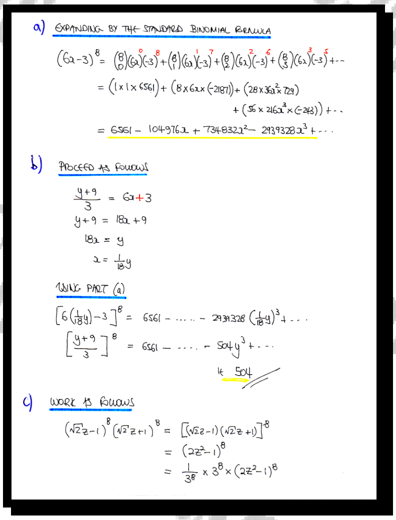
a) Find the first four terms, in ascending powers of x, of the binomial expansion of $(6x-3)^8$, simplifying fully each coefficient.

b) Hence find the coefficient of …

i. … y^3 in the binomial expansion of $\left(\frac{y+9}{3}\right)^8$.

ii. … z^6 in the binomial expansion of $\left(\sqrt{2}z-1\right)^8\left(\sqrt{2}z+1\right)^8$.

$\boxed{6561-104976x+734832x^2-2939328x^3+\ldots}$, $\boxed{\left[y^3\right]=504}$, $\boxed{\left[z^6\right]=-448}$



a) EXPANDING BY THE STANDARD BINOMIAL FORMULA

$$(6x-3)^8 = \binom{8}{0}(6x)^0(-3)^8+\binom{8}{1}(6x)^1(-3)^7+\binom{8}{2}(6x)^2(-3)^6+\binom{8}{3}(6x)^3(-3)^5+\ldots$$
$$= (1\times1\times6561)+(8\times6x\times(-2187))+(28\times36x^2\times729)$$
$$+(56\times216x^3\times(-243))+\ldots$$
$$= 6561-104976x+734832x^2-2939328x^3+\ldots$$

b) PROCEED AS FOLLOWS

$$\frac{y+9}{3} = 6x+3$$
$$y+9 = 18x+9$$
$$18x = y$$
$$x = \frac{1}{18}y$$

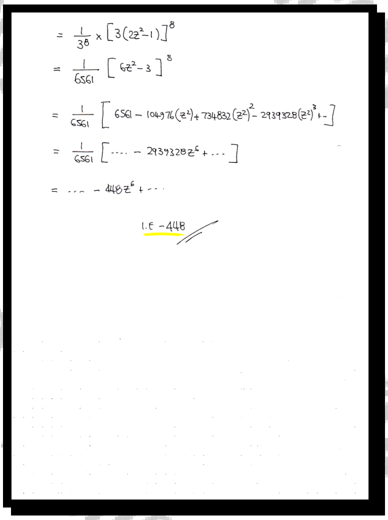
USING PART (a)

$$\left[6\left(\tfrac{1}{18}y\right)-3\right]^8 = 6561-\ldots\ldots-2939328\left(\tfrac{1}{18}y\right)^3+\ldots$$
$$\left[\frac{y+9}{3}\right]^8 = 6561-\ldots\ldots-504y^3+\ldots$$

ie 504

c) WORK AS FOLLOWS

$$(\sqrt{2}z-1)^8(\sqrt{2}z+1)^8 = \left[(\sqrt{2}z-1)(\sqrt{2}z+1)\right]^8$$
$$= (2z^2-1)^8$$
$$= \frac{1}{3^8}\times3^8\times(2z^2-1)^8$$



$$= \frac{1}{3^8}\times\left[3(2z^2-1)\right]^8$$
$$= \frac{1}{6561}\left[6z^2-3\right]^8$$
$$= \frac{1}{6561}\left[6561-104976(z^2)+734832(z^2)^2-2939328(z^2)^3+\ldots\right]$$
$$= \frac{1}{6561}\left[\ldots-2939328z^6+\ldots\right]$$
$$= \ldots-448z^6+\ldots$$

I.E -448

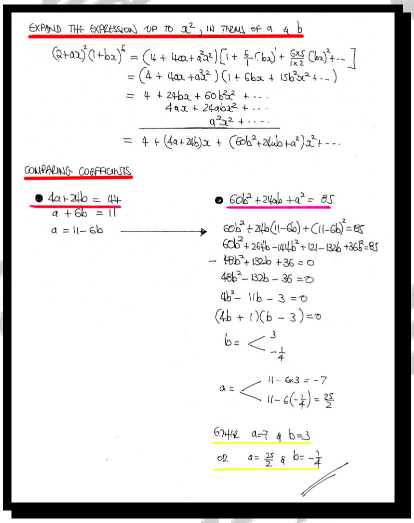
Question 61 (****+)

$$(2+ax)^2(1+bx)^6 = 4+44x+85x^2+\dots,$$

where a and b are integers.

Find the possible values of a and the possible values of b.

$\boxed{[a,b]=[-7,3] \cup \left[\frac{25}{2},-\frac{1}{4}\right]}$



EXPAND THE EXPRESSION UP TO x^2, IN TERMS OF a & b

$$(2+ax)^2(1+bx)^6 = (4+4ax+a^2x^2)\left[1+\frac{6}{1}(bx)^1+\frac{6\times5}{1\times2}(bx)^2+\dots\right]$$

$$= (4+4ax+a^2x^2)(1+6bx+15b^2x^2+\dots)$$

$$\begin{aligned} = 4 + 24bx &+ 60b^2x^2 + \dots \\ 4ax &+ 24abx^2 + \dots \\ &\quad a^2x^2 + \dots \end{aligned}$$

$$= 4+(4a+24b)x+(60b^2+24ab+a^2)x^2+\dots$$

COMPARING COEFFICIENTS

- $4a+24b=44$

$a+6b=11$

$a=11-6b \longrightarrow$

- $60b^2+24ab+a^2=85$

$60b^2+24b(11-6b)+(11-6b)^2=85$

$60b^2+264b-144b^2+121-132b+36b^2=85$

$-48b^2+132b+36=0$

$48b^2-132b-36=0$

$4b^2-11b-3=0$

$(4b+1)(b-3)=0$

$b = \begin{cases} 3 \\ -\frac{1}{4} \end{cases}$

$a = \begin{cases} 11-6\times3=-7 \\ 11-6\left(-\frac{1}{4}\right)=\frac{25}{2} \end{cases}$

EITHER $a=-7$ & $b=3$

OR $a=\frac{25}{2}$ & $b=-\frac{1}{4}$

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Question 62 (****+)

a) Given that c is a non zero constant, determine the first four terms, in ascending powers of x, in the binomial expansion of $(1+cx)^6$.

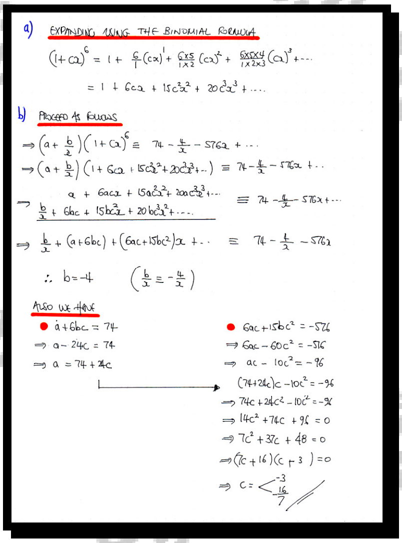
It is further given that

$$\left(a+\frac{b}{x}\right)(1+cx)^6 \equiv -\frac{4}{x}+74-576x+\dots,$$

where a and b are non zero constants.

b) Show that one of the two possible values of c is -3, and find the other.

$\boxed{}$, $\boxed{1+6cx+15c^2x^2+20c^3x^3+\dots}$, $\boxed{c=-\frac{16}{7}}$



a) EXPANDING USING THE BINOMIAL FORMULA

$(1+cx)^6 = 1 + \frac{6}{1}(cx)^1 + \frac{6\times5}{1\times2}(cx)^2 + \frac{6\times5\times4}{1\times2\times3}(cx)^3 + \dots$

$= 1 + 6cx + 15c^2x^2 + 20c^3x^3 + \dots$

b) PROCEED AS FOLLOWS

$\Rightarrow \left(a+\frac{b}{x}\right)(1+cx)^6 \equiv 74 - \frac{4}{x} - 576x + \dots$

$\Rightarrow \left(a+\frac{b}{x}\right)(1+6cx+15c^2x^2+20c^3x^3+\dots) \equiv 74 - \frac{4}{x} - 576x + \dots$

$\Rightarrow \begin{array}{l} a + 6acx + 15ac^2x^2 + 20ac^3x^3 + \dots \\ \frac{b}{x} + 6bc + 15bc^2x + 20bc^3x^2 + \dots \end{array} \equiv 74 - \frac{4}{x} - 576x + \dots$

$\Rightarrow \frac{b}{x} + (a+6bc) + (6ac+15bc^2)x + \dots \equiv 74 - \frac{4}{x} - 576x$

$\therefore b = -4 \quad \left(\frac{b}{x} \equiv -\frac{4}{x}\right)$

ALSO WE HAVE

- $a + 6bc = 74$
  - $\Rightarrow a - 24c = 74$
  - $\Rightarrow a = 74 + 24c$
- $6ac + 15bc^2 = -576$
  - $\Rightarrow 6ac - 60c^2 = -576$
  - $\Rightarrow ac - 10c^2 = -96$

$(74+24c)c - 10c^2 = -96$

$\Rightarrow 74c + 24c^2 - 10c^2 = -96$

$\Rightarrow 14c^2 + 74c + 96 = 0$

$\Rightarrow 7c^2 + 37c + 48 = 0$

$\Rightarrow (7c+16)(c+3) = 0$

$\Rightarrow c = \begin{cases} -3 \\ -\frac{16}{7} \end{cases}$

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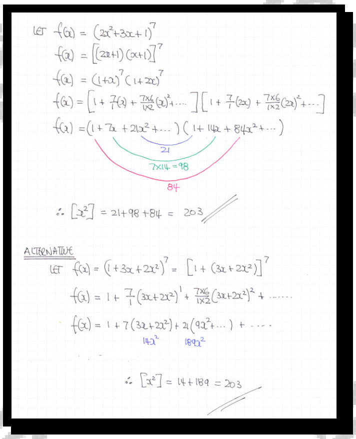
Question 63 **(****+)**

Find the coefficient of x^2 in the binomial expansion of

$$\left(2x^2+3x+1\right)^7 .$$

$\boxed{\text{B}}, \boxed{\left[x^2\right] = 203}$



LET $f(x) = \left(2x^2+3x+1\right)^7$

$f(x) = \left[(2x+1)(x+1)\right]^7$

$f(x) = (1+x)^7(1+2x)^7$

$f(x) = \left[1+\frac{7}{1}(x)+\frac{7\times6}{1\times2}(x)^2+\dots\right]\left[1+\frac{7}{1}(2x)+\frac{7\times6}{1\times2}(2x)^2+\dots\right]$

$f(x) = \left(1+7x+21x^2+\dots\right)\left(1+14x+84x^2+\dots\right)$

21

$7\times14=98$

84

$\therefore \left[x^2\right] = 21+98+84 = 203$

ALTERNATIVE

LET $f(x) = \left(1+3x+2x^2\right)^7 = \left[1+\left(3x+2x^2\right)\right]^7$

$f(x) = 1+\frac{7}{1}\left(3x+2x^2\right)^1+\frac{7\times6}{1\times2}\left(3x+2x^2\right)^2+\dots\dots$

$f(x) = 1+\underbrace{7\left(3x+2x^2\right)}_{14x^2}+\underbrace{21\left(9x^2+\dots\right)}_{189x^2}+\dots$

$\therefore \left[x^2\right] = 14+189 = 203$

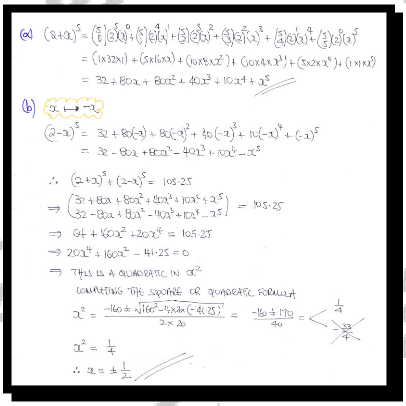
Question 64 **(****+)**

a) Find the binomial expansion of $(2+x)^5$.

b) Hence solve the equation

$$(2+x)^5+(2-x)^5=105.25\,.$$

$$\boxed{(2+x)^5=32+80x+80x^2+40x^3+10x^4+x^5}\ ,\ \boxed{x=\pm\frac{1}{2}}$$



(a) $(2+x)^5=\binom{5}{0}(2)^5(x)^0+\binom{5}{1}(2)^4(x)^1+\binom{5}{2}(2)^3(x)^2+\binom{5}{3}(2)^2(x)^3+\binom{5}{4}(2)^1(x)^4+\binom{5}{5}(2)^0(x)^5$

$=(1\times32\times1)+(5\times16\times x)+(10\times8\times x^2)+(10\times4\times x^3)+(5\times2\times x^4)+(1\times1\times x^5)$

$=32+80x+80x^2+40x^3+10x^4+x^5$

(b) $x\longmapsto -x$

$(2-x)^5=32+80(-x)+80(-x)^2+40(-x)^3+10(-x)^4+(-x)^5$

$=32-80x+80x^2-40x^3+10x^4-x^5$

$\therefore\ (2+x)^5+(2-x)^5=105.25$

$\Rightarrow \begin{pmatrix}32+80x+80x^2+40x^3+10x^4+x^5\\32-80x+80x^2-40x^3+10x^4-x^5\end{pmatrix}=105.25$

$\Rightarrow 64+160x^2+20x^4=105.25$

$\Rightarrow 20x^4+160x^2-41.25=0$

$\Rightarrow$ THIS IS A QUADRATIC IN x^2

COMPLETING THE SQUARE OR QUADRATIC FORMULA

$x^2=\frac{-160\pm\sqrt{160^2-4\times20\times(-41.25)}}{2\times20}=\frac{-160\pm170}{40}=\begin{cases}\frac{1}{4}\\ \cancel{-\frac{33}{4}}\end{cases}$

$x^2=\frac{1}{4}$

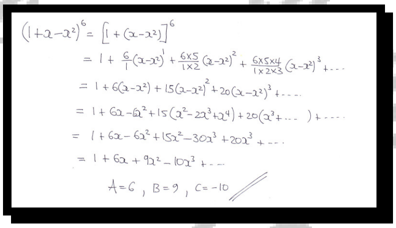
$\therefore\ x=\pm\frac{1}{2}$

Question 65 **(****+)**

$$\left(1+x-x^2\right)^6=1+Ax+Bx^2+Cx^3+\ldots$$

Determine the value of each of the constants A, B and C.

$$\boxed{\quad}\ ,\ \boxed{A=6}\ ,\ \boxed{B=-21}\ ,\ \boxed{C=-10}$$



$(1+x-x^2)^6=[1+(x-x^2)]^6$

$=1+\frac{6}{1}(x-x^2)^1+\frac{6\times5}{1\times2}(x-x^2)^2+\frac{6\times5\times4}{1\times2\times3}(x-x^2)^3+\ldots$

$=1+6(x-x^2)+15(x-x^2)^2+20(x-x^2)^3+\ldots$

$=1+6x-6x^2+15(x^2-2x^3+x^4)+20(x^3+\ldots)+\ldots$

$=1+6x-6x^2+15x^2-30x^3+20x^3+\ldots$

$=1+6x+9x^2-10x^3+\ldots$

$A=6,\ B=9,\ C=-10$

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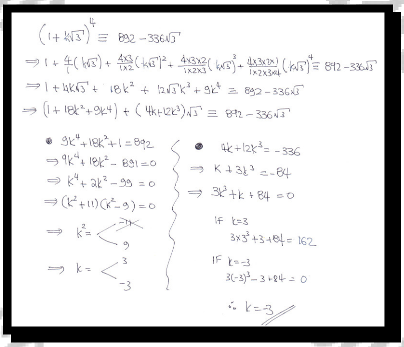
Question 66 (****+)

It is given that if k is a constant then

$$\left(1+k\sqrt{3}\right)^4 \equiv 892-336\sqrt{3}\,.$$

Determine the value of k.

$\boxed{k=-3}$



$\left(1+k\sqrt{3}\right)^4 \equiv 892-336\sqrt{3}$

$\Rightarrow 1+\frac{4}{1}\left(k\sqrt{3}\right)+\frac{4\times3}{1\times2}\left(k\sqrt{3}\right)^2+\frac{4\times3\times2}{1\times2\times3}\left(k\sqrt{3}\right)^3+\frac{4\times3\times2\times1}{1\times2\times3\times4}\left(k\sqrt{3}\right)^4 \equiv 892-336\sqrt{3}$

$\Rightarrow 1+4k\sqrt{3}+18k^2+12\sqrt{3}k^3+9k^4 \equiv 892-336\sqrt{3}$

$\Rightarrow \left(1+18k^2+9k^4\right)+\left(4k+12k^3\right)\sqrt{3} \equiv 892-336\sqrt{3}$

- $9k^4+18k^2+1=892$

$\Rightarrow 9k^4+18k^2-891=0$

$\Rightarrow k^4+2k^2-99=0$

$\Rightarrow (k^2+11)(k^2-9)=0$

$\Rightarrow k^2 = \cancel{-11}$ or 9

$\Rightarrow k = 3$ or -3

- $4k+12k^3=-336$

$\Rightarrow k+3k^3=-84$

$\Rightarrow 3k^3+k+84=0$

IF $k=3$

$3\times3^3+3+84=162$

IF $k=-3$

$3(-3)^3-3+84=0$

$\therefore k=-3$

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Question 67 **(****+)**

If A, k and n are constants, with $n \in \mathbb{N}$, then

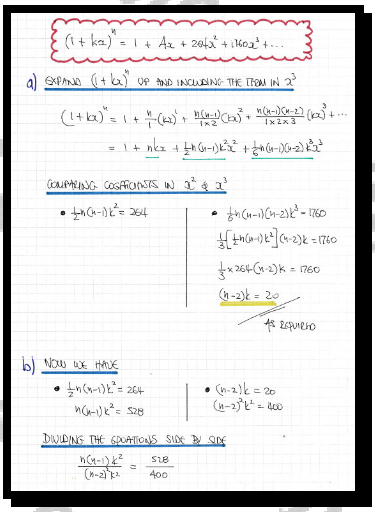
$$(1+kx)^n = 1 + Ax + 264x^2 + 1760x^3 + \ldots$$

a) Show that

$$(n-2)k = 20.$$

b) Determine the value of A.

$\boxed{}$, $\boxed{A = 24}$



$$\boxed{(1+kx)^n = 1 + Ax + 264x^2 + 1760x^3 + \ldots}$$

a) EXPAND $(1+kx)^n$ UP AND INCLUDING THE TERM IN x^3

$$(1+kx)^n = 1 + \frac{n}{1}(kx)^1 + \frac{n(n-1)}{1\times2}(kx)^2 + \frac{n(n-1)(n-2)}{1\times2\times3}(kx)^3 + \cdots$$

$$= 1 + nkx + \tfrac{1}{2}n(n-1)k^2x^2 + \tfrac{1}{6}n(n-1)(n-2)k^3x^3$$

COMPARING COEFFICIENTS IN x^2 & x^3

- $\frac{1}{2}n(n-1)k^2 = 264$
- $\frac{1}{6}n(n-1)(n-2)k^3 = 1760$

$\frac{1}{3}\left[\frac{1}{2}n(n-1)k^2\right](n-2)k = 1760$

$\frac{1}{3}\times 264(n-2)k = 1760$

$(n-2)k = 20$

AS REQUIRED

b) NOW WE HAVE

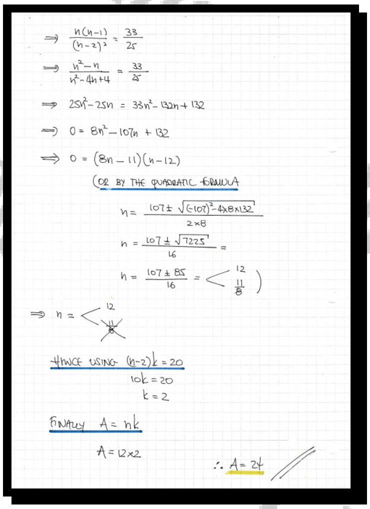
- $\frac{1}{2}n(n-1)k^2 = 264$

  $n(n-1)k^2 = 528$
- $(n-2)k = 20$

  $(n-2)^2k^2 = 400$

DIVIDING THE EQUATIONS SIDE BY SIDE

$$\frac{n(n-1)k^2}{(n-2)^2k^2} = \frac{528}{400}$$



$$\Rightarrow \frac{n(n-1)}{(n-2)^2} = \frac{33}{25}$$

$$\Rightarrow \frac{n^2-n}{n^2-4n+4} = \frac{33}{25}$$

$$\Rightarrow 25n^2 - 25n = 33n^2 - 132n + 132$$

$$\Rightarrow 0 = 8n^2 - 107n + 132$$

$$\Rightarrow 0 = (8n-11)(n-12)$$

(OR BY THE QUADRATIC FORMULA

$$n = \frac{107 \pm \sqrt{(-107)^2 - 4\times8\times132}}{2\times8}$$

$$n = \frac{107 \pm \sqrt{7225}}{16} =$$

$$n = \frac{107 \pm 85}{16} = \begin{cases} 12 \\ \frac{11}{8} \end{cases})$$

$$\Rightarrow n = \begin{cases} 12 \\ \cancel{\frac{11}{8}} \end{cases}$$

HENCE USING $(n-2)k = 20$

$10k = 20$

$k = 2$

FINALLY $A = nk$

$A = 12\times2$

$\therefore A = 24$

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Question 68 **(*****)**

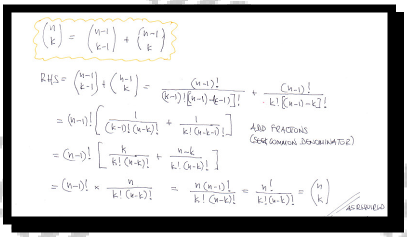
The binomial coefficients are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},\ k \in \mathbb{N},\ n \in \mathbb{N} \cup \{0\}.$$

Show directly from the above definition that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\text{RHS} = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\left[(n-1)-(k-1)\right]!} + \frac{(n-1)!}{k!\left[(n-1)-k\right]!}$$

$$= (n-1)!\left[\frac{1}{(k-1)!(n-k)!} + \frac{1}{k!(n-k-1)!}\right]$$

ADD FRACTIONS (SEEK COMMON DENOMINATOR)

$$= (n-1)!\left[\frac{k}{k!(n-k)!} + \frac{n-k}{k!(n-k)!}\right]$$

$$= (n-1)! \times \frac{n}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

AS REQUIRED

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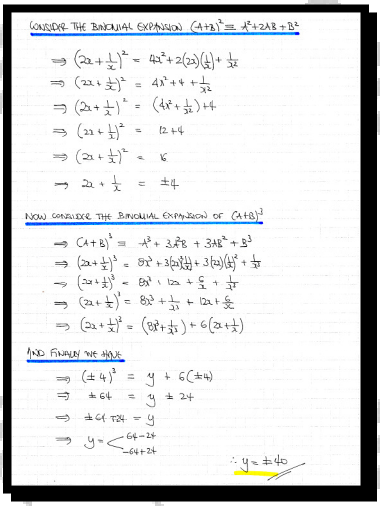
Question 69 **(*****)**

Find the value, or possible values, of y if

$$y = 8x^3 + \frac{1}{x^3} \qquad \text{and} \qquad 4x^2 + \frac{1}{x^2} = 12.$$

$\boxed{\quad}$, $\boxed{y = \pm 40}$



CONSIDER THE BINOMIAL EXPANSION $(A+B)^2 \equiv A^2 + 2AB + B^2$

$$\Rightarrow \left(2x + \frac{1}{x}\right)^2 = 4x^2 + 2(2x)\left(\frac{1}{x}\right) + \frac{1}{x^2}$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^2 = 4x^2 + 4 + \frac{1}{x^2}$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^2 = \left(4x^2 + \frac{1}{x^2}\right) + 4$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^2 = 12 + 4$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^2 = 16$$
$$\Rightarrow 2x + \frac{1}{x} = \pm 4$$

NOW CONSIDER THE BINOMIAL EXPANSION OF $(A+B)^3$

$$\Rightarrow (A+B)^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^3 = 8x^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \frac{1}{x^3}$$
$$\rightarrow \left(2x + \frac{1}{x}\right)^3 = 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^3 = 8x^3 + \frac{1}{x^3} + 12x + \frac{6}{x}$$
$$\Rightarrow \left(2x + \frac{1}{x}\right)^3 = \left(8x^3 + \frac{1}{x^3}\right) + 6\left(2x + \frac{1}{x}\right)$$

AND FINALLY WE HAVE

$$\Rightarrow (\pm 4)^3 = y + 6(\pm 4)$$
$$\Rightarrow \pm 64 = y \pm 24$$
$$\Rightarrow \pm 64 \mp 24 = y$$
$$\Rightarrow y = \begin{cases} 64 - 24 \\ -64 + 24 \end{cases}$$

$$\therefore y = \pm 40$$

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Question 70 (*****)

It is given that

$$f(x) = \sum_{r=0}^{n} \left[\binom{n}{r} x^r \left(1 + x + x^2\right)^{n-r} \right],$$

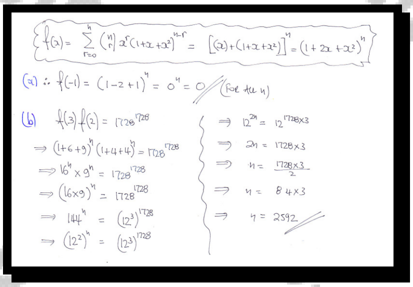
where n is a positive integer constant.

a) Evaluate $f(-1)$.

b) Find the value of n that satisfies the equation

$$f(3)\,f(2) = 1728^{1728}.$$

$\boxed{}$, $\boxed{f(-1) = 0}$, $\boxed{n = 2592}$



$$f(x) = \sum_{r=0}^{n} \binom{n}{r} x^r (1+x+x^2)^{n-r} = \left[(x) + (1+x+x^2)\right]^n = (1+2x+x^2)^n$$

(a) $\therefore f(-1) = (1-2+1)^n = 0^n = 0$ (For all n)

(b) $f(3)f(2) = 1728^{1728}$

$\Rightarrow (1+6+9)^n(1+4+4)^n = 1728^{1728}$

$\Rightarrow 16^n \times 9^n = 1728^{1728}$

$\Rightarrow (16\times9)^n = 1728^{1728}$

$\Rightarrow 144^n = (12^3)^{1728}$

$\Rightarrow (12^2)^n = (12^3)^{1728}$

$\Rightarrow 12^{2n} = 12^{1728\times3}$

$\Rightarrow 2n = 1728\times3$

$\Rightarrow n = \frac{1728\times3}{2}$

$\Rightarrow n = 864\times3$

$\Rightarrow n = 2592$

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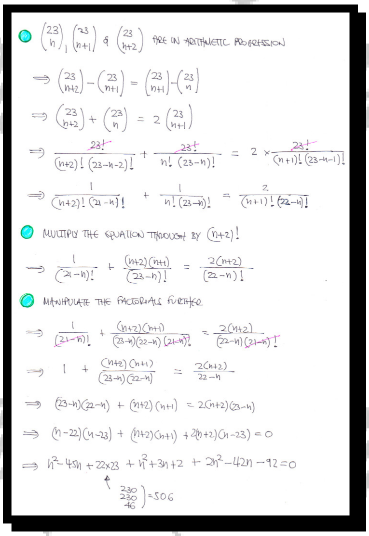
Question 71 (*****)

The coefficients of x^n, x^{n+1} and x^{n+2} in the binomial expansion of $(1+x)^{23}$ are in arithmetic progression.

Determine the possible values of n.

[illegible], $\boxed{n = 8,\ 13}$



● $\binom{23}{n}, \binom{23}{n+1}$ & $\binom{23}{n+2}$ ARE IN ARITHMETIC PROGRESSION

$$\Rightarrow \binom{23}{n+2} - \binom{23}{n+1} = \binom{23}{n+1} - \binom{23}{n}$$

$$\Rightarrow \binom{23}{n+2} + \binom{23}{n} = 2\binom{23}{n+1}$$

$$\Rightarrow \frac{\cancel{23!}}{(n+2)!\,(23-n-2)!} + \frac{\cancel{23!}}{n!\,(23-n)!} = 2 \times \frac{\cancel{23!}}{(n+1)!\,(23-n-1)!}$$

$$\Rightarrow \frac{1}{(n+2)!\,(21-n)!} + \frac{1}{n!\,(23-n)!} = \frac{2}{(n+1)!\,(22-n)!}$$

● MULTIPLY THE EQUATION THROUGH BY $(n+2)!$

$$\Rightarrow \frac{1}{(21-n)!} + \frac{(n+2)(n+1)}{(23-n)!} = \frac{2(n+2)}{(22-n)!}$$

● MANIPULATE THE FACTORIALS FURTHER

$$\Rightarrow \frac{1}{\cancel{(21-n)!}} + \frac{(n+2)(n+1)}{(23-n)(22-n)\cancel{(21-n)!}} = \frac{2(n+2)}{(22-n)\cancel{(21-n)!}}$$

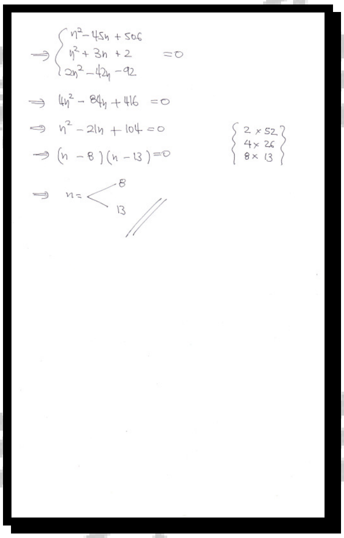
$$\Rightarrow 1 + \frac{(n+2)(n+1)}{(23-n)(22-n)} = \frac{2(n+2)}{22-n}$$

$$\Rightarrow (23-n)(22-n) + (n+2)(n+1) = 2(n+2)(23-n)$$

$$\Rightarrow (n-22)(n-23) + (n+2)(n+1) + 2(n+2)(n-23) = 0$$

$$\Rightarrow n^2 - 45n + 22\times 23 + n^2 + 3n + 2 + 2n^2 - 42n - 92 = 0$$

$$\begin{matrix}230\\230\\46\end{matrix}\Bigg\} = 506$$



$$\Rightarrow \left\{\begin{matrix} n^2 - 45n + 506 \\ n^2 + 3n + 2 \\ 2n^2 - 42n - 92 \end{matrix}\right. = 0$$

$$\Rightarrow 4n^2 - 84n + 416 = 0$$

$$\Rightarrow n^2 - 21n + 104 = 0 \qquad \left\{\begin{matrix} 2\times 52 \\ 4\times 26 \\ 8\times 13 \end{matrix}\right\}$$

$$\Rightarrow (n-8)(n-13) = 0$$

$$\Rightarrow n = \left\langle\begin{matrix} 8 \\ 13 \end{matrix}\right. \quad //$$

Question 72 (*****)

It is given that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=k^n,$$

where n and k are positive integer constants.

a) By considering the binomial expansion of $(1+x)^n$, determine the value of k.

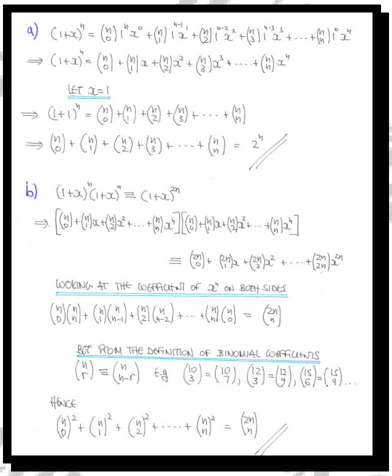
b) By considering the coefficient of x^n in

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

simplify fully

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$

$\boxed{\quad\quad}$, $\boxed{k=2}$



a) $(1+x)^n = \binom{n}{0}1^n x^0 + \binom{n}{1}1^{n-1}x^1 + \binom{n}{2}1^{n-2}x^2 + \binom{n}{3}1^{n-3}x^3 + \ldots + \binom{n}{n}1^0 x^n$

$\Rightarrow (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n$

LET $x=1$

$\Rightarrow (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$

$\Rightarrow \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = 2^n$ //

b) $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$

$\Rightarrow \left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n\right]\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n\right]$

$\equiv \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{2n}x^{2n}$

LOOKING AT THE COEFFICIENT OF x^n ON BOTH SIDES

$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$

BUT FROM THE DEFINITION OF BINOMIAL COEFFICIENTS

$\binom{n}{r} \equiv \binom{n}{n-r}$ e.g. $\binom{10}{3}=\binom{10}{7}$, $\binom{12}{3}=\binom{12}{9}$, $\binom{15}{6}=\binom{15}{9}\ldots$

HENCE

$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$ //

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Question 73 (*****)

It is given that

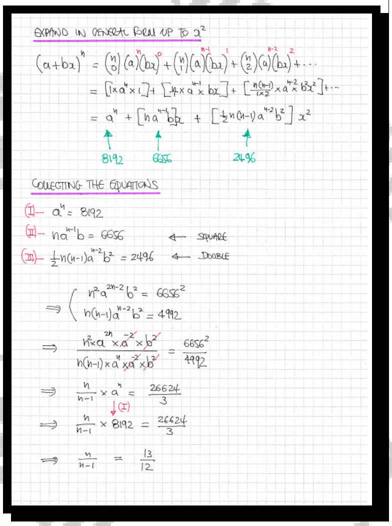
$$(a+bx)^n = 8192 + 6656x + 2496x^2 + \ldots ,$$

where a, b and n are non zero constants.

Use algebra to determine the values of a, b and n.

No credit will be given to solutions by inspection and/or verification

$\boxed{[a,b,n] = \left[2,\frac{1}{8},13\right]}$



EXPAND IN GENERAL FORM UP TO x^2

$(a+bx)^n = \binom{n}{0}(a)^n(bx)^0 + \binom{n}{1}(a)^{n-1}(bx)^1 + \binom{n}{2}(a)^{n-2}(bx)^2 + \cdots$

$= [1 \times a^n \times 1] + [n \times a^{n-1} \times bx] + \left[\frac{n(n-1)}{1\times 2} \times a^{n-2} \times b^2x^2\right] + \cdots$

$= \underset{\uparrow\,8192}{a^n} + \underset{\uparrow\,6656}{[na^{n-1}b]}x + \underset{\uparrow\,2496}{\left[\frac{1}{2}n(n-1)a^{n-2}b^2\right]}x^2$

COLLECTING THE EQUATIONS

(I) – $a^n = 8192$

(II) – $na^{n-1}b = 6656$ ← SQUARE

(III) – $\frac{1}{2}n(n-1)a^{n-2}b^2 = 2496$ ← DOUBLE

$\Rightarrow \left(\begin{array}{l} n^2a^{2n-2}b^2 = 6656^2 \\ n(n-1)a^{n-2}b^2 = 4992 \end{array} \right.$

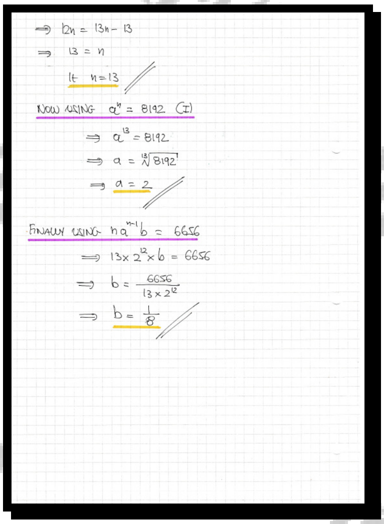
$\Rightarrow \frac{n^2 \times a^{2n} \times \cancel{a^{-2}} \times \cancel{b^2}}{n(n-1) \times a^n \times \cancel{a^{-2}} \times \cancel{b^2}} = \frac{6656^2}{4992}$

$\Rightarrow \frac{n}{n-1} \times a^n = \frac{26624}{3}$

$\downarrow$ (I)

$\Rightarrow \frac{n}{n-1} \times 8192 = \frac{26624}{3}$

$\Rightarrow \frac{n}{n-1} = \frac{13}{12}$



$\Rightarrow 12n = 13n - 13$

$\Rightarrow 13 = n$

$\therefore n = 13$ //

NOW USING $a^n = 8192$ (I)

$\Rightarrow a^{13} = 8192$

$\Rightarrow a = \sqrt[13]{8192}$

$\Rightarrow a = 2$ //

FINALLY USING $na^{n-1}b = 6656$

$\Rightarrow 13 \times 2^{12} \times b = 6656$

$\Rightarrow b = \frac{6656}{13 \times 2^{12}}$

$\Rightarrow b = \frac{1}{8}$ //

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Question 74 (*****)

It is given that

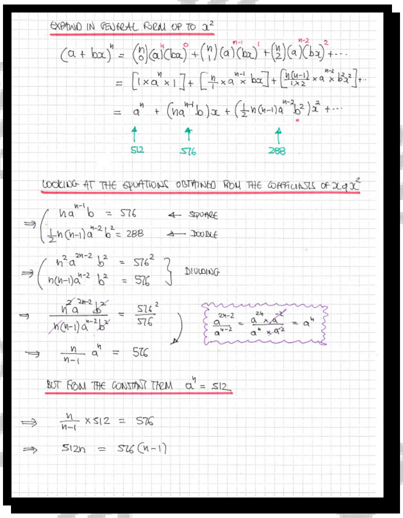
$$(a+bx)^n = 512 + 576x + 288x^2 + \dots ,$$

where a, b and n are non zero constants.

Use algebra to determine the values of a, b and n.

No credit will be given to solutions by inspection and/or verification

$$\boxed{[a,b,n] = \left[2, \frac{1}{4}, 9\right]}$$



EXPAND IN GENERAL FORM UP TO x^2

$$(a+bx)^n = \binom{n}{0}(a)^n(bx)^0 + \binom{n}{1}(a)^{n-1}(bx)^1 + \binom{n}{2}(a)^{n-2}(bx)^2 + \dots$$

$$= \left[1 \times a^n \times 1\right] + \left[\frac{n}{1} \times a^{n-1} \times bx\right] + \left[\frac{n(n-1)}{1\times 2} \times a^{n-2} \times b^2x^2\right] + \dots$$

$$= \underset{512}{\underset{\uparrow}{a^n}} + \underset{576}{\underset{\uparrow}{(na^{n-1}b)}}x + \underset{288}{\underset{\uparrow}{\left(\tfrac{1}{2}n(n-1)a^{n-2}b^2\right)}}x^2 + \dots$$

LOOKING AT THE EQUATIONS OBTAINED FROM THE COEFFICIENTS OF x & x^2

$$\Rightarrow \left(\begin{array}{l} na^{n-1}b = 576 \quad \leftarrow \text{SQUARE} \\ \frac{1}{2}n(n-1)a^{n-2}b^2 = 288 \quad \leftarrow \text{DOUBLE} \end{array}\right.$$

$$\Rightarrow \left(\begin{array}{l} n^2a^{2n-2}b^2 = 576^2 \\ n(n-1)a^{n-2}b^2 = 576 \end{array}\right\} \text{DIVIDING}$$

$$\Rightarrow \frac{n^{\cancel{2}}a^{2n-2}\cancel{b^2}}{\cancel{n}(n-1)a^{n-2}\cancel{b^2}} = \frac{576^2}{576}$$

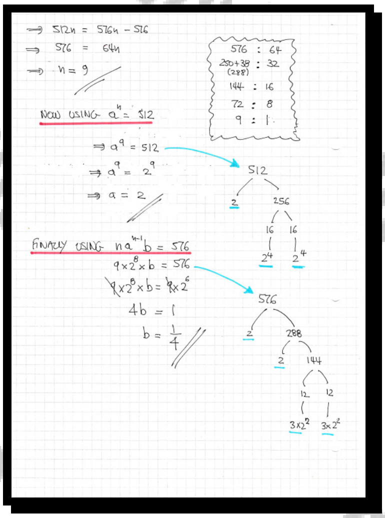
$$\boxed{\frac{a^{2n-2}}{a^{n-2}} = \frac{a^{2n} \times \cancel{a^{-2}}}{a^n \times \cancel{a^{-2}}} = a^n}$$

$$\Rightarrow \frac{n}{n-1}a^n = 576$$

BUT FROM THE CONSTANT TERM $a^n = 512$

$$\Rightarrow \frac{n}{n-1} \times 512 = 576$$

$$\Rightarrow 512n = 576(n-1)$$



$$\Rightarrow 512n = 576n - 576$$

$$\Rightarrow 576 = 64n$$

$$\Rightarrow n = 9$$

$$\boxed{\begin{array}{c} 576 : 64 \\ \underset{(288)}{250+38} : 32 \\ 144 : 16 \\ 72 : 8 \\ 9 : 1 \end{array}}$$

NOW USING $a^n = 512$

$$\Rightarrow a^9 = 512$$

$$\Rightarrow a^9 = 2^9$$

$$\Rightarrow a = 2$$

512

2 256

16 16

2^4 2^4

FINALLY USING $na^{n-1}b = 576$

$$9 \times 2^8 \times b = 576$$

$$\cancel{9} \times 2^8 \times b = \cancel{9} \times 2^6$$

$$4b = 1$$

$$b = \frac{1}{4}$$

576

2 288

2 144

12 12

3×2^2 3×2^2

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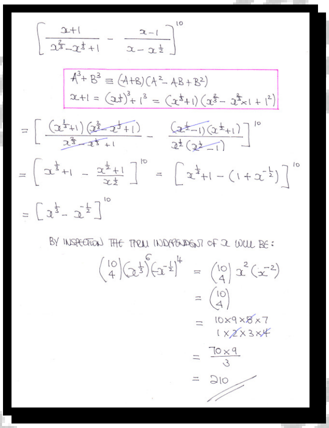
Question 75 (*****)

Find the term independent of x in the expansion of

$$\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}} \right)^{10}.$$

$\boxed{}$, $\boxed{210}$



$$\left[\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}} \right]^{10}$$

$$\boxed{\begin{array}{l} A^3 + B^3 \equiv (A+B)(A^2 - AB + B^2) \\ x+1 = (x^{\frac{1}{3}})^3 + 1^3 = (x^{\frac{1}{3}}+1)(x^{\frac{2}{3}} - x^{\frac{1}{3}} \times 1 + 1^2) \end{array}}$$

$$= \left[\frac{(x^{\frac{1}{3}}+1)\cancel{(x^{\frac{2}{3}} - x^{\frac{1}{3}}+1)}}{\cancel{x^{\frac{2}{3}} - x^{\frac{1}{3}}+1}} - \frac{\cancel{(x^{\frac{1}{2}}-1)}(x^{\frac{1}{2}}+1)}{x^{\frac{1}{2}}\cancel{(x^{\frac{1}{2}}-1)}} \right]^{10}$$

$$= \left[x^{\frac{1}{3}} + 1 - \frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}} \right]^{10} = \left[x^{\frac{1}{3}} + 1 - (1 + x^{-\frac{1}{2}}) \right]^{10}$$

$$= \left[x^{\frac{1}{3}} - x^{-\frac{1}{2}} \right]^{10}$$

BY INSPECTION THE TERM INDEPENDENT OF x WILL BE:

$$\binom{10}{4}(x^{\frac{1}{3}})^6(-x^{-\frac{1}{2}})^4 = \binom{10}{4} x^2 (x^{-2})$$

$$= \binom{10}{4}$$

$$= \frac{10 \times 9 \times \cancel{8} \times 7}{1 \times \cancel{2} \times 3 \times \cancel{4}}$$

$$= \frac{70 \times 9}{3}$$

$$= 210$$

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Question 76 (*****)

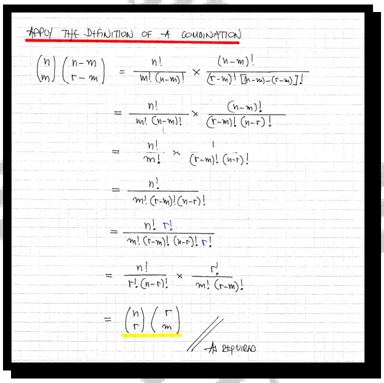
The binomial coefficients are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \ k \in \mathbb{N}, \ n \in \mathbb{N} \cup \{0\}.$$

Show directly from the above definition that if $n \geq r \geq m$, then

$$\binom{n}{m}\binom{n-m}{r-m} = \binom{n}{r}\binom{r}{m}.$$

$\boxed{}$, $\boxed{\text{proof}}$



APPLY THE DEFINITION OF A COMBINATION

$$\binom{n}{m}\binom{n-m}{r-m} = \frac{n!}{m!(n-m)!} \times \frac{(n-m)!}{(r-m)!\,[(n-m)-(r-m)]!}$$

$$= \frac{n!}{m!(n-m)!} \times \frac{(n-m)!}{(r-m)!\,(n-r)!}$$

$$= \frac{n!}{m!} \times \frac{1}{(r-m)!\,(n-r)!}$$

$$= \frac{n!}{m!\,(r-m)!\,(n-r)!}$$

$$= \frac{n!\,r!}{m!\,(r-m)!\,(n-r)!\,r!}$$

$$= \frac{n!}{r!\,(n-r)!} \times \frac{r!}{m!\,(r-m)!}$$

$$= \binom{n}{r}\binom{r}{m} \quad //$$

AS REQUIRED